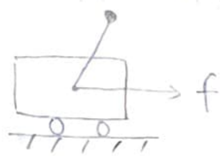
Today: Acrobat, Cart-Poles, Quadrotors III

- Partial Feedback Linearization
  - Strong inertial coupling
- Energy-shaping w/ Lyapunov
- Differential Flatness

Last time: PFL of Cart-Pole

collocated PFL



$$\ddot{x} = \ddot{x}^d$$

$$\ddot{\theta} = \underline{\text{nonlinear}}$$

Non-collocated PFL

$$\ddot{\theta} = \ddot{\theta}^d \quad \text{almost always}$$

$$\ddot{x} = \text{nonlinear}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{matrix} \leftarrow \text{passive} \\ \leftarrow \text{actuated} \end{matrix}$$

$B \in \mathbb{R}^{n \times m}$ — n: num dof, m: num actuators

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 = \tau_1(q,\dot{q})$$

$$M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 = \tau_2(q,\dot{q}) + u$$

$$T = \frac{1}{2}\dot{q}^T M(q)\,\dot{q} \qquad M(q) > 0$$

$$\Rightarrow \quad M_{11} > 0\ , \quad M_{22} > 0$$

$$\ddot{q}_1 = M_{11}^{-1}\left[\tau_1 - M_{12}\ddot{q}_2\right]$$

$$M_{21}\left(M_{11}^{-1}\left[\tau_1 - M_{12}\ddot{q}_2\right]\right) + M_{22}\ddot{q}_2 = \tau_2 + u$$

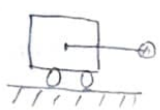
$$\underbrace{\left(M_{22} - M_{21}M_{11}^{-1}M_{12}\right)}_{\text{Schur complement} \,>\, 0}\ddot{q}_2 + M_{21}M_{11}^{-1}\tau_1 - \tau_2 = u$$

$$\ddot{q}_2 = \ddot{q}_2^{\,d}$$

$$\ddot{q}_2 = M_{12}^{\cancel{-1}\ \#}\left(\tau_1 - M_{11}\ddot{q}_1\right)$$

Pseudo-inverse $\quad \mathbb{R}^{(n-m)\times m}$

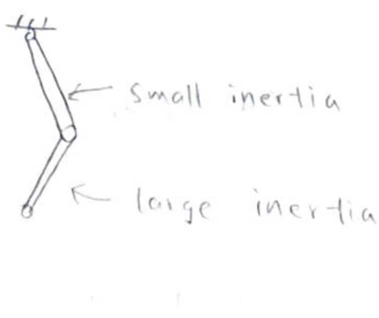
has a unique ~~solution~~ soln when $\underbrace{\mathrm{rank}(M_{12})}_{\text{can be state dependent}} = n-m$



"inertial coupling"

Example:

Acrobat, small inertia on up part, ~~targer~~ larger inertia on lower part



Task - space "output"

$$z = h(q) \;\; = -l\cos\theta$$ e.g. example

$$H = \frac{\partial h}{\partial q} \qquad H_1 = \frac{\partial h}{\partial q_1} \qquad H_2 = \frac{\partial h}{\partial q_2}$$

$$H = [H_1, H_2]$$

Theorem 7.1 - Task Space PFL

If the actuated joints are commanded so that

$$\ddot{q}_2 = \bar{H}^{+}\left[\ddot{y}^d - \dot{H}\dot{q} - H_1 M_{11}^{-1}\tau_1\right],$$

where $\bar{H} = H_2 - H_1 M_{11}^{-1} M_{12}$ and $\bar{H}^{+}$ is the right Moore - Penrose pseudo - inverse,

$$\bar{H}^{+} = \bar{H}^{T}(\bar{H}\bar{H}^{T})^{-1},$$

then we have

$$\ddot{y} = \ddot{y}^d$$

subject to

$$\operatorname{rank}(\bar{H}) = p,$$

proof sketch.

Differentiating the output function we have

$$\dot{y} = H\dot{q}$$

$$\ddot{y} = \dot{H}\dot{q} + H_1\ddot{q}_1 + H_2\ddot{q}_2$$

Solving 3.2 for the dynamics of the unactuated joints we have:

$$\ddot{q}_1 = M_{11}^{-1} (\tau_1 - M_{12} \ddot{q}_2)$$

Substituting, we have

$$\ddot{y} = \dot{H} \dot{q} + H_1 M_{11}^{-1} \tau_1 + \underbrace{(H_2 - H_1 M_{11}^{-1} M_{12})}_{\bar{H}} \ddot{q}_2$$

take $\ddot{q}_2 = \bar{H}^{+} [\ddot{y}^d - \dot{H}\dot{q} - H_1 M_{11}^{-1} \tau_1]$

