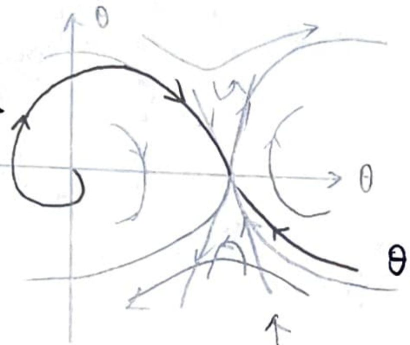
# Today: Trajectory Stabilization

Idea: Linearize along a trajectory

trajectory optimization →




$\theta_0 = \pi,\ \dot{\theta}_0 = 0,\ u_0 = 0$

linear near fixed point

$\theta_0 = -\frac{3\pi}{4}$

$\dot{\theta} = 1$

$u = 1$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}\left[u - b\dot{\theta} - mgl\sin\theta\right] \end{bmatrix} \quad \begin{matrix} m = l = 1 \\ b = 0 \\ g = 10 \end{matrix} = \begin{bmatrix} \dot{\theta} \\ u - 10\sin\theta \end{bmatrix}$$

$$\dot{x} \approx f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_0 \\ u = u_0}} (x - x_0) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x_0 \\ u = u_0}} (u - u_0)$$

$$= \begin{bmatrix} \dot{\theta}_0 \\ u_0 - 10\sin\theta_0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos\theta_0 & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_{1,2} = \pm\sqrt{-10\cos\theta_0} \qquad V_{1,2} = \begin{bmatrix} 1 \\ \pm\sqrt{-10\cos\theta_0} \end{bmatrix}$$

$\theta_0 = \pi$ $\quad \sqrt{-10\cos\theta_0}$ $\quad$ max

$\theta_0 \to \frac{\pi}{2}$ $\quad \sqrt{-10\cos\theta_0} \searrow$ $\quad$ vector flats

$\frac{\pi}{2} > \theta_0 > 0$ $\quad$ imaginary eigenvectors

Moving coordinate system

$x_0(t), \ V_0(t)$ ( e.g. from trajectory optimization )

consistent w/ $\dot{x}_0(t) = f(x_0(t), V_0(t))$

$$\bar{x} = x - x_0(t) \qquad \bar{u}(t) = u - u_0(t)$$

$$\dot{x} = \overset{\dot{x}_0(t)}{\cancel{f(x_0, u_0)}} + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0)$$

$$\dot{\bar{x}} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_0(t) \\ u = u_0(t)}}}_{A(t)} \bar{x} + \underbrace{\frac{\partial f}{\partial u}}_{B(t)} \bar{u}$$

Time-vary Linearization

$$\dot{\bar{x}} = A(t)\,\bar{x} + B(t)\,\bar{u}$$

$$\bar{x} \to 0 \Rightarrow x \to x_0(t)$$

Time-varying LQR

$$\min_{\bar{u}(\cdot)} \int_0^{\infty / t_f} \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u} \, dt$$

infinite time or not $\qquad Q = Q^T \geq 0, \ R = R^T > 0$

$$J(\bar{x}, t) = \bar{x}^T S(t)\, \bar{x}$$

$S(t) = S^T(t) > 0$ $\qquad \bar{x}^T \dot{S}(t)\, \bar{x}$

$$0 = \min_{\bar{u}} \left[ g(\bar{x}, \bar{u}) + \frac{\partial J}{\partial \bar{x}} f(\bar{x}, \bar{u}) + \frac{\partial J}{\partial t} \right]$$

$$\bar{u} = -R^{-1} B(t)\, S(t)\, \bar{x} = -K(t)\, \bar{x}$$

to compute $S(t)$, makes it consistent

$$-\dot{S}(t) = S(t) A(t) + A^T(t) S(t) \cancel{- S(t) B(t)}$$

$$- S^T(t) B(t) R^{-1} B^T(t) S(t) + Q$$

$$S(t_f) = ?$$

put a cost ~~of~~ on final position

$$\min_{\bar{u}(\cdot)} \int_0^{t_f} \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u} + \bar{x}^T(t_f)\, Q_f\, \bar{x}(t_f)$$

then

$$S(t_f) = Q_f$$

another version

turn on LQR near the fixed point

$$\min_{\bar{u}(\cdot)} \int_0^{t_f} \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\, dt + \int_{t_f}^{\infty} \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\, dt$$

$$\dot{x} = Ax + Bu$$

$$S(t_f) = Q_f = S$$

$S$ is from LQR at $t_f$

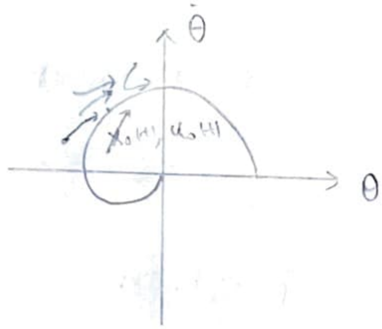
Integrate previous equation backward in time

have $S(t)$. have $K(t)$ for control

$$-\dot{S}(t) = S(t)A(t) + A^T(t)S(t)$$
$$- S^T(t)B(t)R^{-1}B^T(t)S(t) + Q$$

differential ~~Ritta~~ Riccati Eq

beautiful in continuous case




cost to minimize to state at clock, may not optimal policy to go

unconstrained Linear → LQR

+ Linear constraints → Linear Model-Predictive Control

$$\min_{u[\cdot]} \sum x^T Q x + u^T R u$$

s.t. $x[n+1] = A[n]x[n] + B[n]u[n]$

$x[0] = x_0$

Direct transcript, use $x[n]$ as decision variables

still linear constraints.

