Quiz 1 for MATH 226g, sections 39553R and 39559

Name: **Date:**

Problem 1: Show that the following equation represents a sphere in $\mathbb{R}^3$, and find its center and radius:

$$x^2 + y^2 + z^2 - 10x + 2z - 3 = 0.$$

(Hint: $x^2 + 2ax = (x + a)^2 - a^2$.)

We complete the squares:

$$\begin{aligned} x^2 + y^2 + z^2 - 10x + 2z - 3 &= 0 \\ \implies \big((x-5)^2 - 25\big) + y^2 + \big((z+1)^2 - 1\big) - 3 &= 0 \\ \implies (x-5)^2 + y^2 + (z+1)^2 &= 29. \end{aligned}$$

The original equation therefore defines a sphere — specifically, the sphere of radius $\sqrt{29}$ centered at $(5, 0, -1)$.

Problem 2: Consider the 2-dimensional vectors $\mathbf{a} = \langle 4, 1 \rangle$ and $\mathbf{b} = \langle 1, -2 \rangle$. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} + \mathbf{b}$, illustrating the triangle law. Compute the length $|\mathbf{a} + \mathbf{b}|$.

To compute the length $|\mathbf{a}+\mathbf{b}|$, we first compute $\mathbf{a}+\mathbf{b} = \langle 5, -1 \rangle$, and then compute $|\mathbf{a}+\mathbf{b}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$.

Quiz 2 for MATH 226, section 39559

Name: **Date:**

Problem 1: Consider the triangle ABC in $\mathbb{R}^3$ with vertices $A = (1, 3, 2)$, $B = (5, -1, 0)$, and $C = (0, 2, -1)$. What is the angle of this triangle at the vertex A?

The angle θ we are after is the angle made by the vectors $\mathbf{AB} = \langle 4, -4, -2\rangle$ and $\mathbf{AC} = \langle -1, -1, -3\rangle$. We can therefore compute θ using the "master identity for the dot product":

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\mathbf{AB}\cdot\mathbf{AC}}{|\mathbf{AB}||\mathbf{AC}|}\right) \\ &= \cos^{-1}\left(\frac{-4+4+6}{\sqrt{36}\sqrt{11}}\right) \\ &= \cos^{-1}\left(\frac{\sqrt{11}}{11}\right). \end{aligned}$$

Problem 2: Compute the area of the triangle ABC defined in Problem 1, and find a vector orthogonal to the plane through A, B, and C.

We can compute the area of this triangle by recognizing that this area is half the area of the parallelogram defined by $\mathbf{AB}$ and $\mathbf{AC}$, and using the fact that the area of this parallelogram is the magnitude of the cross product of these vectors:

$$\begin{aligned} \text{area of } ABC &= \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| \\ &= \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & -2 \\ -1 & -1 & -3 \end{vmatrix}\right\| \\ &= \frac{1}{2}\big|\langle 10, 14, -8\rangle\big| \\ &= 3\sqrt{10}. \end{aligned}$$

Finally, a vector orthogonal to the plane containing A, B, and C is given by the cross product we just computed: $\langle 10, 14, -8\rangle$.

Quiz 3 for MATH 226, section 39559

20 minutes; use back if necessary

Name: **Date:**

Problem 1: Define L to be the line through $P = (3, 1, 2)$ and $Q = (-1, 0, 1)$. Express L in terms of either a vector equation or parametric equations. Where does L intersect the plane defined by $x - y + z = 3$?

The vector $\mathbf{PQ}$ is parallel to L, and we have $\mathbf{PQ} = \langle -4, -1, -1 \rangle$. Moreover, L contains the point $P = (3, 1, 2)$. It follows that we can express L in terms of the following parametric equations:

$$x = -4t + 3, \quad y = -t + 1, \quad z = -t + 2.$$

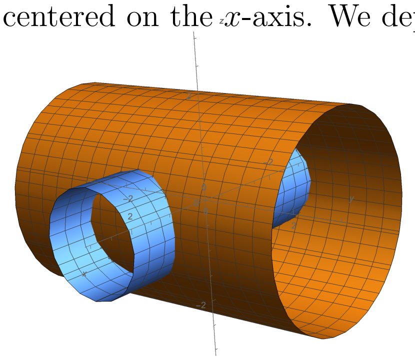
To compute the intersection point of L with the given plane, we substitute the component functions $x(t), y(t), z(t)$ into the equation $x - y + z = 3$:

$$\begin{aligned} x(t) - y(t) + z(t) = 3 &\iff (-4t + 3) - (-t + 1) + (-t + 2) = 3 \\ &\iff -4t + 4 = 3 \\ &\iff t = \tfrac{1}{4}. \end{aligned}$$