$$\ddot{y} = \dot{H}\dot{q} + H_1 M_{11}^{-1} \tau_1 + \bar{H}\bar{H}^{+} [\ddot{y}^d - \dot{H}\dot{q} - H_1 M_{11}^{-1} \tau_1]$$

$$= \dot{H}\dot{q} + H_1 M_{11}^{-1} \tau_1 + \ddot{y}^d - \dot{H}\dot{q} - H_1 M_{11}^{-1} \tau_1$$

$$= \ddot{y}^d$$

Note that the last line required the rank condition that rank $(\bar{H}) = p$ be full ~~row~~ ~~column~~ row rank. the rows of $\bar{H}$ are linearly independent, allowing $\bar{H}\bar{H}^{+} = I$

$$\text{null}(\bar{H}\bar{H}^T) = \text{null}(\bar{H}^T)$$

if $\bar{H}^T x = 0$, then $\bar{H}\bar{H}^T x = 0$

if $\bar{H}\bar{H}^T x = 0$, then $\bar{H}^T x \in \text{null}(\bar{H})$

$\text{ran}(\bar{H}^T) \perp \text{null}(\bar{H})$

$\bar{H} y = 0$, $y \perp$ row vectors of $\bar{H}$

$y \perp$ column vectors of $\bar{H}^T$

if $y = \bar{H}^T x$

$$y^T y = y^T \bar{H}^T x = x^T \bar{H} y = 0$$

then $y = 0$

if $\text{ran } A \perp \text{ran } B$, $x \in \text{ran } A \cap \text{ran } B$

$$x^T x = 0, \quad x = 0$$

right pseudo inverse, full row rank, ~~each~~ rows are linearly independent

$$\ddot{q}_2 = \ddot{q}_2^d + K_p (q_2 - q_2^d) + K_d (\dot{q}_2 - \dot{q}_2^d)$$

Energy shaping for the Cart - Pole

Idea: Regulate pole to it's homoclinic orbit using collocated PFL

$$\ddot{x} = u$$

$$\ddot{\theta} = -u\underset{\cos\theta}{c} - \underset{\sin\theta}{s}$$

$$E = \frac{1}{2}\dot{\theta}^2 - \cos\theta$$

$E^d = 1$ ← homoclinic orbit of pendulum

$$V(\ ) = \frac{1}{2}(\underbrace{E - E^d}_{\tilde{E}})^2$$

$$\tilde{E} = E - E^d$$

$$\dot{\tilde{E}} = \dot{E} - \cancel{\dot{E}^d} = \dot{\theta}\ddot{\theta} + \sin\theta \cdot \dot{\theta}$$

$$= \dot{\theta}(-uc - s) + \dot{\theta} s$$

$$= -u\dot{\theta}\cos\theta$$

$$\dot{V}(x) = \tilde{E}\dot{\tilde{E}}$$

$$= (E - E^d)(-u\dot{\theta}\cos\theta)$$

$$= -u\dot{\theta}\cos\theta\,(E - E^d)$$

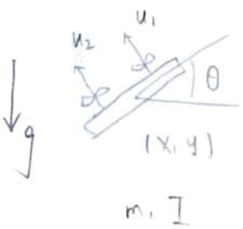
Choose $u = k(\cos\theta)\dot{\theta}\tilde{E}$

$$\Rightarrow \dot{V} = -k\dot{\theta}^2\cos^2\theta\,\tilde{E}^2$$

Regulation on cart, cart at the center middle, and zero velocity

$$V(x) = \frac{1}{2}(E - E^d)^2 + \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$$

Quadrotor has larger range where LQR can be used



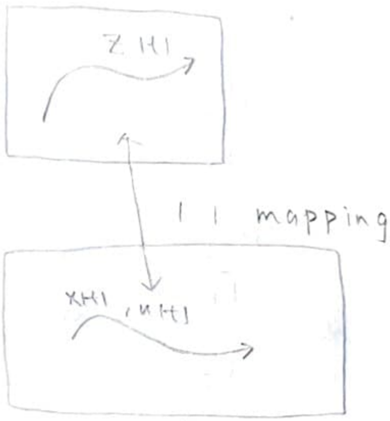
$$m\ddot{x} = -(u_1 + u_2)\sin\theta$$

$$m\ddot{y} = (u_1 + u_2)\cos\theta - mg$$

$$I\ddot{\theta} = r(u_1 - u_2)$$

Differential flatness

$$Z = h(q)$$



$$Z = \begin{bmatrix} x \\ y \end{bmatrix}$$

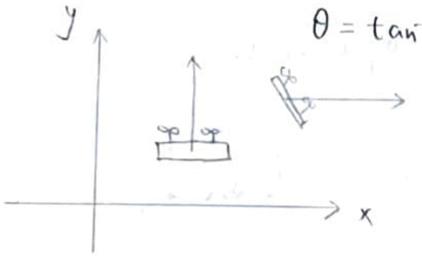
Claim : If you give me $\forall t \in [t_0, t_f]$

