Contents

1 Review of Propositional Logic

Task: Recall enough propositional logic to see how it matches up with set theory.

Definition: A proposition is any declarative sentence that is either true or false.

1.1 Connectives

	Connectives		Notation in Maths
and	$\wedge$		
or	$\vee$	"Inclusive or"	
not	$\neg$	Sometimes denoted $\sim$	
implies	$\rightarrow$	if/then; called implication	$\Rightarrow$
if and only if	$\leftrightarrow$	Called equivalence	$\Leftrightarrow$

1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	P$\wedge$Q
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	P$\vee$Q
F	F	F
F	T	T
T	F	T
T	T	T

P	$\neg$P
F	T
T	F

P	Q	P$\rightarrow$Q
F	F	T
F	T	T
T	F	F
T	T	T

P	Q	P$\leftrightarrow$Q(biconditional)
F	F	T
F	T	F
T	F	F
T	T	T

Priority of the Connectives

Highest to Lowest: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

1.2 Important Tautologies

$$\begin{array}{rcl} (P \rightarrow Q) & \leftrightarrow & (\neg P \vee Q) \\ (P \leftrightarrow Q) & \leftrightarrow & [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\ \neg(P \wedge Q) & \leftrightarrow & (\neg P \vee \neg Q) \\ \neg(P \vee Q) & \leftrightarrow & (\neg P \wedge \neg Q) \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{De Morgan Laws}$$

As a result, $\neg$ and $\vee$ together can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Less obvious: One connective called the sheffer stroke P|Q (which stands for "not both P and Q" or "P nand Q") can be used to represent all of $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ since $\neg$P $\leftrightarrow$ P|P and P $\vee$ Q $\leftrightarrow$ (P|P) | (Q|Q).

Recall if P$\rightarrow$Q is a given implication, Q$\rightarrow$P is called the converse or P$\rightarrow$Q. $\neg$Q $\rightarrow$ $\neg$P.

1.3 Logical Equivalences

De Morgan's Laws: 1.$\neg(p \vee q) \equiv \neg p \wedge \neg q$ 2.$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Figure 1: Logical Equivalences

$$p \to q \equiv \neg p \lor q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \lor q \equiv \neg p \to q$$
$$p \land q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \land \neg q$$
$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$
$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$
$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$
$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Figure 2: Involving Conditional Statements

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Figure 3: Involving Biconditional Statements

1.3.1 Satisfiability

satisfiable: if there's an assignment of truth values to its variables that makes it true(tautology/contingency)

unsatisfiable: iff. the negation of a compound proposition is tautology

solution: an assignment makes a compound proposition true

1.4 Indirect Arguments/Proofs by Contradiction/Reductio as absurdum

Based on the tautology (P→Q) ↔ (¬Q → ¬P)

Example: Famous argument that $\sqrt{2}$ is irrational.

Proof:

Suppose $\sqrt{2}$ is rational, then it can be expressed is fraction form $\frac{a}{b}$. Let us **assume** that our fraction is in the lowest term, **i.e.** their only common divisor is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by B^2 yields

$$2b^2 = a^2$$

Since $a^2 b^2$, we can conclude that a^2 is even because whatever the value of b^2 has to be multiplied by 2. If a^2 is even, then a is also even. Since a is even, no matter what the value of a is, we can always find an integer that if we divide a by 2, it is equal to that integer. If we let that integer be k, then $\frac{a}{b} = k$ which means that $a = 2k$.

Substituting the value of $2k$ to a, we have $2b^2 = (2k)^2$ which means that $2b^2 = 4k^2$. dividing both sides by 2 we have $b^2 = 2k^2$. That means that the value b^2 is even, since whatever the value of k you have to multiply it by 2. Again, is b^2 is even, then b is even.

This implies that both a and b are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that $\frac{a}{b}$ has no common divisor except 1. Since we found a contradiction, our assumption is, therefore, false. Hence the theorem is true.

qed

2 Predicate logic and Quantifiers

Task: Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variable x, y, z, so their truth value may change depending on which values these variables assume: $P(x), Q(x, y), R(x, y, z)$

Jargon: domain of discourse(domain)

2.1 Introduce quantifiers

2.1.1 $\exists$ existential quantifier

Syntax: $\exists x P(x)$

Definition: $\exists x P(x)$ is true if $P(x)$ is true or some value of x; it is false otherwise.

2.1.2 $\forall$ universal quantifier

Syntax: $\forall x P(x)$

Definition: $\forall x P(x)$ is true if $P(x)$ is true for all allowable values of x. It is false otherwise.

2.1.3 ∃! for one and only one: Uniqueness Quantifier

Syntax: $\exists! x P(x) or \exists_1$

Definition: $\exists! x P(x)$ is true if $P(x)$ is true for exactly one value of x and false for all often values of x; otherwise, $\exists! x P(x)$ is false.

