COMP90038 Algorithms and Complexity

Lecture 12: More Divide-and-Conquer Algorithms (with thanks to Harald Søndergaard)

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Divide and Conquer

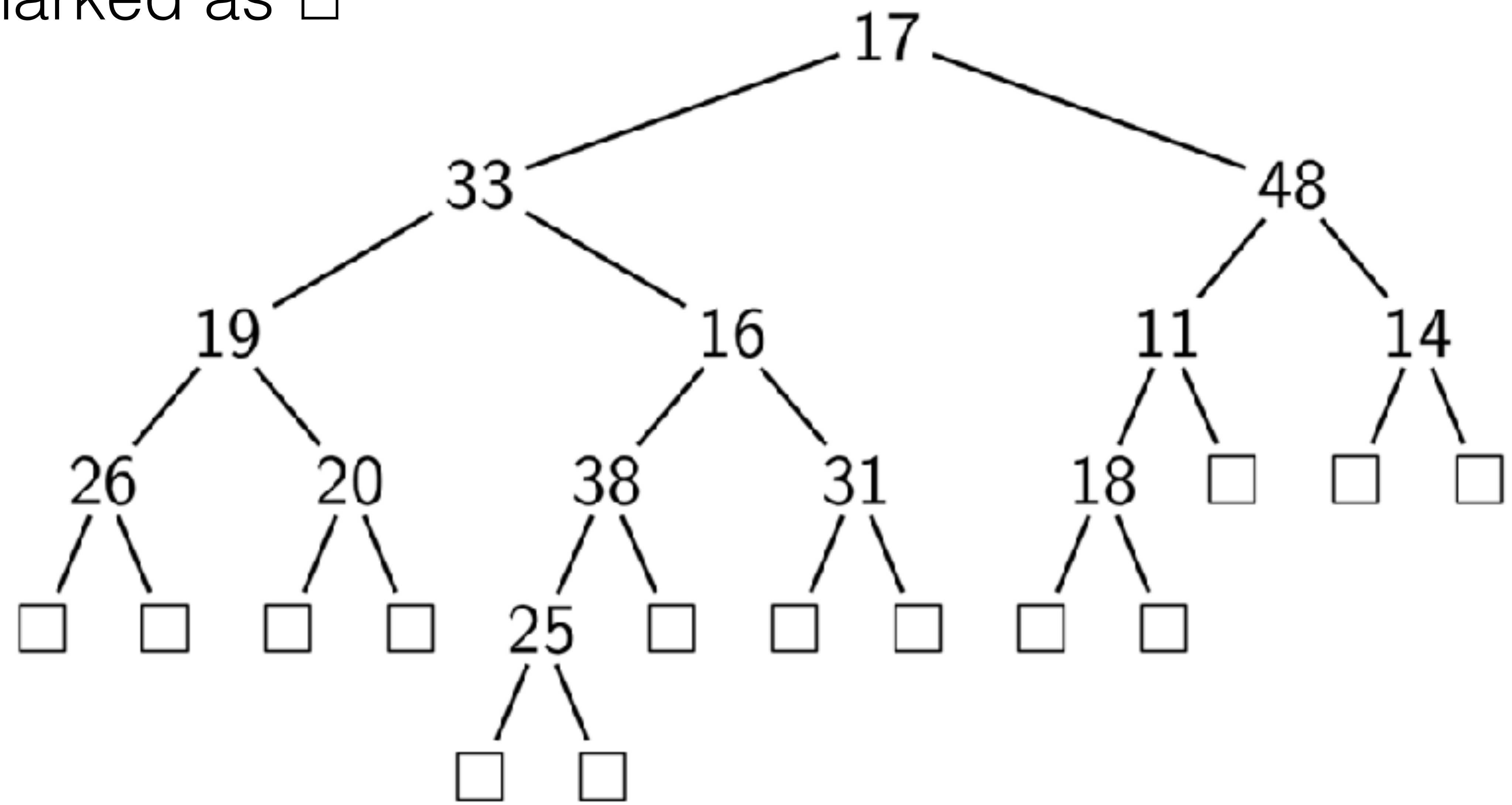
- In the last lecture we studied the archetypal divide-and-conquer sorting algorithms: mergesort and quicksort.

- We also introduced the powerful **master theorem**, providing solutions to a large class of recurrence relations, for free.
  - allows us to quickly determine the complexity of these divide-and-conquer algorithms

- Now we shall look at tree traversal, and then a final example of divide-and-conquer, giving a better solution to the closest-pair problem.

Binary Trees

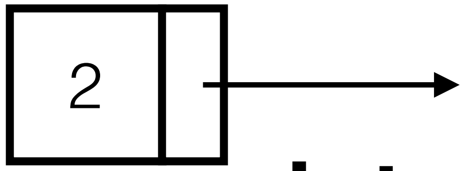
- An example of a **binary tree**, with empty subtrees marked as □



- This tree has **height** 4, the empty tree having height -1

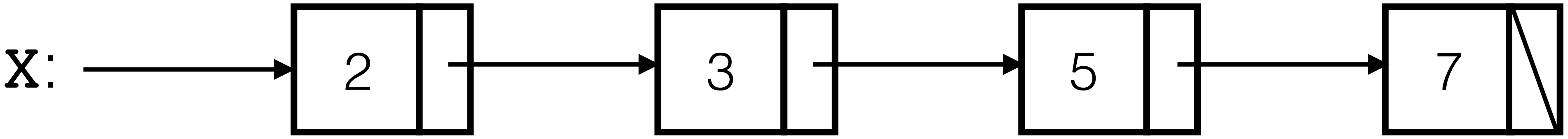
Review of Linked List Terminology

node

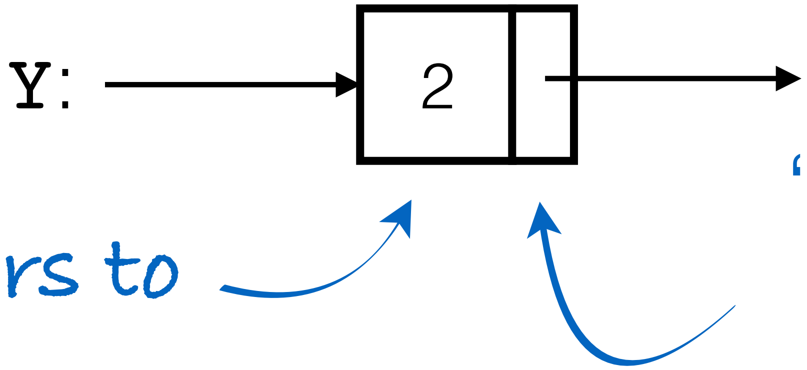


pointer

(in Java: “reference”)



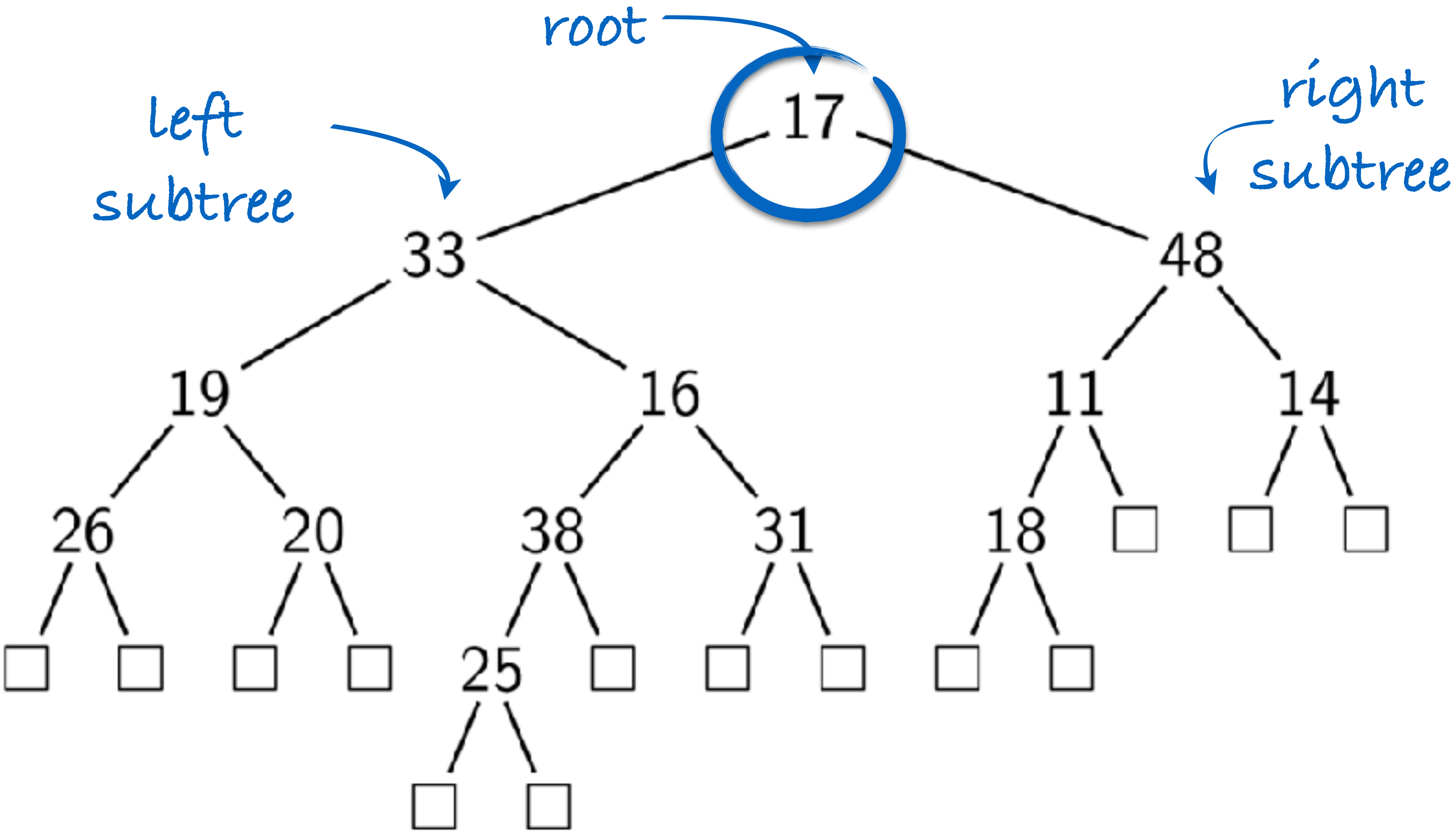
`X` is (a pointer to) the **head node** of the list



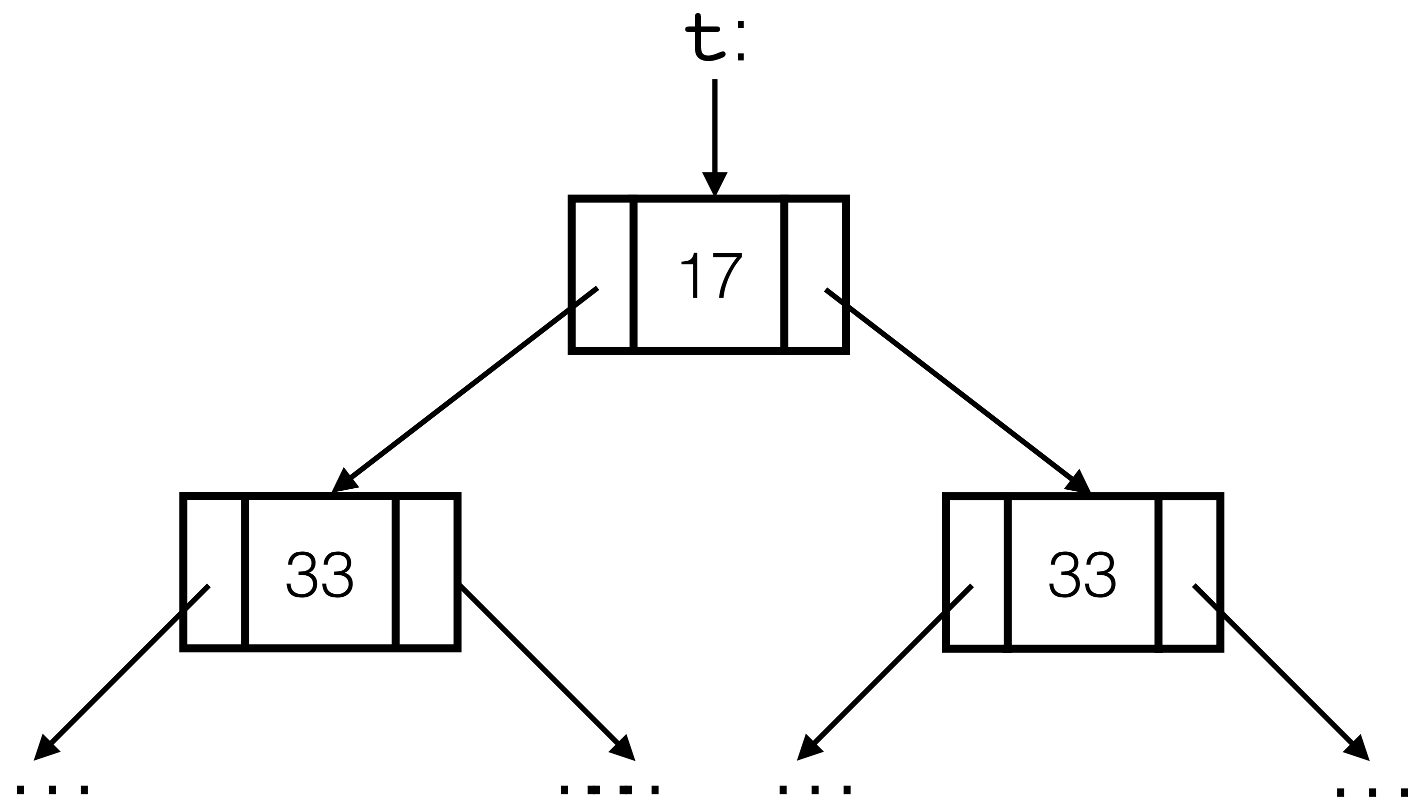
“`Y.val`” *refers to*

“`Y.next`” *refers to*

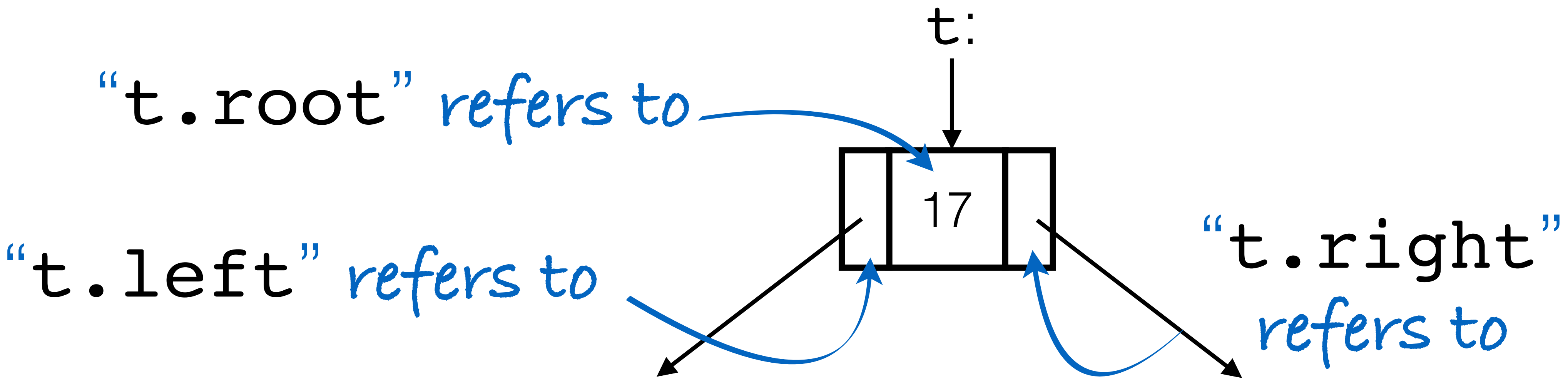
Tree Terminology



Tree Terminology



`t` is (a pointer to) the **root node** of the tree



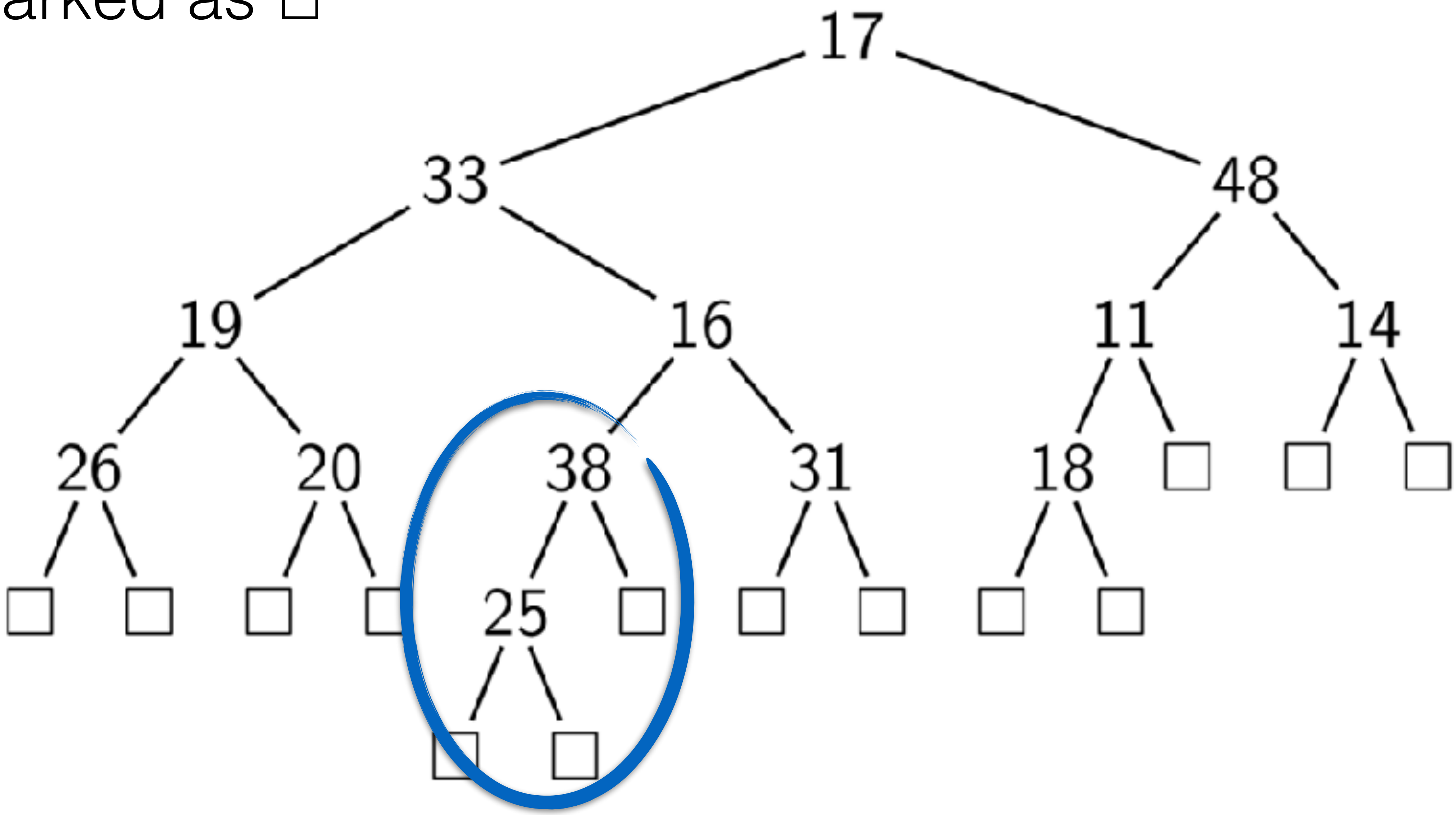
“`t.root`” *refers to*

“`t.left`” *refers to*

“`t.right`” *refers to*

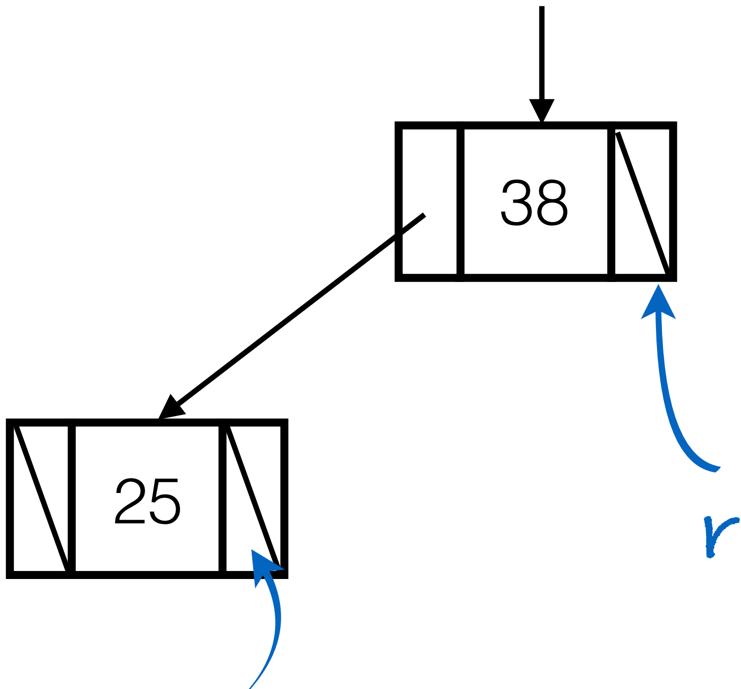
Binary Trees

- An example of a **binary tree**, with empty subtrees marked as □



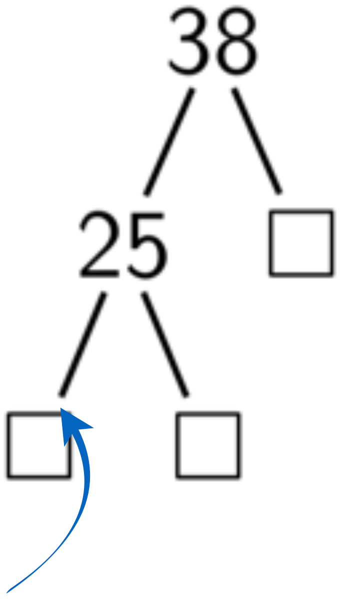
- This tree has **height** 4, the empty tree having height -1

Empty Nodes



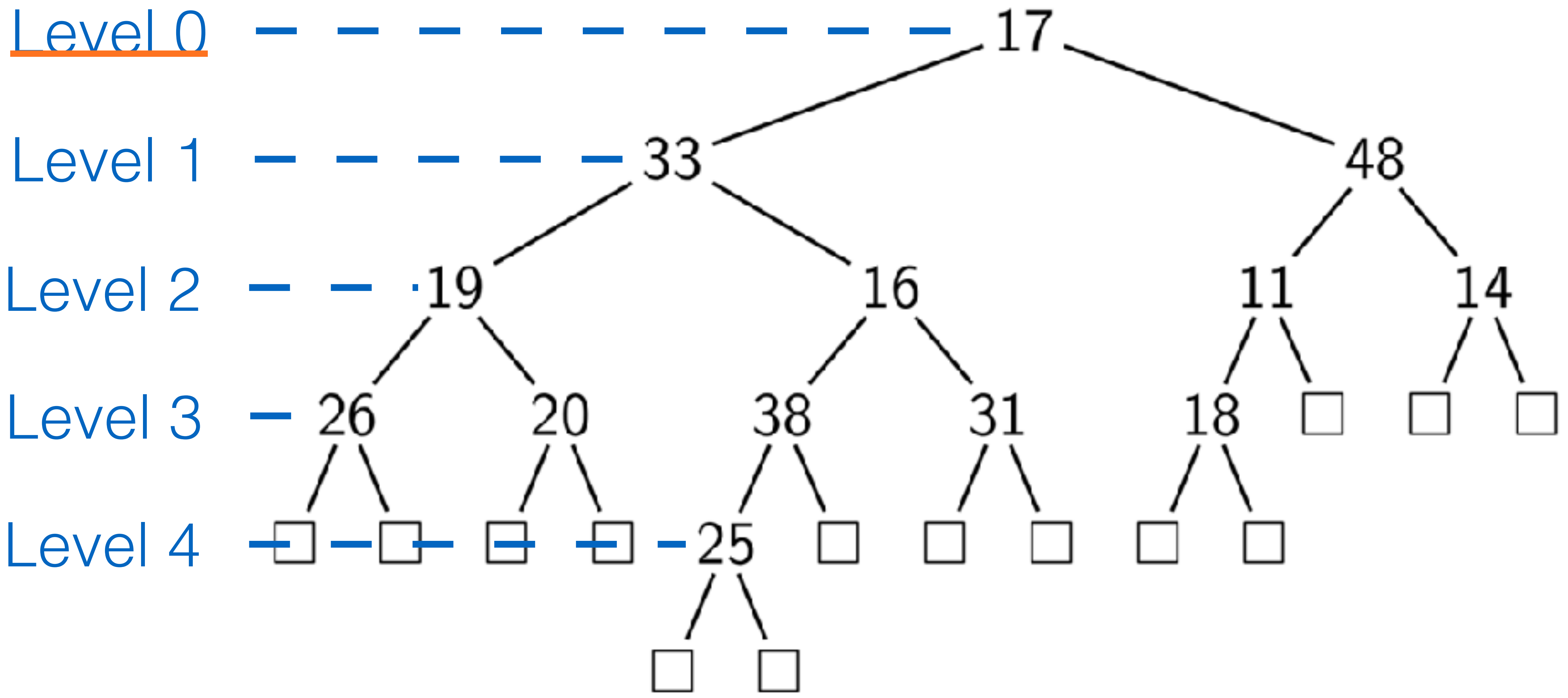
right subtree is empty

empty trees are just null pointers



null pointer

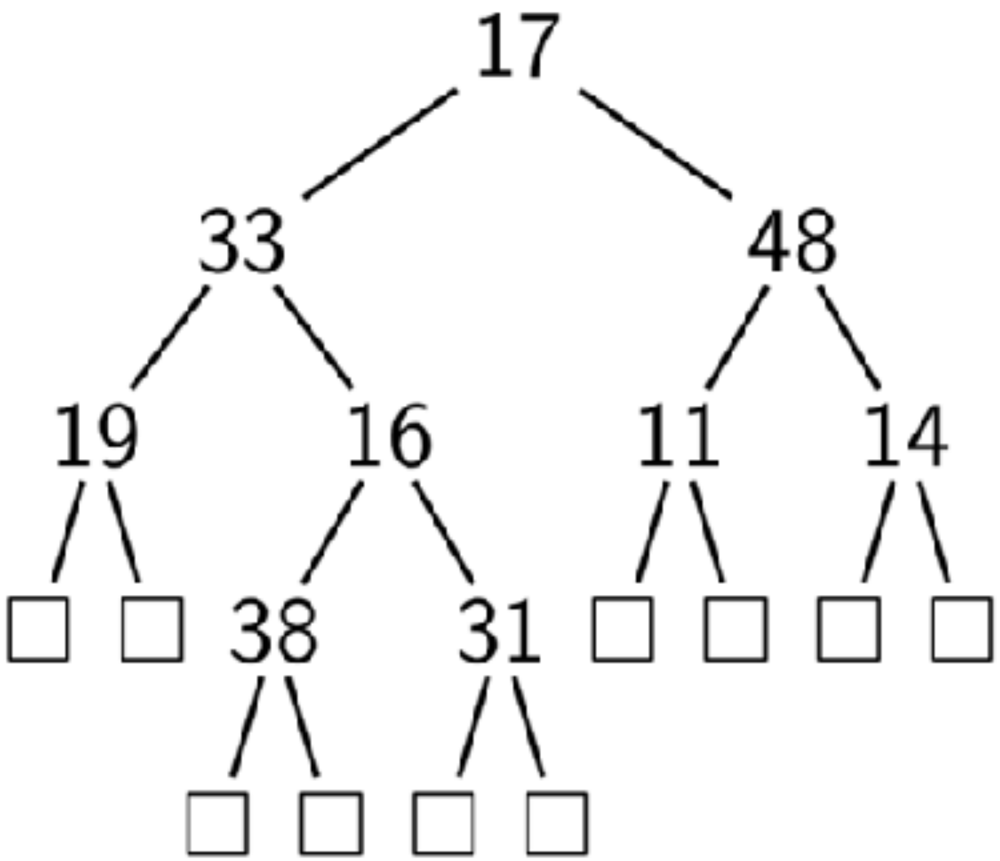
Levels and Height



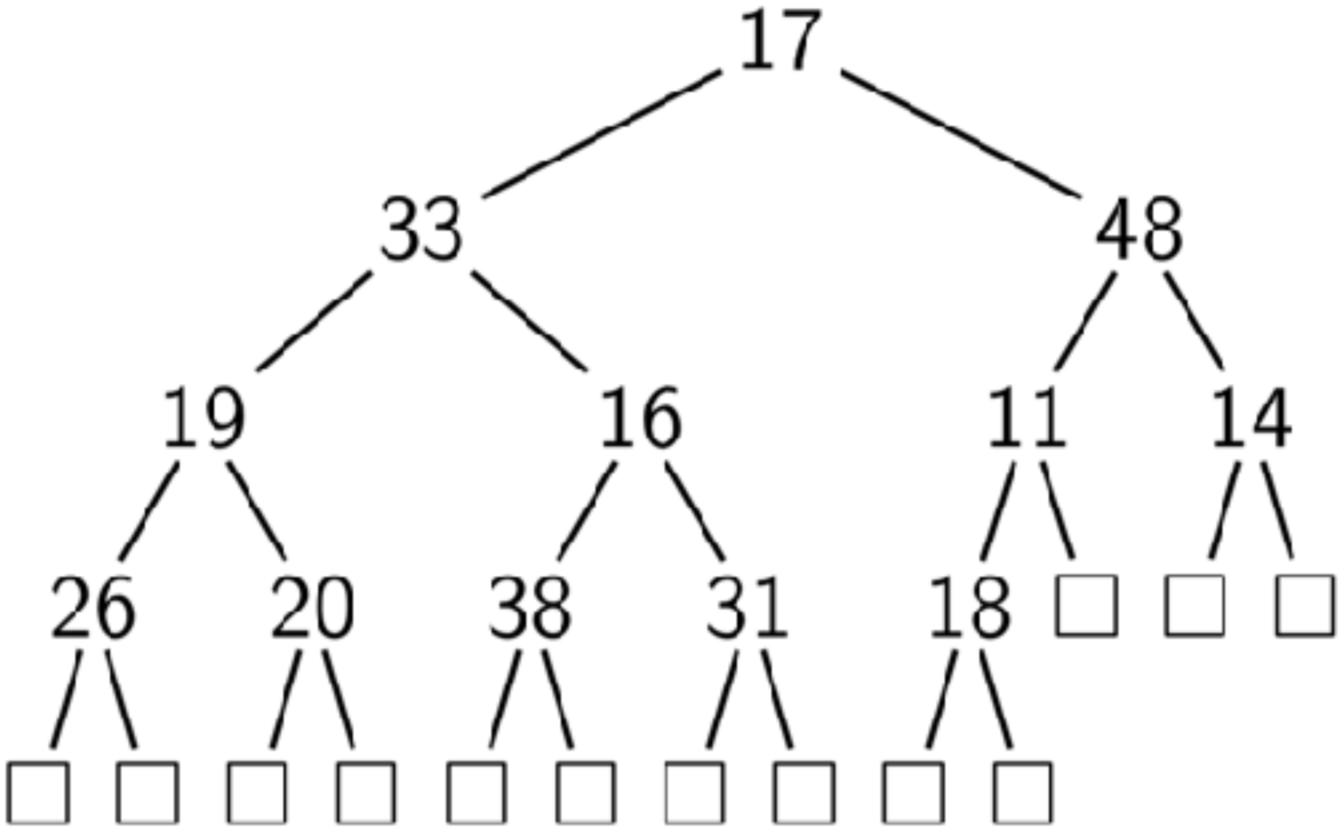
So the tree has **height** 4 (its **maximum level**)