$Z(t) \Rightarrow x(t), u(t)$

must be four times differentiable

$$\frac{-m\ddot{x}}{m\ddot{y} + mg} = \frac{(u_1+u_2)\sin\theta}{(u_1+u_2)\cos\theta} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{-\ddot{x}}{g}\right)$$



$$\ddot{\theta} = f\left(\frac{d^2\ddot{x}}{dt^2}, \frac{d^2\ddot{y}}{dt^2}\right) = f\left(\frac{d^4x}{dt^4}, \frac{d^4y}{dt^4}\right)$$

given a dynamic system

$$\dot{x} = f(x, u)$$

$\cancel{Z = h(x, u, \dot{u}, \ldots)}$

Differential flatness in z

$$z = h(x, u, \dot{u}, \dots, \frac{d^k u}{dt^k})$$

$$x = x(z, \dot{z}, \dots, \frac{d^k z}{dt^k})$$

$$u = u(z, \dot{z}, \dots, \frac{d^k z}{dt^k})$$

3D Version

$$z = [x, y, z, \theta_{yaw}]$$

motors are not pure force, has torque or inertia coupling. ~~orther~~ otherwise cannot stabilize at some state

moment produced by the rotating props

Note:

Linearizing the manipulator equations

$$\dot{x} = f(x,u) \approx f(x^*, u^*) + \left[\frac{\partial f}{\partial x}\right]_{x=x^*, u=u^*} (x - x^*)$$

$$+ \left[\frac{\partial f}{\partial u}\right]_{x=x^*, u=u^*} (u - u^*)$$

$$f(x,u) = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left[-C(q,\dot{q})\dot{q} + \tau_g(q) + Bu\right] \end{bmatrix}$$

$$\approx A_{lin}(x - x^*) + B_{lin}(u - u^*)$$

we define $\bar{x} = x - x^*$, $\bar{u} = u - u^*$, and write

$$\dot{\bar{x}} = A_{lin}\,\bar{x} + B_{lin}\,\bar{u}$$

$$A_{lin} = \begin{bmatrix} 0 & I \\ M^{-1}\dfrac{\partial \tau_g}{\partial q} + \sum_j M^{-1}\dfrac{\partial B_j}{\partial q} u_j & 0 \end{bmatrix}_{x=x^*,\ u=u^*}$$

note $$\frac{\partial}{\partial q}\left\{ M^{-1}(q)\left[\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q}\right]\right\}$$

$$= \frac{\partial}{\partial q} M^{-1}(q) \cdot \left[\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q}\right]$$

$$+ M^{-1}(q)\left[\frac{\partial \tau_g(q)}{\partial q} + \frac{\partial B(q)}{\partial q} u - \frac{\partial C(q,\dot{q})}{\partial q}\cdot\dot{q}\right]$$

at equilibrium point, we have $\dot{q} = 0$ and

$$\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q} = 0.$$

$$\frac{\partial}{\partial q}\left\{ M^{-1}(q)\left[\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q}\right]\right\}$$

$$= M^{-1}(q)\left(\frac{\partial \tau_g(q)}{\partial q} + \frac{\partial B(q)}{\partial q} u\right)$$

$$\frac{\partial}{\partial \dot{q}} \left\{ M^{-1}(q) \left[\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q} \right] \right\}$$

$$= -M^{-1}(q) \left(\frac{\partial C(q,\dot{q})}{\partial \dot{q}} \dot{q} + C(q \cdot \dot{q}) I \right]$$

when $\dot{q} = 0$, $C(q,\dot{q}) = 0$ ~~at~~

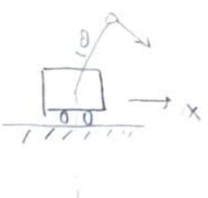
$$\frac{\partial}{\partial \dot{q}} \left\{ M^{-1}(q) \left[\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q} \right] \right\} = 0$$

take derivative with respect to u

$$\frac{\partial}{\partial u} \left\{ M^{-1}(q) \left[\tau_g(q) + B(q)u - C(q,\dot{q})\dot{q} \right] \right\}$$

$$= M^{-1}(q) B(q)$$

Manipulator equation of cart-pole system



$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[(\dot{x} + \dot{\theta} l \cos\theta)^2 + (\dot{\theta} l \sin\theta)^2 \right]$$

$$U = mgl\cos\theta$$

$$L = T - U$$

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2 (\dot{x} + \dot{\theta} l \cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 (\dot{x} + \dot{\theta} l \cos\theta) l \cos\theta + m_2 \dot{\theta} l \sin\theta \cdot l \sin\theta$$