Substituting $t = \frac{1}{4}$ back into component functions, we see that the intersection point of L with the plane $x - y + z = 0$ is $\mathbf{r}(\frac{1}{4}) = \langle 2, \frac{3}{4}, \frac{7}{4} \rangle$.

Problem 2: Draw the surfaces defined by $x^2 + z^2 = 4$ and $y^2 + z^2 = 1$ in $\mathbb{R}^3$. Draw their curve(s) of intersection, and find vector function(s) that represent these curve(s).

We begin by noting that these equations define two cylinders: one of radius 2, centered on the y-axis, and one of radius 1, centered on the x-axis. We depict this below.



We see from this figure, together with the defining equations, that there are two curves. Moreover, the projection of each to the yz-plane is the entire unit circle, which we can parametrize by $y(t) = \cos t$, $z(t) = \sin t$. Now we use the equation $x^2 + z^2 = 4$ to deduce $x(t) = \pm\sqrt{4 - \sin^2 t}$. We can therefore parametrize these two curves by the vector functions $\gamma_{\pm}(t) = \big\langle \pm \sqrt{4 - \sin^2 t}, \cos t, \sin t \big\rangle$.

Quiz 4 for MATH 226, section 39559

15 minutes; use back if necessary

Name: NSB **Date:**

Problem 1: Consider $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$. Compute the arclength function $s(t)$ starting from $t = 0$. Reparametrize $\mathbf{r}$ with respect to arclength, again starting from $t = 0$.

$$\begin{aligned} |\mathbf{r}'(t)| &= \left|\langle e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t \rangle\right| \\ &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} \\ &= e^t \sqrt{(\cos^2 t - 2\cos t \sin t + \sin^2 t) + (\cos^2 t + 2 \cos t \sin t + \sin^2 t)} \\ &= \sqrt{2} e^t. \end{aligned}$$

$$s(t) = \int_0^t |\mathbf{r}'(u)|\, du = \int_0^t \sqrt{2} e^u \, du = \sqrt{2}(e^t - 1).$$

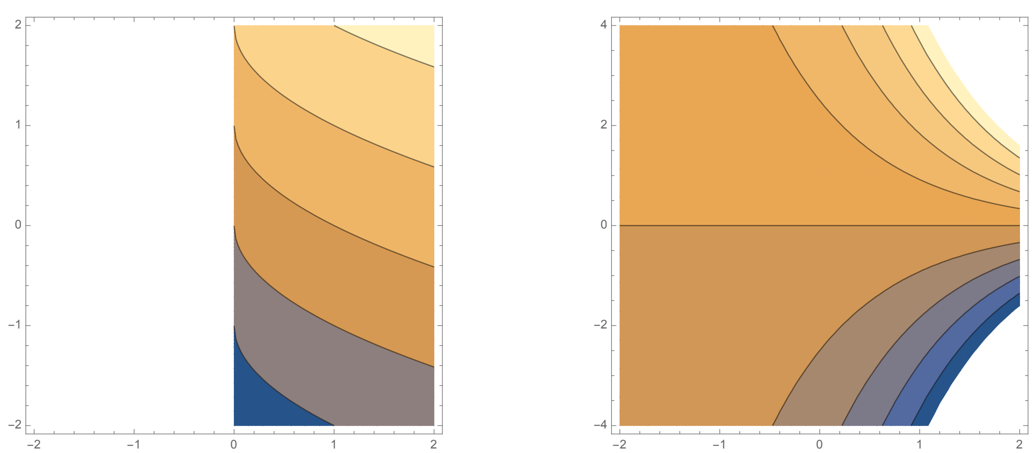
$$s = \sqrt{2}(e^t - 1) \implies e^t = \sqrt{2}s/2 + 1 \implies t = \log\bigl(\sqrt{2}s/2 + 1\bigr).$$