Binary Tree Concepts

- Special trees have their **external nodes** □ only at level *h* and *h*+1 for some *h*:



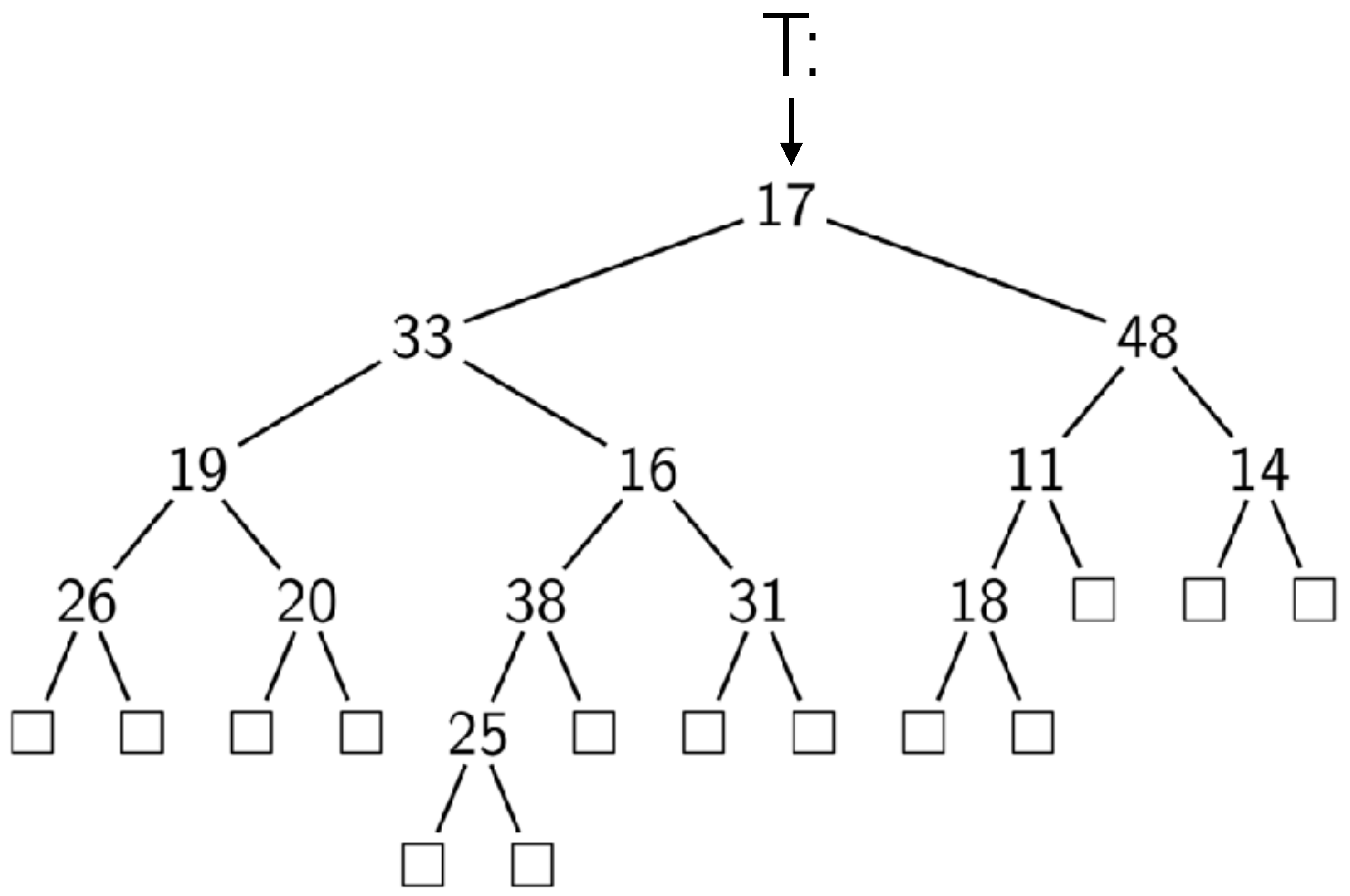
A **full** binary tree: Each node has 0 or 2 (non-empty) children.



A **complete** tree: Each level filled left to right. (Every level except perhaps the last is completely filled.)

Calculating the Height

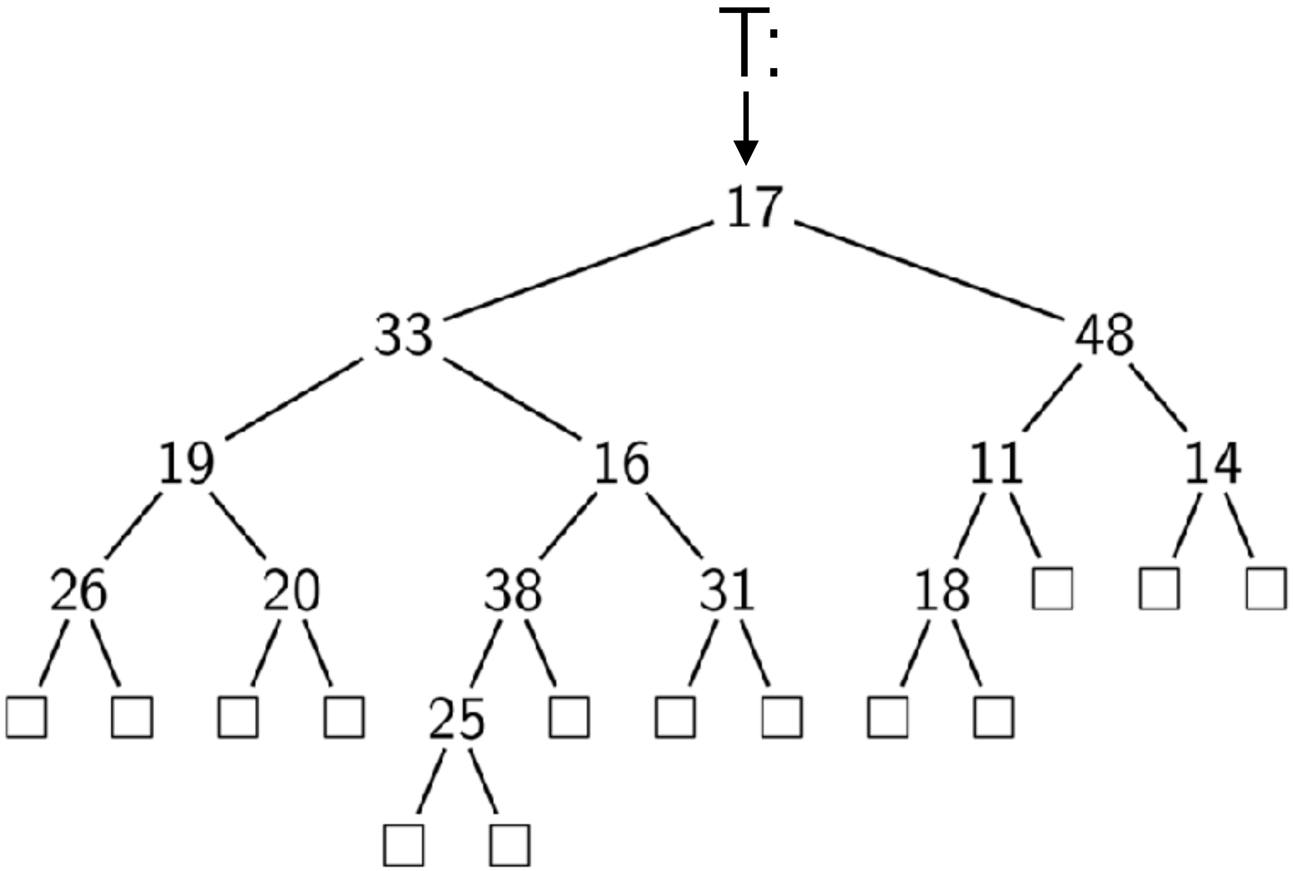
- Recursion is the natural way of calculating the **height**:



```
function HEIGHT($T$)
    if $T = null$ then
        return $-1$
    else
        return $max(\text{HEIGHT}(T.left), \text{HEIGHT}(T.right)) + 1$
```

Height Complexity

- Input size: number *n* of (internal) nodes (e.g. for T *n* is 13)
- Number of external nodes **always** *n*+1 (e.g. for T *x* is 14)
- The function HEIGHT makes one tree comparison (is T null/empty?) per node (internal and external), so altogether 2n + 1 comparisons.



Binary Tree Traversal

- **Preorder** traversal visits the root, then the left subtree, and finally the right subtree.

- **Inorder** traversal visits the left subtree, then the root, and finally the right subtree.

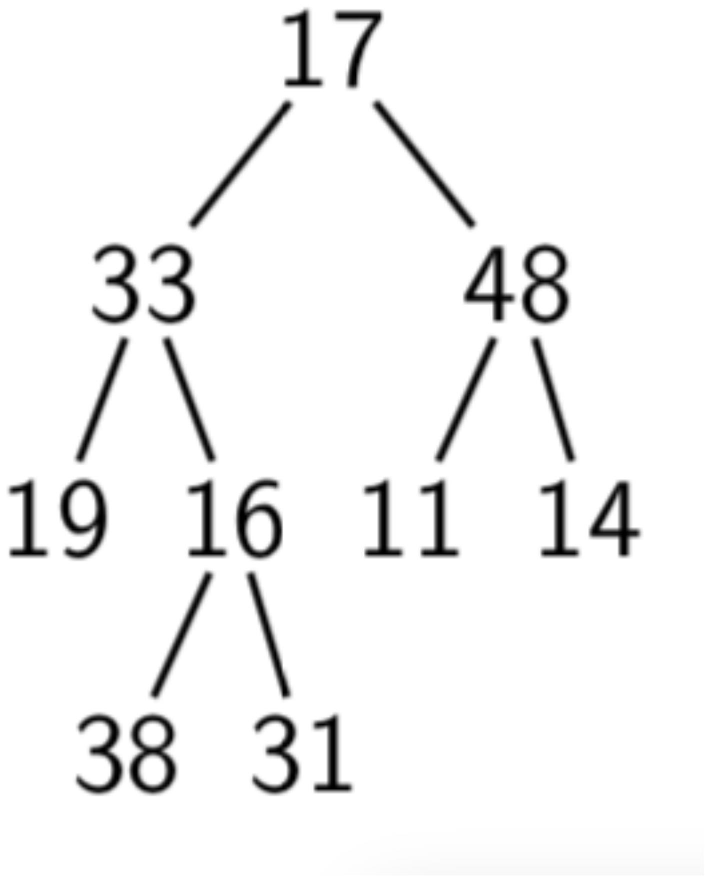
- **Postorder** traversal visits the left subtree, the right subtree, and finally the root.

- **Level-order** traversal visits the nodes, level by level, starting from the root.

Preorder Traversal

Visit order:

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



先左后右

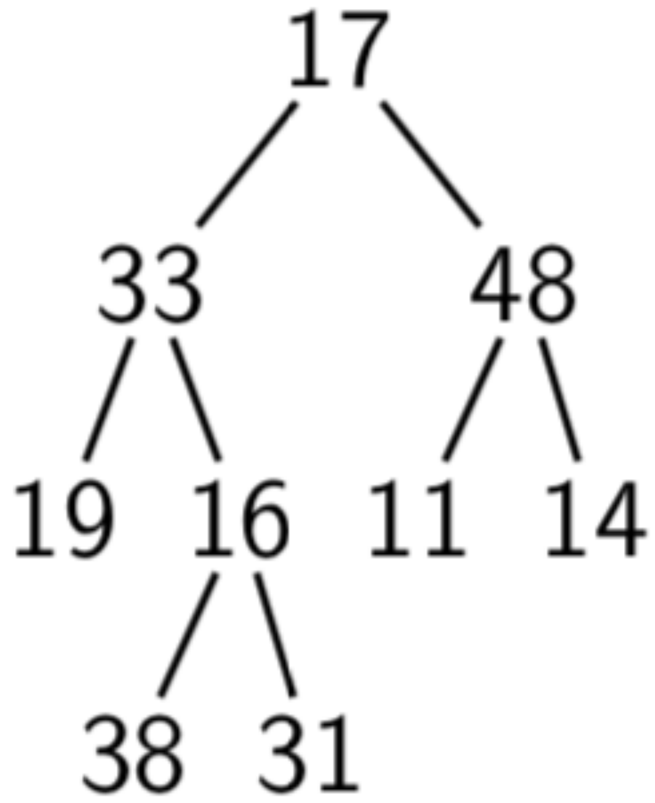
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17

```
procedure PREORDERTRAVERSE($T$)
    if $T \neq null$ then
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        PREORDERTRAVERSE($T.left$)
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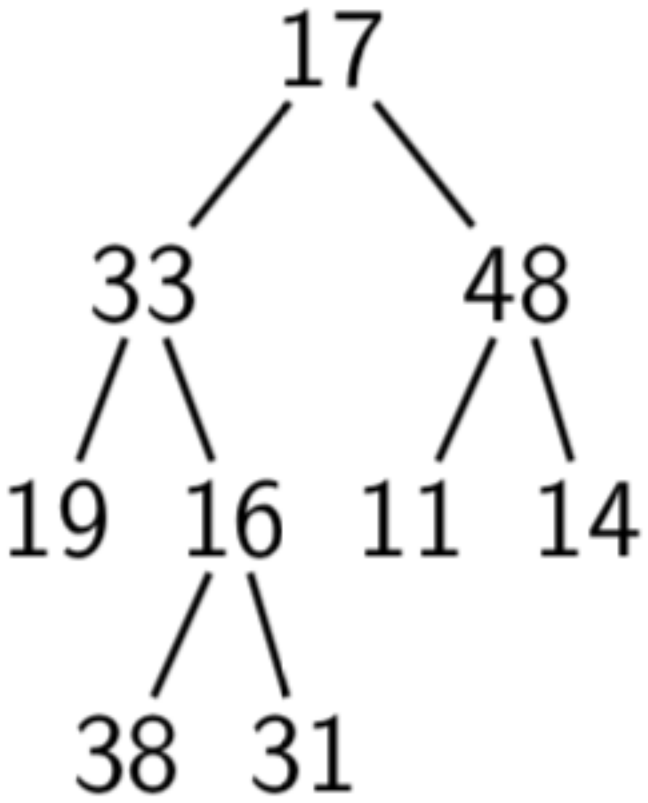


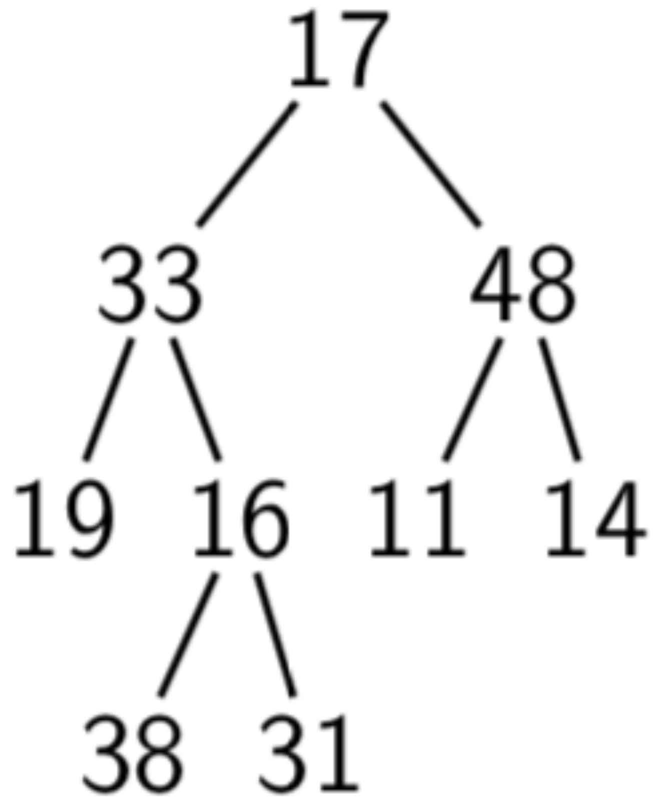
PREORDERTRAVERSE(33)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



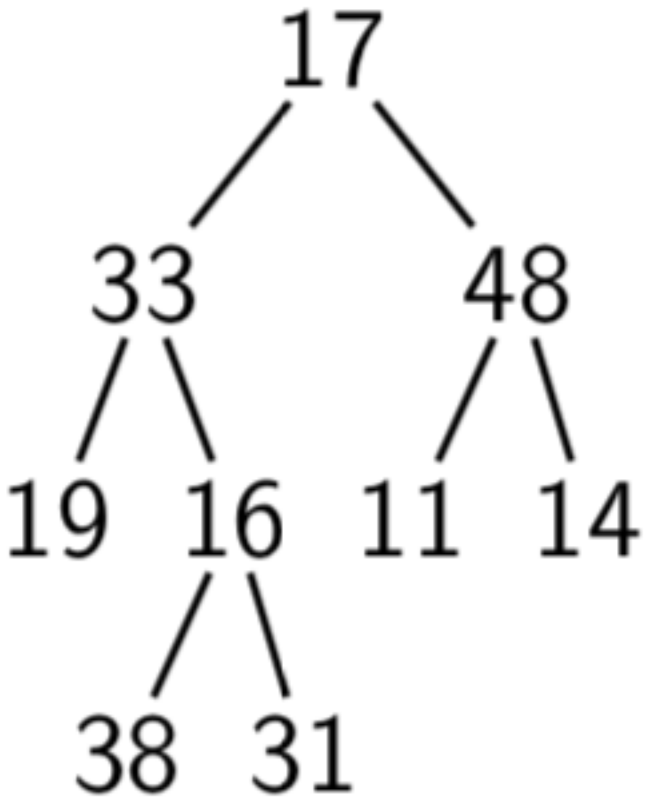
PREORDERTRAVERSE(33)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33

```
procedure PREORDERTRAVERSE(T)
    if T ≠ null then
        visit T.root
        PREORDERTRAVERSE(T.left)
        PREORDERTRAVERSE(T.right)
```



PREORDERTRAVERSE(19)

PREORDERTRAVERSE(33)

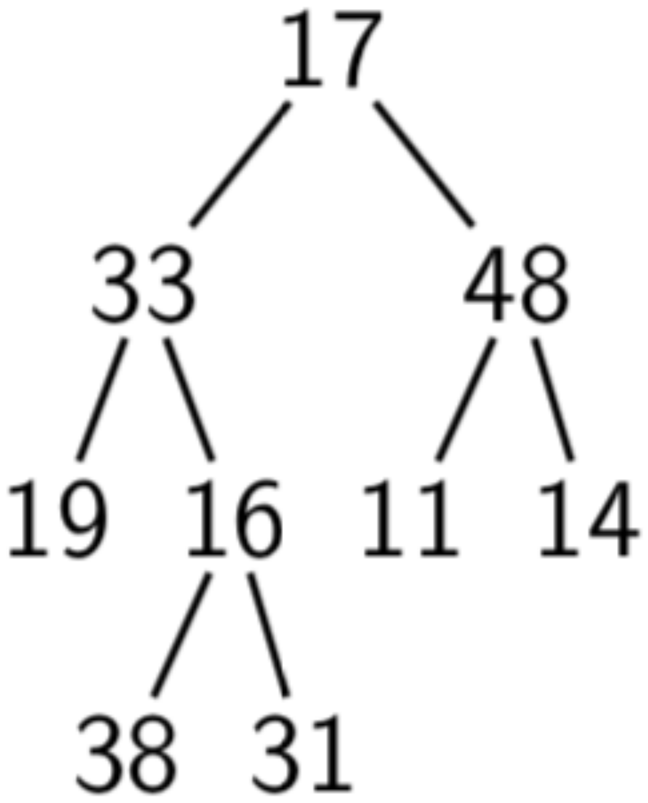
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



PreorderTraverse(19)
PreorderTraverse(33)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

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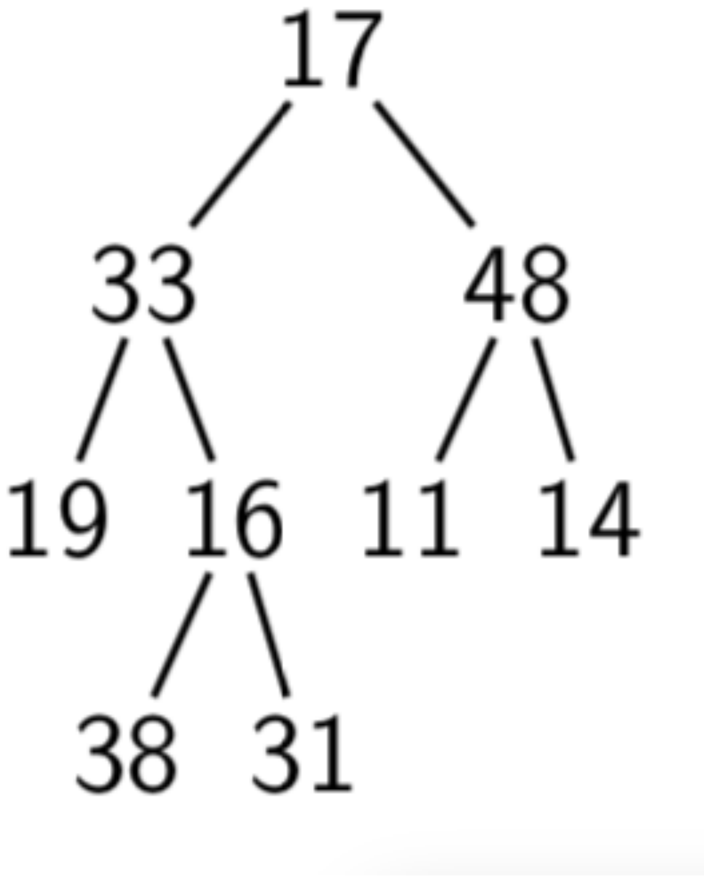

PreorderTraverse(null)

PreorderTraverse(19)

PreorderTraverse(33)

PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

```
procedure PREORDERTRAVERSE($T$)
    if $T \neq null$ then
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        PREORDERTRAVERSE($T.left$)
        PREORDERTRAVERSE($T.right$)
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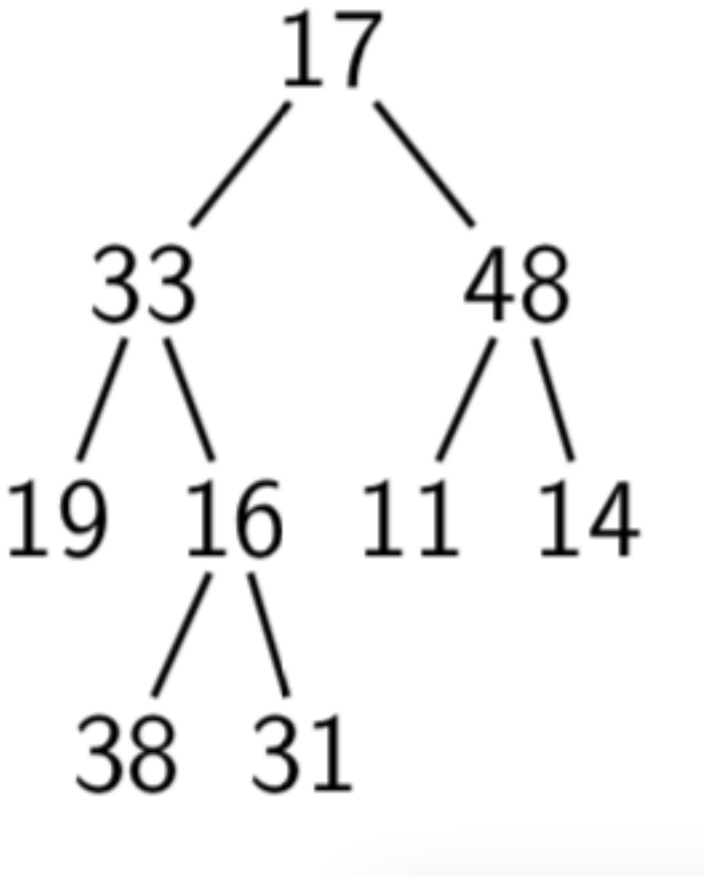

PREORDERTRAVERSE(19)

PREORDERTRAVERSE(33)

PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

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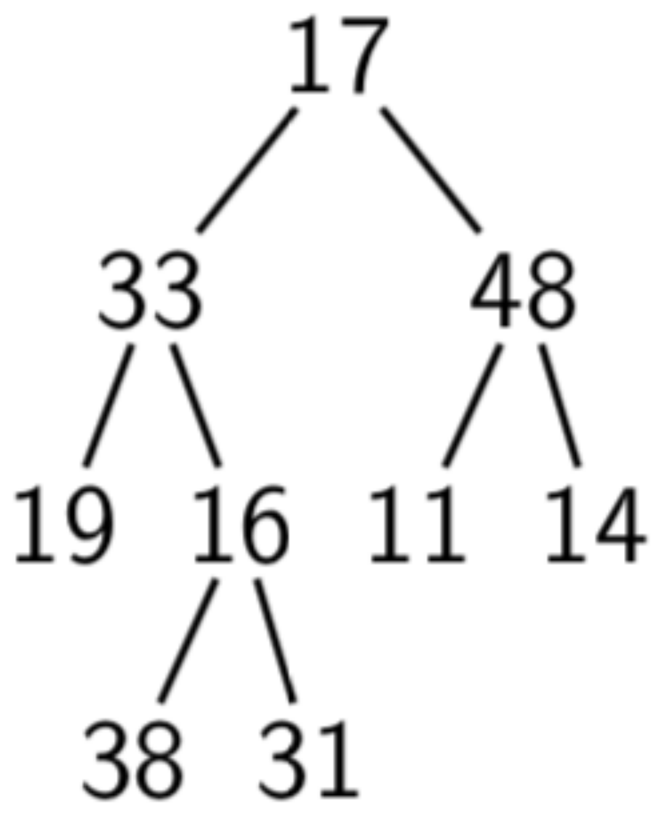

PreorderTraverse(null)

PreorderTraverse(19)

PreorderTraverse(33)

PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

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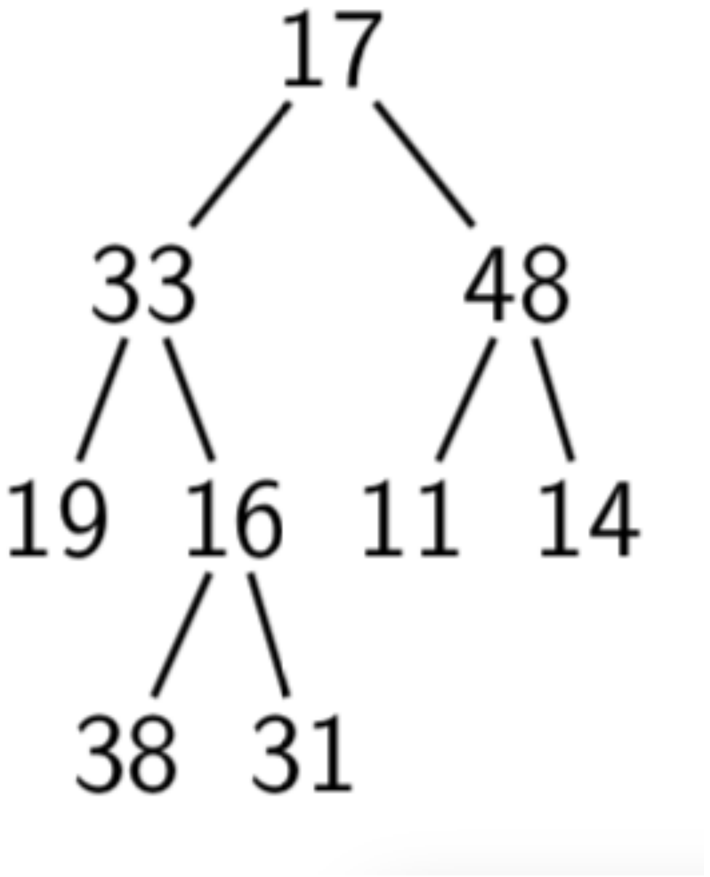

PreorderTraverse(19)
PreorderTraverse(33)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

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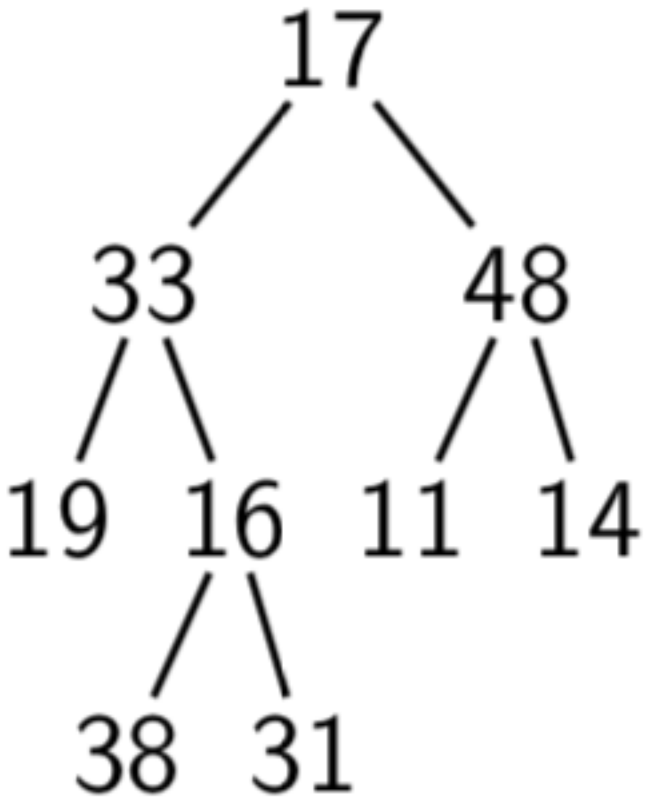


PreorderTraverse(33)

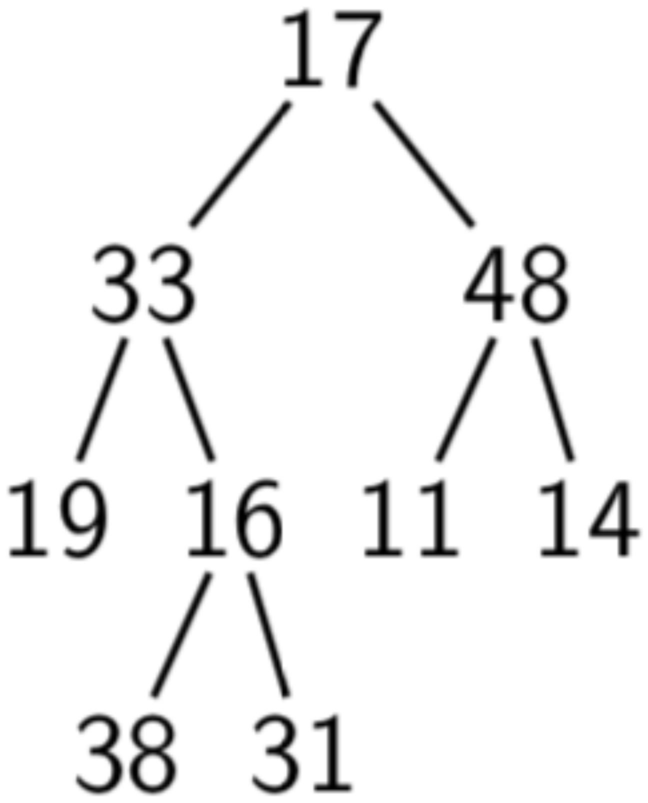
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



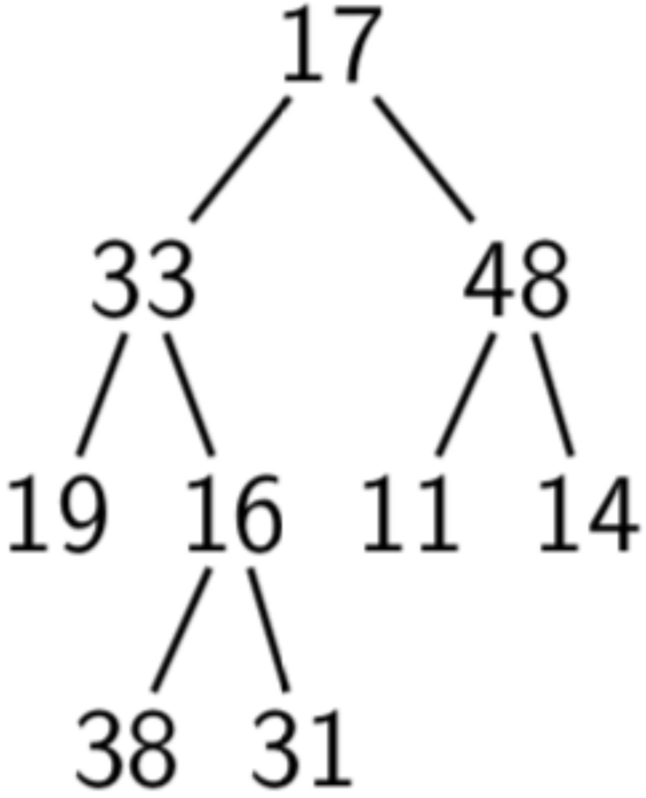
PreorderTraverse(16)
PreorderTraverse(33)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



PreorderTraverse(16)

PreorderTraverse(33)

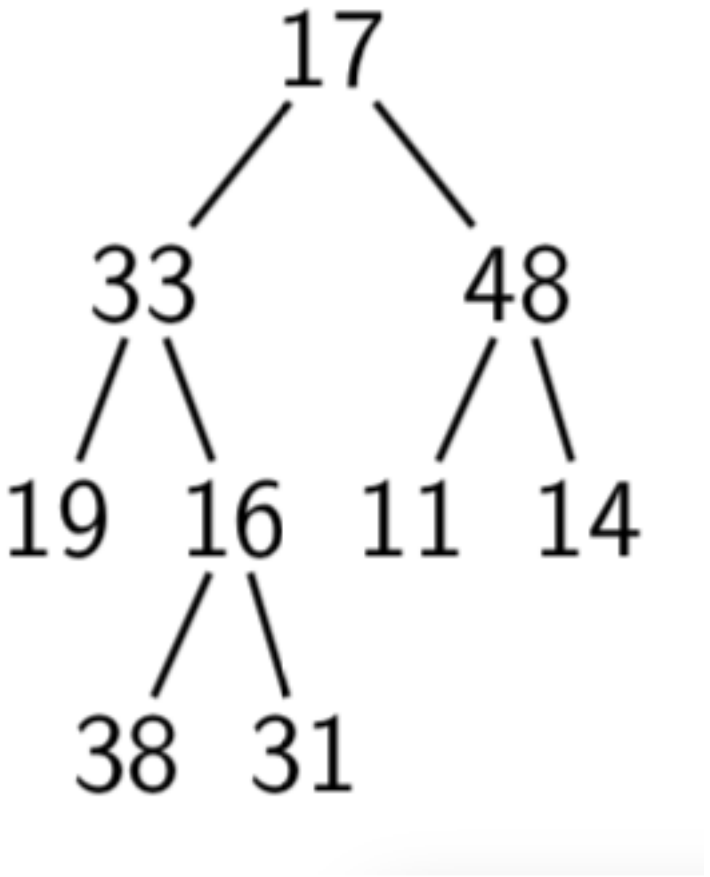
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



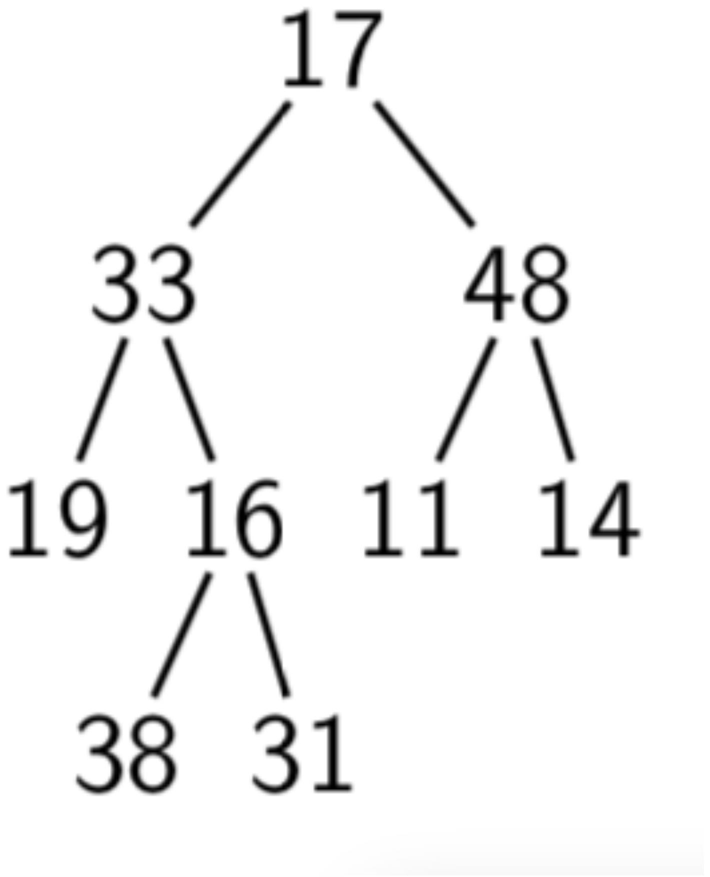
PreorderTraverse(38)
PreorderTraverse(16)
PreorderTraverse(33)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



PreorderTraverse(38)

PreorderTraverse(16)

PreorderTraverse(33)

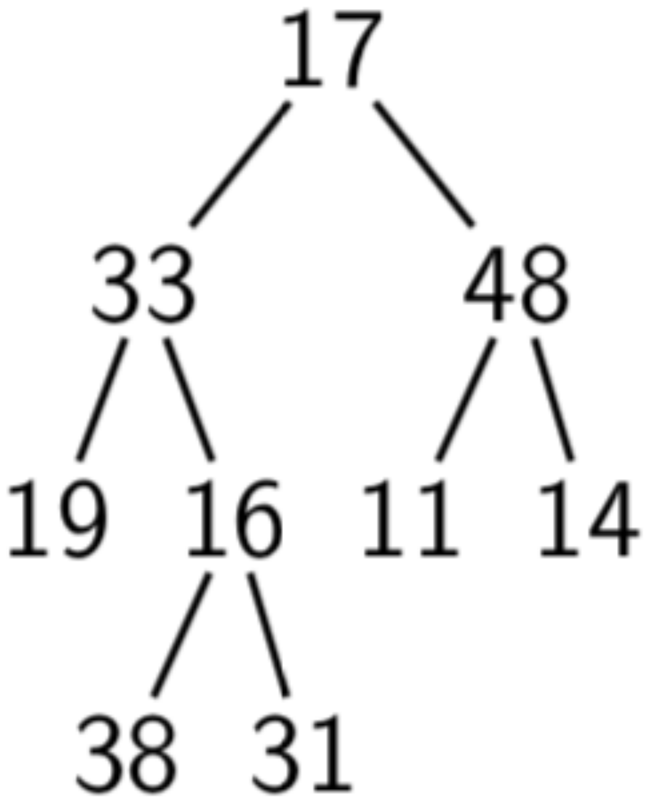
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38

```
procedure PREORDERTRAVERSE($T$)
    if $T \neq null$ then
        visit $T.root$
        PREORDERTRAVERSE($T.left$)
        PREORDERTRAVERSE($T.right$)
```



PREORDERTRAVERSE(38)

PREORDERTRAVERSE(16)

PREORDERTRAVERSE(33)

PREORDERTRAVERSE(17)

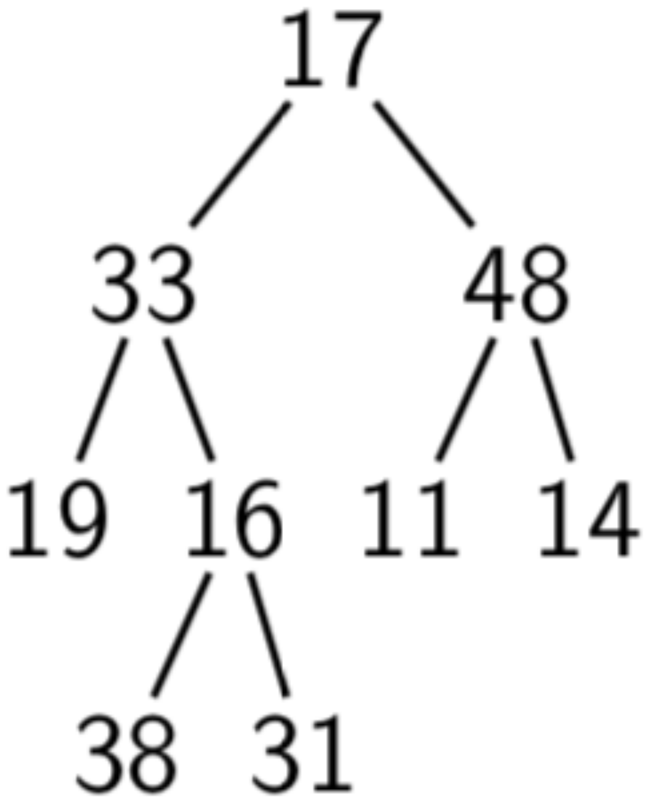
Call Stack

(…skipping the calls to PREORDERTRAVERSE(null)…)

Preorder Traversal

Visit order: 17 33 19 16 38

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



PreorderTraverse(16)

PreorderTraverse(33)

PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38

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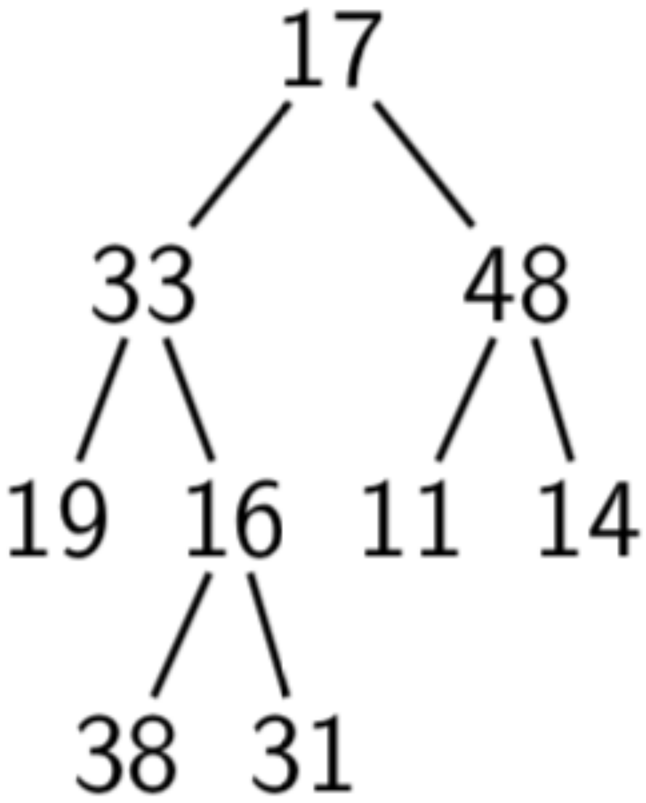


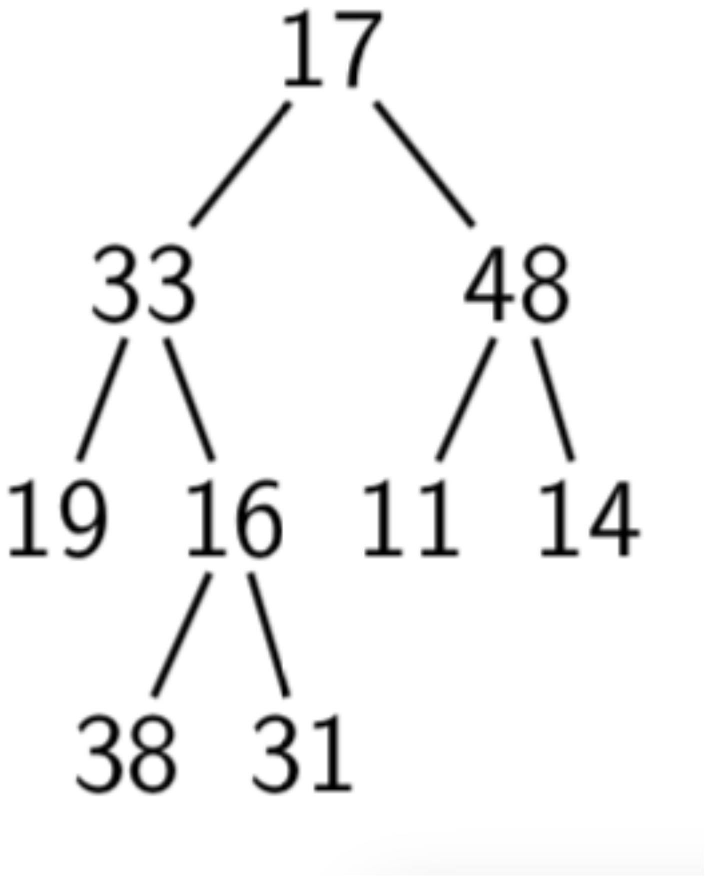
PreorderTraverse(31)
PreorderTraverse(16)
PreorderTraverse(33)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31

```
procedure PREORDERTRAVERSE(T)
    if $T \neq null$ then
        visit T.root
        PREORDERTRAVERSE(T.left)
        PREORDERTRAVERSE(T.right)
```



PREORDERTRAVERSE(31)

PREORDERTRAVERSE(16)

PREORDERTRAVERSE(33)

PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31

```
procedure PREORDERTRAVERSE($T$)
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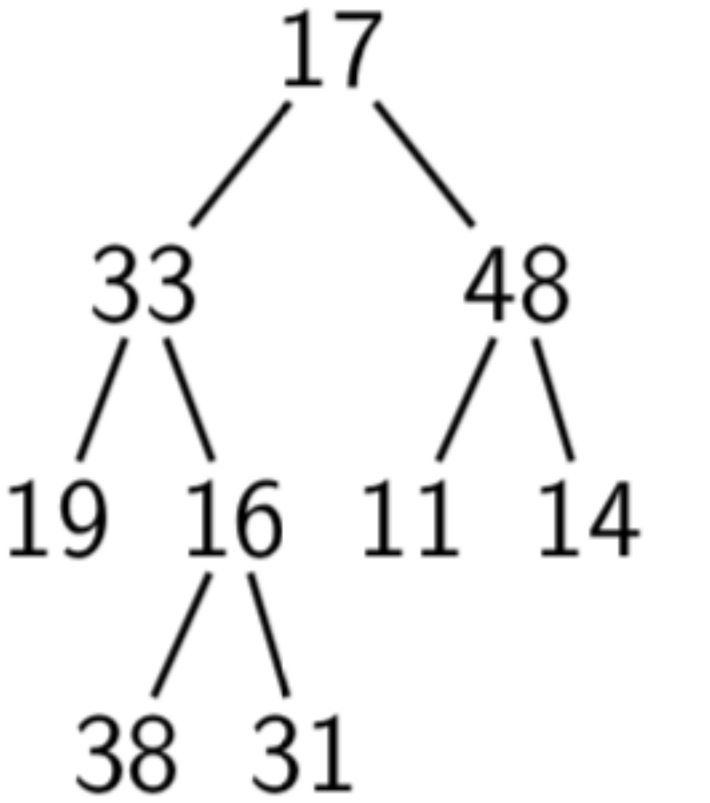


PREORDERTRAVERSE(31)

PREORDERTRAVERSE(16)

PREORDERTRAVERSE(33)

PREORDERTRAVERSE(17)

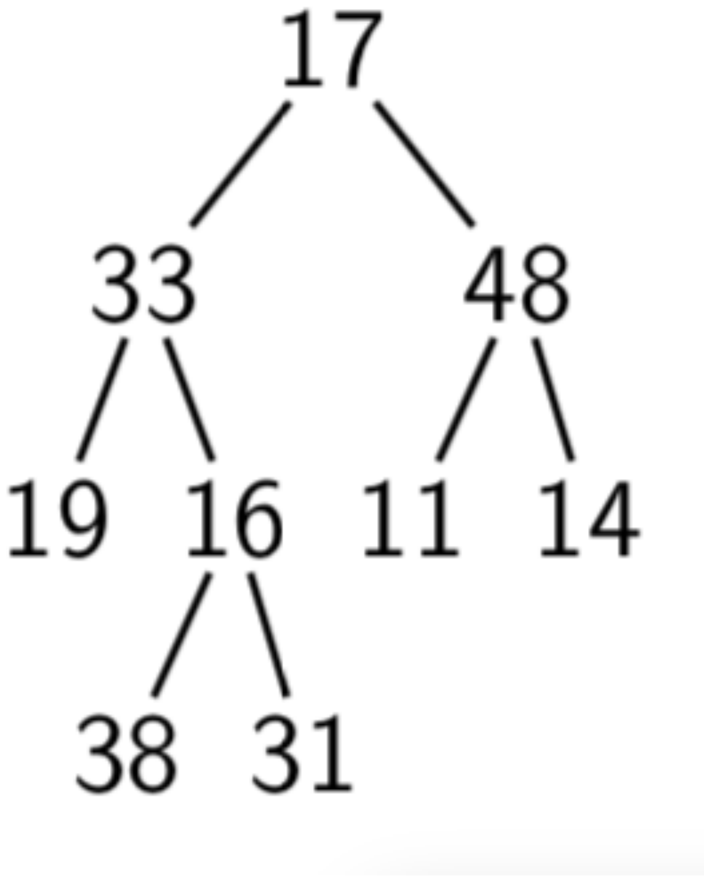
Call Stack

(…skipping the calls to PREORDERTRAVERSE(null)…)

Preorder Traversal

Visit order: 17 33 19 16 38 31

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



PreorderTraverse(16)

PreorderTraverse(33)

PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31

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procedure PreorderTraverse(T)
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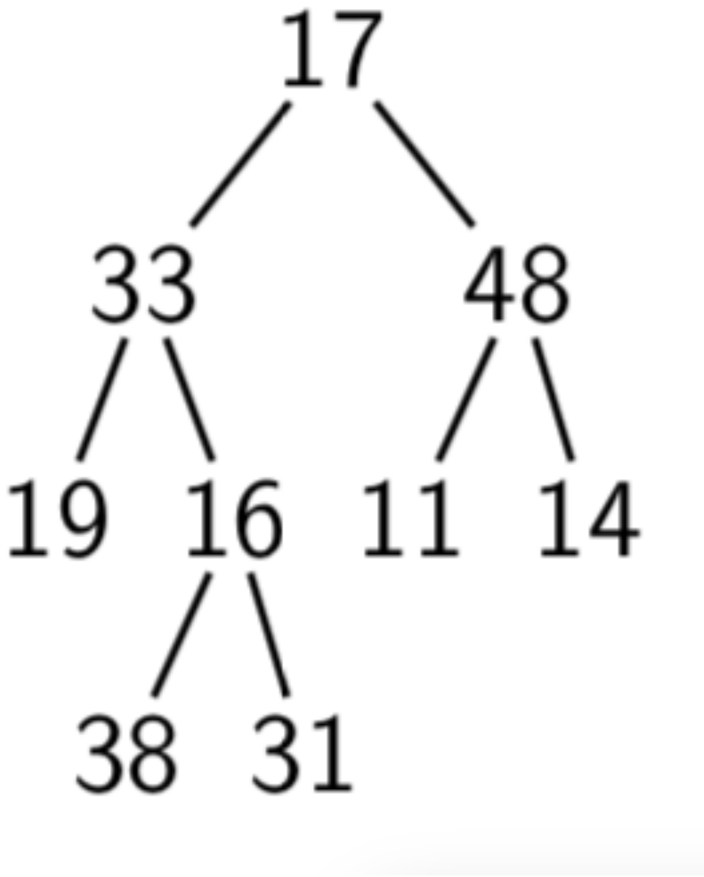

PREORDERTRAVERSE(33)

PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
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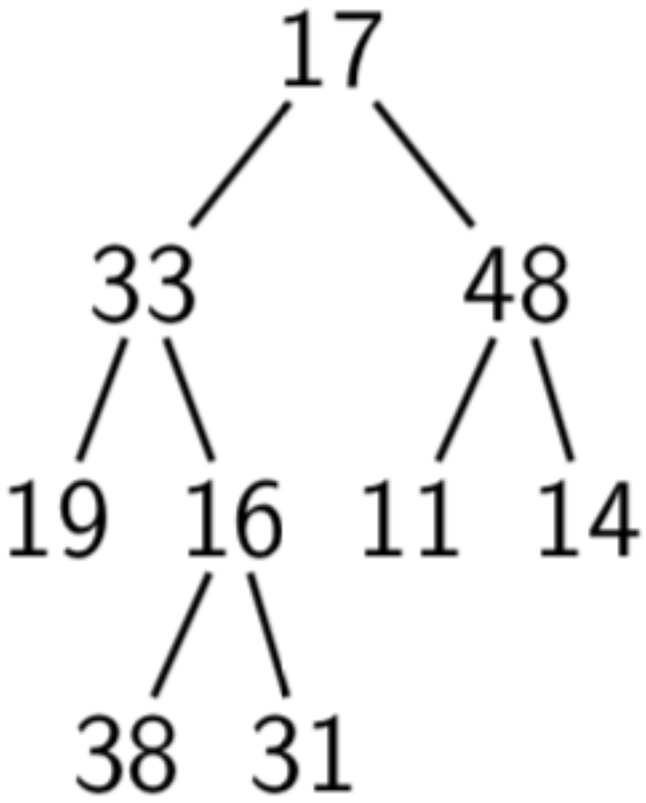

PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
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        PreorderTraverse($T.left$)
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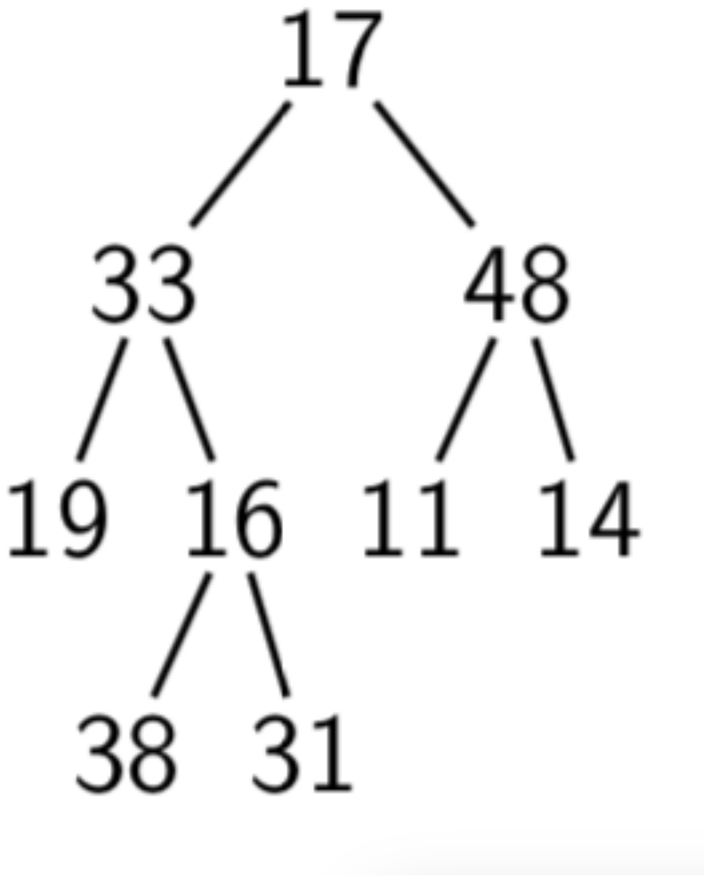


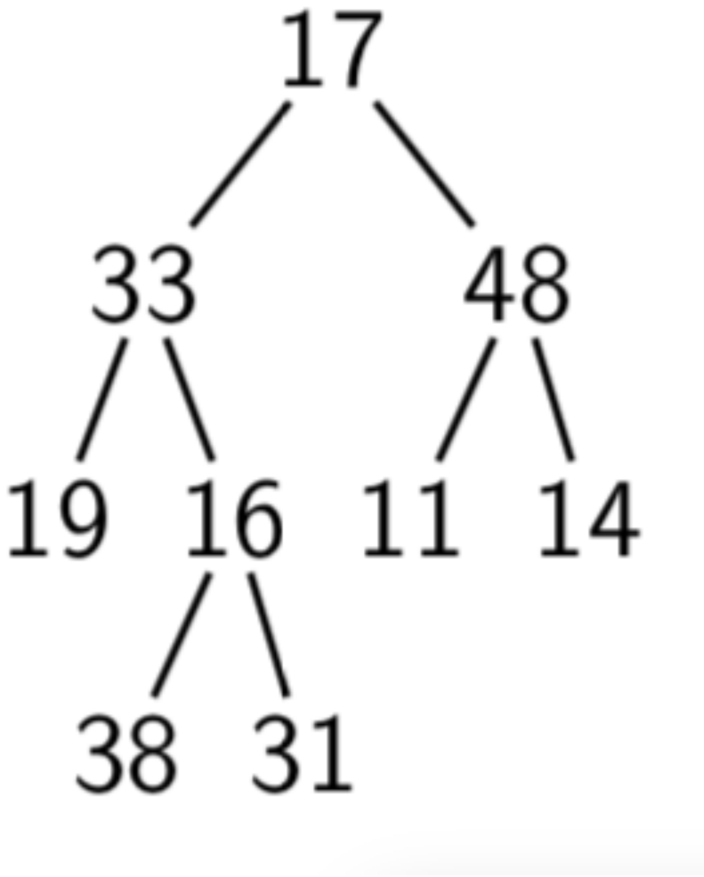
PreorderTraverse(48)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



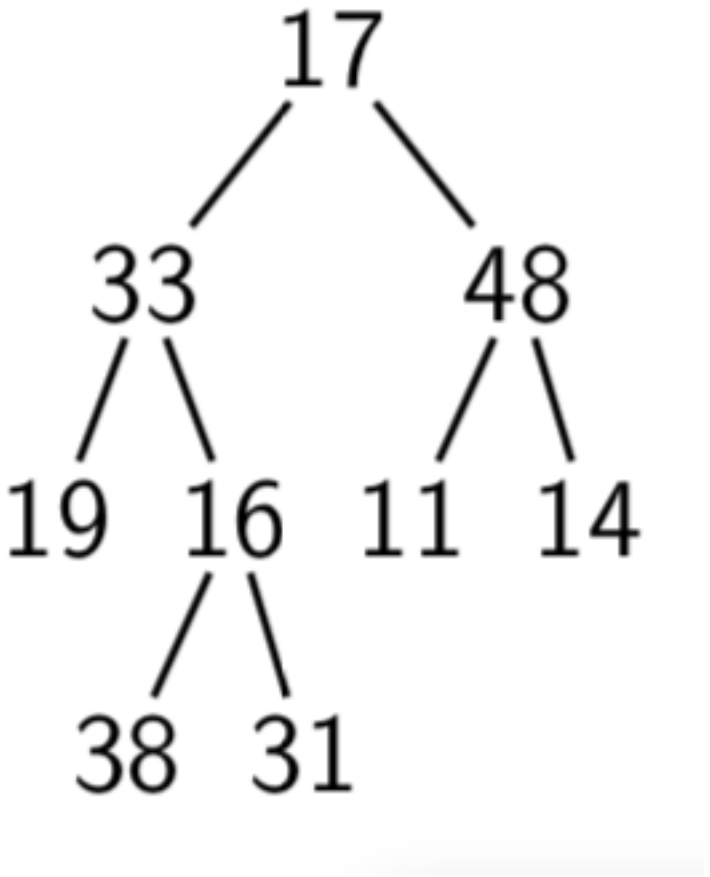
PREORDERTRAVERSE(48)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



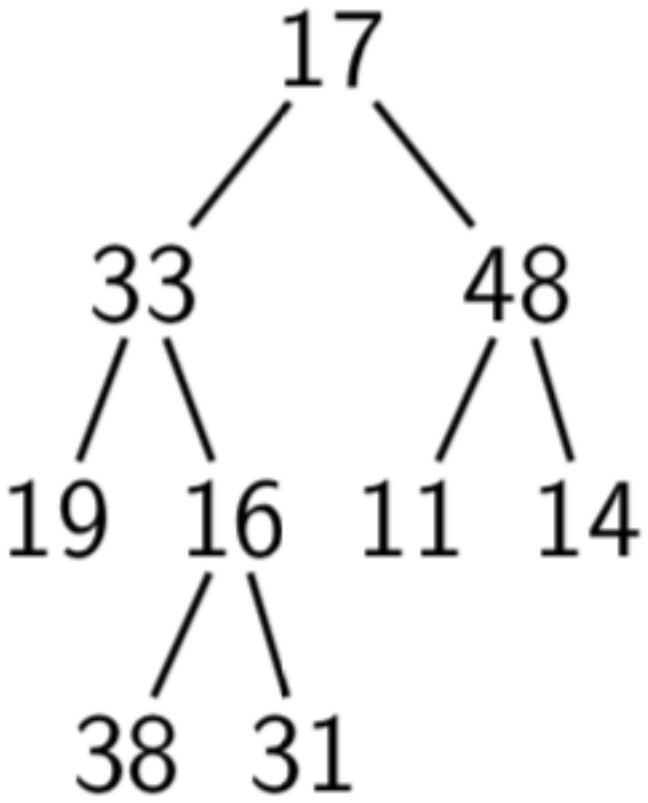
PreorderTraverse(11)
PreorderTraverse(48)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11

```
procedure PREORDERTRAVERSE($T$)
    if $T \neq null$ then
        visit $T.root$
        PREORDERTRAVERSE($T.left$)
        PREORDERTRAVERSE($T.right$)
```



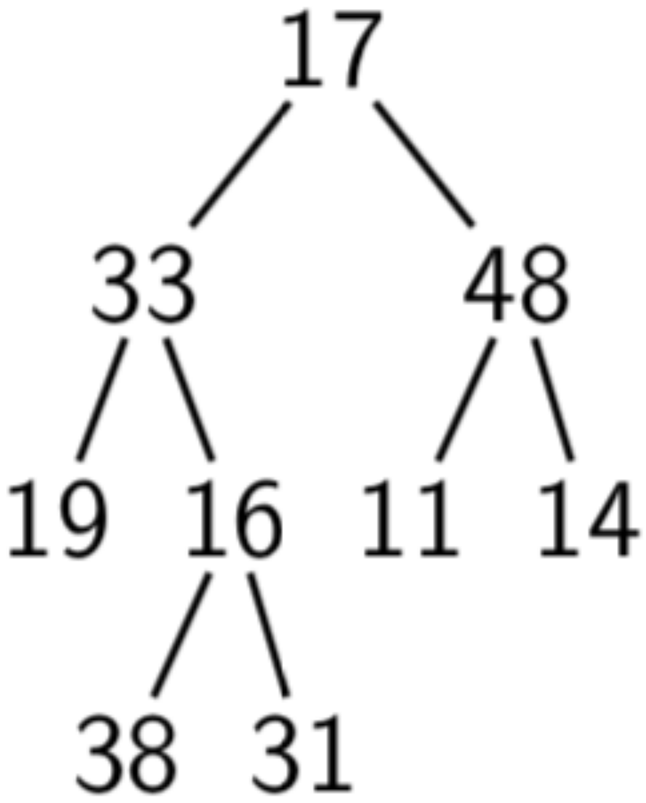
PREORDERTRAVERSE(11)
PREORDERTRAVERSE(48)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



PreorderTraverse(11)
PreorderTraverse(48)
PreorderTraverse(17)

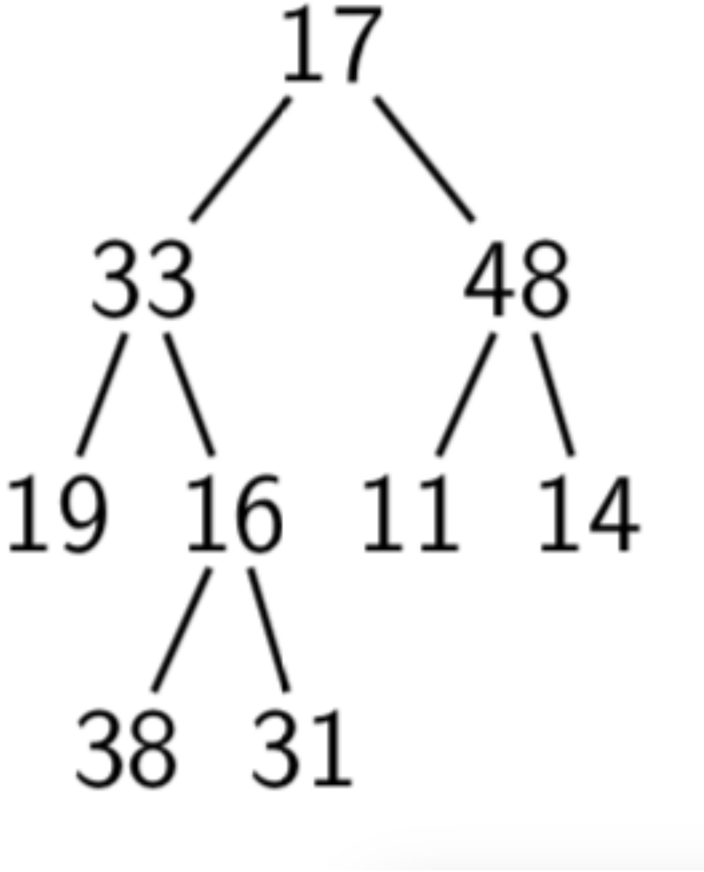
Call Stack

(…skipping the calls to PreorderTraverse(null)…)

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11

```
procedure PREORDERTRAVERSE($T$)
    if $T \neq null$ then
        visit $T.root$
        PREORDERTRAVERSE($T.left$)
        PREORDERTRAVERSE($T.right$)
```



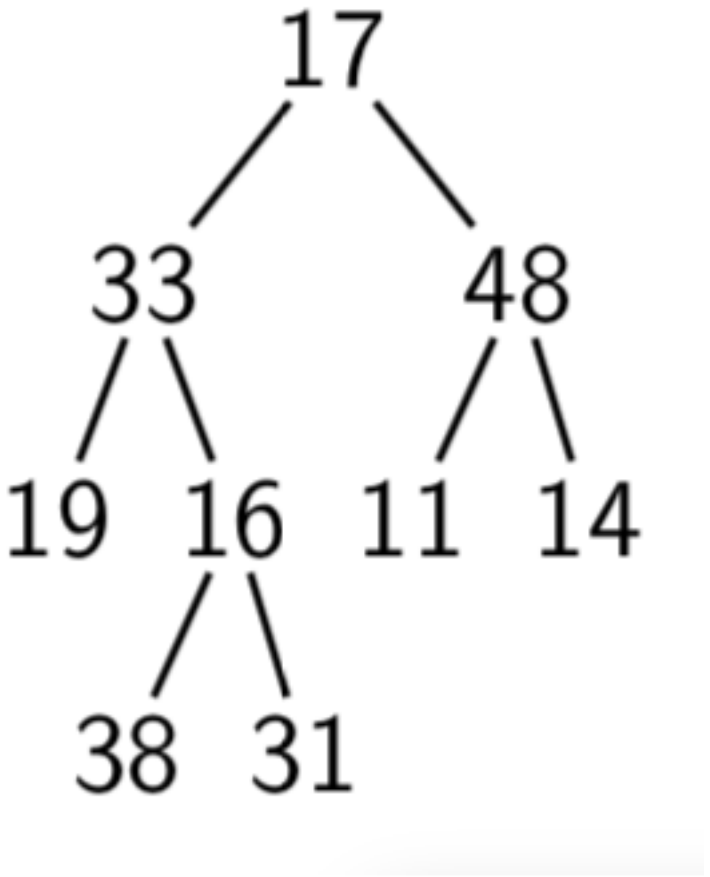
PREORDERTRAVERSE(48)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



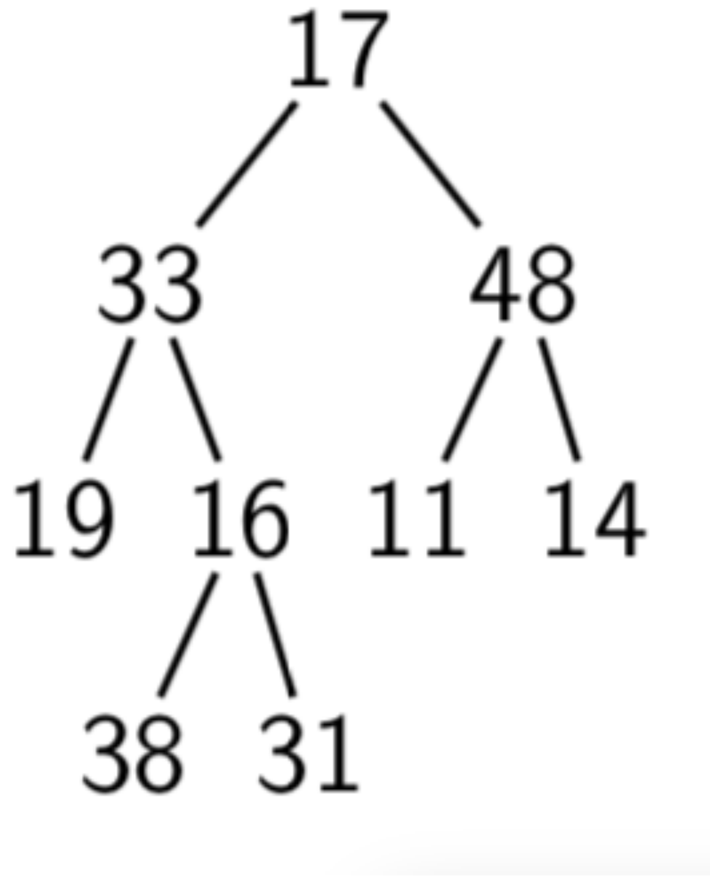
PREORDERTRAVERSE(14)
PREORDERTRAVERSE(48)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11 14

```
procedure PREORDERTRAVERSE(T)
    if $T \neq null$ then
        visit T.root
        PREORDERTRAVERSE(T.left)
        PREORDERTRAVERSE(T.right)
```



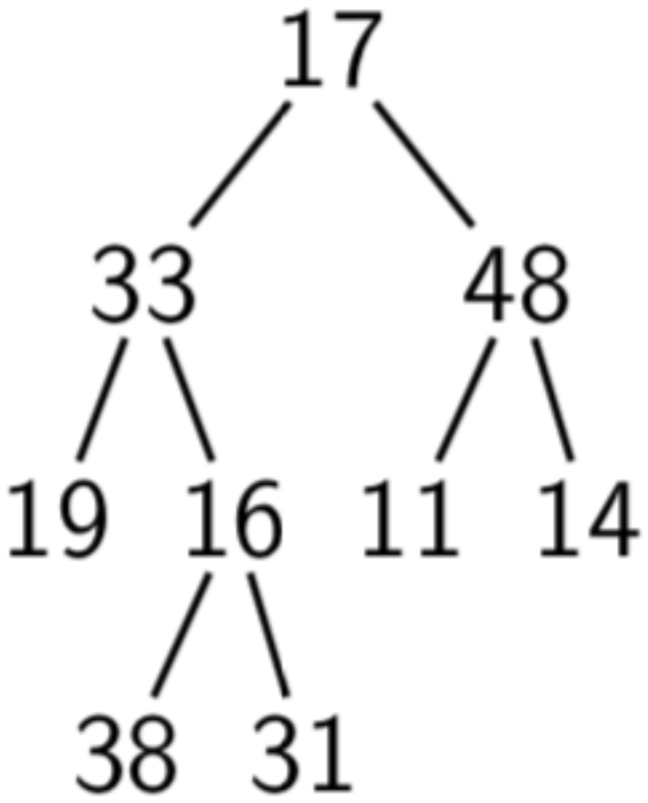
PREORDERTRAVERSE(14)
PREORDERTRAVERSE(48)
PREORDERTRAVERSE(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11 14

```
procedure PreorderTraverse($T$)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse($T.left$)
        PreorderTraverse($T.right$)
```



PreorderTraverse(14)
PreorderTraverse(48)
PreorderTraverse(17)

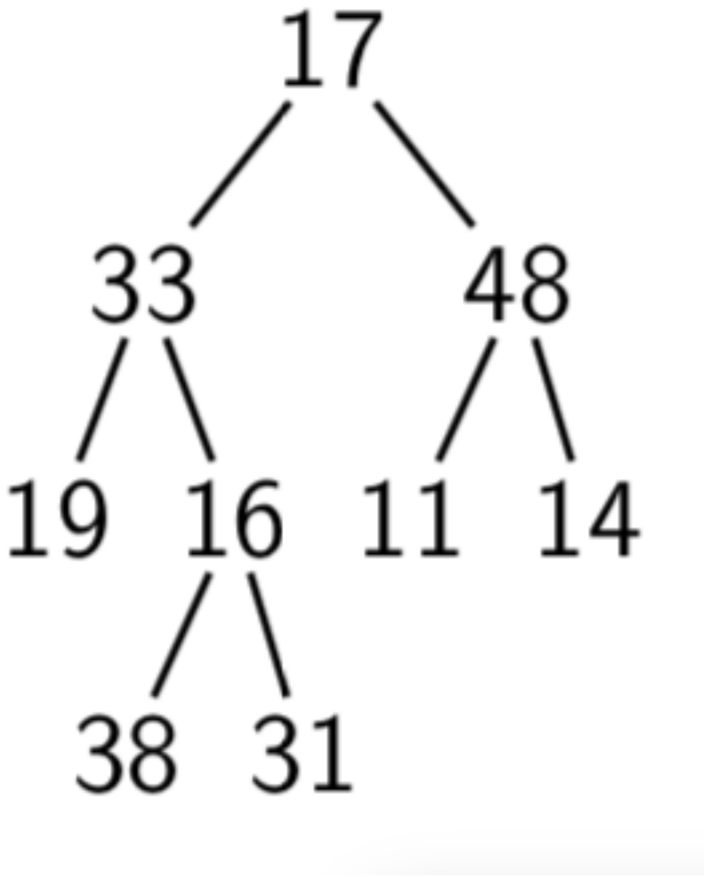
Call Stack

(…skipping the calls to PreorderTraverse(null)…)

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11 14

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit T.root
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



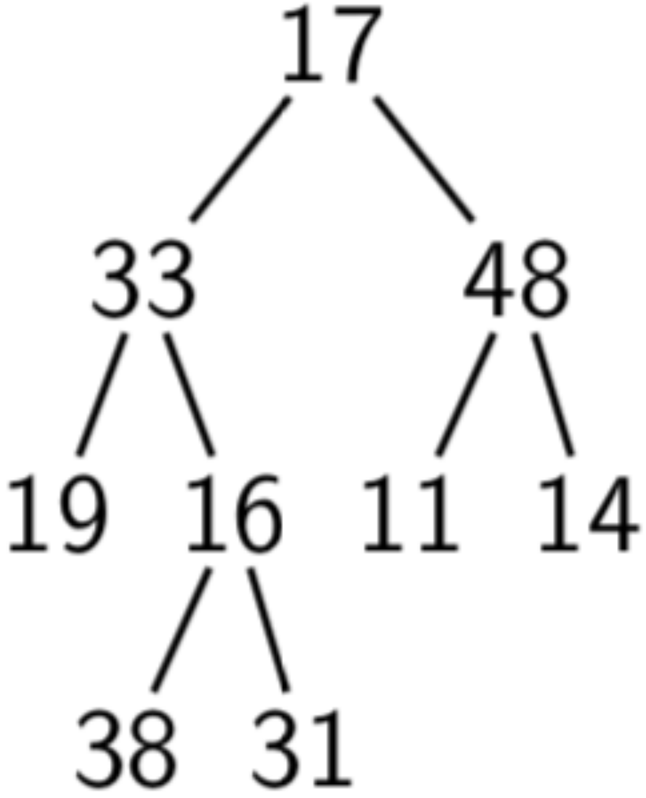
PreorderTraverse(48)
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11 14

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```



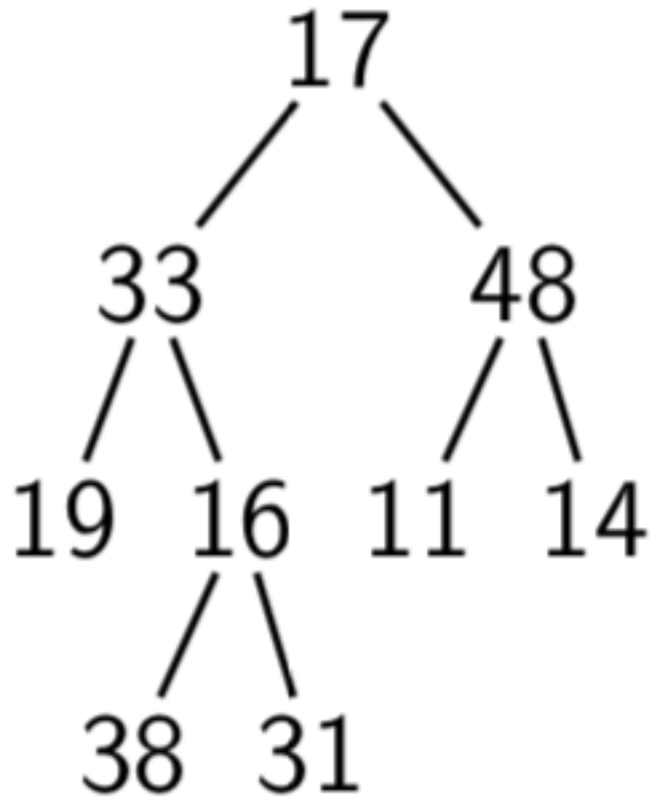
PreorderTraverse(17)

Call Stack

Preorder Traversal

Visit order: 17 33 19 16 38 31 48 11 14