$\cancel{\partial t}$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \theta} = m_2 (\dot{x} + \dot{\theta} l \cos\theta)(-\dot{\theta} l \sin\theta) + m_2 \dot{\theta} l \sin\theta \, \dot{\theta} l \cos\theta - m_2 g l \sin\theta$$

$$= -m_2 \dot{x} \dot{\theta} l \sin\theta - m_2 \dot{\theta}^2 l^2 \sin\theta\cos\theta + m_2 \dot{\theta}^2 l^2 \sin\theta \cos\theta - m_2 g l \sin\theta$$

$$= -m_2 (\dot{x}\dot{\theta} + g) l \sin\theta$$

Lagrange equation

$$(m_1+m_2)\ddot{x} + m_2 \ddot{\theta} l \cos\theta - m_2 l \dot{\theta}^2 \sin\theta = f$$

$$m_2 \ddot{x} l \cos\theta + m_2 l^2 \ddot{\theta} + m_2 g l \sin\theta = 0$$

$$M(q) = \begin{bmatrix} m_1 + m_2 & m_2 l \cos\theta \\ m_2 l \cos\theta & m_2 l^2 \end{bmatrix} \qquad C(q,\dot{q}) = \begin{bmatrix} 0 & -m_2 l \dot{\theta} \sin\theta \\ 0 & 0 \end{bmatrix}$$

$$\tau_g(q) = \begin{bmatrix} 0 \\ -m_2 g l \sin\theta \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\ddot{x} = \frac{1}{m_c + m_p \sin^2\theta} \left[f + m_p \sin\theta \,(l\dot{\theta}^2 + g\cos\theta) \right]$$

$$\ddot{\theta} = \frac{1}{l(m_c + m_p \sin^2\theta)} \left[-f\cos\theta - m_p l \dot{\theta}^2 \cos\theta \sin\theta - (m_c + m_p) g \sin\theta \right]$$

$$A_{lin} = \begin{bmatrix} 0 & I \\ M^{-1}\dfrac{\partial \tau_g}{\partial q} + \sum_j M^{-1}\dfrac{\partial B_j}{\partial q}u_j & 0 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(m_1+m_2)m_2 l^2 - m_2^2 l^2 \cos^2\theta}\begin{bmatrix} m_2 l^2 & -m_2 l\cos\theta \\ -m_2 l\cos\theta & m_1+m_2 \end{bmatrix}$$

$$= \frac{1}{(m_1 + m_2 \sin^2\theta)m_2 l^2}\begin{bmatrix} m_2 l^2 & -m_2 l\cos\theta \\ -m_2 l\cos\theta & m_1+m_2 \end{bmatrix}$$

$$\frac{\partial \tau_g}{\partial q} = \begin{bmatrix} 0 & 0 \\ 0 & -m_2 g l\cos\theta \end{bmatrix}$$

$$\frac{\partial B}{\partial q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{lin} = \begin{bmatrix} 0 & I \\ M^{-1}\dfrac{\partial \tau_g}{\partial q} & 0 \end{bmatrix}$$

$$M^{-1}\frac{\partial \tau_g}{\partial q} = \frac{1}{(m_1+m_2\sin^2\theta)m_2 l^2}\begin{bmatrix} m_2 l^2 & -m_2 l\cos\theta \\ -m_2 l\cos\theta & m_1+m_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -m_2 g l\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & m_2^2 g l^2\cos^2\theta \\ 0 & -(m_1+m_2)m_2 g l\cos\theta \end{bmatrix}\frac{1}{(m_1+m_2\sin^2\theta)m_2 l^2}$$

$$= \begin{bmatrix} 0 & \dfrac{m_2 g\cos^2\theta}{m_1+m_2\sin^2\theta} \\ 0 & -\dfrac{(m_1+m_2)g\cos\theta}{(m_1+m_2\sin^2\theta)l} \end{bmatrix}$$

$$B_{lin} = \begin{bmatrix} 0 \\ M^{-1} B \end{bmatrix}_{x = x^*,\ u = u^*}$$

$$M^{-1} B = \frac{1}{(m_1 + m_2 \sin^2\theta)\, m_2 l^2} \begin{bmatrix} m_2 l^2 & -m_2 l \cos\theta \\ -m_2 l \cos\theta & m_1 + m_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{m_1 + m_2 \sin^2\theta} \begin{bmatrix} 1 \\ -\frac{\cos\theta}{l} \end{bmatrix}$$

Note we linearize near $x = 0$, $\theta = \pi$

$$M^{-1} \frac{\partial \tau_g}{\partial q} = \begin{bmatrix} 0 & \frac{m_2}{m_1} g \\ 0 & \frac{m_1 + m_2}{m_1} \frac{g}{l} \end{bmatrix}$$

$$M^{-1} B = \begin{bmatrix} \frac{1}{m_1} \\ -\frac{\cos\theta}{m_1 l} \end{bmatrix}$$