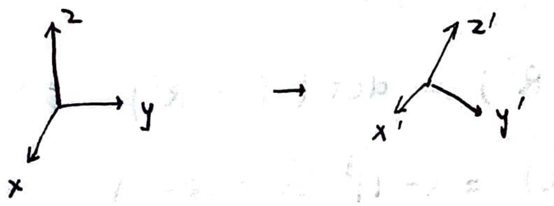
Euler's rotation theorem

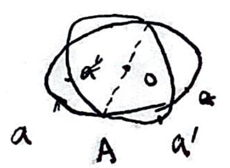
(Theorem) in 3D, any displacement of a rigid body that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through that fixed point.

Geometric intuition

if we find two points remain fix, each point should remain fixed after rotation



we imagine xOy and x'Oy' planes have intersection



then point A in x'y'z' frame is mapped from a* in the frame xyz, A is mapped to a' in x'y'z'

then we find a point O on the sphere such that its distance is equal to a, A, a'

then the rigid aOA on sphere should be mapped to rigid AOa', point O does not change

Matrix proof

consider a rotation matrix, it must map a set of basis to another basis, then it must be orthogonal matrix, for matrix R, we have $R^T R = RR^T = I$

we want to show, $\exists v \in \mathbb{R}^3$, such that $Rv = v$, which means R has eigenvalue $\lambda = 1$

we first note $\det(R^T R) = \det(R^T)\det(R) = \det(R)^2 = \det(I)$

then $\det(R) = \pm 1$, we have $\det(R) = 1$ for rotation matrix (right hand property)

$$\det(R - I) = \det(R^T) \cdot \det(R - I)$$

$$= \det(R^T R - R^T) = \det(I - R^T)$$

$$= \det(I - R) = (-1)^3 \det(R - I)$$

then we have

$$\det(R - I) = 0$$

there exists n such that $Rn = n$

Rigid body motion

in $\mathbb{R}^3$, rigid body is a set of points such that

$$d(x, y) = c$$

in its motion

We want to find the set of functions such that

this property is preserved

$$d(f(x), f(y)) = d(x, y)$$

then

$$(f(x) - f(y))^T (f(x) - f(y)) = (x-y)^T (x-y)$$