⇒ Q.P.

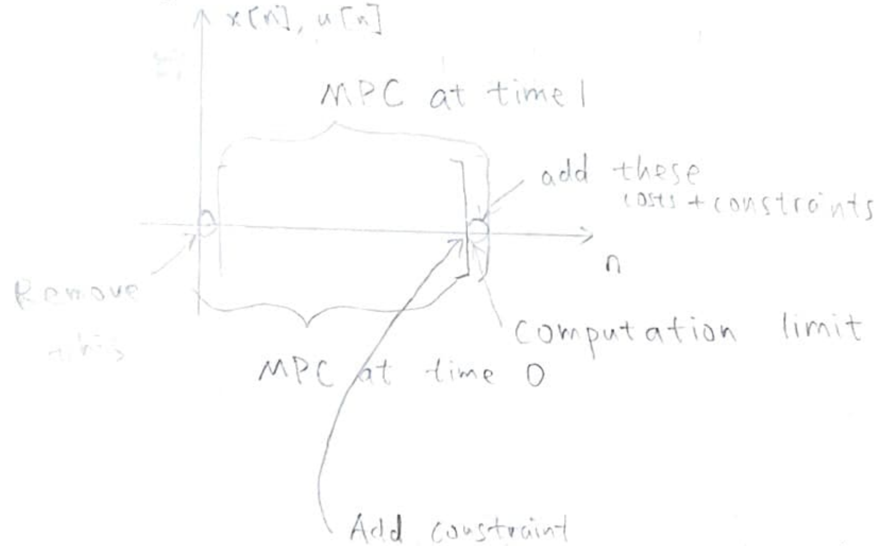
TVLQR - locally stabilizes $x_0(t)$, $u_0(t)$

LMPC - locally stabilize constrained system (if you get the details right)

Can't solve QP for $N \to \infty$

Need finite horizon

Receding horizon control




$x[N] = x_G$

- Guard againt infeasibility
- Cost added at end smaller than the initial cost that was re removed

$$V(x,k) = \min_u \sum_{n=k}^{k+N} x^T Q x \quad \text{s.t.} \ \dots$$

↑ constraints

① stay at $x_G$ is feasible

(need argument)

② add cost is 0

$$V(x, k+1) \leq V(x,k)$$

stability is guaranteed

Iterative LQR

(for trajectory optimization)

Initial guess $x_0(t),\ u_0(t)$

Stabilize w/ LQR

but using quadratic approx. of nonlinear cost

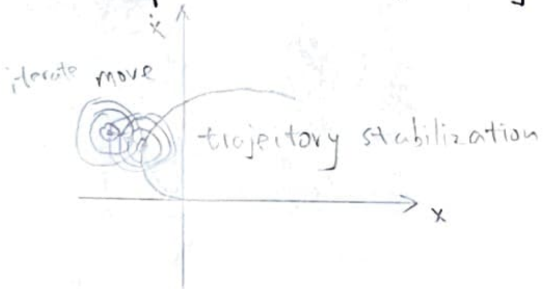
$$g(x,u) \approx g(x_0, u_0) + \frac{\partial g}{\partial x}(x - x_0) + \frac{\partial g}{\partial u}(u - u_0)$$

$$+ (x - x_0)^T \frac{\partial^2 g}{\partial x^2}(x - x_0) + \dots$$

the trajectory stabilization problem becomes

$$\min_u \int_0^T x^T Q x + q^T x + q_0 + u^T R u + r^T u + r_0 + u^T P x \, dt$$

Stablization center is not at ( $x_0(t)$, $u_0(t)$ ) somewhere close to optimal trajectory with respect to $g(x,u)$




works reasonable, converge with good speed

there exists $J(\bar{x}, t) = \bar{x}^T S(t) \bar{x}$ that is a Lyapunov function candidate, but not sure. Can perform sum of squares optimization to decide if it is stable, create funnel from Lyapunov function?

Dimensionless Analysis

- Bird or plane ...
  - with mass $m$, wing area $S$, operating in a fluid with density $\rho$
  - ~~with~~ which requires a distance $x$ to slow from $V_0$ to $V_f$
- Distance - averaged drag coefficient, $C_D$:

$$\langle C_D \rangle = \frac{2m}{\rho S x} \ln\left(\frac{V_0}{V_f}\right)$$

Perching problem

open loop does not work

LQR - Trees

no one trajectory, many trajectories fill in the space

but $v \to 0$. controllability $\downarrow$