```
procedure PreorderTraverse(T)
    if $T \neq null$ then
        visit $T.root$
        PreorderTraverse(T.left)
        PreorderTraverse(T.right)
```

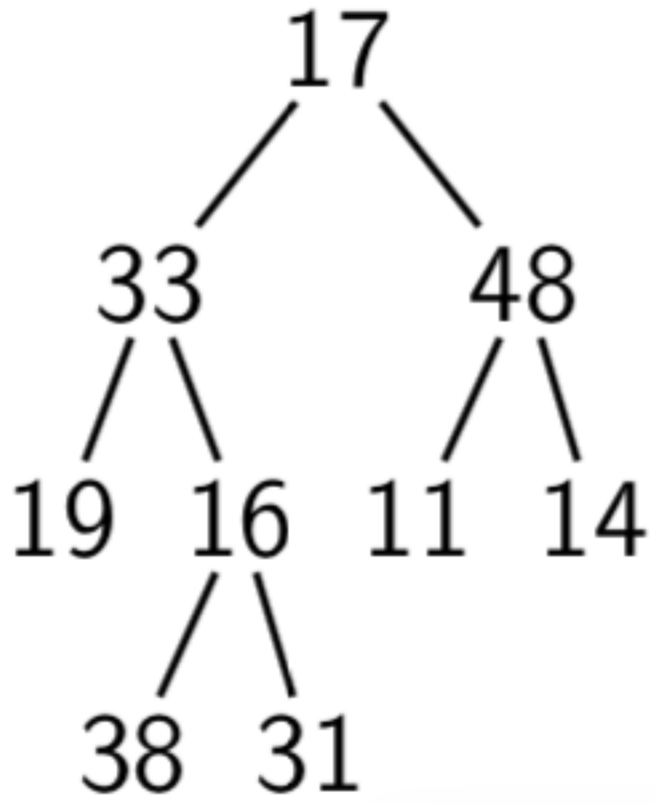


Call Stack

Inorder Traversal

Visit order:

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



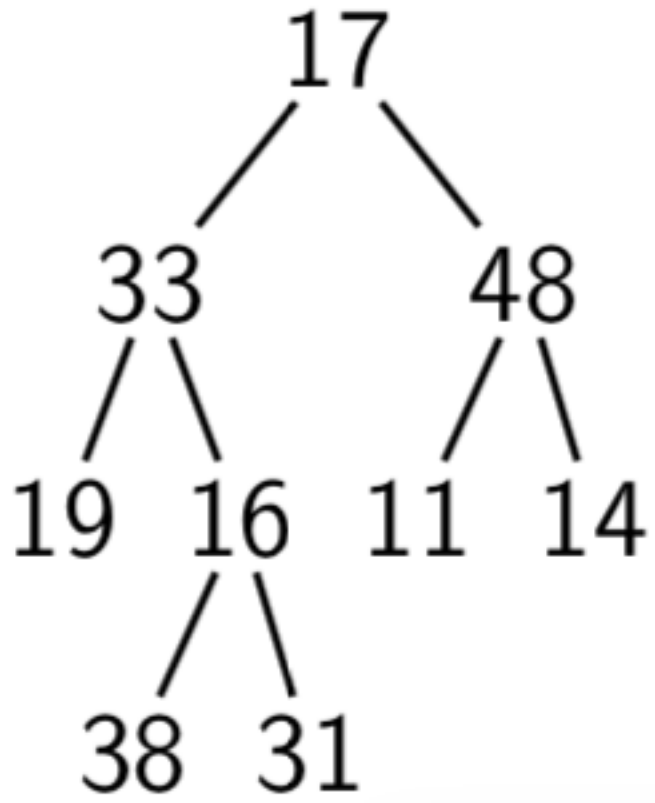
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order:

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



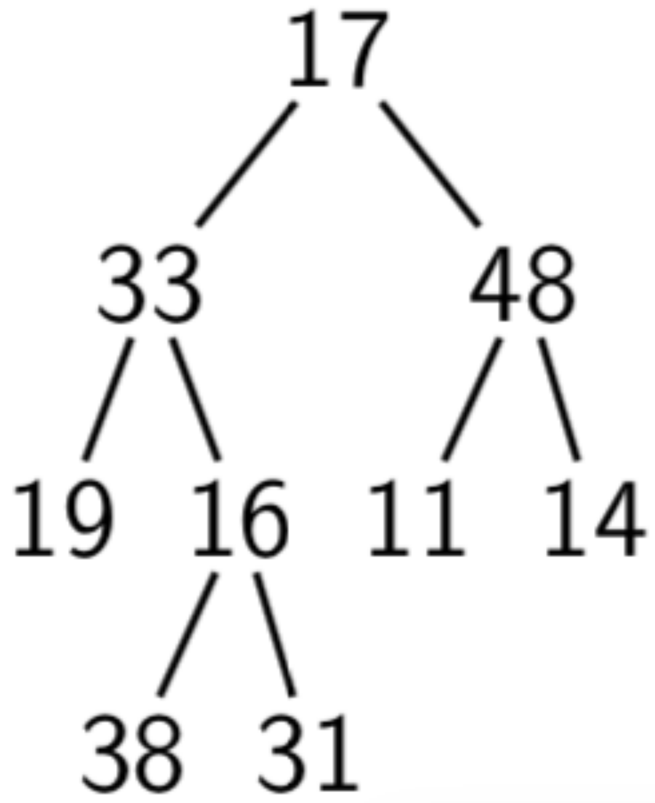
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order:

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



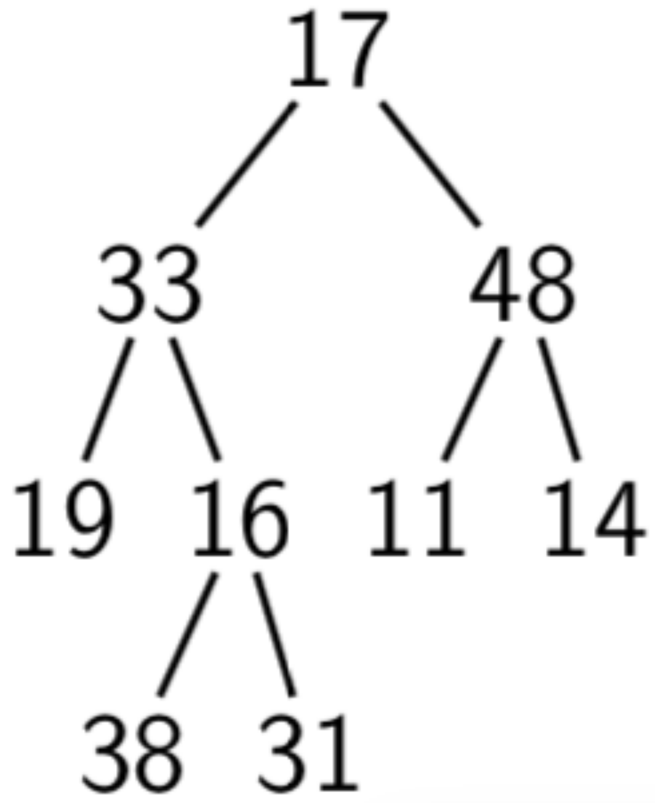
InorderTraverse(19)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order:

```
procedure INORDERTRAVERSE($T$)
    if $T \neq null$ then
        INORDERTRAVERSE($T.left$)
        visit $T.root$
        INORDERTRAVERSE($T.right$)
```



INORDERTRAVERSE(null)

INORDERTRAVERSE(19)

INORDERTRAVERSE(33)

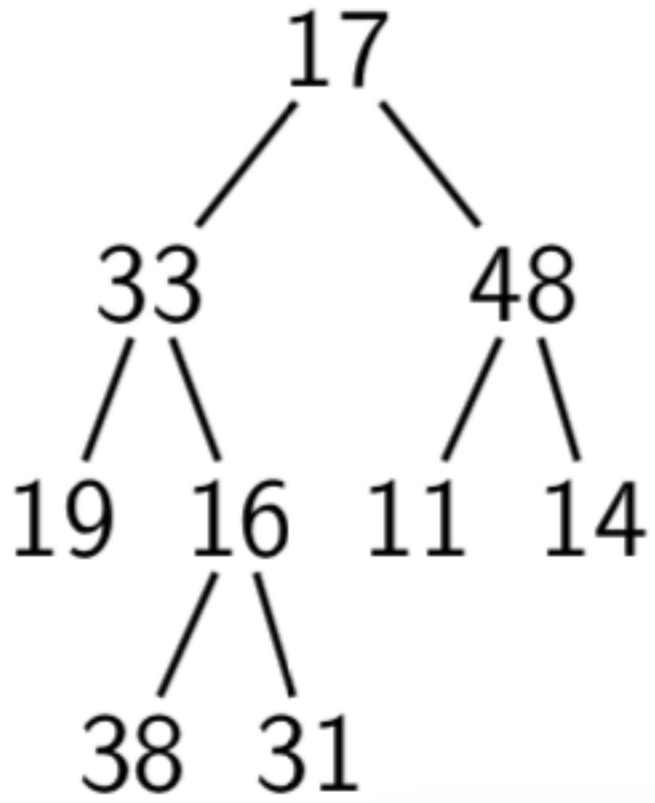
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order:

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



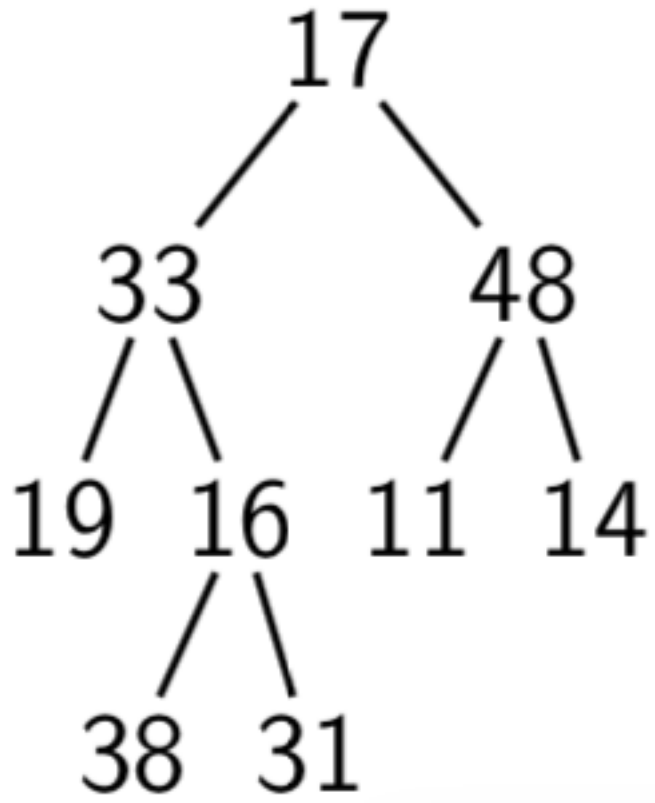
InorderTraverse(19)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19

```
procedure INORDERTRAVERSE($T$)
    if $T \neq null$ then
        INORDERTRAVERSE($T.left$)
        visit $T.root$
        INORDERTRAVERSE($T.right$)
```



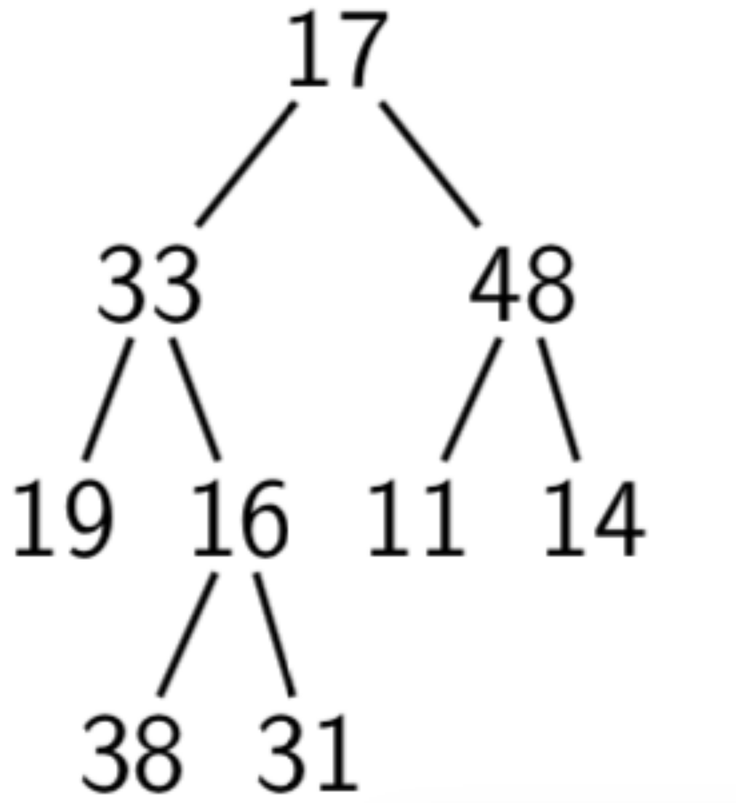
INORDERTRAVERSE(19)
INORDERTRAVERSE(33)
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



InorderTraverse(null)

InorderTraverse(19)

InorderTraverse(33)

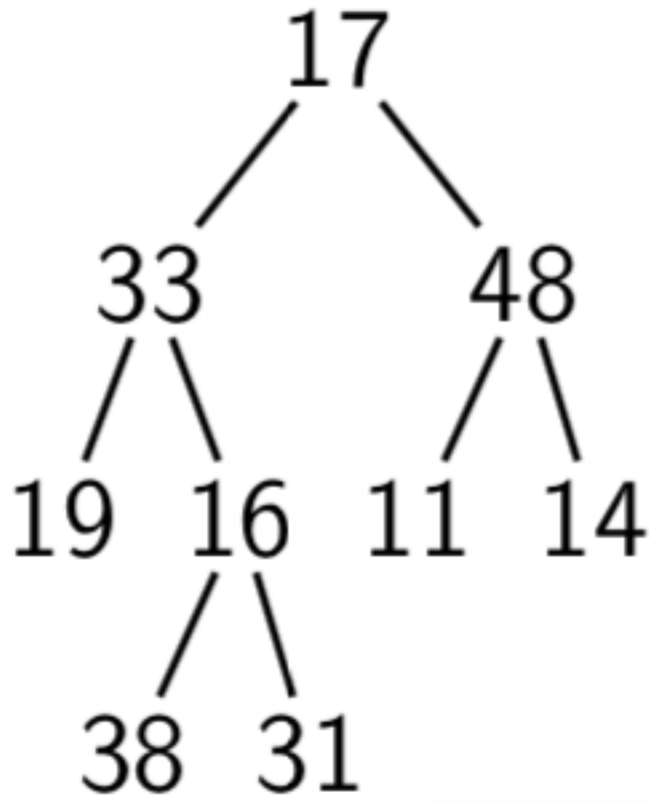
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19

```
procedure INORDERTRAVERSE($T$)
    if $T \neq null$ then
        INORDERTRAVERSE($T.left$)
        visit $T.root$
        INORDERTRAVERSE($T.right$)
```



INORDERTRAVERSE(19)

INORDERTRAVERSE(33)

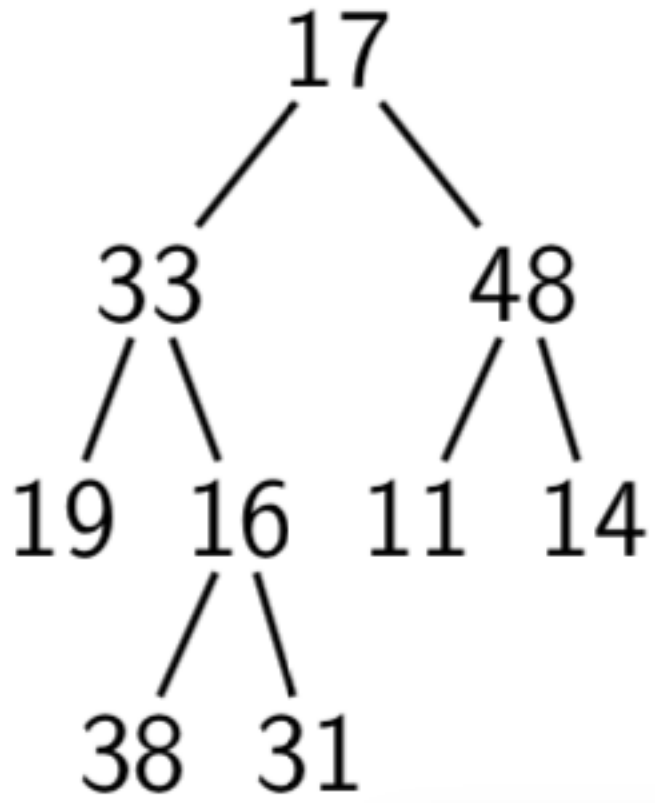
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



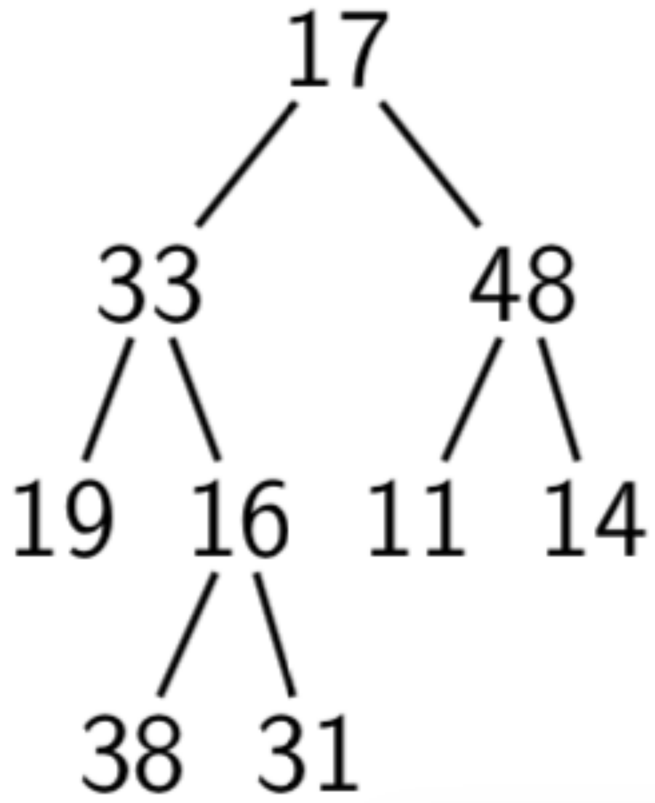
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



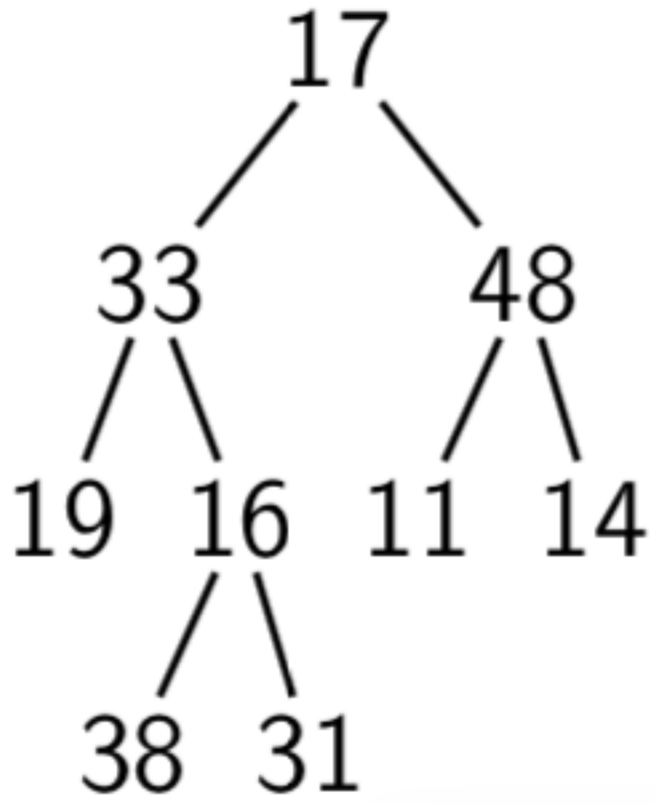
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



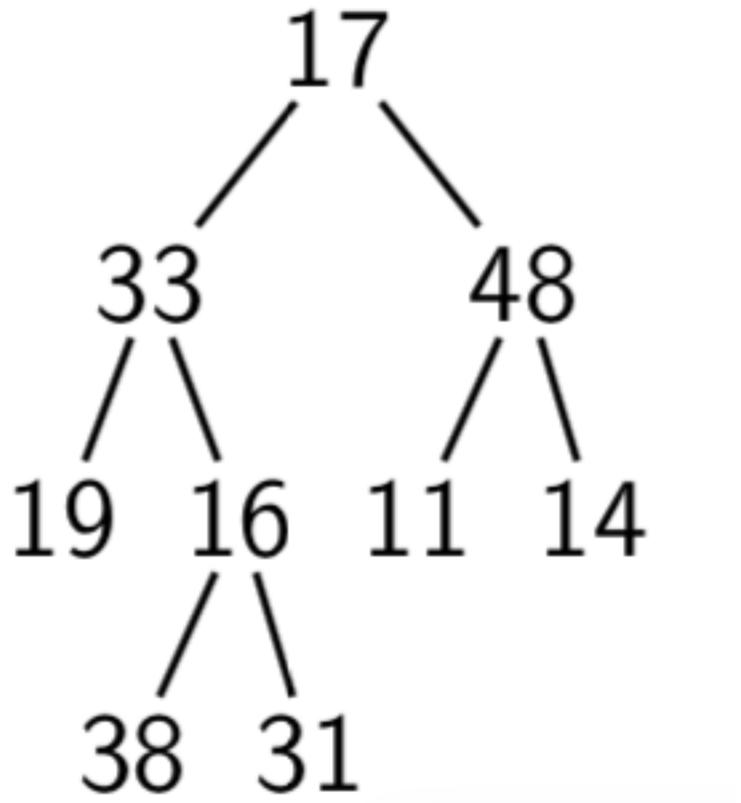
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



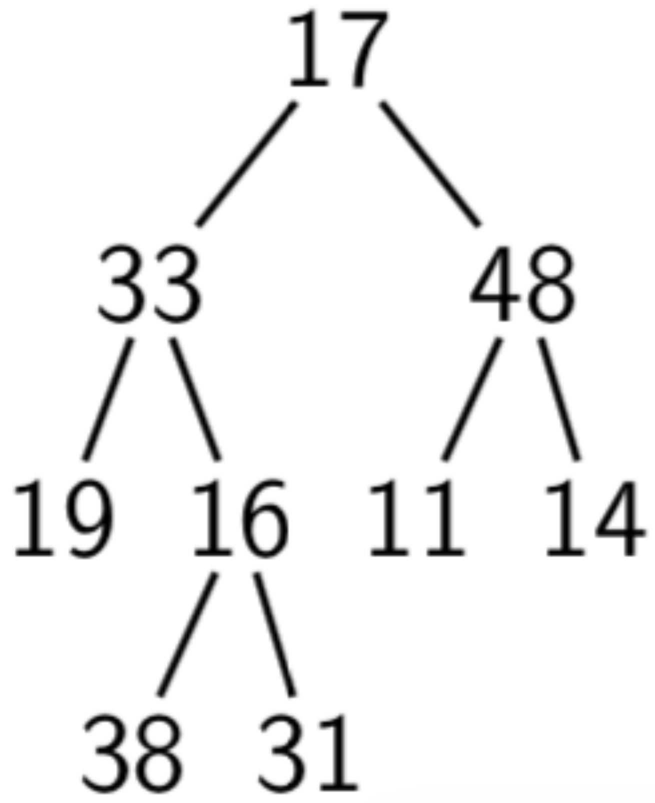
InorderTraverse(38)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33

```
procedure INORDERTRAVERSE(T)
    if $T \neq null$ then
        INORDERTRAVERSE(T.left)
        visit T.root
        INORDERTRAVERSE(T.right)
```



INORDERTRAVERSE(null)

INORDERTRAVERSE(38)

INORDERTRAVERSE(16)

INORDERTRAVERSE(33)

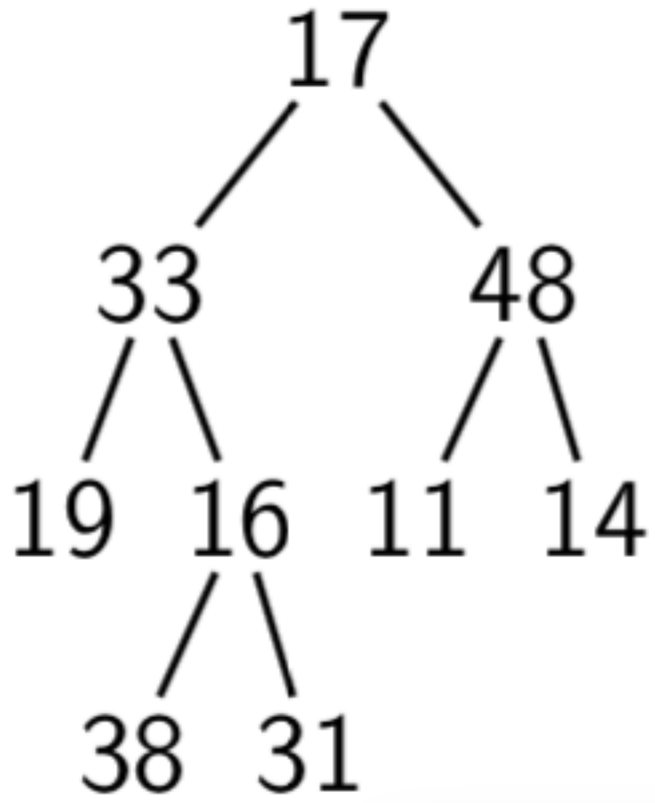
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



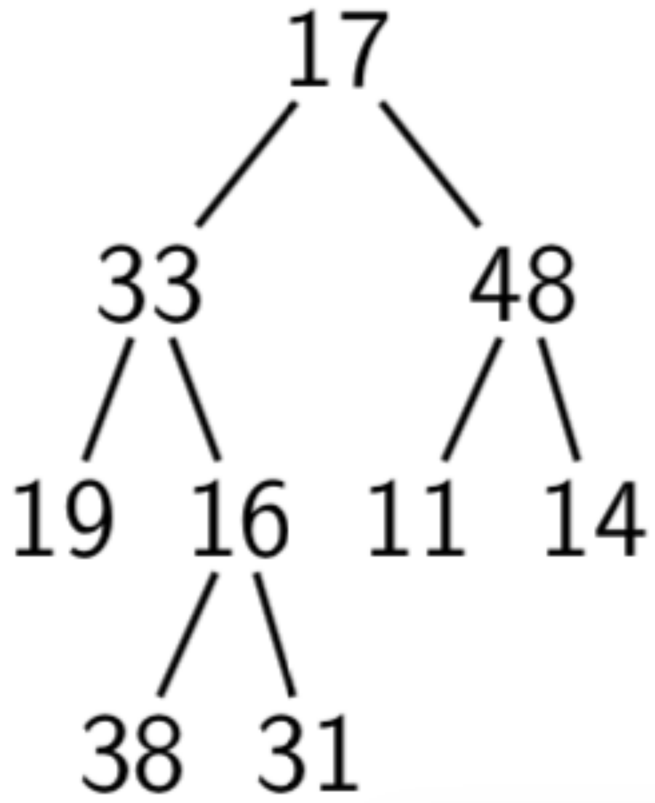
INORDERTRAVERSE(38)
INORDERTRAVERSE(16)
INORDERTRAVERSE(33)
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



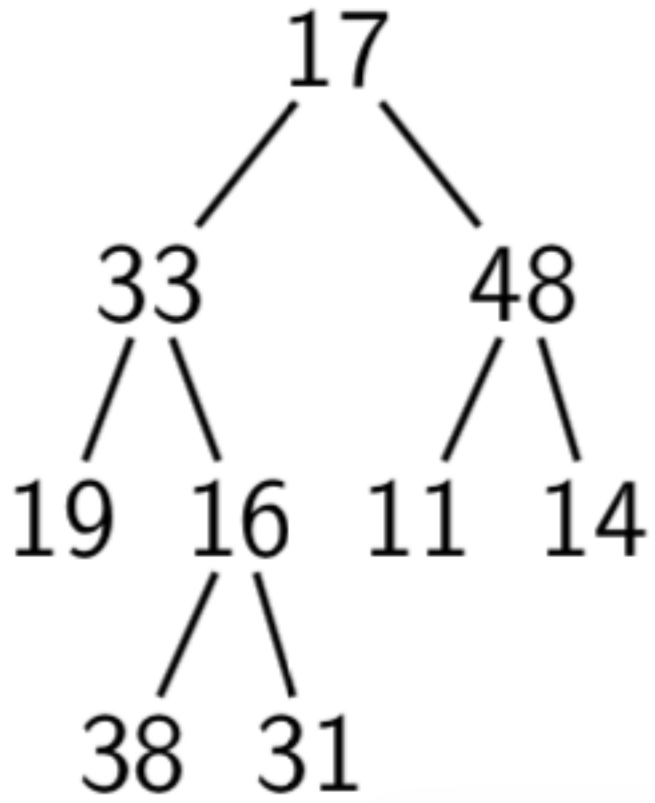
InorderTraverse(38)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



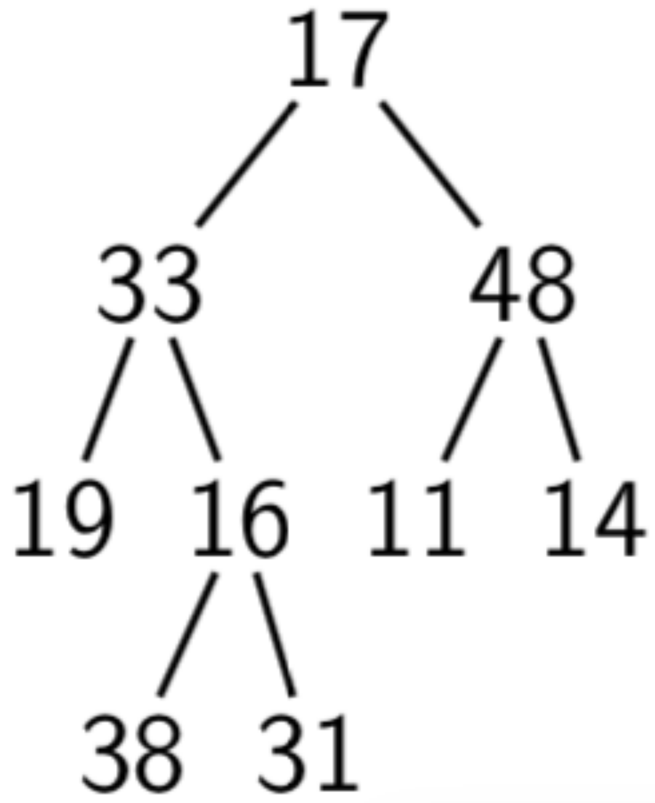
InorderTraverse(null)
InorderTraverse(38)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



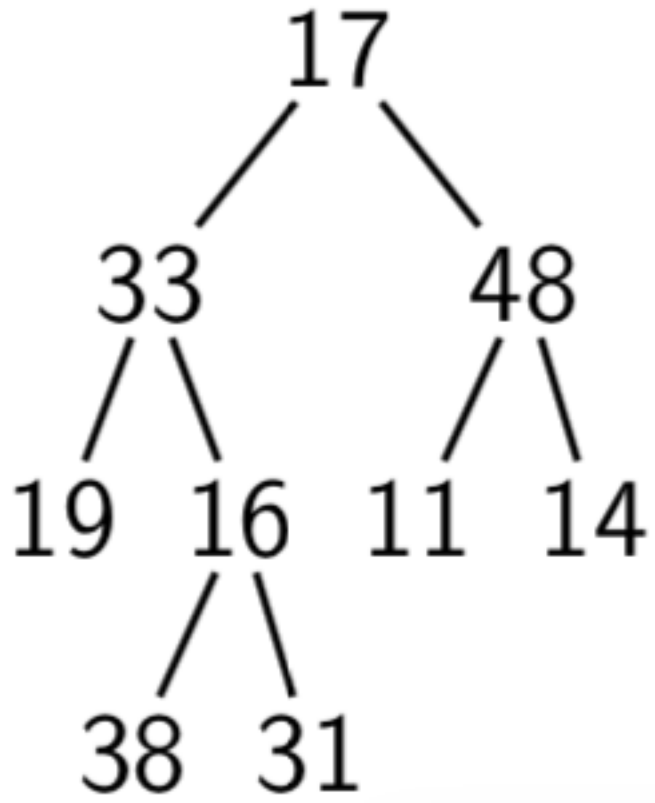
InorderTraverse(38)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



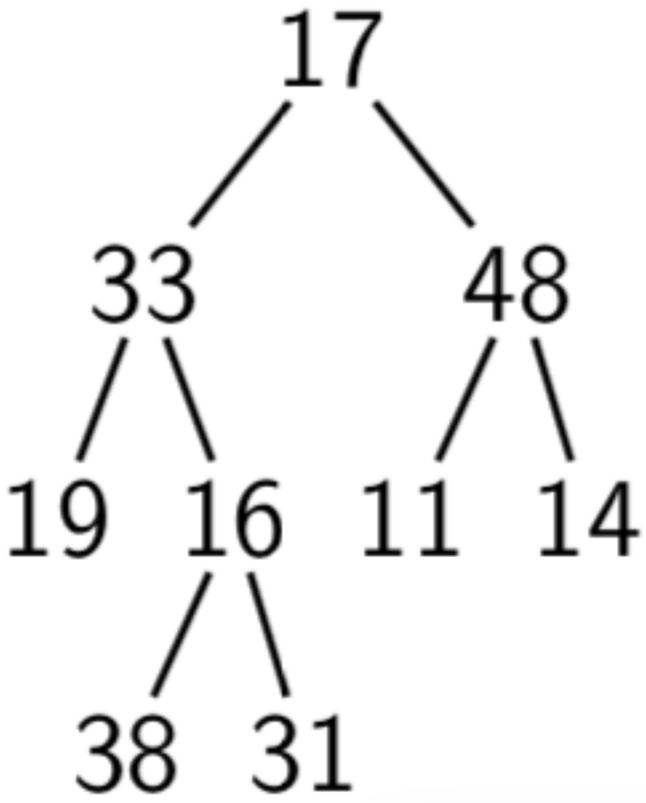
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



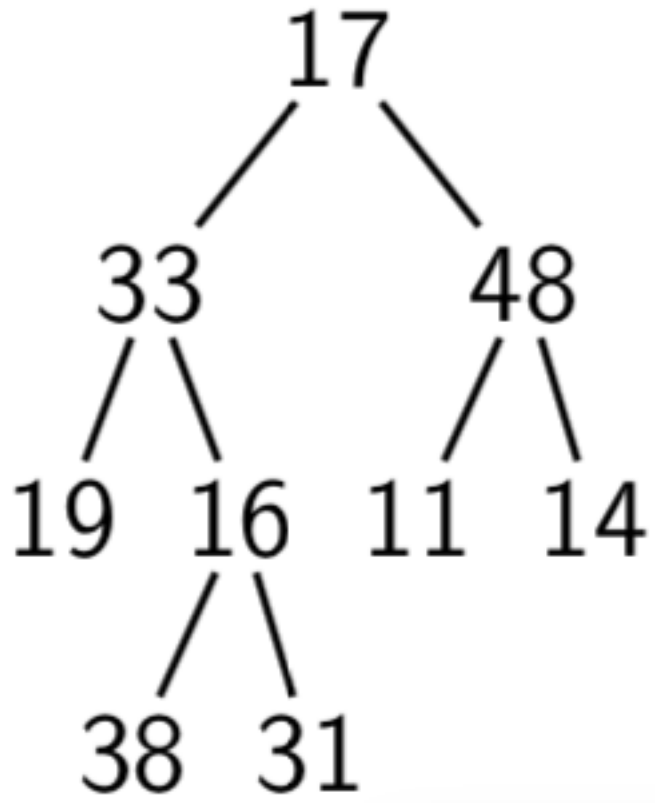
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



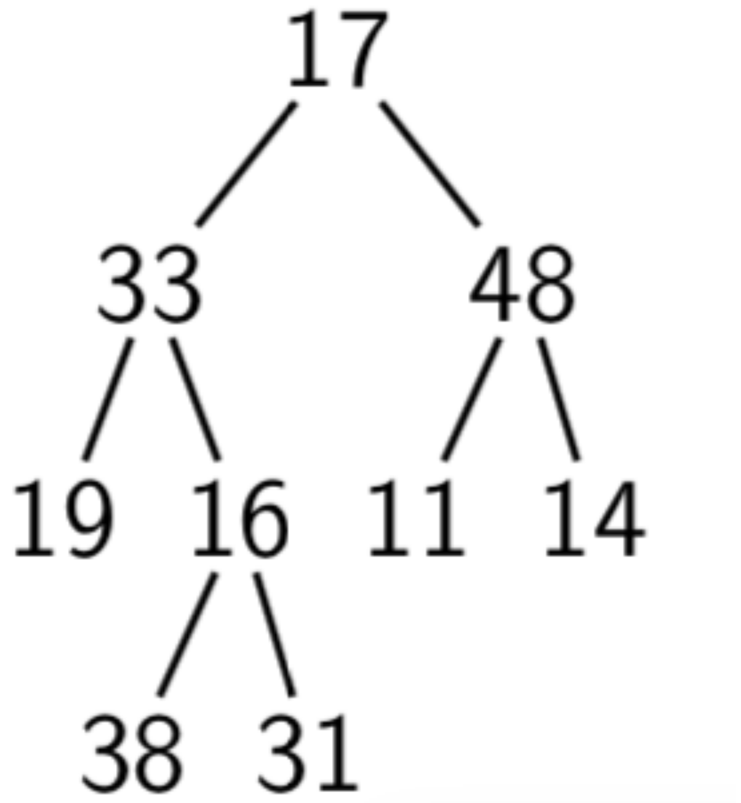
InorderTraverse(31)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16

