# Lyapunov functions

KYP YouTube

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## The system

Consider the time-invariant system

$$\dot{x} = f(x)$$

where $f: \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz

$x = 0 \in \mathbb{D}$ is an equilibrium point of the system.

## Lyapunov function candidate

Let $V: \mathbb{D} \to \mathbb{R}$ be a continuously differentiable ($C^1$) function

The derivative of $V$ along the system trajectories is

$$\dot{V} = \frac{dV(x)}{dt} = \frac{dV(x)}{dx} f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

## Definition (Lyapunov function)

$V$ is a Lyapunov function for $x = 0$ iff

i) $V$ is $C^1$

ii) $\left.\begin{array}{l} V(0) = 0 \\ V(x) > 0 \quad \text{in } \mathbb{D} \setminus \{0\} \end{array}\right\}$ $V$ is positive definite in $D$

iii) $\dot{V}(0) = 0$

$\dot{V}(x) \leq 0$ in $\mathbb{D} \setminus \{0\}$ $\Rightarrow$ Minimum at the origin

$V$ is positive definite in $D$

$\Downarrow$

level surfaces closed near the equilibrium point

iii) $\dot{V}(0) = 0$

$\left.\begin{array}{l} \dot{V}(0) = 0 \\ \dot{V}(x) \leq 0 \quad \text{in } \mathbb{D} \setminus \{0\} \end{array}\right\}$ $\dot{V}$ is negative semi-definite in $\mathbb{D}$

$\uparrow$

$f(x)$

If, moreover,

$\left.\begin{array}{l} \dot{V}(0) = 0 \\ \dot{V}(x) < 0 \quad \text{in } \mathbb{D} \setminus \{0\} \end{array}\right\}$ $\dot{V}$ is negative definite in $\mathbb{D}$

then $V$ is a strict Lyapunov function for $x = 0$.

Lyapunov's direct method

Theorem 4.1

If $\exists$ Lyapunov function for $x=0$, then $x=0$ is stable

If $\exists$ strict Lyapunov function for $x=0$, then $x=0$ is ~~asympt~~ asymptotically stable

# How to apply Lyapunov's direct method

1) Choose a Lyapunov function ~~cand~~ candidate $V(x)$
   - Electrical / mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2} x^T P x$
     - $V(x) = \frac{1}{2} ( x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 )$
     - some methods exist for choosing $V(x)$

2) Determine whether $V(x)$ is a Lyapunov function a strict Lyapunov function for the equilibrium point.

3) If the answer is yes:

   The equilibrium point is stable / asymptotically stable

   If the answer is no:

# Application of Lyapunov's direct method

Pendulum without friction

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{g}{l} \sin x_1$

1) $V(x) = V_{pot} + V_{kin}$

$= \int_0^{x_1} -\frac{g}{l} \sin y \, dy + \frac{1}{2} x_2^2$

$= \frac{g}{l} (1 - \cos x_1) + \frac{1}{2} x_2^2$

2) $V(x) = \frac{g}{l} (1 - \cos x_1) + \frac{1}{2} x_2^2$

i) $V$ is $C^1$

ii) choose domain that only contains one local minima

$\mathbb{D} = \{ x \in \mathbb{R}^2 : |x_1| < 2\pi \}$

iii) $\dot{V}(x) = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2$

$= \frac{g}{l} \sin x_1 \cdot x_2 + x_2 \cdot -\frac{g}{l} \sin x_1 = 0$

$\forall x \in \mathbb{R}^2$

$V : \mathbb{D} = \{ x \in \mathbb{R}^2 : |x_1| < 2\pi \} \to \mathbb{R}$ is a Lyapunov function for $x = 0$

$\frac{dV}{dx} \neq 0$

$\frac{dV}{dt}$ trajectory is closed curve. derivative along this trajectory is 0

Pendulum with friction

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m} x_2 \qquad m = 1$$

1) $V(x) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x_2^2$

2) i) $C^1$

ii) $V(x)$ is possitive definite in $\mathbb{D} = \{x \in \mathbb{R}^2 : |x_1| < 2\pi\}$

iii) $\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$

$$= \frac{g}{l}\sin x_1 \cdot x_2 + x_2\left(-\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\right)$$

$$= -\frac{k}{m}x_2^2$$ not negative definite

$\dot{V}$ negative semidefinite in $\mathbb{D}$

$\dot{V}(0) = 0$

$\dot{V}(x) \leq 0 \qquad \forall x \in \mathbb{D}\setminus\{0\}$

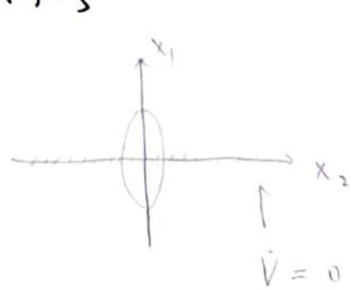
$\dot{V}$ negative definite in $\mathbb{D}$

$\dot{V}(0) = 0$

$\dot{V}(x) < 0 \qquad \forall x \in \mathbb{D}\setminus\{0\}$

$\dot{V}(0) = 0$

$\dot{V}(x_1, 0) = 0 \quad \forall x_1$




$\dot{V}(x)$ is negative semidefinite in $\mathbb{D}$

Th 4.1

$\Rightarrow$ $x = 0$ is stable

Lecture 3 part 4

How do we analyze the Lyapunov stability properties?

