Lecture 20:

* Robust Constrained Control
* Polytopes and polytopic Trajectories
* Non-deterministic Systems
* Robust Controlled Invariance

Discrete time system:

$$x_{t+1} = f_t(x_t, u_t) + w_t, \qquad w_t \in W_t, \quad \forall t \in \mathbb{N}$$

ξ - trajectory $(x_0, u_0), (x_1, u_1), \ldots$

$\xi \in \Sigma$

$\Psi \subset \Sigma \rightarrow$ set of ~~trajectory~~ trajectories

$\downarrow$

acceptable trajectoryies

1) $x_t \in X, \quad u_t \in U \qquad t = 0, \ldots, T-1$

$x_T \in X_{Goal} \qquad x_{init} \rightarrow x_T$

2) $x_t \in X, \quad u_t \in U, \quad \forall t \in \mathbb{N}$

3) $(x_t, u_t) \in \mathbb{H}_t, \quad t = 0, \ldots, T-1$ joint constraints on state and control

$x_T \in X_{Goal}$

$$u_t = \pi_t(x_0, \ldots, x_t, u_0, \ldots, u_{t-1})$$

$$\pi = (\pi_t)_{t=0,1,\ldots}$$

$$\phi(x_{init}, \pi) = \{\xi \in \Sigma \mid \exists w_t \in W, t \in \mathbb{N}, x_{t+1} = f(x_t, u_t) + w_t, x_0 = x_{init}\}$$

x_{init}

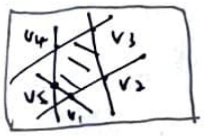
Problem : Design π

Such that $\phi(x_{init}, \pi) \subset \psi$

$$x_{t+1} = A_t x_t + B_t u_t + w_t \quad , \quad w_t \in W_t$$

H-~~Polytope~~ Polyhedron $\quad \mathbb{P} = \{ x \in \mathbb{R}^n \mid \underbrace{H}_{\in \mathbb{R}^{q \times n}} x \le \underbrace{h}_{\in \mathbb{R}^q} \}$

$\mathbb{R}^n$



$\ker(H) = \{0\} \Rightarrow$ H-polytope

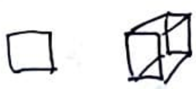
$$\text{V-polytope} = \text{Convex-hull}(\{v_1, v_2, \dots, v_N\})$$

$$= \{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{N} \lambda_i v_i \, , \, \lambda_i \ge 0 \, , \, \sum_{i=1}^{N} \lambda_i = 1 \}$$

in high-dim H-polytope preferred

solve for solution to check membership

n-dim $\quad 2n$ # surfaces



2^n # vertice

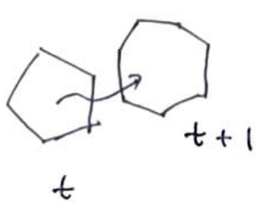
Minkowski sum

$S_1, S_2 \subset \mathbb{R}^n$

$$S_1 \oplus S_2 = \{ s_1 + s_2 \mid s_1 \in S_1 , \; s_2 \in S_2 \}$$

$$x_{t+1} = A_t x_t + B_t u_t + \underbrace{w_t}_{c_t} \qquad w_t \in W_t$$

$$W_t = \{c_t\}$$



$$\mathbb{P}_t = \bar{x}_t + G_t Q$$

$$Q = \{ q \in \mathbb{R}^n \mid H_q q \le h_q \} = [-1, 1]^{n_q}$$

$G_t \in \mathbb{R}^{n \times n_q}$ $\quad$ Zonotope $\quad \cancel{Z}$ Z $(\bar{x}, G)$

$\bar{x}_t \in \mathbb{R}^n$

Zonotopes are symmetric

$n_q \uparrow$ more flexible representation

Controller :

$$x = \bar{x}_t + G_t q \quad , \quad q \in Q$$

$$u_t = \bar{u}_t + \theta_t q$$

$$x_{t+1} = A_t x_t + B_t u_t + c_t = A_t (\bar{x}_t + G_t q) + B_t (\bar{u}_t + \theta_t q)$$

$$= A_t \bar{x}_t + B_t \bar{u}_t + c_t + (A_t G_t + B_t \theta_t) q$$

$$\mathbb{P}_{t+1} = \underbrace{(A_t \bar{x}_t + B_t \bar{u}_t + c_t)}_{\bar{x}_{t+1}} + \underbrace{(A_t G_t + B_t \theta_t)}_{G_{t+1}} Q$$