```
procedure INORDERTRAVERSE(T)
    if $T \neq null$ then
        INORDERTRAVERSE(T.left)
        visit T.root
        INORDERTRAVERSE(T.right)
```



INORDERTRAVERSE(null)

INORDERTRAVERSE(31)

INORDERTRAVERSE(16)

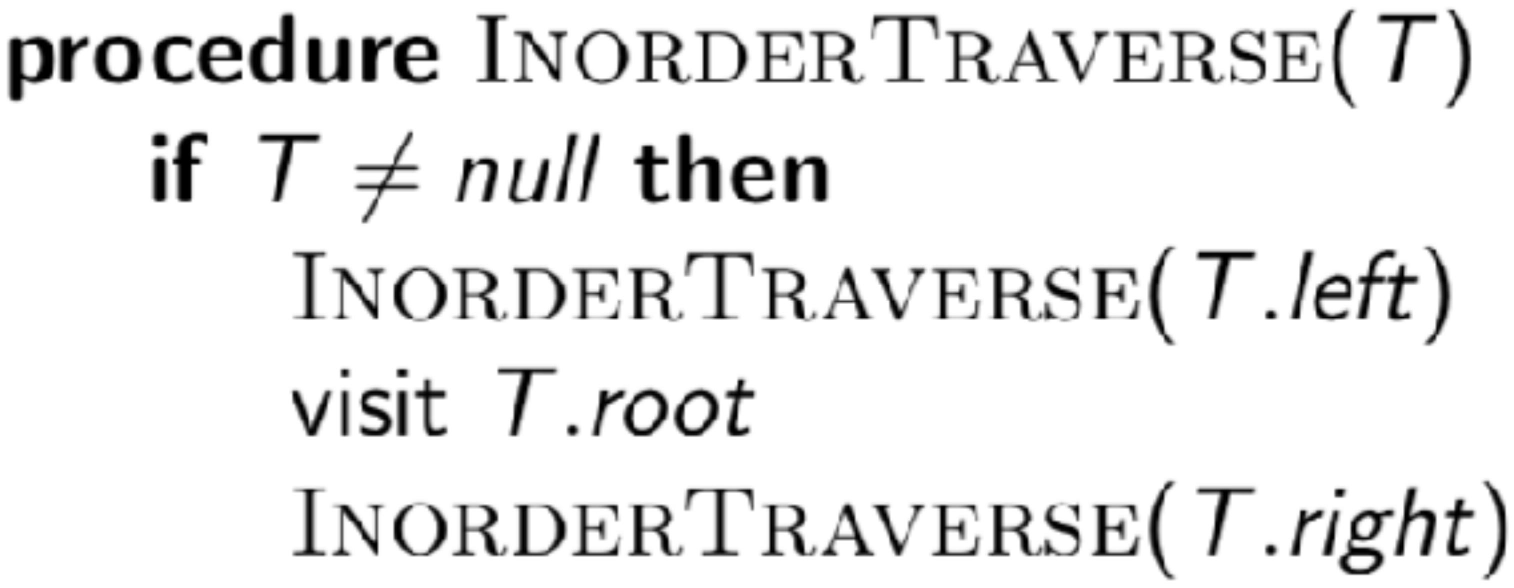
INORDERTRAVERSE(33)

INORDERTRAVERSE(17)

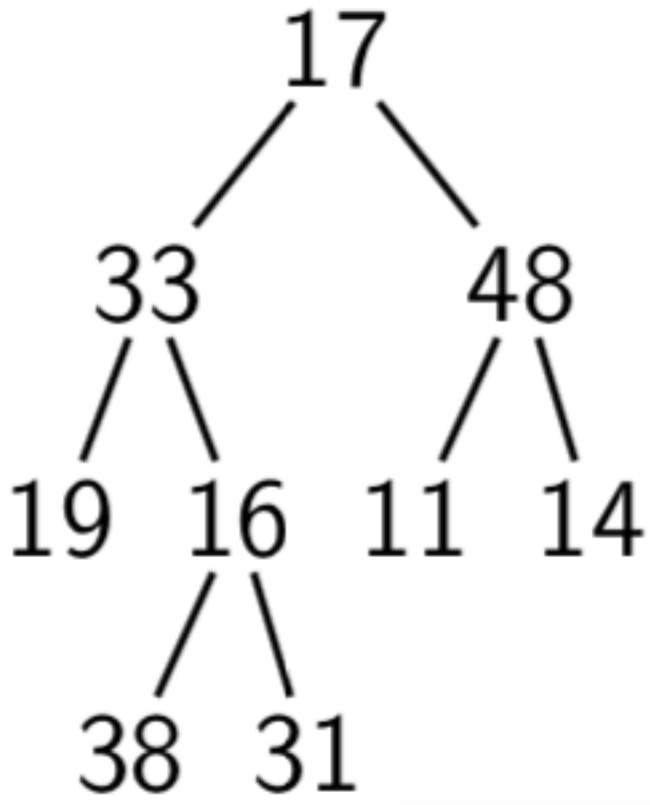
Call Stack

Inorder Traversal

Visit order: 19 33 38 16



```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



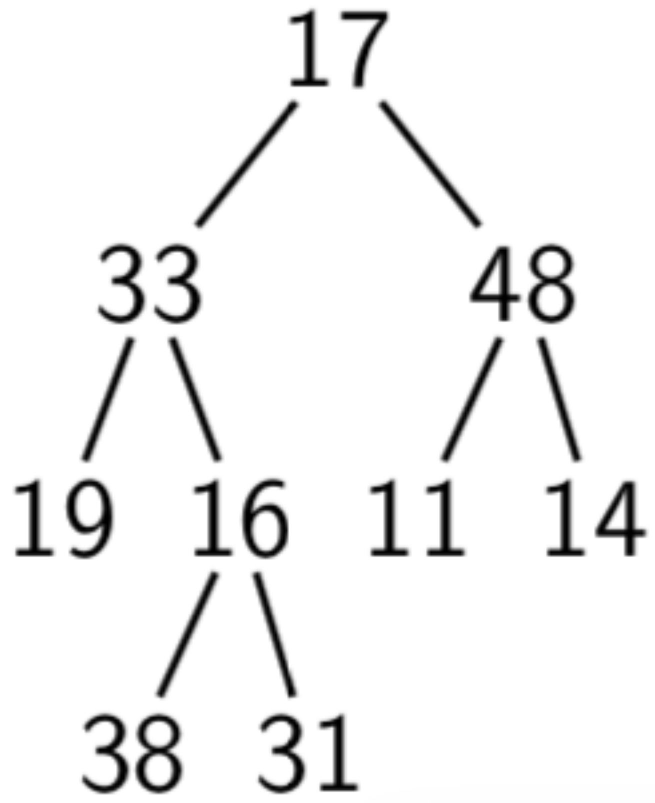
InorderTraverse(31)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31

```
procedure INORDERTRAVERSE(T)
    if $T \neq null$ then
        INORDERTRAVERSE(T.left)
        visit T.root
        INORDERTRAVERSE(T.right)
```



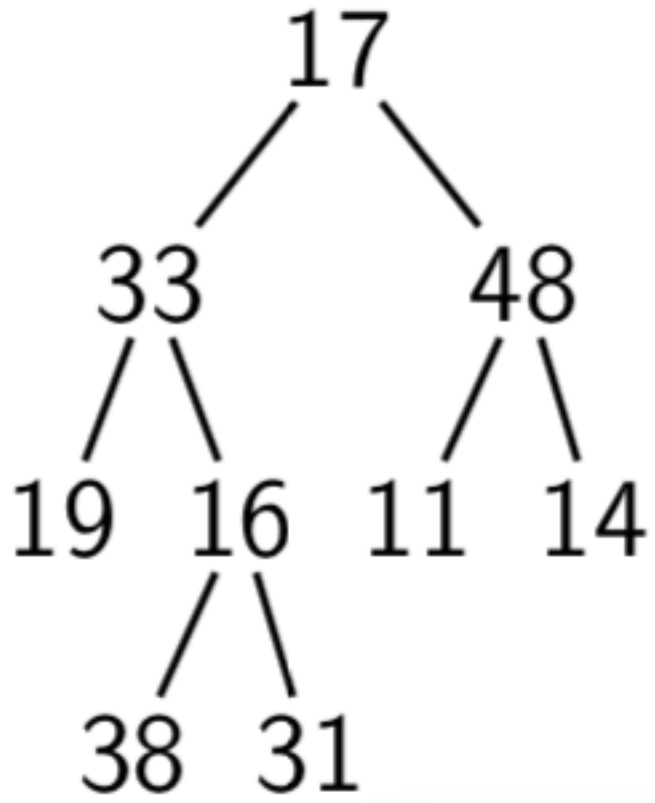
INORDERTRAVERSE(31)
INORDERTRAVERSE(16)
INORDERTRAVERSE(33)
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31

```
procedure InorderTraverse(T)
    if T ≠ null then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



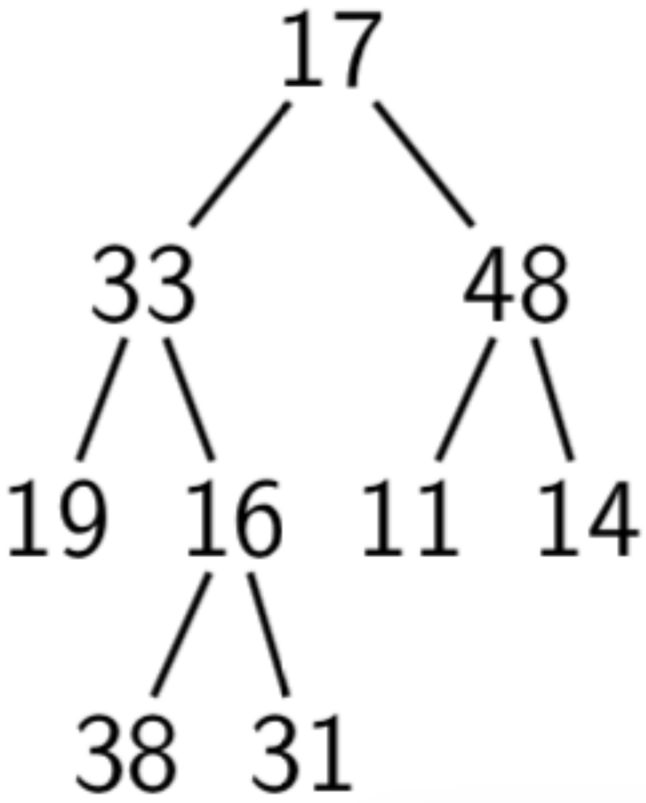
InorderTraverse(null)
InorderTraverse(31)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31

```
procedure InorderTraverse(T)
    if T ≠ null then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



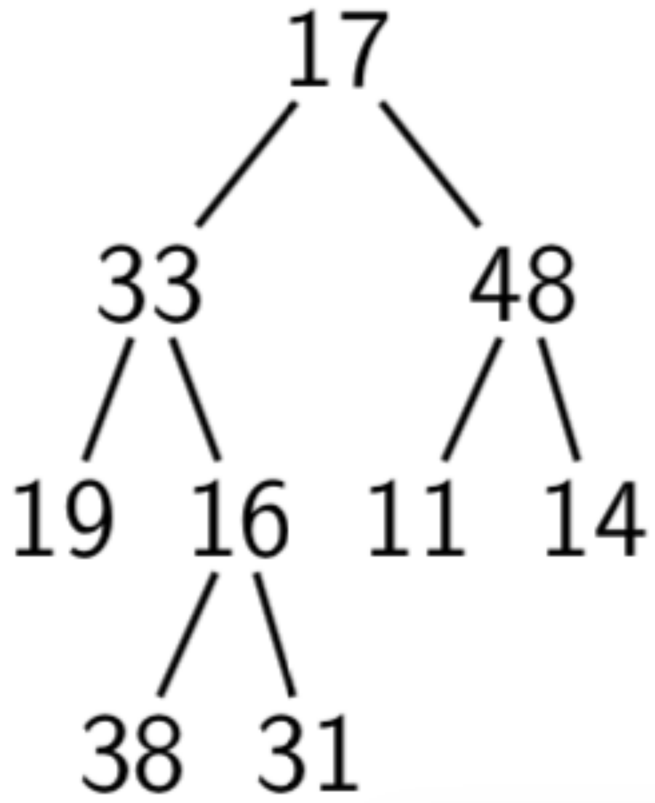
InorderTraverse(31)
InorderTraverse(16)
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



InorderTraverse(16)

InorderTraverse(33)

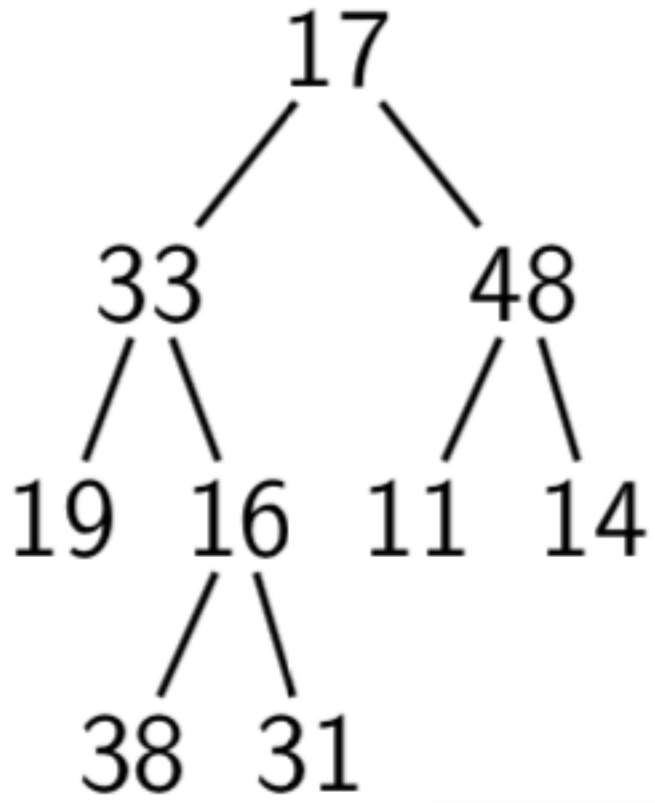
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



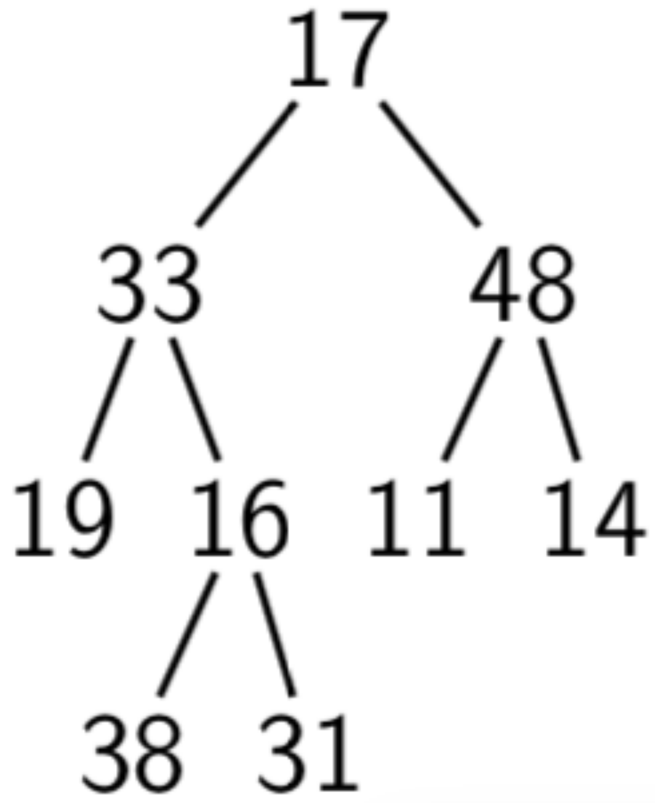
InorderTraverse(33)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



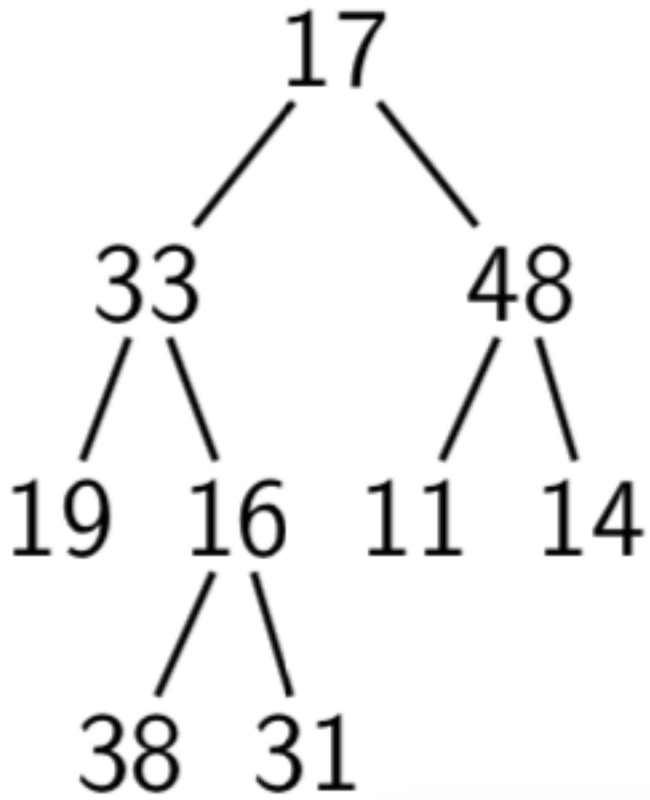
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



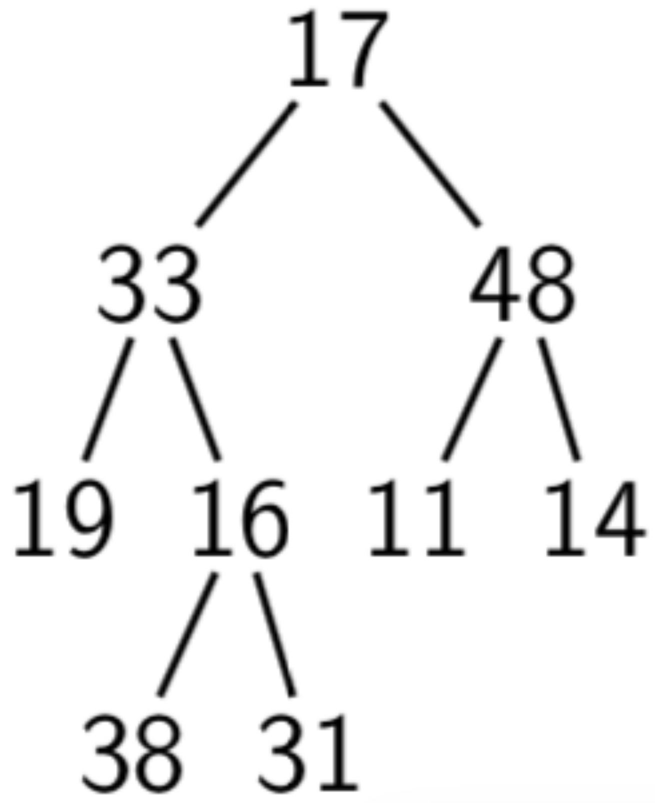
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



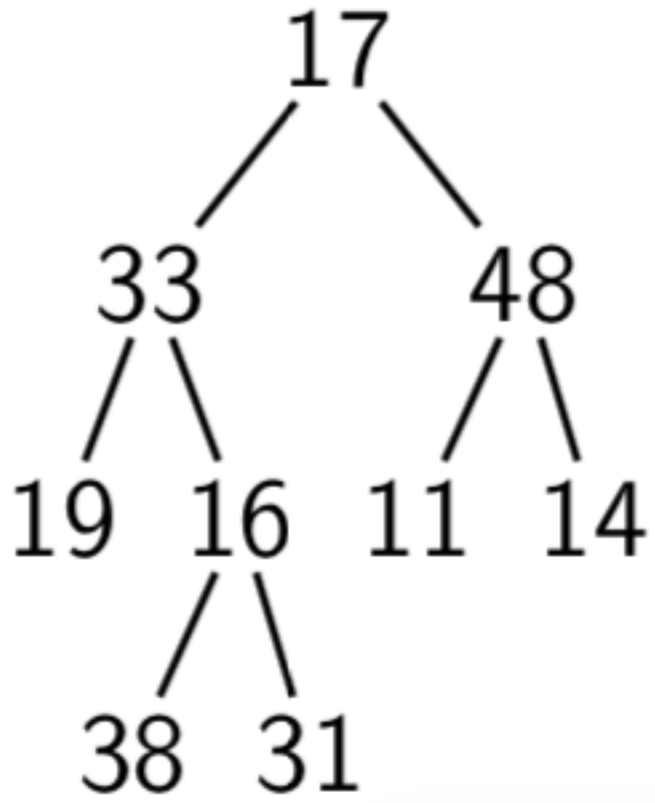
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



