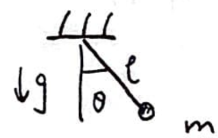
Lecture 2: Nonlinear Dynamics

- Definitions
- Graphical Analysis
- Goal Build intuition (Love dynamics)

The Simple Pendulum



"simple" ⇒ point mass

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = -mgl\cos\theta$$

$$L = T - U$$

$$ml^2\ddot{\theta} + mgl\sin\theta = Q = \underbrace{-b\dot{\theta}}_{\text{damping}} + u$$

$$ml\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$$

Given $\theta(0)$, $\dot{\theta}(0)$

Solve $\theta(t)$

What happens as $t \to \infty$

Does it ever enter ~~a failure state~~ some region?

$$\phi(0) \geq 0$$

6 Tools:
1) Linearization

2) Graphical Analysis

Even simpler $u = 0$

b large.

$b\dot{\theta} \gg ml^2\ddot{\theta}$

$ml^2\ddot{\theta}$ is $\frac{kg\ m^2}{s^2}$ "dimensional analysis"

b $\frac{kg\ m^2}{s^2}$

$b\,0 \gg ml^2$

s ?

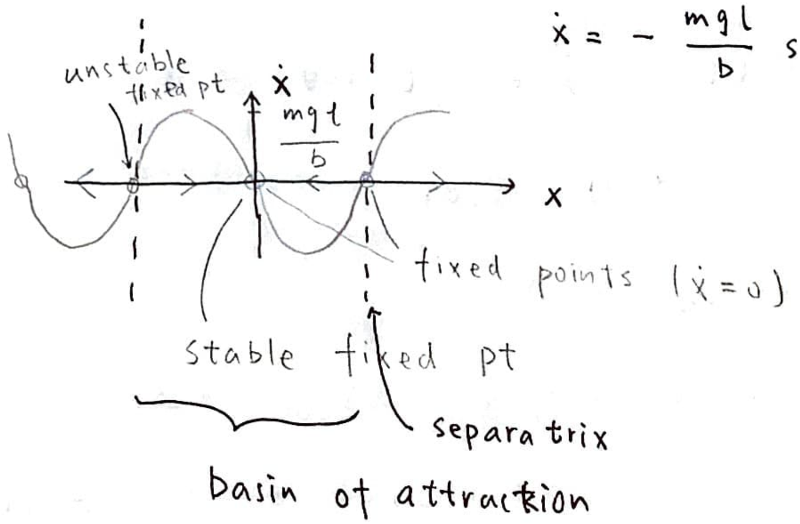
$b\sqrt{\frac{l}{g}} \gg ml^2$

$ml^2\ddot{\theta} + b\dot{\theta} \approx b\dot{\theta}$

$b\dot{\theta} + mgl\sin\theta = 0$

$b\dot{x} + mgl\sin x = 0 \qquad x \in [0, 2\pi]$

$\dot{x} = -\frac{mgl}{b}\sin x$



"regions"

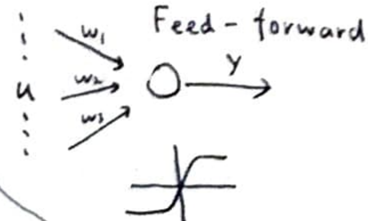
Autapse (shallow neural network)

RNN (recurrent neural net)

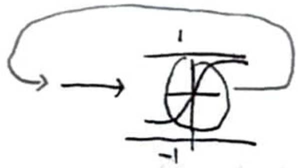
$y = f(u)$

$\dot{x} = f(x, u)$

$y = g(x, u)$

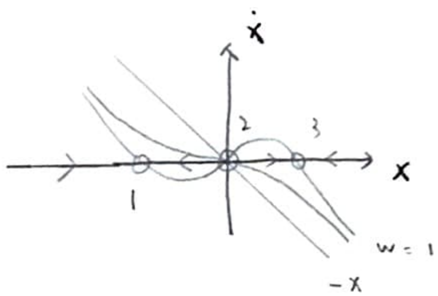


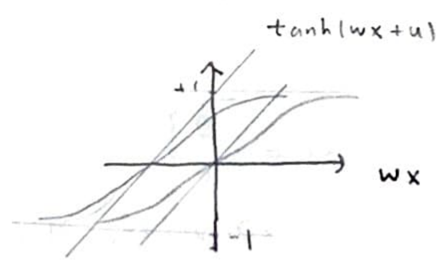
$\dot{x} = -x + \tanh(w \cdot x)$



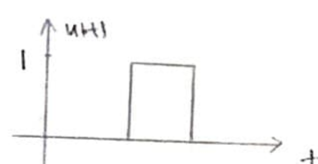
Autapse

Long Short-term Memory (LSTM)





$\dot{x} = -x + \tanh(wx + u)$



turn on, 1 disappears, 3 does not change much, states move to 3

turn off. stable at 3

turn on different direction, 3 disappears, states go to 1

8 "Forget gate"

$$\dot{x} = (f-1)x + \tanh(wx+u)$$

$\uparrow$

$f \in \{0,1\}$

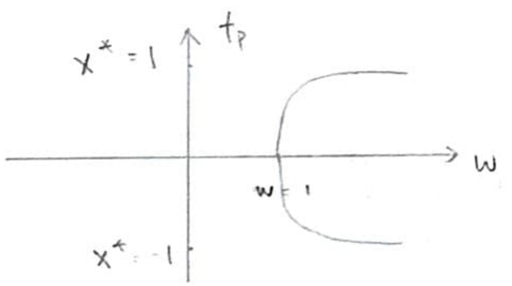
if $f=1$, fixed points 1 and 3 ~~dis app~~ disappear

only $\tanh(wx+u)$

Tracking fixed points

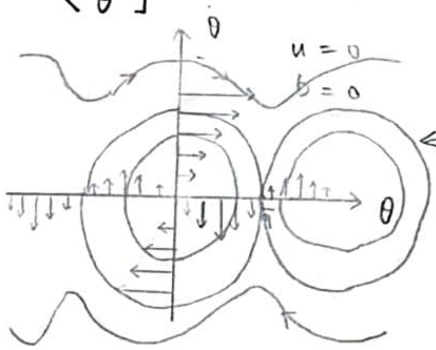
"bifurcation analysis"

$$\dot{x} = -x + \tanh(wx)$$



x^* for fixed pt

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \dot{x} = f(x) = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$



2D vector field

homoclinic Orbit

"heteroclinic"

A fixed pt x^* is locally Stable in the sense of Lyapunov (is L.)

if for any $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\|x(0) - x^*\|^2 < \delta$

then $\forall t > 0$ $\|x(t) - x^*\|^2 < \varepsilon$

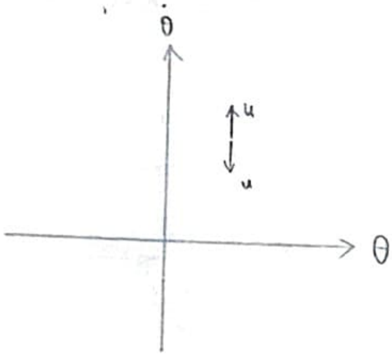
Asymptotically stable if $\| x(0) - x^* \| < \varepsilon$,

$$\lim_{t \to \infty} \| x(t) - x^* \| = 0$$

Exponentially stability $\| x(t) - x^* \| \leq C e^{-\alpha t}$, $C, \alpha > 0$

With fraction, often not asymptotically stability

nor exponentially stability



control changes

the vector field

with vector on the second

axis $\dot{\theta}$