2.1.4 Alternation of Quantifiers: Nested Quantifiers

$\forall x \exists y \forall z \quad P(x, y, z)$
NB: The order cannot be exchanged as it might modify the truth values of the statement (think of examples with two quantifiers).

Example: domain: real numbers, P(x,y):= x + y = y + x
$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$

2.1.5 Negation of Quantifiers

$\neg(\exists x P(x)) \quad \leftrightarrow \quad \forall x \neg P(x)$
$\neg(\forall x P(x)) \quad \leftrightarrow \quad \exists x \neg P(x)$

2.1.6 Precedence of Quantifiers

The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus. $\forall x P(x) \vee Q(x) \equiv (\forall x P(x)) \vee Q(x)$
When the domain of a quantifier if finite, quantified statements can be expressed using propositional logic.

2.1.7 Order of Quantifiers

∀∀: The order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.
Like x,y real numbers, P(x,y):= x + y = y + x, s.t. $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$

∀∃/∃∀: Like x,y real numbers, Q(x,y):= x + y = 0
$\forall x \exists y Q(x, y)$: y can depend on x; $\exists y \forall x$: y is a constant independent of x. It's like the order $\forall, \exists,$ ...from the smallest scope to the largest scope.

2.1.8 Null Quantifiers

$(\forall x P(x)) \vee A \equiv \forall x (P(x) \vee A) \quad (\exists x P(x)) \vee A \equiv \exists x (P(x) \vee A)$
$(\forall x P(x)) \wedge A \equiv \forall x (P(x) \wedge A) \quad (\exists x P(x)) \wedge A \equiv \exists x (P(x) \wedge A)$
$(\forall x P(x)) \wedge A \rightarrow \forall x (P(x) \wedge A) \quad (\exists x P(x)) \wedge A \rightarrow \exists x (P(x) \wedge A)$

2.2 Binding variables

bound: A quantifier is used on a variable x, which we say x is **bound**

free: No quantifier or set bounds a variable, which we say x is **free**

scope: The part of a logical expression a quantifier is applied, which we say the part is the **scope** of the quantifier

Binding variables: the same letter is often used of represent variables bound by different quantifiers with scopes that do **not overlap**.
Substitution: $\exists(P(x) \wedge Q(x)) \vee \forall y R(y) \xrightarrow{y \to x} \exists(P(x) \wedge Q(x)) \vee \forall x R(x)$

2.3 Logical Equavalences Involving Quantifiers

Distributive Law:
$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
$\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

2.4 Rules of Inference

argument: a sequence of statements that end with a conclusion

valid: the conclusion must follow from true **premises** or a tautology

fallacy: leads to invalid argument, some forms of incorrect resoning

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Figure 4: Rule of Inference

Constructive dilemma: $(A \Rightarrow B), (C \Rightarrow D), (A \vee C) \Leftrightarrow (B \vee D)$

Clauses: To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and conclusion must be expressed as **clauses**, where a clause is a disjuction of variables or negations of these variables.

Example of Resolution: Premises$(p \wedge q) \vee r$, $r \rightarrow s$, conclusion $p \vee s$

2.4.1 Rules of Inference for Quantified Statements

(UI)Universal instantiation: "All women are wise" that "Lisa is wise", where Lisa is a member of the domain of all women.

(UG)Universal generalization: The premise c must arbitrarily picked without additional assumptions.

(EI)Existential instantiation: We have no knowledge of what c is, only that is exists. Because it exists, we may give it a name (c) and continue the argument.

(EG)Existential generalization: We know one element c in the domain for which p(c) is true, then we know that $\exists x P(x)$ is true.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Figure 5: Rule of Inference for Quantified Statements

3 Proofs

3.1 Intro

Theorem: Always omits a universal quantifier when it states. Its proofs often have "obviously"/"clearly' indicating that stpes have been omitted the author expects the reader to be able to fill in without hints.

3.1.1 Direct Proofs

For conditional statement $p \to q$. If p is true, q cannot be false.

Def1: $\exists k \in \mathbb{Z}, n = 2k$ and n is *even*. n is odd if $n = 2k + 1$. No integer between even or odd.

3.1.2 Proof by Contraposition

To prove $p \to q$ is equal to prove $\neg q \to \neg p$.

3.1.3 Vacuous and Trivial Proofs

Vacuous: $p \to q$ is always true if p is false. Trivial: $p \to q$ is true if q is true already.

Def2: Rational number r. $r \in \mathbb{R}$ if $\exists p, q \in \mathbb{Z}$ where $q \neq 0, s.t.\ r = \frac{p}{q}$. A number is *irrational* which isn't a rational.

3.1.4 Proofs by Contradiction

(omitted)

3.2 Method and Strategy

3.2.1 Exhaustive Proof

Those proofs proceed by exhausting all possibilities(aka. Proof by Covering all Cases).