$$\mathbf{r}(t(s)) = \Bigl\langle \bigl(\sqrt{2}s/2 + 1\bigr) \cos \log\bigl(\sqrt{2}s/2 + 1\bigr), \bigl(\sqrt{2}s/2 + 1\bigr) \sin \log\bigl(\sqrt{2}s/2 + 1\bigr) \Bigr\rangle.$$

Problem 2: Consider $f(x, y) = \sqrt{x} + y$ and $g(x, y) = ye^x$. What are the domain and range of f and of g? Draw a contour plot for each function that shows a few level curves. For each level curve, provide the corresponding equation.

Domain of f is all $(x, y) \in \mathbb{R}^2$ with $x \geq 0$ (argument of $\sqrt{-}$ must be nonnegative). Domain of g is all $(x, y) \in \mathbb{R}^2$. The range of both functions is $\mathbb{R}$ (one way to see this is to note that $f(0, y) = g(0, y) = y$, so as y varies over $\mathbb{R}$, $f(0, y)$ resp. $g(0, y)$ varies over all of $\mathbb{R}$).

For any $k \in \mathbb{R}$, level curves of f are $\sqrt{x} + y = k$, i.e. $y = -\sqrt{x} + k$; level curves of g are $ye^x = k$, i.e. $y = ke^{-x}$. Here's what they look like:



Quiz 5 for MATH 226, section 39559

15 minutes; use back if necessary

Name: **Date:**

Problem 1: If $\cos(xyz) = 1 + x^2y^2 + z^2$, find $\partial z/\partial x$ and $\partial z/\partial y$.

First, we implicitly differentiate with respect to x:

$$(yz + xyz_x)(-\sin(xyz)) = 2xy^2 + 2zz_x \implies (2z + xy\sin(xyz))z_x = -yz\sin(xyz) - 2xy^2$$
$$\implies z_x = -\frac{yz\sin(xyz) + 2xy^2}{2z + xy\sin(xyz)}.$$

We do the same, but with respect to y:

$$(xz + xyz_y)(-\sin(xyz)) = 2x^2y + 2zz_y \implies (2z + xy\sin(xyz))z_y = -xz\sin(xyz) - 2x^2y$$
$$\implies z_y = -\frac{xz\sin(xyz) + 2x^2y}{2z + xy\sin(xyz)}.$$

Problem 2: Compute the linearization of $f(x, y, z) = x^2e^{xy+z^2}$ at $(x, y, z) = (1, 0, 0)$. Use this to estimate $f(0.9, -0.1, -0.2)$.

We compute the partial derivatives of f, then evaluate them at $(1, 0, 0)$:

$$f_x = 2xe^{xy+z^2} + x^2ye^{xy+z^2}, \quad f_y = x^3e^{xy+z^2}, \quad f_z = 2x^2ze^{xy+z^2}$$
$$\implies f_x(1, 0, 0) = 2, \quad f_y(1, 0, 0) = 1, \quad f_z = 0.$$

We can use this to form the linear approximation to f at $(1, 0, 0)$:

$$L(x, y, z) = 1 + 2(x - 1) + y.$$

We therefore have:

$$f(0.9, -0.1, -0.2) \approx L(0.9, -0.1, -0.2) = 1 + 2(-0.1) + (-0.1) = 0.7.$$

Quiz 6 for MATH 226, section 39559

15 minutes; use back if necessary. Due October 9, 2020.

Name: **Date:**

Problem 1: Using whichever method you like, find the point on the plane $x - y + 3z = 1$ closest to the origin.