- Definitions
  - If we have solution $x(t) = \dots$ OK
- Phase plane analysis (dim $x = 2$)
  - Phase portrait
  - Local analysis

can be generalized

Phase portrait of linearized system $\overset{Re(\lambda) \neq 0}{\Longleftrightarrow}$ Local phase portrait of nonlinear system

- New method: Lyapunov's indirect method (dim $x = n$)

Theorem 4.7 (Lyapunov's indirect method)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \quad f: \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n \text{ is } C^1$$

1) Linearize the system about $x = 0$, $\dot{x} = Ax$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=0}$$

2)

2) Find the eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$

3) a) $\forall i \quad Re(\lambda_i) < 0 \Rightarrow x = 0$ is locally asymptotically stable

b) $\exists i \quad Re(\lambda_i) > 0 \Rightarrow x = 0$ is unstable

c) $\forall i \quad Re(\lambda_i) \leq 0$, $\exists i \quad Re(\lambda_i) = 0 \Rightarrow$ No conclusion

Corollary 4.3

$\forall i \quad Re(\lambda_i) < 0 \Rightarrow x = 0$ is locally exponentially stable

Comments

+ Simple to use

÷ Not always conclusive

÷ Only <u>local</u> results

Example

Given

$$\dot{x} = ax - x^3.$$

Analyze the ~~stab~~ stability properties of the equilibrium point $x = 0$ using Lyapunov's indirect method.

(if $a>0$, another two at $x = \sqrt{a}$, $x = -\sqrt{a}$)

$$\frac{\partial f}{\partial x} = a - 3x^2 \Big|_{x=0} = a$$

Eigenvalue $\lambda = a$

if $a > 0$, ~~sta~~ unstable

it $a < 0$, stable

if $a = 0$, no conclusion

$f : \mathbb{R}^1 \to \mathbb{R}^1 \quad C^1$

1) Linearize about $x = 0$

$$\frac{\partial f}{\partial x}\Big|_{x=0} = a - 3x^2\Big|_{x=0} = a$$

$$\dot{x} = ax$$

2) $\lambda = a$

3) $a < 0$ $x = 0$ is (locally) asymptotically stable

$a > 0$ $x = 0$ is unstable

$a = 0$ $x = 0$ is ?

$\dot{x} = -x^3$ no conclusion

Corollary 4.3, Sec. 4.7

Let $x = 0$ be an equilibrium point for

$\dot{x} = f(x)$ $f : D \to \mathbb{R}^n$ is $C^1$

$\underbrace{\forall i \ \mathrm{Re}(\lambda_i) < 0}_{A \text{ is Hurwitz}} \iff x = 0$ is (locally) ~~exponent~~ exponentially stable

$\dot{x} = ax - x^3$ $\quad a < 0 \quad x = 0$ is (locally) exponentially stable

$a = 0$

$\dot{x} = -x^3 \quad x = 0$ cannot be exponentially stable

Example

Given $\dot{x} = -x^3$

$\dot{x} = ax - x^3$ $\quad a < 0 \quad x = 0$ is locally exponentially stable

$a > 0 \quad x = 0$ is unstable

$a = 0$ no conclusion

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov direct method

1) $V(x) = \frac{1}{2}x^2 \qquad V = \frac{1}{2}x^T P x \qquad P = I$

Continuously differentiable

$x =$ velocity $v$

$F = -v^3$

$E = E_{kin} = \frac{1}{2}v^2$

2) $V(0) = 0 \qquad V(x) > 0 \qquad x \neq 0$

↳ $V$ is positive definite in $\mathbb{D} = \mathbb{R}$

3) $\dot{V}(x) = \frac{dV(x)}{dx} \cdot \dot{x}$

$= x \cdot -x^3$

$= -x^4$

$\dot{V}$ is negative definite in $\mathbb{D} = \mathbb{R}$

$x = 0 \qquad \dot{V}(x) = 0$

$x \neq 0 \qquad \dot{V}(x) < 0$

$\overset{\text{Th4.1}}{\Rightarrow}$ $x = 0$ is (locally) asymptotically stable

Theorem 4.2 Global asymptotic stability

If

$\mathbb{D} = \mathbb{R}^n$

- $\exists$ a strict Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}$ for $x=0$

and

- $V$ is radially unbounded

then $x=0$ is globally asymptotically stable

Definition

$V(x)$ is radially unbounded iff

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

$V(x) = \frac{1}{2} x^2 = \frac{1}{2} \|x\|^2$ $V(x)$ is radially unbounded

$\xRightarrow{\text{Th } 4.2}$ $x=0$ is globally asymptotically stable

Necessary for global results

For $C^1$ functions $V$:

- Positive definite $\Rightarrow$ Level surfaces are closed for small values of $c$

- Radial unboundedness $\Rightarrow$ Level surfaces are closed $\forall c$

If the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$

level surfaces $V(x) = c$ is closed

$$\Updownarrow$$

$\Omega_c$ is bounded


Example

$$V(x) = \frac{x_1^2}{(1+x_1^2)} + x_2^2 \quad \text{Positive definite}$$

$V$ is not radially unbounded

$x_1 \to \infty \quad x_2 = 0 \quad V(x) \to 1$