So far:

Manipulator Eqs.

Feedback Linearization

Optimal Control.

Value Iteration.

"Model's Underactuated Systems"

Cart-Pole. Systems.

Acrobot.

Pendubot.

Furuta Pendulum

Reaction-wheel Pendulum

Ball-and-Bean

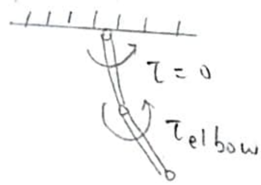
Planar VTOL

(Quadrator)

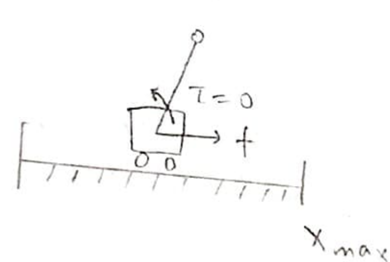
Havercraft.

Bicycle models.

Acrobot



Cart-Pole



Eqs of motion for both

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_{grav}(q) + Bu$$

Acrobot

$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ $u = \tau_{elbow}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

torque limit

$|u| \leq u_{max}$

Cart-Pole

$q = \begin{bmatrix} x_{cart} \\ \theta_{pend} \end{bmatrix}$, $u = f_{cart}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$|u| \leq u_{max}$

$\cancel{|x| \leq x_{max}}$

$|x_{cart}| \leq x_{max}$

Value iteration? moh?

Linear Quadratic Regulator $\Leftarrow$ Let's see.

LQR

$\dot{x} = Ax + Bu$ $J = \int_0^\infty dt \; x^T Q x \cancel{+ u^T Q} + u^T R u$

$\Rightarrow J^* = x^T S x$. $u = -Kx$

$\dot{x} = f(x,u)$

Local linear approximation around (x_0, u_0)

~~Ta~~

Taylor expansion.

$$\cancel{\dot{x}} \quad \dot{x} \approx \underbrace{f(x_0, u_0)}_{\dot{x}_0} + \overbrace{\left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_0 \\ u = u_0}}}^{A} (x - x_0)$$

$$+ \overbrace{\left.\frac{\partial f}{\partial u}\right|_{\substack{x = x_0 \\ u = u_0}}}^{B} (u - u_0)$$

$$\dot{x} - \dot{x}_0 = A(x - x_0) + B(u - u_0)$$

Change coordinates

$$\bar{x} = x - x_0 \qquad \bar{u} = u - u_0$$

$$\dot{\bar{x}} = x - \dot{x}_0$$

↑ want this to be zero to have a fixed coordinate ~~sy~~ system

↖ a fixed point

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = \tau$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad u = \tau$$

Linearize around $\theta = \pi$, $\dot{\theta} = 0$, $u = 0$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}(\tau - b\dot{\theta} - mgl\sin\theta) \end{bmatrix} \approx 0 + \begin{array}{cc} \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \dot{\theta}} \end{array} \left.\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & -\frac{b}{ml^2} \end{bmatrix}\right|_{\theta = \pi}$$

$$+ \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

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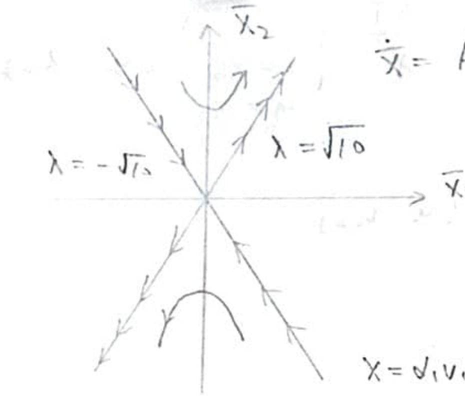
$$= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \bar{u}$$

$m = 1,\ l = 1,\ b = 0,\ g = 10$ $\qquad A = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}$



