1. (a) low = 1, high = 10

low = 6, high = 10

low = 9, high = 10

low = 9, high = 9

return value = -1

(b) The first 9 lines of the function are all constant time

$\therefore T(n) = \cancel{\Theta(\frac{n}{2}) + c_1} \; T(\frac{n}{2}) + c_1 = T(\frac{n}{4}) + c_1 + c_2$

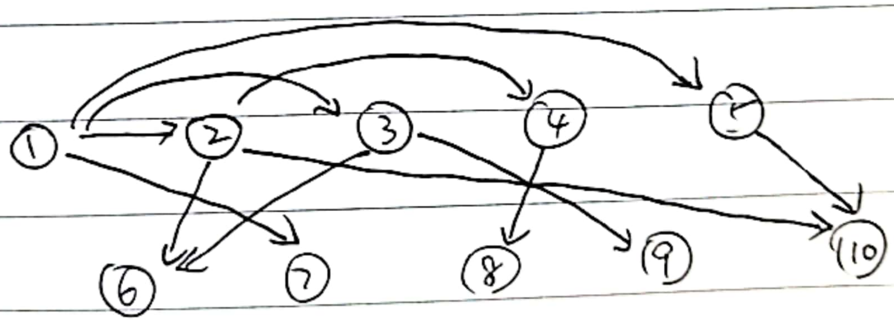
$= \log n \cdot T(1) + c = \Theta(\log n)$ where $n = high - low$

(c) $T(n) = c_1 n + c_2 + T(\frac{n}{2}) = T(\frac{n}{4}) + c_3 \cdot \frac{n}{2} + c_4 + c_1 n + c_2$

$= \cdots = \log n \cdot T(1) + c_1 \cdot n \log n + c_2$

$\therefore$ The worst case runtime is $\Theta(n \log n)$

2. (a)



(b) The edges to add: $\underbrace{\langle 1,1 \rangle, \langle 2,2 \rangle, \cdots \langle 10,10 \rangle}_{\text{self-loops (reflexive)}}$

$\underbrace{\langle 1,4 \rangle, \langle 1,6 \rangle, \langle 1,9 \rangle, \langle 1,10 \rangle, \langle 2,8 \rangle}_{\text{transitivity}}$

(c) reflexive: $p|p$ certainly true

transitive: if $p|q$ and $q|s$, then $p|s$ ✓

antisymmetric: if $p|q$ and $q|p$, then $p=q$ ✓

If not, then one of p and q will be larger than the other, resulting in the invalidity of one of the divisibility.

∴ Divisibility is partial order on $\mathbb{Z}^{30}$

3. Assume G is not connected, then there are more than 1 connected components in the graph. It is guaranteed ~~than~~ that one of the component has ~~at least~~ less than $\lfloor \frac{n}{2} \rfloor$ vertices because if not, then the total number of vertices in every component will exceed n.

∴ Let's consider the particular component. Each vertex has a degree of at most $\frac{n}{2}-1$ since it cannot connect to itself. Contradiction!

∴ G is connected.

4. (a) Based on Menger's Theorem, edge connectivity = maximum number of edge-disjoint paths from s to t, which is 2 (the 2 edges that extend out from s)

∴ Edge connectivity is 2.

~~(b) Loop invariant: after removing u, the remaining graph is 3-connected. at least 2-connected~~

(b) Loop invariant: When removing 2 vertices (one of them is u), the graph is 3-connected.
excluding the previously discovered u's.

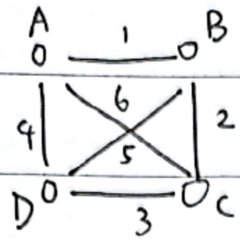
(C) Base Case: No vertex removed. Vacuously true.

IH: P(u): Loop invariant in (b)

IS: Consider another undiscovered vertex u. Let's pair it with all the available v's. If all the subgraph after removing u and v is 3-connected, then the graph including u and excluding previously discovered u's is 3-connected. According to IH, all the graphs including previously found u's are connected.

Correctness: After discovering all the vertices of the graph and pairing them with all the other vertices, we can safely determine that it's 3-connected.

5. (a)



Kruskal's algorithm:

First, add the edge AB, then add BC, then add CD.

All the adding processes do not create cycles and 1, 2, 3 are the smallest weights.

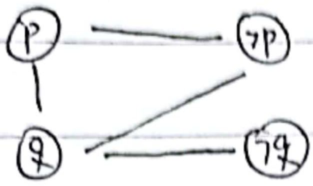
(b) Base Case: $n=2$ $\quad o\overset{1}{—}o \quad \frac{2\cdot 1}{2}=1$ ✓

IH: For K_n, there exists an assignment that the cost is $\frac{n(n-1)}{2}$

IS: Consider K_{n+1}. Remove a vertex V from K_{n+1} and its incident n edges. According to IH, MST cost is equal to $\frac{n(n-1)}{2}$. When we add back the edges and the vertex, since we only need one more edge to traverse the whole set of vertices (every vertex is connected to every other vertex), we assign weight n to one of the added-back n edges. Thus,

MST cost $= \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$. ✓

6. (a) No. If $(p \oplus q) \wedge (\neg p \oplus q)$ is true, then both $p \oplus q$ and $\neg p \oplus q$ ~~is~~ are true, but it's impossible since p and $\neg p$ are opposite, which cannot have both expressions be true.



Let's color p true, then $\neg p$ and q must all be false. However, $\neg p$ and q are connected. ∴ Not 2-colorable.

(b) Yes. $p = \text{True}$, $q = \text{False} \Longrightarrow p \oplus q = \text{True}$ and $\neg p \oplus \neg q$ is true.

color p true, then $\neg p$ and q are false, and $\neg q$ is true ∴ Is 2-colorable.

(c)