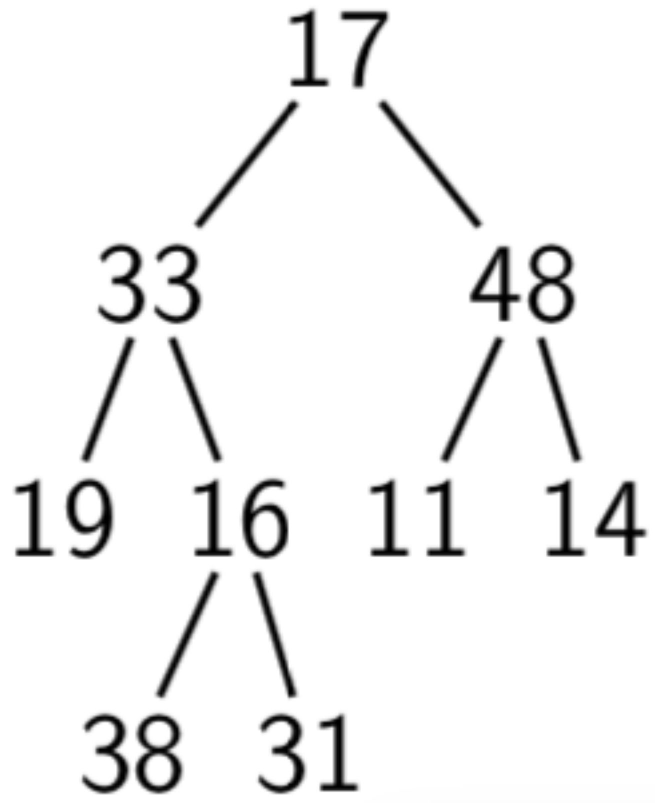
InorderTraverse(11)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



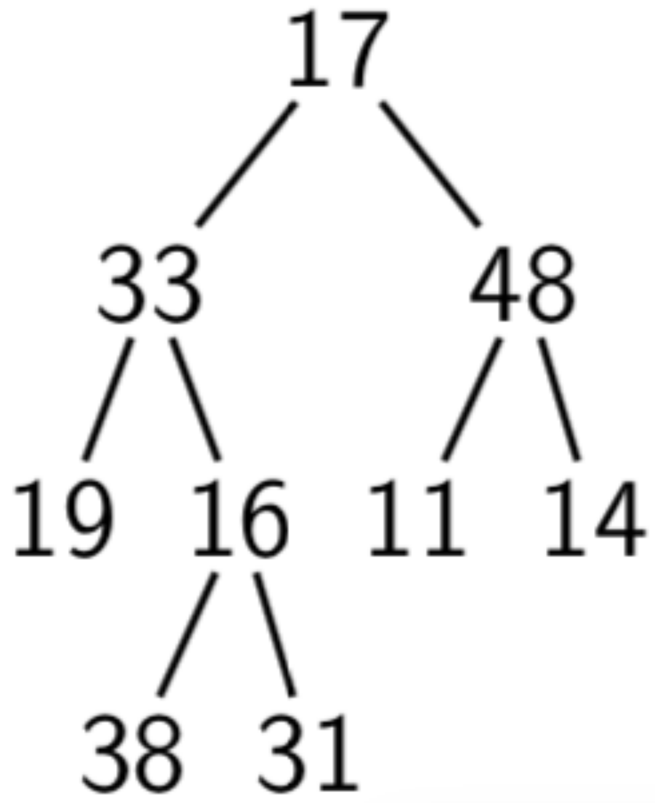
InorderTraverse(null)
InorderTraverse(11)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



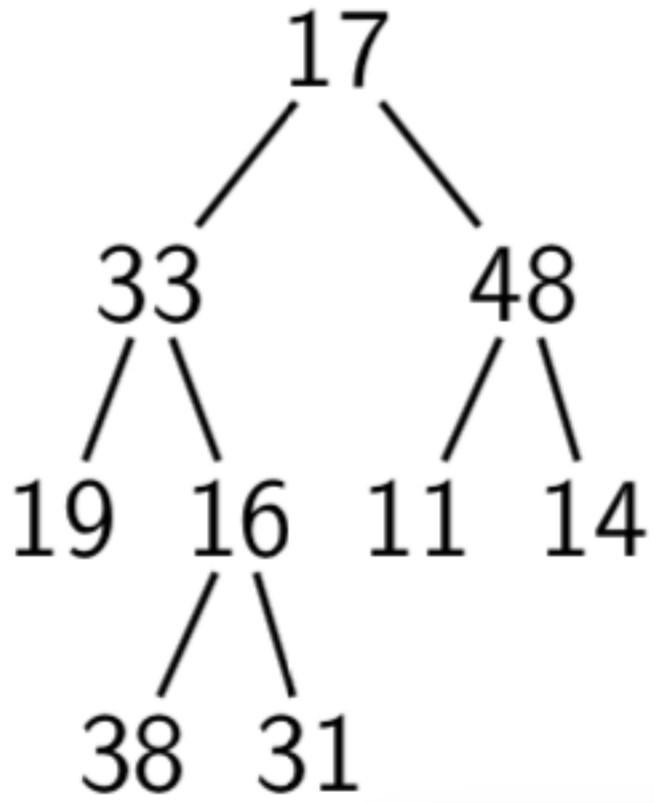
InorderTraverse(11)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



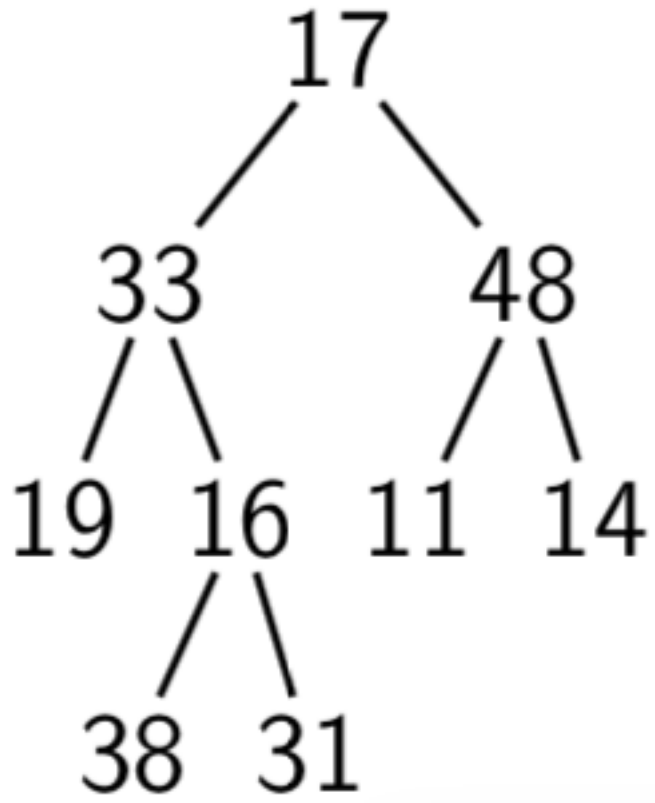
InorderTraverse(11)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11

```
procedure INORDERTRAVERSE($T$)
    if $T \neq null$ then
        INORDERTRAVERSE($T.left$)
        visit $T.root$
        INORDERTRAVERSE($T.right$)
```



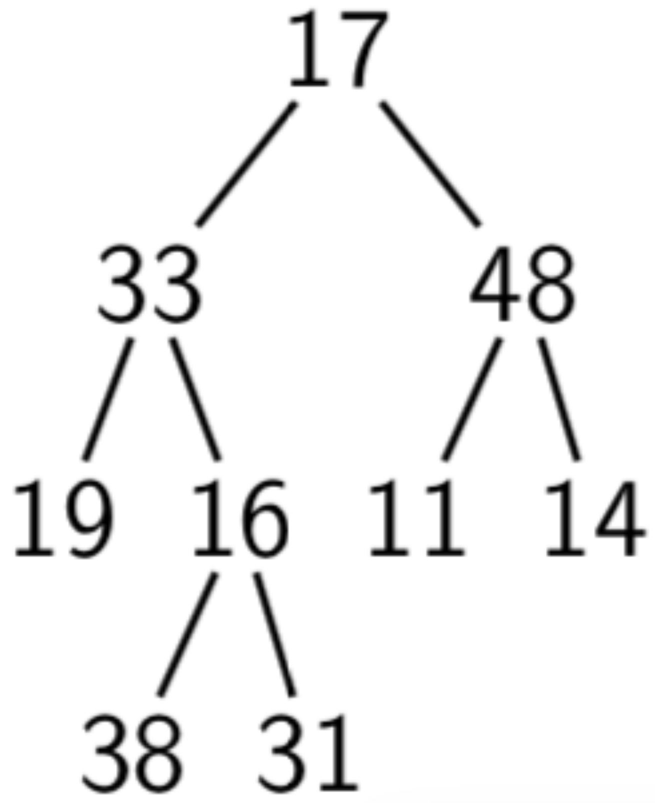
INORDERTRAVERSE(null)
INORDERTRAVERSE(11)
INORDERTRAVERSE(48)
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11

```
procedure INORDERTRAVERSE($T$)
    if $T \neq null$ then
        INORDERTRAVERSE($T.left$)
        visit $T.root$
        INORDERTRAVERSE($T.right$)
```



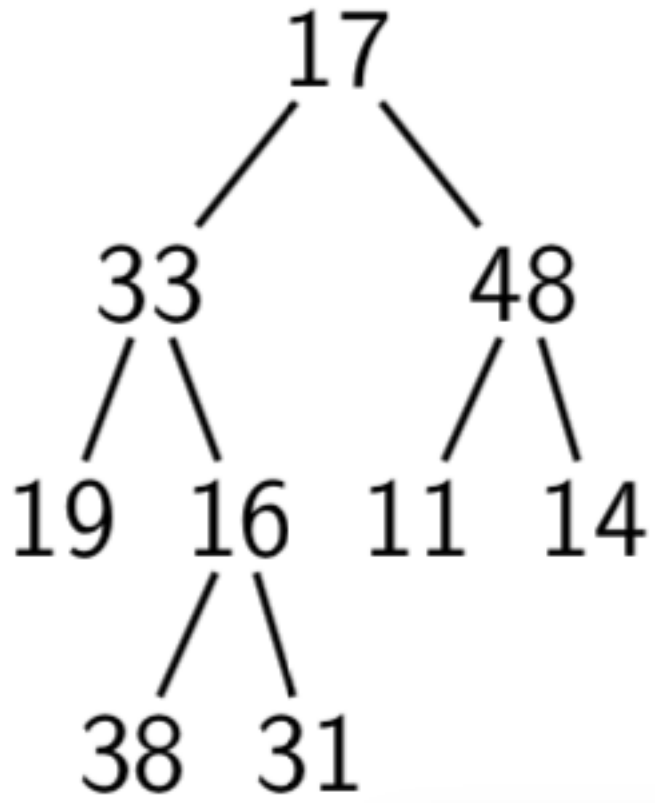
INORDERTRAVERSE(11)
INORDERTRAVERSE(48)
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



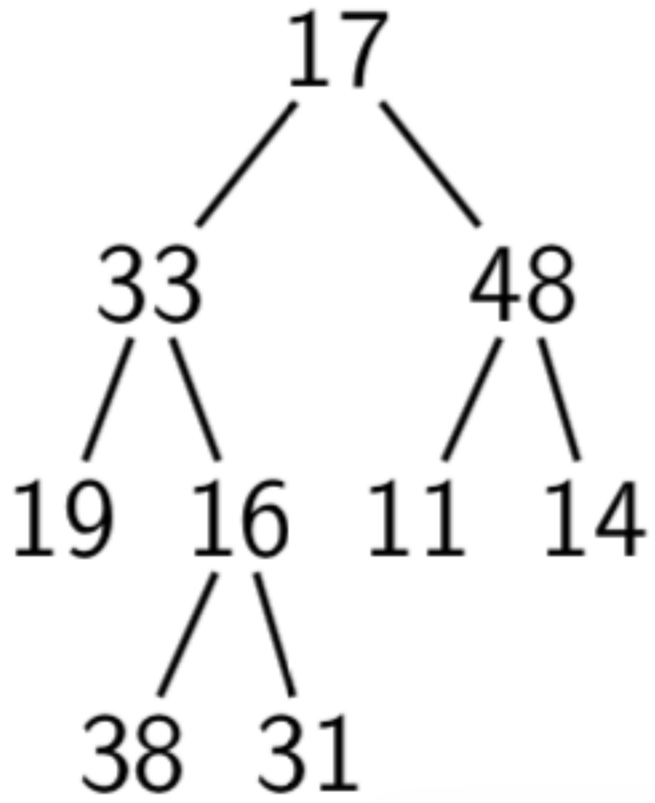
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48

```
procedure INORDERTRAVERSE($T$)
    if $T \neq null$ then
        INORDERTRAVERSE($T.left$)
        visit $T.root$
        INORDERTRAVERSE($T.right$)
```



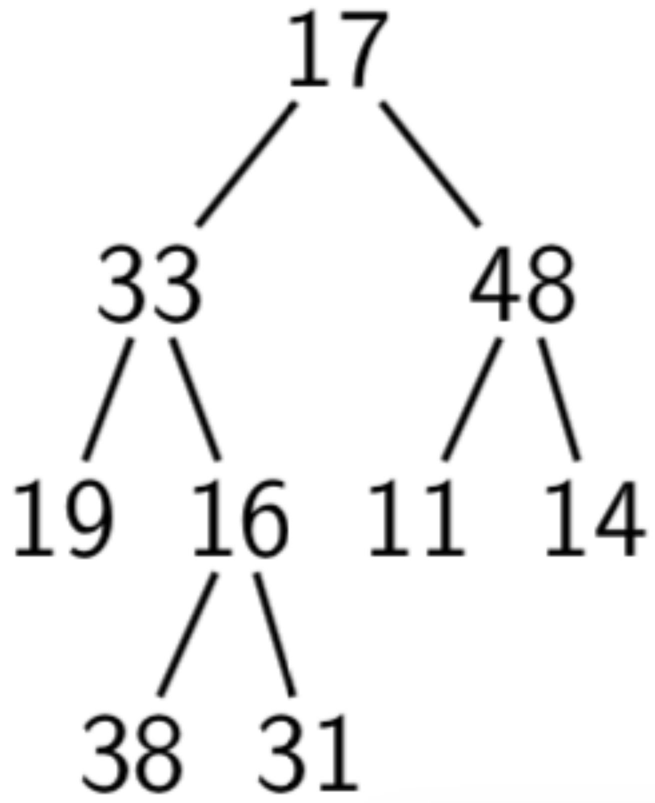
INORDERTRAVERSE(48)
INORDERTRAVERSE(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



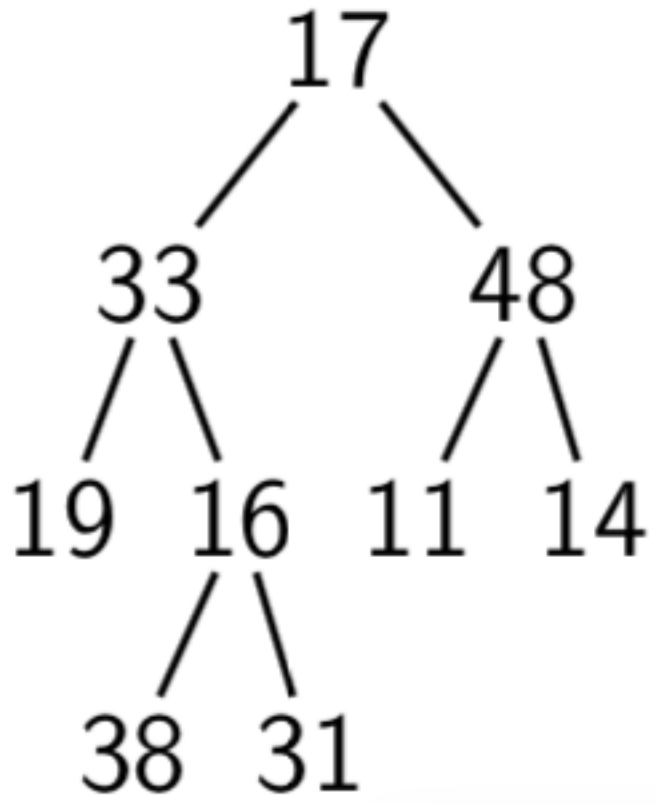
InorderTraverse(14)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



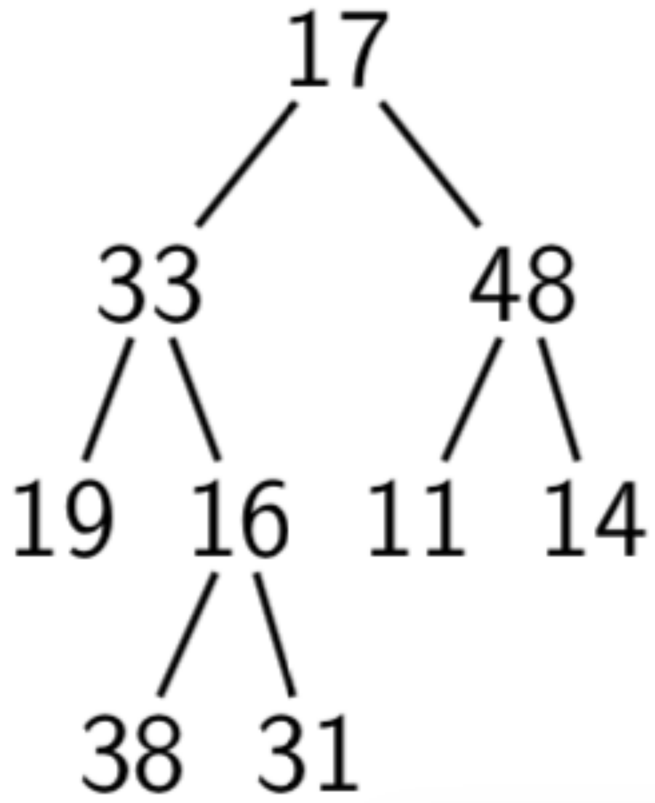
InorderTraverse(null)
InorderTraverse(14)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```



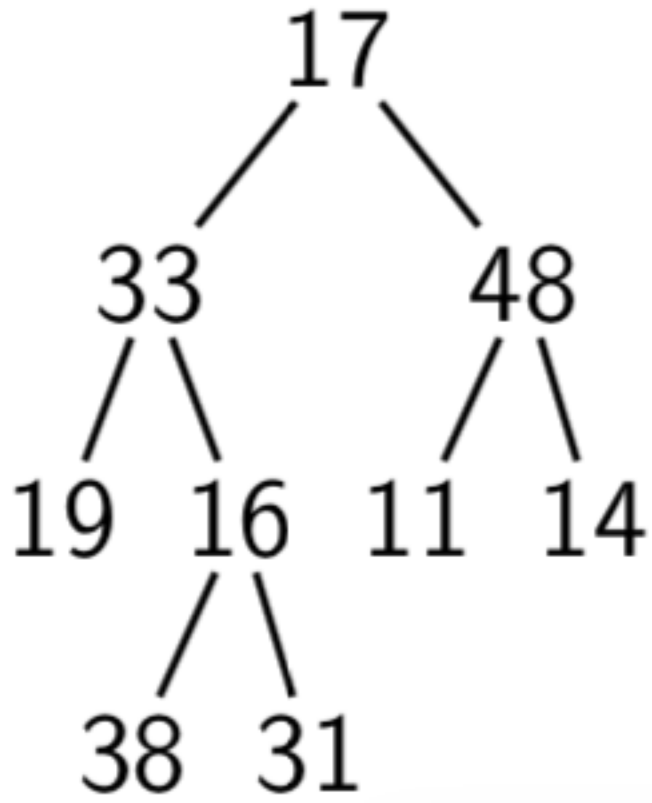
InorderTraverse(14)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48 14

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



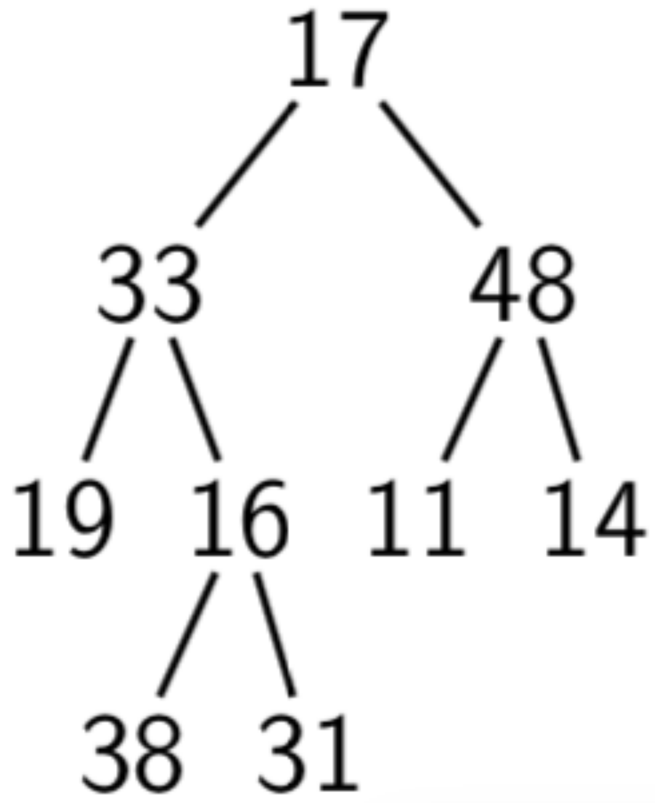
InorderTraverse(14)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48 14

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



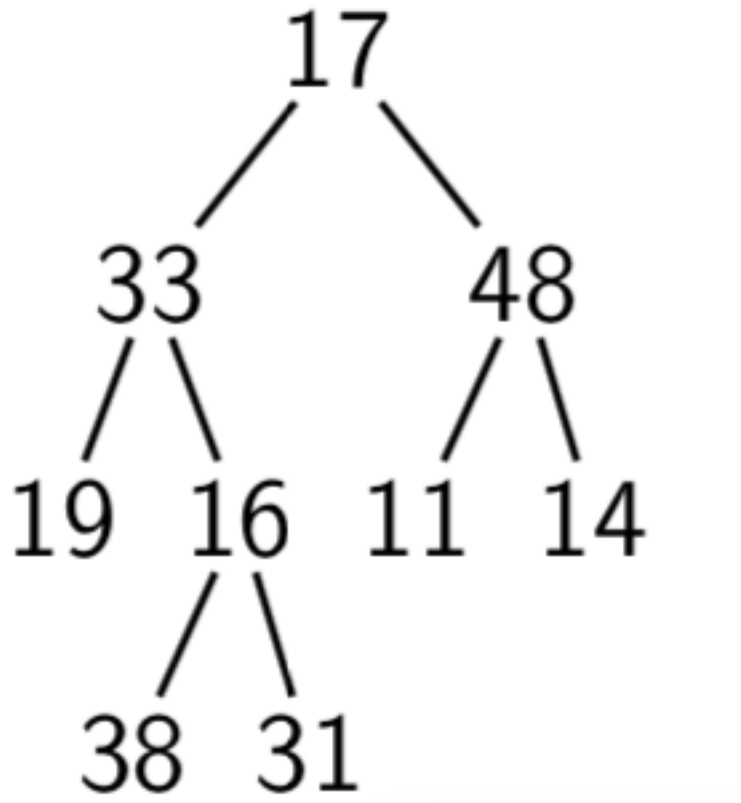
InorderTraverse(null)
InorderTraverse(14)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48 14

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



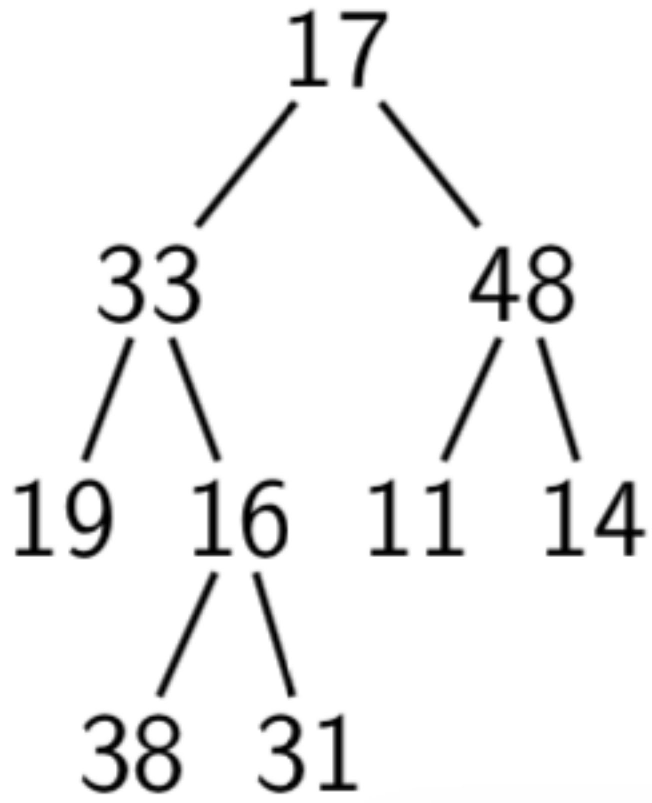
InorderTraverse(14)
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48 14

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



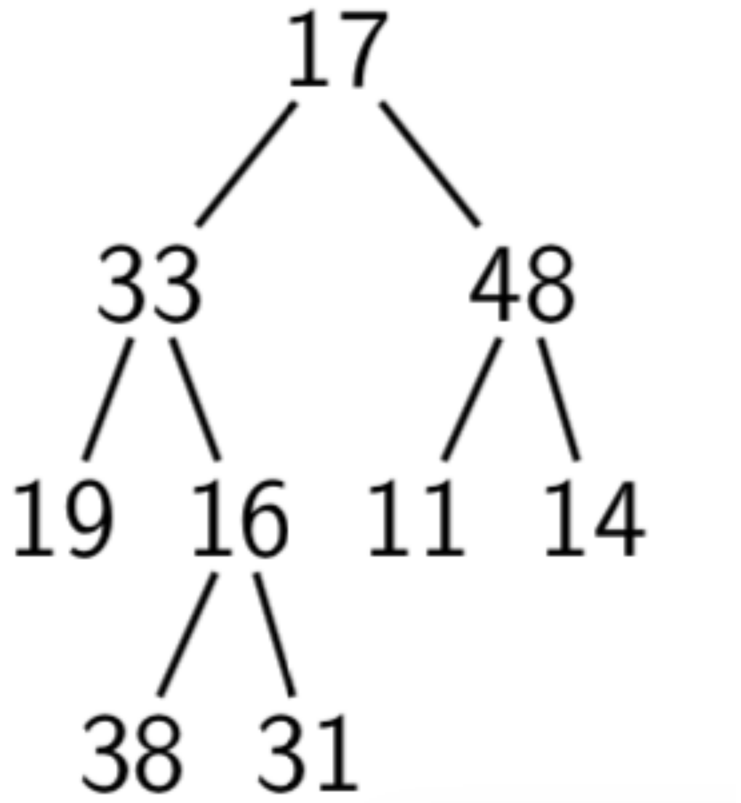
InorderTraverse(48)
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48 14

```
procedure InorderTraverse($T$)
    if $T \neq null$ then
        InorderTraverse($T.left$)
        visit $T.root$
        InorderTraverse($T.right$)
```



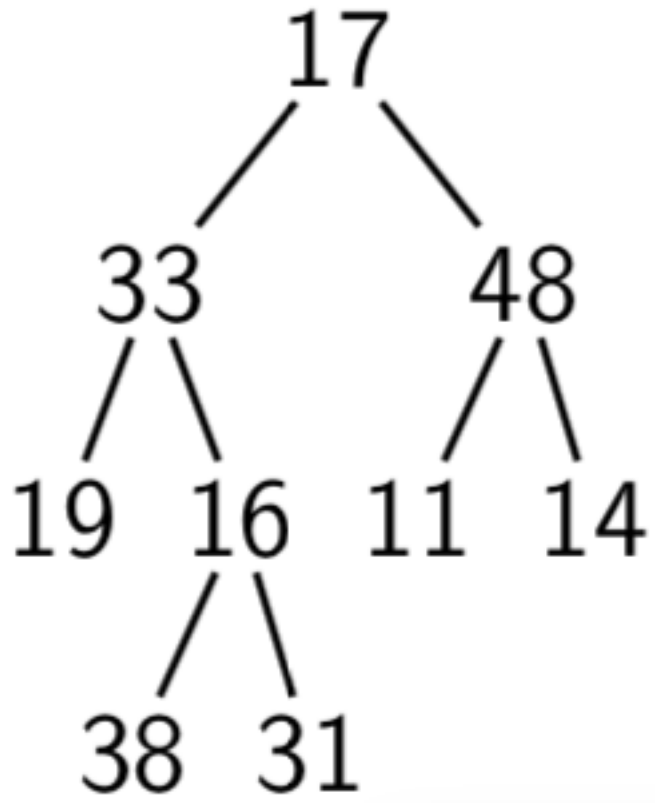
InorderTraverse(17)

