407 Midterm Solutions

1)

a)

$P(Y_1=1, Y_2=1) = 1/12 \Rightarrow Y_1+Y_2=2$
$\qquad Y_1-Y_2=0$

$P(Y_1=1, Y_2=0) = \frac{1}{3}\frac{3}{4} = \frac{1}{4} \Rightarrow Y_1+Y_2=1$
$\qquad Y_1-Y_2=1$

$P(Y_1=0, Y_2=1) = \frac{2}{3}\frac{1}{4} = \frac{1}{6} \Rightarrow Y_1+Y_2=1$
$\qquad Y_1-Y_2=-1$

$P(Y_1=0, Y_2=0) = \frac{2}{3}\frac{3}{4} = \frac{1}{2} \Rightarrow Y_1+Y_2=0$
$\qquad Y_1-Y_2=0$

So the joint distribution table is

		Y_1+Y_2		
		0	1	2
Y_1-Y_2	-1	0	1/6	0
	0	1/2	0	1/12
	1	0	1/4	0

b)

$P(Y_1 - Y_2 = -1) = 1/6$

$P(Y_1 - Y_2 = 0) = 7/12$

$P(Y_1 - Y_2 = 1) = 1/4$

Thus $E(Y_1 - Y_2) = 1/4 - 1/6 = 1/12$

$Var(Y_1 - Y_2) = E[(Y_1 - Y_2)^2] - [E(Y_1 - Y_2)]^2$

$= 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{4} - \left(\frac{1}{12}\right)^2 = 59/144$

So $SD(Y_1 - Y_2) = \sqrt{59}/12$

$\boxed{\text{Note}}$: It's not true that

$Var(Y_1 - Y_2) = Var(Y_1) - Var(Y_2)$

2)

a) Let X = number of heads

So $E(X) = \frac{1}{2}(60) = 30$

$SD(X) = \sqrt{60 \frac{1}{2} \frac{1}{2}} = \sqrt{15}$

$P(28.5 \le X \le 32.5)$

$= P\left(\frac{28.5-30}{\sqrt{15}} \le X \le \frac{32.5-30}{\sqrt{15}}\right)$

$\approx P(-.387 \le Z \le .645)$ standard normal

$= \Phi(.645) - \Phi(-.387)$

$= \Phi(.645) - (1 - \Phi(.387))$

$\approx .39$

b) Similarly, (details omitted)

$P(29 \le X \le 32) \approx .298$

3)

a) $X = X_1 + \cdots + X_n$ where $X_i = \begin{cases} 1 & i \text{ fixed} \\ 0 & \text{else} \end{cases}$

Now $E(X_i) = P(X_i = 1) = \frac{(n-1)!}{n!}$ $\leftarrow$ number of permutations fixing i

total number of permutations $\rightarrow$

$= 1/n.$

So $E(X) = n\left(\frac{1}{n}\right) = 1$

b) X is approximately Poisson (1), so

$$P(X=3) \approx \frac{1}{e}\,\frac{1}{3!} = \frac{1}{6e}$$

4) a) X is binomial (n,p) with $n = 250$, $p = 2/500$

So
$$\frac{P(X=1)}{P(X=2)} = \frac{\binom{n}{1} p (1-p)^{n-1}}{\binom{n}{2} p^2 (1-p)^{n-2}}$$

$$= \frac{n}{n(n-1)/2}\,\frac{1-p}{p} = 2$$

b) $P(X=1 \mid X=1 \text{ or } 2)$

$= P(X=1, X=1 \text{ or } 2) / P(X=1 \text{ or } 2)$

$= P(X=1) / P(X=1 \text{ or } 2)$

$= P(X=1) / [P(X=1) + P(X=2)]$

$$= \frac{P(X=1)/P(X=2)}{P(X=1)/P(X=2)+1} \underset{\substack{\uparrow \\ \text{by part} \\ a}}{=} 2/3$$

c) $P(X=1) = \binom{2}{1}\binom{498}{249} \Big/ \binom{500}{250}$

$P(X=2) = \binom{2}{2}\binom{498}{248} \Big/ \binom{500}{250}$

$$\Rightarrow \frac{P(X=1)}{P(X=2)} = 500/249$$

So arguing as in part b), we get $\frac{500}{749}$

$= .667..$