$\dot{\bar{x}} = A\bar{x}$

eigen-analysis of A $\qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Av = \lambda v$$

$$\det(A - \lambda I) = 0$$

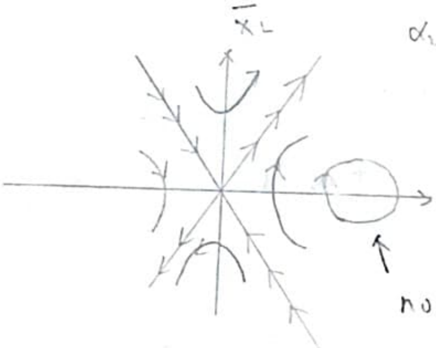
$X = \alpha_1 v_1 + \alpha_2 v_2$

$$\det \begin{bmatrix} -\lambda & 1 \\ 10 & -\lambda \end{bmatrix} = 0$$

$\dot{x} = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2$

Stability requires all $\lambda < 0$ $\quad \max(\lambda) \leq 0$

$\lambda_1 = \sqrt{10} \qquad V_1 = \begin{bmatrix} 1 \\ \sqrt{10} \end{bmatrix}$



$\lambda_2 = -\sqrt{10} \qquad V_2 = \begin{bmatrix} -1 \\ \sqrt{10} \end{bmatrix}$

non-linear system

A, B $\qquad$ ~~st~~

K, S = Linear Quadratic Regulator (A, B, Q, R)

cost matrices

K_{sys} = Linear Quadratic Regulator (system, Context, Q, R)

(x_0, u_0)

in ~~Matlab~~ matlab

[K, s] = lqr [A, B, Q, R)

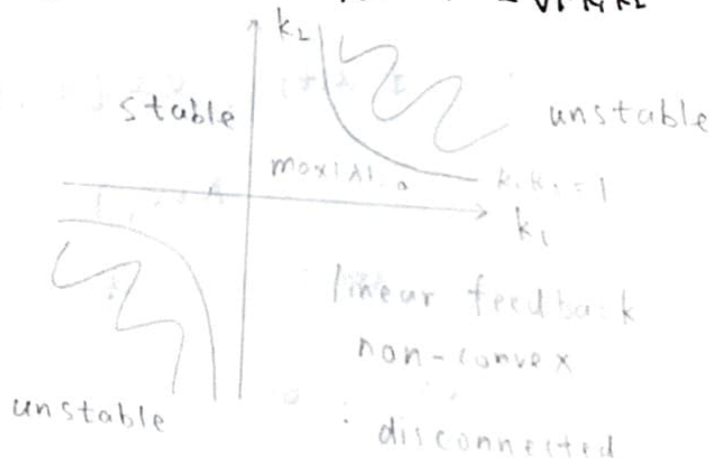
care about close loop behavior

$$\dot{x} = (A - BK)x \quad \text{is stable}$$

can be very hard to tune K directly

$$A - BK = \begin{bmatrix} -1 & k_1 \\ k_2 & -1 \end{bmatrix}$$

eigenvalues $\lambda = -1 \pm \sqrt{+k_1 k_2}$



for any $Q = Q^T \succeq 0$

$R = R^T \succ 0$

will return a stabilizing K

not every stable controller is from

LQR, Youla parameters

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Scale out the units}$$

$$R = [1]$$

Local linear approximations (good)

$\neq$

Feedback linearization (bad)

Controllability vs. Underactuated

def: controllable. if

if $\forall x(0)$ $\exists u(t)$ $t \in [t_0, t_f]$

$x(t_f) = 0$

Acrobot is underactuated

but controllable

Linearizing the manipulator eqs

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$$

Linearize around fixed pt $(q,\dot{q},u)$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}\frac{\partial \tau_g}{\partial q} & -M^{-1}C \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}$$

if linearize at $f(x_0, u_0) \neq 0$,

control is affine

$$\dot{x} = Ax + Bu + C$$

more work to find stable controllers