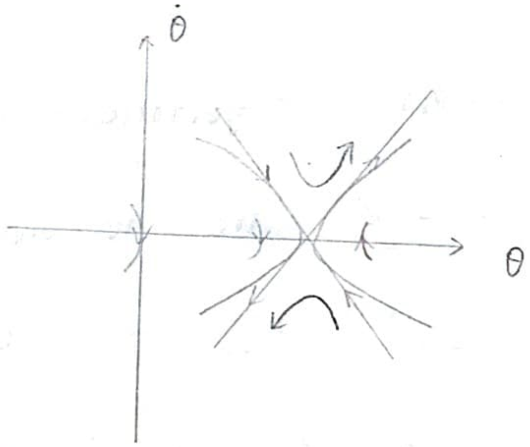
Today: Acrobot & Cart - Pole II (Swing - Up)

- Stability analysis w/ Lyapunov functions + control design
- Enery shaping
- Partial Feedback Linearization



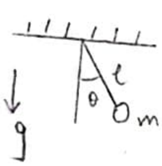
pendulumn linearlzation at fixed point

- Throw - back to "hand - designed" control

Are not going to be optimal

Better (in practice, in many cases)

Stability of the damped pendulum



$$ml^2 \ddot{\theta} + mgl \sin\theta = -b\dot{\theta} \quad , \; b > 0$$

$\theta \to 0$ as $t \to \infty$

How do we prove it ?

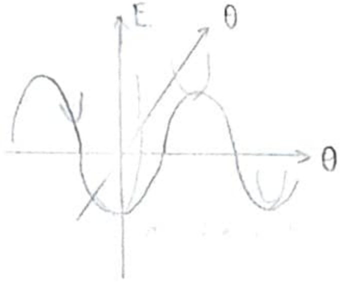
Total energy E

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

$$\frac{d}{dt}E = ml^2\dot\theta\ddot\theta + mgl\sin\theta\,\dot\theta = \dot\theta(-b\dot\theta - \cancel{mgl\sin\theta}) \cancel{+\dot\theta mgl\sin\theta}$$

$$= -b\dot\theta^2 < 0 \quad \text{when} \quad \dot\theta \neq 0$$

also know $E \geq -mgl$



this correct ?

Key idea: Look at solns (trajectories) of the system

$\theta = k\pi$. $k \in \mathbb{Z}$ are the only

<u>invariant sets</u> with $\dot E = 0$

Invariant set G $\quad X(0) \in G \Rightarrow \forall t > 0 \quad X(t) \in G$

Prove that $E \Rightarrow mgl\cos(k\pi)$

$\theta \to k\pi, \ \dot\theta \to 0$

Lyapunov functions

$\dot x = f(x)$

Prove $x = 0$ is a stable fixed point

Produce $V(x)$ $\quad \forall x \neq 0, \ V(x) > 0, \ V(0) = 0.$

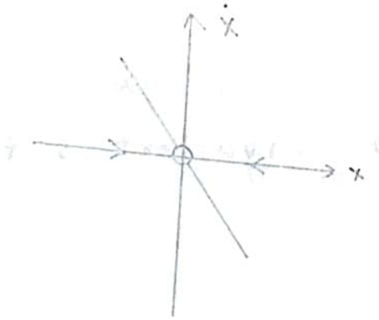
Lyapunov ~~Lyapounv~~ function. $\quad \dot V(x) = \frac{\partial V}{\partial x} f(x) \quad \forall x \neq 0 \quad \dot V(x) < 0, \ \dot V(0) = 0$

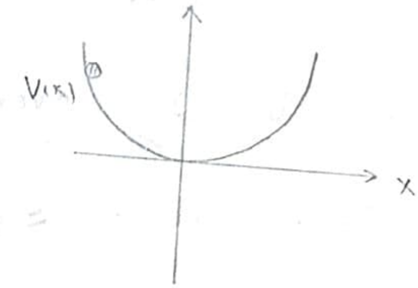
"radially unbounded" $\quad V(x) \to \infty$ as $\|x\| \to \infty$

then as $t \to \infty$, $V \to 0$, $x \to 0$.

"globally asymptotic stability"

$\dot{x} = -x$





$V(x) = x^2$

$\alpha = 0$

$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = -2x \cdot x = -2x^2 < 0 \quad \forall x \neq 0$

Global exponential stability

if

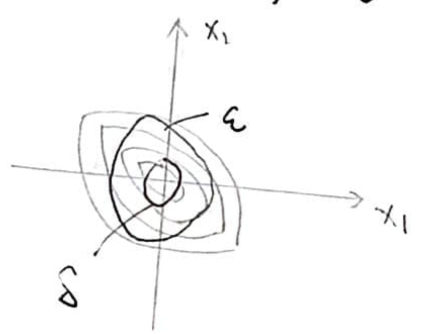
$\dot{V}(x) < -\alpha V(x), \quad \alpha > 0$

$V(x(t)) \leq V(x(0)) e^{-\alpha t}$

is ~~Lyapounv~~ Lyapunov ?

if $\dot{V}(x) \leq 0$

$V(x(0)) = c \qquad \forall t \quad V(x(t)) \leq c$



$\| x(0) - x^* \| < \delta \Rightarrow \forall t \quad \| x(t) - x^* \| < \varepsilon$.