We could solve this using methods from §10 — specifically, by projecting the vector from the origin to any point on this plane onto a normal vector to this plane. Instead, in the spirit of §11, let's minimize a certain function. The z-coordinate of any point on this plane is given by $z = -\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}$, so we can write the squared distance from the origin to an arbitrary point on this plane as:

$$d^2 = x^2 + y^2 + \left(-\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}\right)^2.$$

It is clear that there is a unique point on the plane that is closest to the origin, so let's find the critical points of the function d^2 — we expect there to be exactly one, and for that point to be the point where d^2 achieves its global minimum.

We compute the x- and y-derivatives:

$$\partial_x\left(x^2 + y^2 + \left(-\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}\right)^2\right) = 2x + 2\left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}\right) = \tfrac{20}{9}x - \tfrac{2}{9}y - \tfrac{2}{9},$$
$$\partial_y\left(x^2 + y^2 + \left(-\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}\right)^2\right) = 2y + 2\left(\tfrac{1}{3}\right)\left(-\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}\right) = -\tfrac{2}{9}x + \tfrac{20}{9}y + \tfrac{2}{9}.$$

We must solve for where these quantities are both zero. The first equation gives $10x - y = 1$, hence $y = 10x - 1$. The second equation gives $-x + 10y = -1$, hence $-x + 10(10x - 1) = -1$, hence $x = \frac{1}{11}$. Hence $y = 10 \cdot \frac{1}{11} - 1 = -\frac{1}{11}$. The z-value at this point is $z = -\frac{1}{3} \cdot \frac{1}{11} + \frac{1}{3}\left(-\frac{1}{11}\right) + \frac{1}{3} = \frac{9}{33}$. The point on the plane closest to the origin is therefore $\left(\frac{1}{11}, -\frac{1}{11}, \frac{9}{33}\right)$.

Problem 2: Compute the volume under the surface $z = x/y$ and above the rectangle $R = [0, 2] \times [1, 3]$ (i.e. $R = \{0 \leq x \leq 2, 1 \leq y \leq 3\}$) in the xy-plane.

We compute this volume as a double integral:

$$V = \int_0^2 \int_1^3 \frac{x}{y}\, dy\, dx = \int_0^2 [x \log y]_{y=1}^{y=3}\, dx = \int_0^2 (\log 3)x\, dx = \left[\frac{\log 3}{2} x^2\right]_{x=0}^{x=2} = 2\log 3.$$

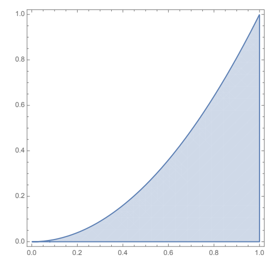
Quiz 7 for MATH 226, section 39559

20 minutes. Due October 16, 2020.

Name: **Date:**

Problem 1: Calculate the iterated integral $I = \int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy$ by first reversing the order of integration.

First, we plot the domain of integration:



With this plot as an aid, we rewrite this domain D as a type-1 region: $D = \{0 \le x \le 1, 0 \le y \le x^2$. We can now compute the given integral:

$$I = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \left[\frac{y^2e^{x^2}}{2x^3}\right]_{y=0}^{y=x^2} dx = \int_0^1 \tfrac{1}{2}xe^{x^2}\,dx = \left[\tfrac{1}{4}e^{x^2}\right]_{x=0}^{x=1} = \frac{e-1}{4}.$$

Problem 2: Evaluate the following two integrals:

(a) $I = \iint_D (x^2+y^2)^{3/2}\,dA$, where D is the region in the first quadrant bounded by $y=0$, $y=x$, and $x^2+y^2=9$.

(b) $I = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3+xy^2)\,dy\,dx$.

We start with the first integral. We can rewrite the domain of integration as the polar rectangle $\{0 \le \theta \le \frac{\pi}{4}, 0 \le r \le 3\}$, which allows us to rewrite I in polar coordinates and compute it:

$$I = \int_0^{\pi/4} \int_0^3 r^4\,dr\,d\theta = \int_0^{\pi/4} \left[\tfrac{1}{5}r^5\right]_{r=0}^{r=3} d\theta = \int_0^{\pi/4} \tfrac{243}{5}\,d\theta = \tfrac{486\pi}{5}.$$

For the second integral, we rewrite the domain as $D = \{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 3\}$. We now compute I:

