Math 407 Final Exam Solutions

1) Let X be uniform on $[0,1]$

then $E(X) = 1/2$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \int_0^1 x^2 dx - 1/4$$

$$= 1/3 - 1/4 = 1/12$$

So $E(A_n) = 1/2$, $Var(A_n) = \frac{1}{n^2} n Var(X)$

$$= \frac{1}{12n}$$

we want n so that

$$.8 = P(A_n < .51)$$

$$= P\left(\frac{A_n - 1/2}{\sqrt{1/12n}} < \frac{.51 - 1/2}{\sqrt{1/12n}} \right)$$

$$= P\left(\frac{A_n - 1/2}{\sqrt{1/12n}} < \frac{.01}{\sqrt{1/12n}} \right)$$

(2)

$$\approx \Phi\left(\frac{.01}{\sqrt{1/12n}}\right)$$

So by the normal table,

$$\frac{.01}{\sqrt{1/(12n)}} \approx .84$$

So $n \approx 588$

2)

a) we know that

$$1 = \int_0^2 cx^2\,dx = cx^3/3 \Big|_0^2 = 8c/3$$

so $c = 3/8$

b) $P(X \le b) = \int_0^b 3/8\, x^2\,dx = b^3/8$

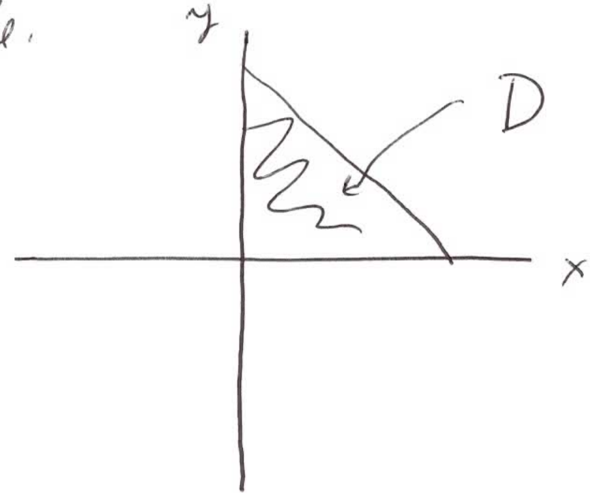
c) $E(X) = \int_0^2 x\, 3/8\, x^2\,dx = 3/2$

d) $E(X^2) = \int_0^2 x^2\, 3/8\, x^2\,dx = 96/40$

So $Var(X) = 96/40 - (3/2)^2 = .15$

So $SD(X) \approx .387$

3) the joint distribution function is $f(x,y)=2$ on the triangle and 0 outside.



$$E(X)=\iint_D x\,f(x,y)\,dx\,dy=\int_0^1\int_0^{1-y}2x\,dx\,dy$$

$$=1/3$$

Similarly (or by symmetry), $E(Y)=1/3$

$$E(XY)=\iint_D xy\,f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^{1-y}2\cancel{x}y\,dx\,dy=1/12$$

So $\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)$

$$=1/12-(1/3)^2=-1/36$$

4) Let D = person has disease
A = person tests positive

Then
$$P(A|D) = .97, \quad P(A|D^c) = .02$$
$$P(D) = .005$$

From Bayes' formula,

$$P(D|A) = \frac{(.97)(.005)}{(.97)(.005) + (.02)(.995)}$$
$$\approx .196$$

5) The conditional density of X given that $Y=1$ is given by

$$\frac{f(x,1)}{f_Y(1)} = \frac{1/2\, e^{-x}}{\int_0^\infty \frac{1}{2} e^{-x}\,dx} = e^{-x}$$

thus

$$E[e^{X/3} \mid Y=1]$$

$$= \int_0^\infty e^{x/3} \frac{f(x,1)}{f_Y(1)} \, dx$$

$$= \int_0^\infty e^{x/3} e^{-x} \, dx$$

$$= \int_0^\infty e^{-2x/3} \, dx = 3/2$$