Lecture 19: Model Systems w/ Stochasity

So far $\dot{x} = f(x,u)$ $\quad x[n+1] = f(x[n], u[n])$

$\dot{x} = f(x,u,w)$ $\quad x[n+1] = f(x[n], u[n], w[n])$

$w(t)$, $w[n]$ output of some random process

$w[n]$

- disturburnces
- model uncertainty

also known as process noise

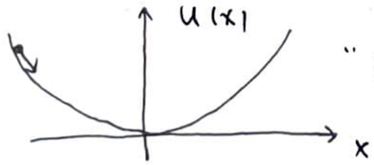
distinct from measurement noise

Additive noise:

$$x[n+1] = f(x[n], u[n]) + B_w w[n]$$

Ex. Particle in a bowl

w/ Brownian motion



"potential well"

$$U(x) = \frac{1}{2}\alpha x^2$$

$$x[n+1] - x[n] = -\left.\frac{\partial U}{\partial x}\right|_{x[n]} + w[n]$$

$$\frac{\partial U(x)}{\partial x} = \alpha x$$

$$x[n+1] = (1-\alpha)x[n] + w[n]$$

$w = 0$ fixed pt $x^* = 0$

Exp stable for $0 < \alpha < 2$

$w[n]$ is Gaussian white noise (iid)

$$E[w[n]] = 0 \qquad E[w[i]w[j]] = \begin{cases} \sigma^2 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Given $x[0]$ what is $x[n]$? what is $x[\infty]$?

$x[n]$ is a random variable

described by $P_{x[n]}(\cdot)$ density function

$$P_w(w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{w^2}{2\sigma^2}}$$

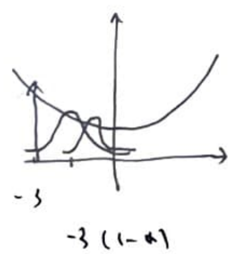
$$P_{x[n+1]|x[n],w[n]}\left(x[n+1] \mid x[n], w[n]\right)$$

$$= P_{x[n+1]|x[n]}\left(x[n+1] \mid x[n]\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-(1-\alpha)x)^2}{2\sigma^2}}$$

"master eq"

$$P_{x[n+1]}(y) = \int_{-\infty}^{\infty} P_{x[n+1]|x[n]}(y \mid x)\, P_{x[n]}(x)\, dx$$

Gaussian $\leftarrow$ Gaussian + if Gaussian



$x[0] = -3$

$P_{x[0]}(x) = \delta(x+3)$

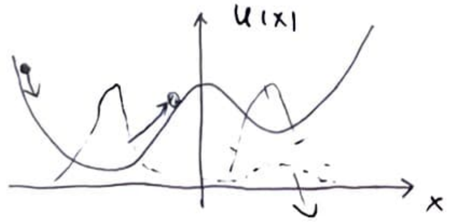
$P_{x[1]}$ Gaussian centered $-3(1-\alpha)$

distribution goes to a fixed point

Steady state distribution $P_{x^*}(\,)$

$$\mu^* = 0, \quad (\sigma^*)^2 = \frac{\sigma^2}{2\alpha - \alpha^2}$$

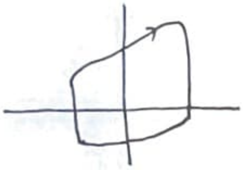
Nonlinear dynamics



"escape attempts"

final distribution

thought experiment:



Van der Por oscillator with Gaussian noise

close to even on steady cycle

rimless wheel on rough terrian

$\alpha[n] \sim$ Gaussian

Apex-to-apex

Poincare map

a range of possible next state

stationary distribution is standing still

no transition out, prob accumulates

"meta-stable" distribution

long-living transients in $P_{x[n]}(\ )$

"mean first passage time" to another region

stability may not be a good metric

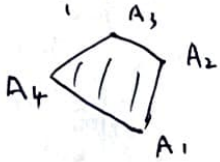
1) Worst case design / analysis

Typically for bounded uncertainty

$P_w(\ )$ has finite domain

e.g. via common Lyapunov function

$\dot{x} = Ax \qquad A = \sum_i \beta_i A_i \qquad \sum_i \beta_i = 1, \quad \beta_i \geq 0$



for all w $\quad \dot{V} < 0$ $\quad$ Can be conservative

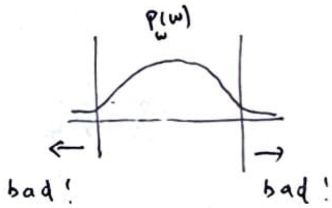
2) Stochastic control

with my objective on $P_{x[n]}(\,)$

most common is expected performance

Risk - aware control
(- constraint)

"chance constraints"



Expected Value

$$J = \sum_{n=0}^{\infty} g(x,u) \qquad \min_{u(\cdot)} E[J] = \sum_{n=0}^{\infty} E[g(x,u)]$$

now J is a random variable $\displaystyle = \sum_{n=0}^{\infty} \left[\int_x g(x,u) P_x(x)\, dx \right]$

compass gait $\rightarrow$ J with stochasity
policy is more robust

LQR problem $\quad J$ may not converge when sum to ∞
if never stabilize ~~around~~ around goal

Stochastic LQR

$$x[n+1] = Ax[n] + Bu[n] + B_w w[n]$$

$\uparrow$ non-zero
Gaussian iid

$$\min_u E\left[\sum_{n=0}^{\infty} x^T Q x + u^T R u \right]$$

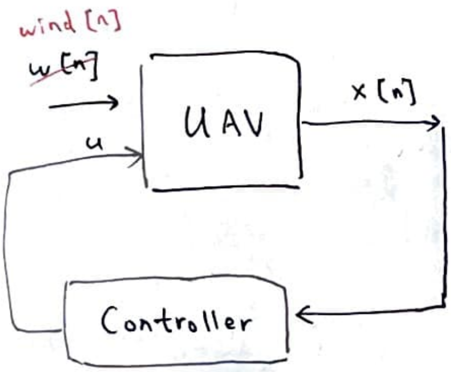
$u = -Kx$ $\quad$ when is the same K as deterministic LQR but control different for non-linear dynamics

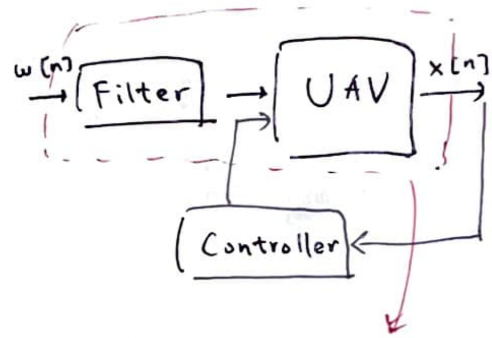
$J = \infty \quad x^T S x + c$ ↗ $\sum_{n=0}^{\infty} E[w[n]\, w^T[n]]$

↑ original S

← stochastic changes control law

Ex. A UAV in wind





system with i.i.d. input

do stochastic LQR

wind is not Gaussian iid

$w[n] \rightarrow \boxed{\text{Filter}} \rightarrow \text{wind}[n]$

Gaussian   low pass ↖ "whitening filter"

$u = -Kx$

↑

low pass filter has state

No guaranteed margins for LQG Regulators

John C. Doyle 1978 paper