Today: Computing Lyapunov functions

Optimization crash course:

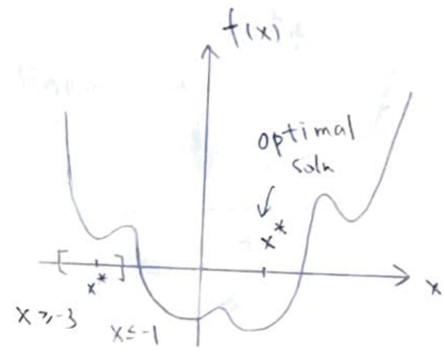
$$\min_{x} f(x)$$

Scalar objective function

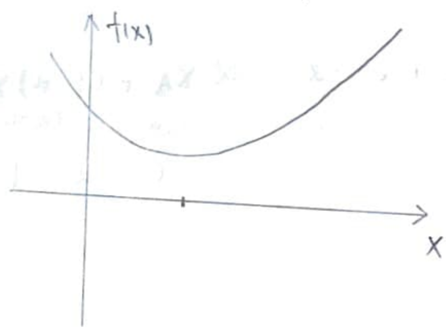
decision variables

s.t. $\forall_i \; g_i(x) \le 0$

constrains



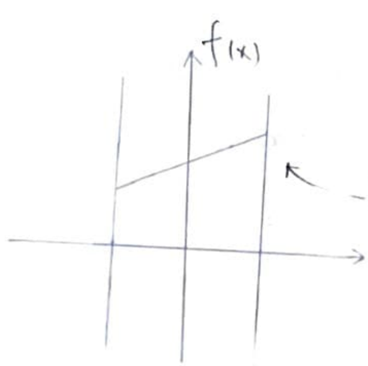
$$\min_{u} \left[g(x,u) \; \dots \right]$$



convex

quadratic cost

(least squares)

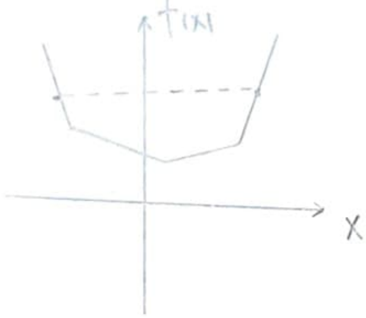


linear cost $\quad c^T x$

Linear Program

Convex optimization

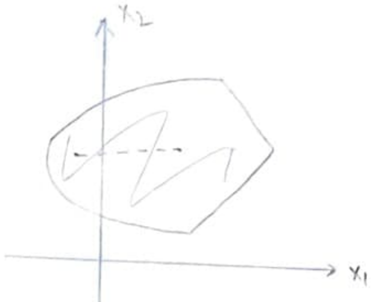
- $f(x)$ is a convex function
- $\forall i \; g_i(x)$ forms a convex set



$\forall \; x_A, x_B$

$\alpha f(x_A) + (1-\alpha) f(x_B) \geq f(\alpha x_A + (1-\alpha) x_B)$

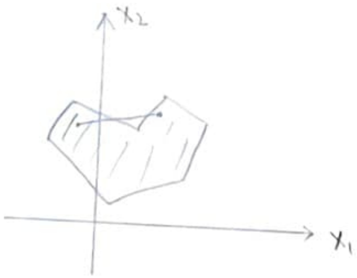
$0 \leq \alpha \leq 1$



$\forall \; x_A, x_B \in G$

$\cancel{\alpha x_A + (1-\alpha}$ $\alpha x_A + (1-\alpha) x_B \in G$

$0 \leq \alpha \leq 1$

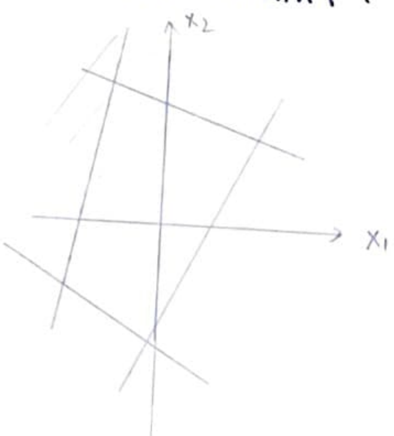


not convex

Linear Program

$\min_x c^T x$

Linear constraint : $Ax \leq \vec{b}$



intersection of linear constraints form convex set

Computing Lyapunov functions
w/ Linear Programming

Idea: Parameterize Lyapunov candidate.