Local stability

only $\dot{V} < 0$ in some ε - ball $\mathcal{B}$ around x^*

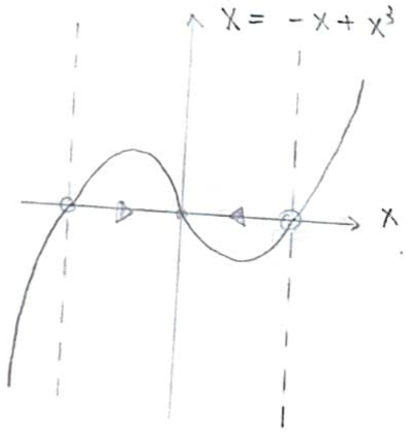
Global $\dot{V} < 0$ everywhere

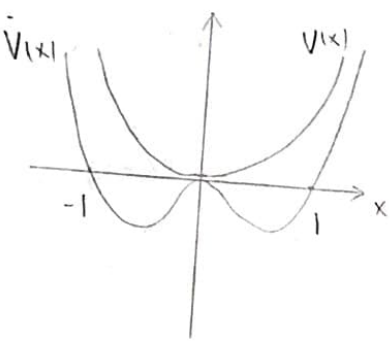
Region stability $\dot{V} < 0$ over an invariant set

e.g. $G = \{ x \neq 0 \mid V(x) \leq \rho \}$

$\uparrow$ $\rho > 0$ is level-set of the ~~Lyaponuv~~ Lyapunov function

$\Rightarrow$ G^* is ~~insi~~ inside the region of attraction of x^*





$V(x) = x^2$

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x) = 2x(-x + x^3) = -2x^2 + 2x^4$$

$$= 2x^2(x^2 - 1) < 0$$

$$\forall \underbrace{x \neq 0 \in (-1, 1)}$$

Prove $x \in (-1, 1)$ is ROA of $x^* = 0$.

La Salle's Theorem

$\dot{V}(x) \leq 0$

$\uparrow$ not $<$

implies that

$\dot{V} \to 0$ as $t \to \infty$ $\Rightarrow$ $x \to$ " largest invariant set w/ $\dot{V} = 0$ ".

the set unions ?

Relationship to D. P.

$$0 = \min_u \left[g(x,u) + \underbrace{\frac{\partial J^*}{\partial x} f(x,u)}_{\frac{dJ^*}{dt}} \right]$$

for $u^* = \pi^*(x)$

$$\frac{dJ^*}{dt} = -g(x, u^*)$$

hard $\Rightarrow$ solve PDE

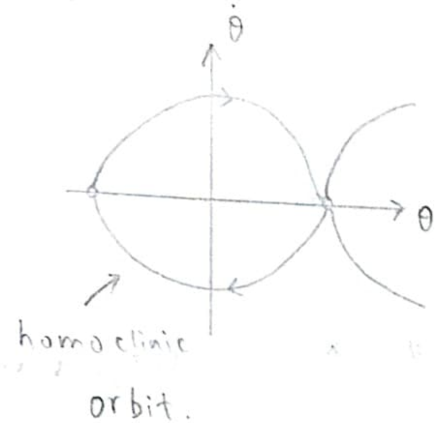
$$\frac{dV}{dt} \preceq 0$$

can often guess a simple soln.

if $g(x, u^*) > 0$ all the time

J^* is ~~Lyapounv~~ Lyapunov function

Swing-up for the pendulum



Goal: swing-up to upright. w/ small torques

Idea: Drive to homoclinic orbit

$$E^{d} = mgl. \quad \text{(}d\text{ = desired)}$$

Driven pendulum $\quad ml^2\ddot{\theta} + mgl\sin\theta = u$

$$\dot{E} = +u\dot{\theta} \qquad (\text{torque} * \text{velocity} \checkmark)$$

$$V(x) = \frac{1}{2}\left(E(x) - E^{d}\right)^2.$$

$$\dot{V}(x) = \dot{E}(x) * \underbrace{\left(E(x) - E^{d}\right)}_{\tilde{E}} = u\dot{\theta}\tilde{E}$$

Pick $\quad u = -k\dot{\theta}\tilde{E} \quad \Rightarrow \quad \dot{V}(x) = -k\dot{\theta}^2\tilde{E}^2$

$k > 0.$

not stable at the top, but apply LQR when near the top.

as long as $k > 0$ is ok, ~~but~~ k can be very small to avoid torque limit damping in the system

$$ml^2 \dot{\theta} + mgl\sin\theta + b\dot{\theta} = u$$

$$\dot{E} = u\dot{\theta} - b\dot{\theta}^2$$

$$\bar{u} = u + b\dot{\theta}$$

~~to~~ ~~ver~~ overcome the dumping energy

Note:

if we have mass wrong, controller is the same

homo clinic orbit

$$mgl = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl\cos\theta$$

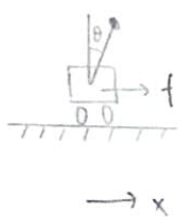
$$\dot{\theta} = \pm\sqrt{\frac{2g}{l}(1+\cos\theta)}$$

Cart - Pole

(w/ all params to 1 even g)

$$2\ddot{x} + \ddot{\theta} c - \dot{\theta} s = f$$

$$\ddot{x} c + \ddot{\theta} + s = 0$$



$$\ddot{\theta} = -\ddot{x} c - s$$

$$f = 2\ddot{x} + (-\ddot{x}c - s)c - \dot{\theta} s$$

Choose $f = \underbrace{(2-c^2)}_{\text{always invertible}} \ddot{x}^d - sc - \dot{\theta}^2 s$

$$\Rightarrow \ddot{x} = \ddot{x}^d$$

Collocated PFL

"partially feedback linearization"

$$\ddot{\theta} = -\ddot{x}^d c - s$$

Can also control $\ddot{\theta}$

Choose $f = (c - \frac{2}{c})\ddot{\theta}^d - 2\tan\theta - \dot{\theta}^2 s$

$\Rightarrow \ddot{\theta} = \ddot{\theta}^d$ (except where $c = 0$)

noncollocated PFL

Collocated PFL: linearization on degree of freedom of the controller

noncollocated PFL: linearization on another degree of freedom of the controller