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \int_0^3 r^2\cos\theta\big(r^2\cos^2\theta + r^2\sin^2\theta\big)\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} r^4\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{5}r^5\cos\theta\right]_{r=0}^{r=3} d\theta = \int_{-\pi/2}^{\pi/2} \tfrac{243}{5}\cos\theta\,d\theta = \left[\tfrac{243}{5}\sin\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2} = \tfrac{486}{5}. \end{aligned}$$

Quiz 8 for MATH 226, section 39553

15 minutes; use back if necessary

Name: **Date:**

Problem 1: Evaluate $I = \iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where E is the portion of the unit ball $x^2+y^2+z^2 \leq 1$ that lies in the first octant (i.e. the region with $x \geq 0, y \geq 0, z \geq 0$).

The region in question can be expressed in spherical coordinates like so:

$$E = \left\{0 \leq \rho \leq 1,\ 0 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi/2\right\}.$$

We use this to compute I:

$$\begin{aligned} I = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \sin\phi\, e^{\rho^3}\, d\phi\, d\theta\, d\rho &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \left(-\rho^2 \cos\phi\, e^{\rho^3}\right) d\phi\, d\theta\, d\rho \\ &= \int_0^1 \int_0^{\pi/2} \rho^2 e^{\rho^3}\, d\theta\, d\rho \\ &= \int_0^1 \tfrac{\pi}{2}\rho^2 e^{\rho^3}\, d\rho = \left[\tfrac{\pi}{6} e^{\rho^3}\right]_{\rho=0}^{\rho=1} = \tfrac{\pi}{6}(e-1). \end{aligned}$$

Problem 2: Evaluate $I = \iiint_E y\,dV$, where E is the solid that lies under $z = 1 - x^2 - y^2$, above the xy-plane, and in the half-space $y \geq 0$.

First, let's note that the upper boundary $\{z = 1 - x^2 - y^2\}$ and the lower boundary $\{z = 0\}$ intersect when $z = 0$, $x^2 + y^2 = 1$. This allows us to write this region in cylindrical coordinates:

$$E = \left\{0 \leq r \leq 1,\ 0 \leq \theta \leq \pi,\ 0 \leq z \leq 1 - r^2\right\}.$$

We can now compute I:

$$\begin{aligned} I = \int_0^1 \int_0^{\pi} \int_0^{1-r^2} r^2 \sin\theta\, dz\, d\theta\, dr &= \int_0^1 \int_0^{\pi} r^2(1-r^2)\sin\theta\, d\theta\, dr \\ &= \int_0^1 \left[-r^2(1-r^2)\cos\theta\right]_{\theta=0}^{\theta=\pi} dr \\ &= \int_0^1 2r^2(1-r^2)\, dr \\ &= \left[\tfrac{2}{3}r^3 - \tfrac{2}{5}r^5\right]_{r=0}^{r=1} = \tfrac{4}{15}. \end{aligned}$$

Quiz 9 for MATH 226, section 39559

15 minutes

Name: **Date:**

Problem 1: Using whichever technique you like, evaluate the integral

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy, \tag{1}$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$.

We'll use Green's theorem. Here $P = \sqrt{1+x^3}$ and $Q = 2xy$, so:

$$Q_x - P_y = 2y - 0 = 0.$$

We therefore have:

$$I = \int_0^1 \int_0^{3x} 2y\,dy\,dx = \int_0^1 [y^2]_{y=0}^{y=3x}\,dx = \int_0^1 9x^2\,dx = [3x^3]_{x=0}^{x=1} = 3.$$

Problem 2: Let $\mathbf{F}$ be a constant vector field $\mathbf{F}(x,y) = \langle a,b\rangle$, for a,b two real numbers. Compute the integral of $\mathbf{F}$ over the circle $x^2+y^2=1$, traversed counterclockwise.

We parametrize the circle by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. Using this, we compute this integral:

$$I = \int_C \mathbf{F}\cdot d\mathbf{r}(t) = \int_0^{2\pi} \langle a,b\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \big[a\cos t, b\sin t\big]_0^{2\pi} = 0. \tag{2}$$