$$\dot{x} = f(x)$$

$$V(x) = \sum_i \alpha_i \phi_i(x)$$ ← nolinear basis functions

$$= \alpha^T \phi(x)$$

$$\dot{V}(x) = \alpha^T \frac{\partial \phi}{\partial x} f(x).$$

find α s.t. $V(0) = 0$ $\quad \forall x \neq 0 \quad V(x) > 0$

$\dot{V}(0) = 0$ $\quad \forall x \neq 0 \quad \dot{V}(x) < 0$

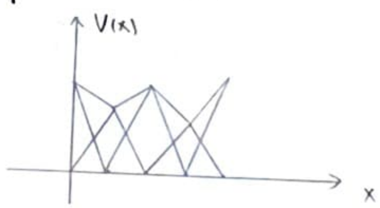
Sample many pts. x_i $\quad V(1) = 1$ ← avoid numerical error, correct scale

$\forall i$ linear constraints: $\alpha^T \phi(x_i) > 0$

linear in decision variables $\quad \alpha^T \frac{\partial \phi}{\partial x} f(x_i) < 0$

Linear program!

mesh functions:



$f(x)$ is pendulum

$$ml^2 \ddot{\theta} + b\dot{\theta} + mgl\sin\theta = 0$$

$$\phi(x) = [1, \cos(\theta), \sin\theta, \dot{\theta}, \cos^2\theta, \sin\theta\cos\theta$$

$f(x)$ is pendulum

$$ml^2 \ddot{\theta} + b\dot{\theta} + mgl\sin\theta = 0$$

$$\phi(x) = [1, \cos\theta, \sin\theta, \dot{\theta},$$
$$\cos^2\theta, \sin\theta\cos\theta, \dot{\theta}\sin\theta, \dot{\theta}\cos\theta, \dot{\theta}^2]$$

$$V(x) = \alpha^T \phi(x)$$

$$\forall x_i \neq 0 \quad V(x_i) \geq \varepsilon x_i^2$$

$$\dot{V}(x_i) \leq -\varepsilon x_i^2 \sin^2\theta$$

$$V(0) = 0, \quad \dot{V}(0) = 0.$$

Can we verify for all x?

idea: $V(x) = \sum_i \alpha_i \phi_i^2(x)$, $\alpha_i \geq 0$

(positive by construction)

Important generalization

$$V(x) = [\phi_1(x) \cdots \phi_k(x)] \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots \\ \alpha_{12} & \ddots & \\ & & \ddots \end{bmatrix}}_{Q = Q^T > 0} \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_k(x) \end{bmatrix}$$

PSD $\forall x \quad x^T Q x \geq 0$

$$\Rightarrow \forall x \quad \phi^T(x) Q \phi(x) \geq 0$$

The set of PSD matrices is a convex set

if $P_1 \geq 0$, $P_2 \geq 0 \Rightarrow \alpha P_1 + (1-\alpha) P_2 \geq 0.$

$$0 \leq \alpha \leq 1$$

$$\forall x \quad x^T P_1 x \geq 0 \quad x^T P_2 x \geq 0.$$

$$x^T P_1 x + x^T P_2 x \geq 0$$

$$\alpha x^T P_1 x + (1-\alpha) x^T P_2 x \geq 0$$

$$x^T (\alpha P_1 + (1-\alpha) P_2) x \geq 0$$

$\min c^T x$ — objective function linear to elements of X

s.t. $X \geq 0$ — semi definite programming (SDP)

X is a matrix

Lyapunov functions w/ SDP

(now $\forall x$).

$\dot{x} = Ax$ is it stable ?

$$V(x) = x^T P x.$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x$$

$$= x^T P A x + x^T A^T P x$$

$$= x^T (A^T P + P A) x$$

$$x(t) = e^{At} x(0)$$

$$\int_0^\infty x^T Q x \, dt = \int_0^\infty (e^{At} x(0))^T Q \, e^{At} x(0) \, dt$$

$$= x^T(0) \underbrace{\int_0^\infty (e^{At})^T Q (e^{At}) \, dt}_{P} \, x(0)$$

find P s.t. $P \geq 0$

$$A^T P + PA \leq 0.$$

Linear Matrix Inequality $-A^T P - PA = Q$

$$Q \geq 0$$

If the solver says there is no solution,

Certificate that the system is not stable

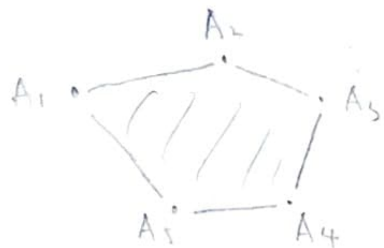
Linear program cannot provide this certificate

Common Lyapunov functions for Robust Stability

$$\dot{x} = Ax$$

elements of A are bounded ~~by~~ but uncertain

$$A = \sum_i \beta_i A_i, \quad 0 \le \beta_i \cancel{\le 1} \qquad \sum \beta_i = 1$$



find $P > 0$

P $\forall i \quad PA_i + A_i^T P < 0$

$$\Rightarrow \quad P\left(\sum_i \beta_i A_i\right) + \left(\sum_i \beta_i A_i\right)^T P < 0$$

sufficient condition to prove the system is stable for A in the set

but, for stable systems A_i, there may not exist a single Lyapunov function