$(x_t, u_t) \in \mathbb{H}_t$

$x_T \in X_{GOAL}$

$\mathbb{L}_t = \bar{u}_t + \theta_t Q$

all possible controls

$$\{\bar{x}_\tau, \bar{u}_\tau, G_\tau, \theta_\tau\}_{\tau = 0, \dots, T-1} = \arg\min \; J$$

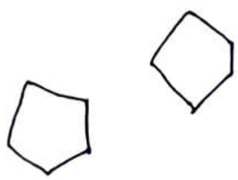
s.t. $\mathbb{P}_t \times \mathbb{L}_t \subseteq \mathbb{H}_t$

$\mathbb{P}_T \subseteq X_{goal}$

Dynamics I

non-stationary noise

$$x_{t+1} = A_t x_t + B_t u_t + w_t \quad , \quad w_t \in W_t = Z(\bar{w}_t, W_t)$$



$\mathbb{P}_{t+1}$

$\mathbb{P}_t$

$$\mathbb{P}_t = Z(\bar{x}_t, G_t)$$

$$x_t = \bar{x}_t + G_t q$$

$$u_t = \bar{u}_t + \theta_t q$$

Dynamics:

$$x_{t+1} = (A_t \bar{x}_t + B_t \bar{u}_t + \bar{w}_t) + (A_t G_t + B_t \theta_t) q + W q'$$

$$q \in [-1,1]^{n_q} \qquad q' \in [-1,1]^{n_w}$$

$\rightarrow$ $(A_t G_t + B_t \theta_t) q + W q'$

$$Z(\bar{x}_1, G_1) \oplus Z(\bar{x}_2, G_2) \quad \text{Minkoski sum}$$

$$= Z(\bar{x}_1 + \bar{x}_2, (G_1 | G_2))$$

$\rightarrow$ main point

$$\bar{x}_{t+1} = A_t \bar{x}_t + B_t \bar{u}_t + \bar{w}_t$$

$$G_{t+1} = (A_t G_t + B_t \theta_t \mid W_t)$$

$\alpha_t \in \mathbb{R}$

possible objective $\max \sum_{\tau=0}^{T} \alpha_\tau^2$

maximize possible disturbance

controller use

$$x_t = \bar{x}_t + G_t q$$

observe x_t, solve q

$$u_t = \bar{u}_t + \theta_t q \quad \swarrow \text{ plug in } u_t$$

if feedback on state

$$u_t = \bar{u}_t + K_t x_t$$

difficult to optimize for K_t same time

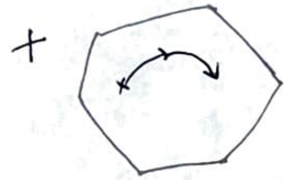
$$x_{t+1} = A_t x_t + B_t(\bar{u}_t + K_t x_t)$$

$$x_{t+2} = (A_t + B_t K_t)[(A_t + B_t K_t) x_t + B_t \bar{u}_t]$$

product $\rightarrow$ nonlinear, nonconvex other parameterization

$$u_t = f(\text{past disturbances}) \qquad w_\tau,\ \tau = 0, \dots, t-1$$

$\mathbb{R}^n$ state-space



state in the region

whatever disturbance there

$X \subseteq \mathbb{R}^n$

$U \subseteq \mathbb{R}^m$

$$x_{t+1} = f(x_t, u_t) + w_t$$

$\theta \in [\theta_{min}, \theta_{max}]$

Ω Robust Control Invariant (RCI) set

$$\forall x \in \Omega,\ \exists u \in U \quad \underline{f(x,u)} + w \in \Omega \quad \forall w \in W$$

$$\Omega = Z(\bar{x}, G) \qquad Ax + Bu$$

$$\Omega^+ = Z(A\bar{x} + B\bar{u} + \bar{w},\ (AG + B\theta \mid W))$$

want: $\Omega = \Omega^+$

trick: $AG + B\theta = 0$, $W = G$: $(0 \mid G)$, $0 = I_W$

① $\bar{x} = A\bar{x} + B\bar{u} + \bar{w}$

② $(AG + B\theta \mid W) = (0 \mid G) = G$

$0 \mid G = G$ same zanotope

Periodic

time varying $A_0, \dots, A_{T-1}$ gets back to G

all $(0 \mid G)$ in the middle

Applications

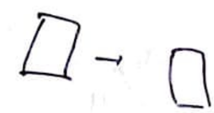
Feedback Control Synthesis for Manipulation

sticking contact / no Concontact

finger left / right

online control

subroutines needed :



1- step MPC (LP / QP)

bound nonlinear system as disturbance

$$f(x_t, u_t) - (A_t x_t + B_t u_t) \in W_t$$

→ very conservative actions