Call Stack

Inorder Traversal

Visit order: 19 33 38 16 31 17 11 48 14

```
procedure InorderTraverse(T)
    if $T \neq null$ then
        InorderTraverse(T.left)
        visit T.root
        InorderTraverse(T.right)
```

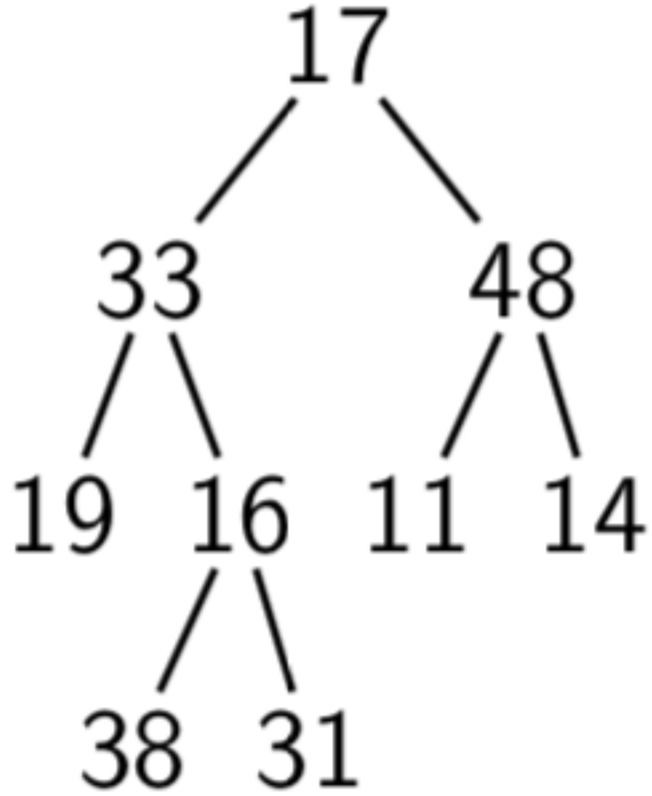


Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



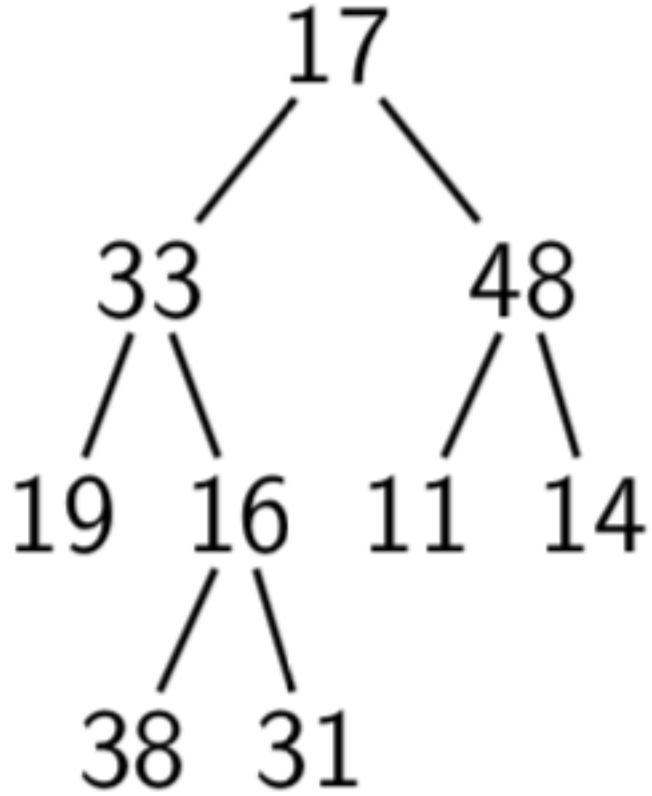
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



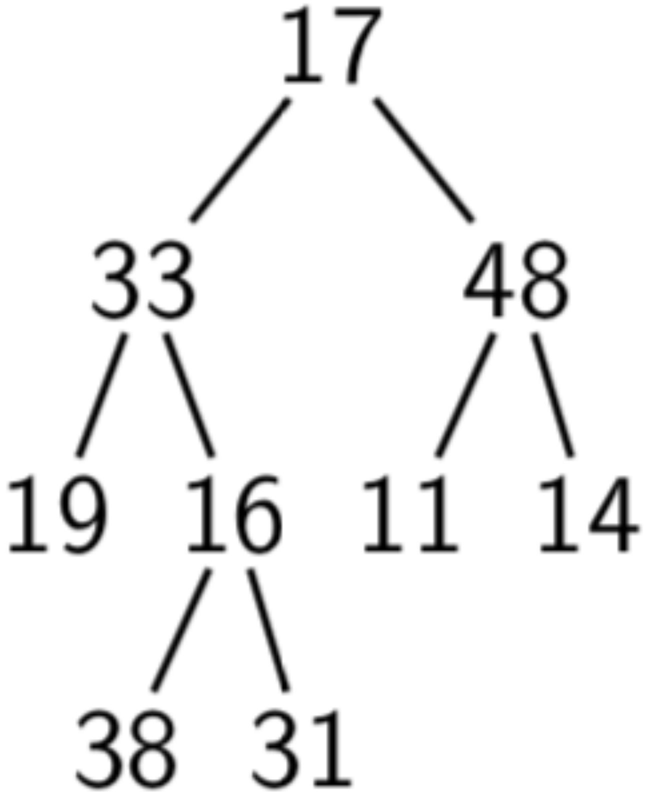
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



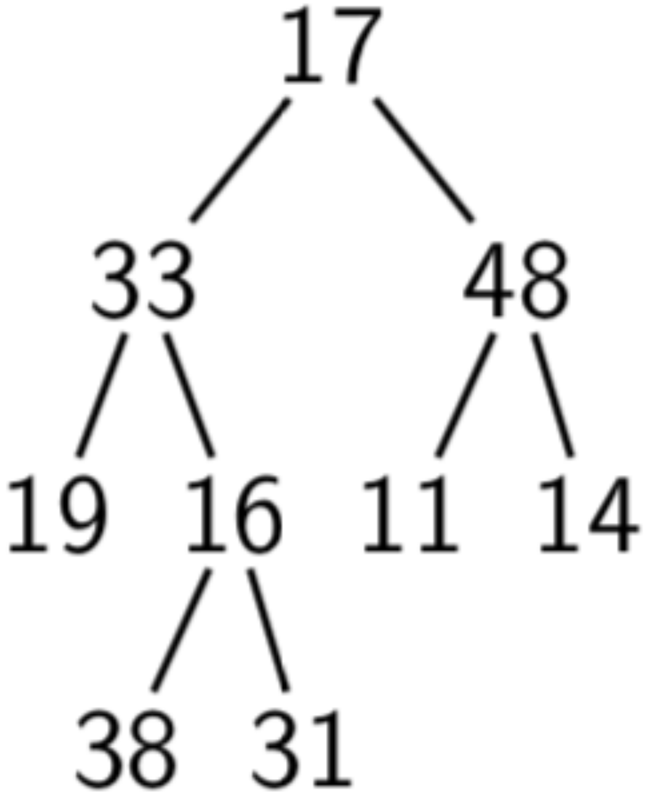
PostorderTraverse(19)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



PostorderTraverse(null)

PostorderTraverse(19)

PostorderTraverse(33)

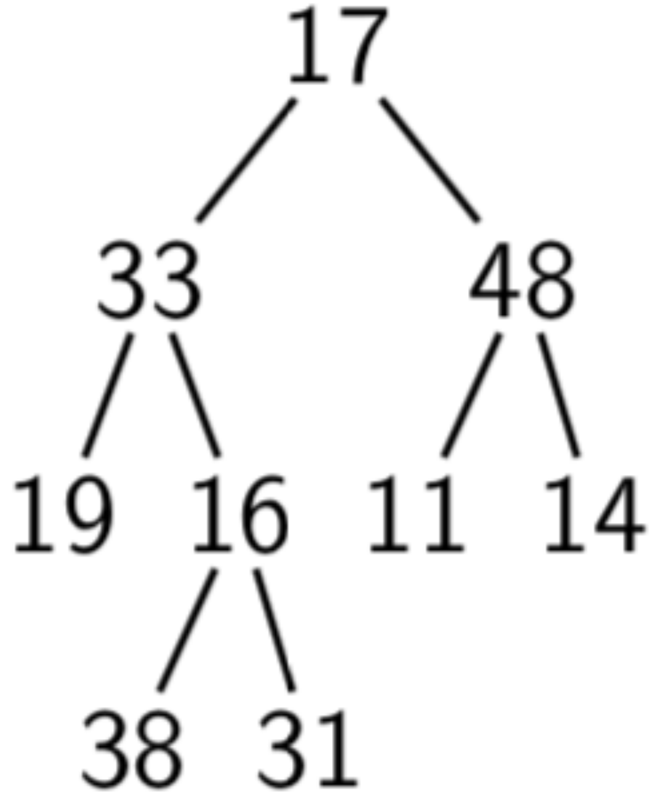
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



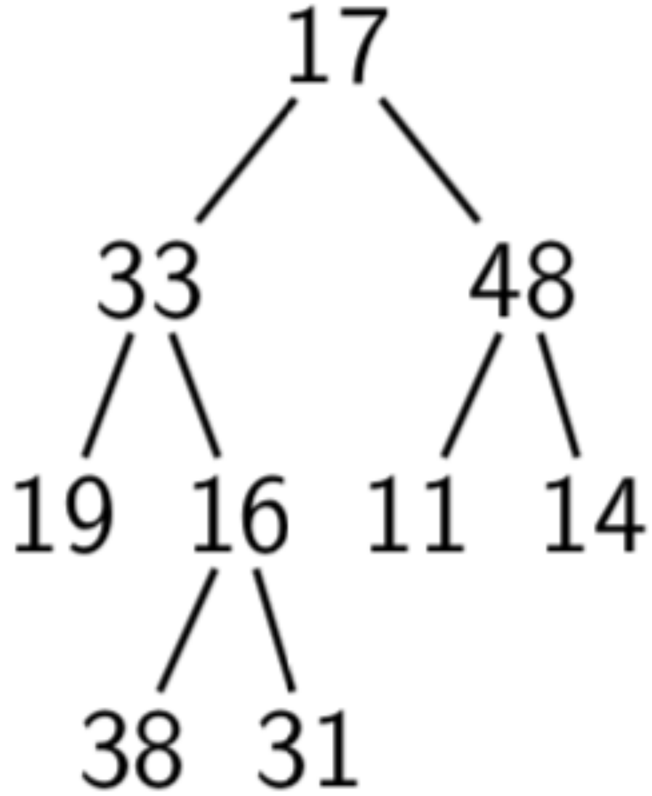
PostorderTraverse(19)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



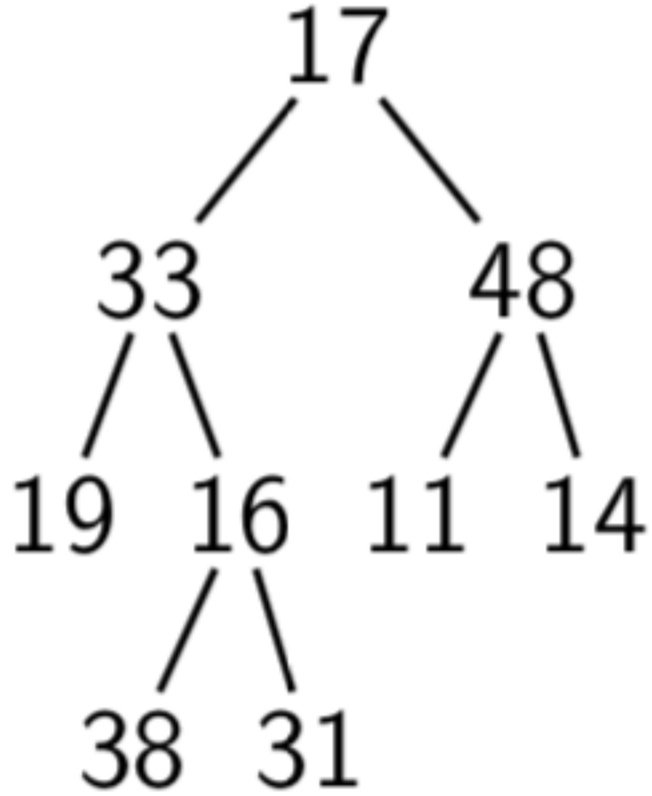
PostorderTraverse(null)
PostorderTraverse(19)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order:

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



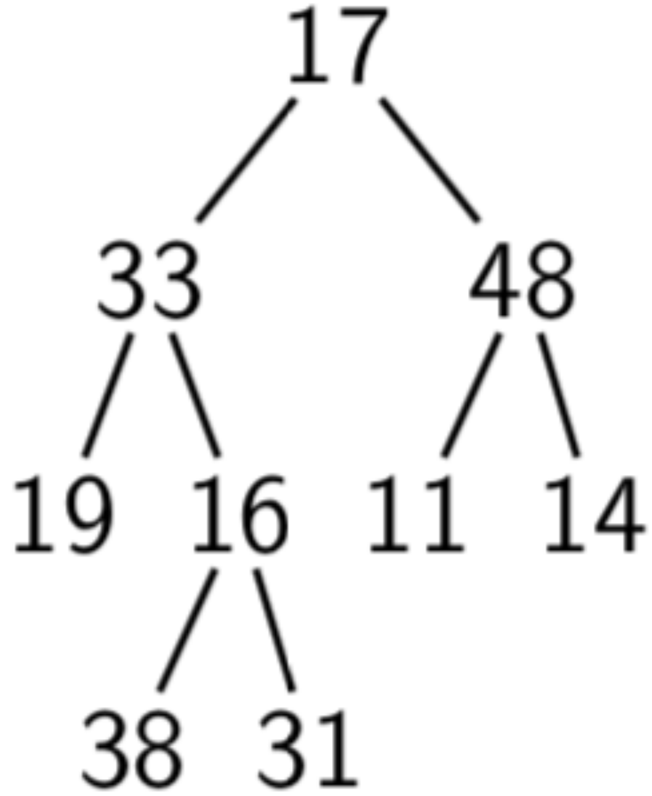
PostorderTraverse(19)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



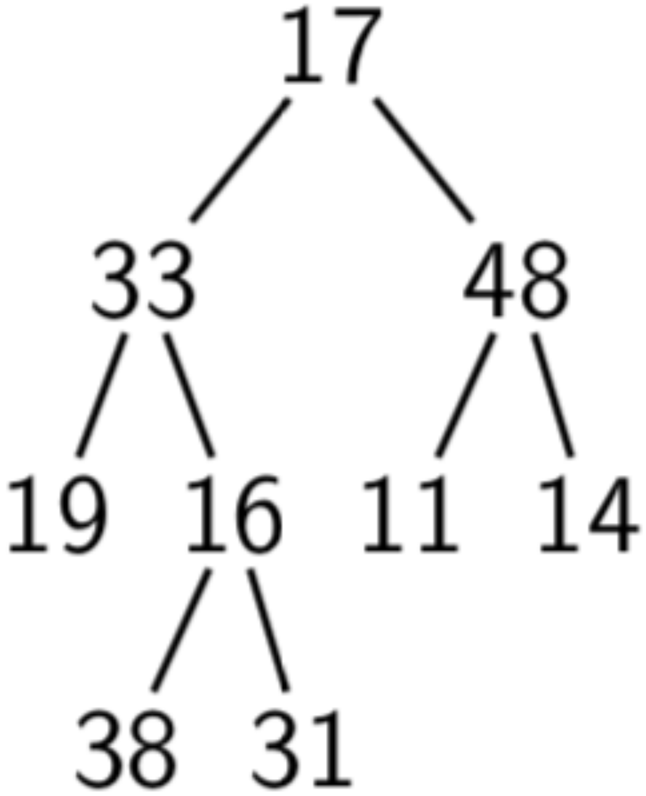
PostorderTraverse(19)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



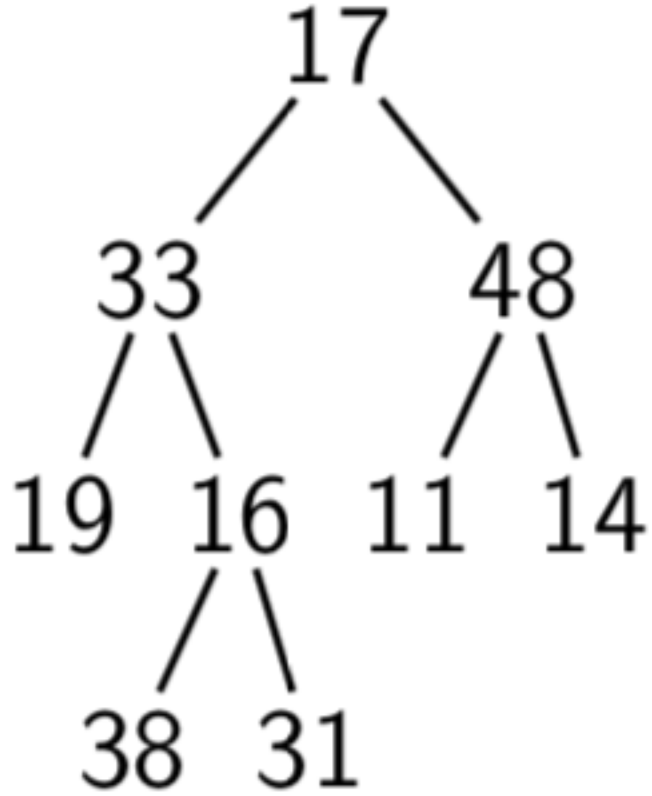
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



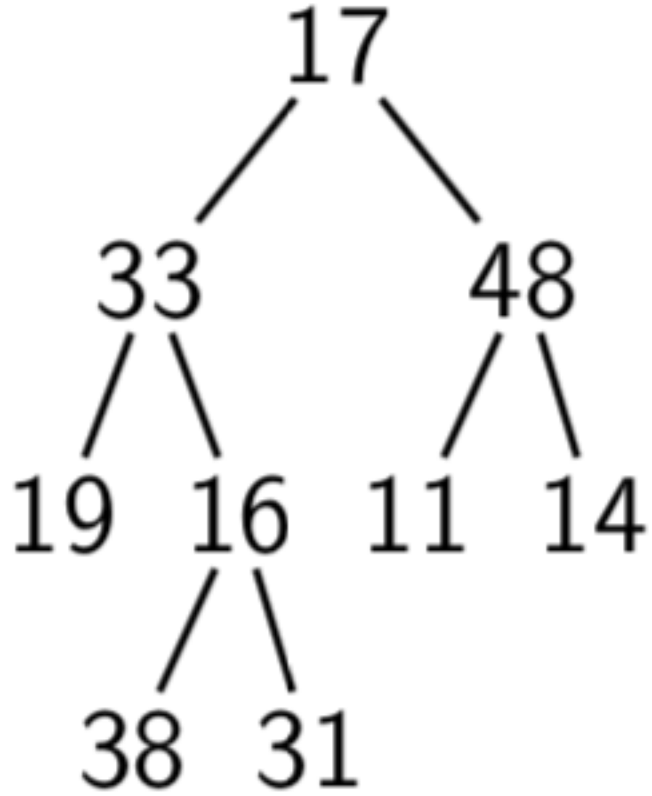
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



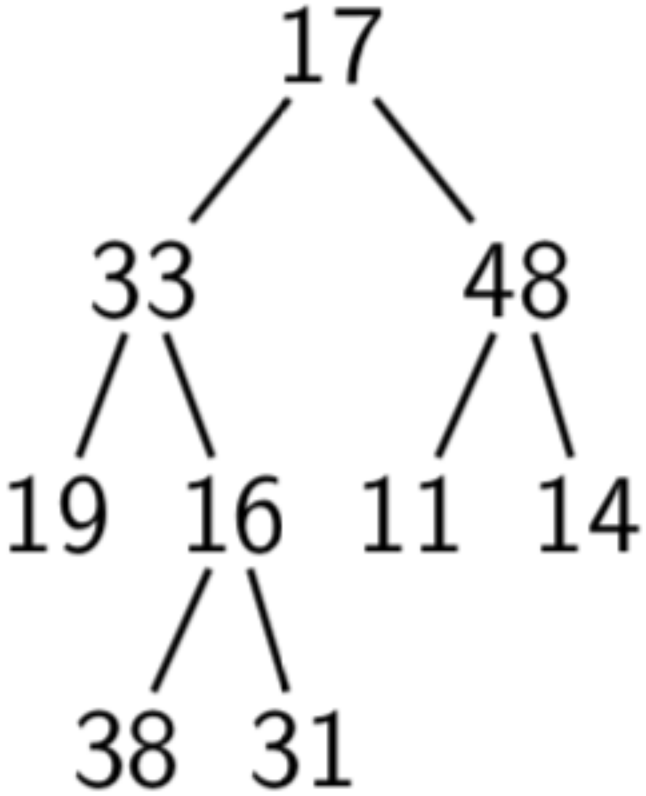
PostorderTraverse(38)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



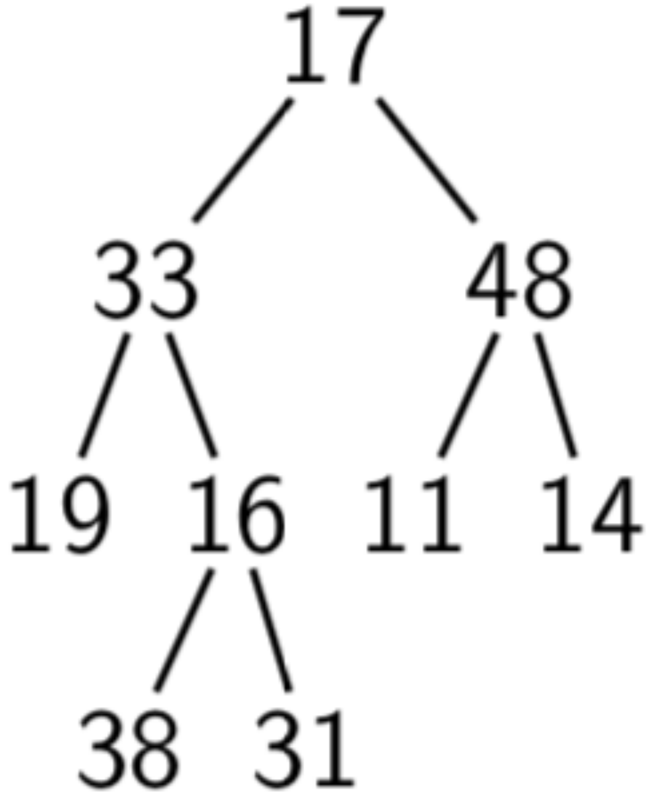
PostorderTraverse(null)
PostorderTraverse(38)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



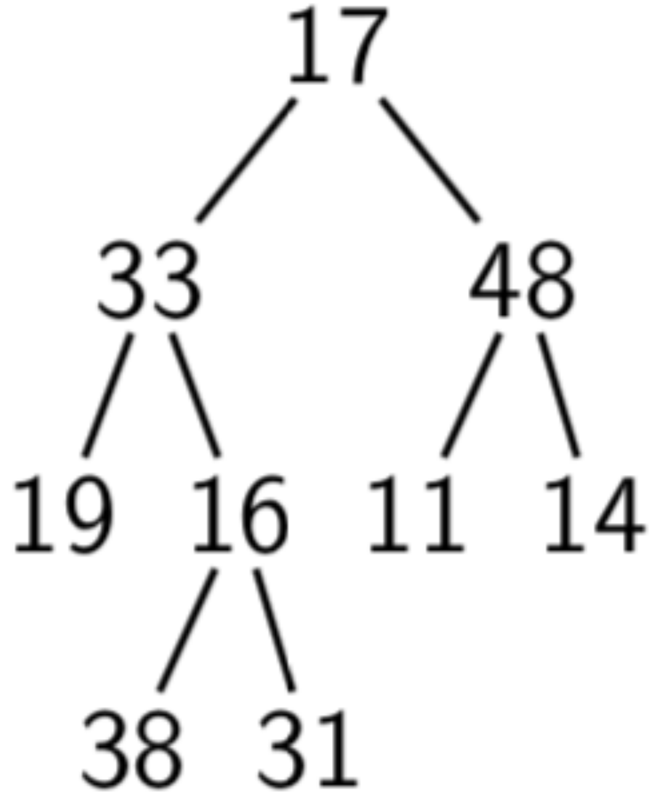
PostorderTraverse(38)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



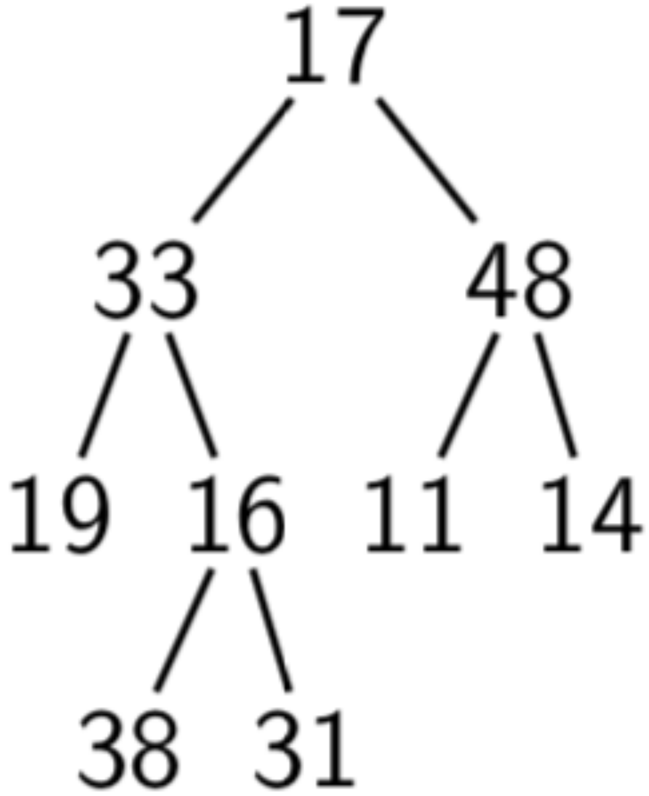
POSTORDERTRAVERSE(null)
POSTORDERTRAVERSE(38)
POSTORDERTRAVERSE(16)
POSTORDERTRAVERSE(33)
POSTORDERTRAVERSE(17)

Call Stack

Postorder Traversal

Visit order: 19

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



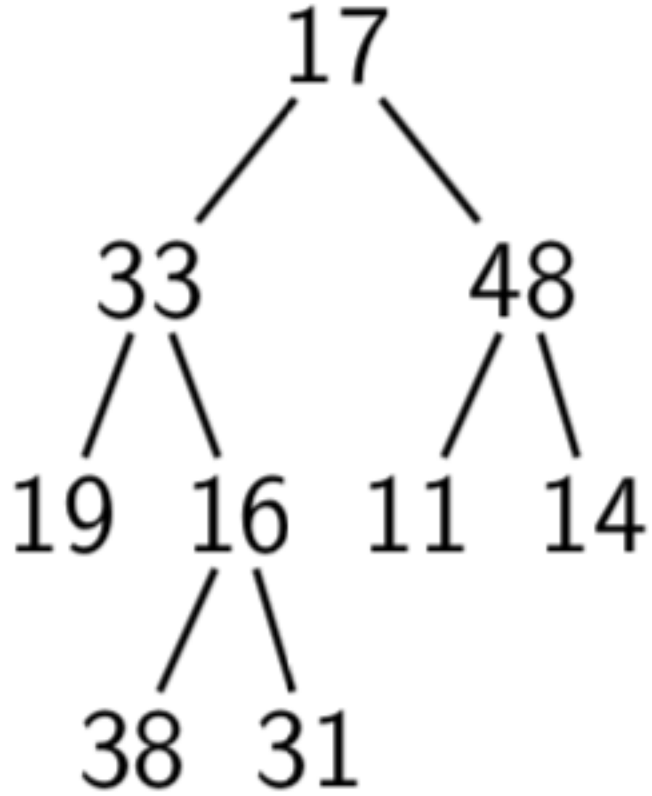
PostorderTraverse(38)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



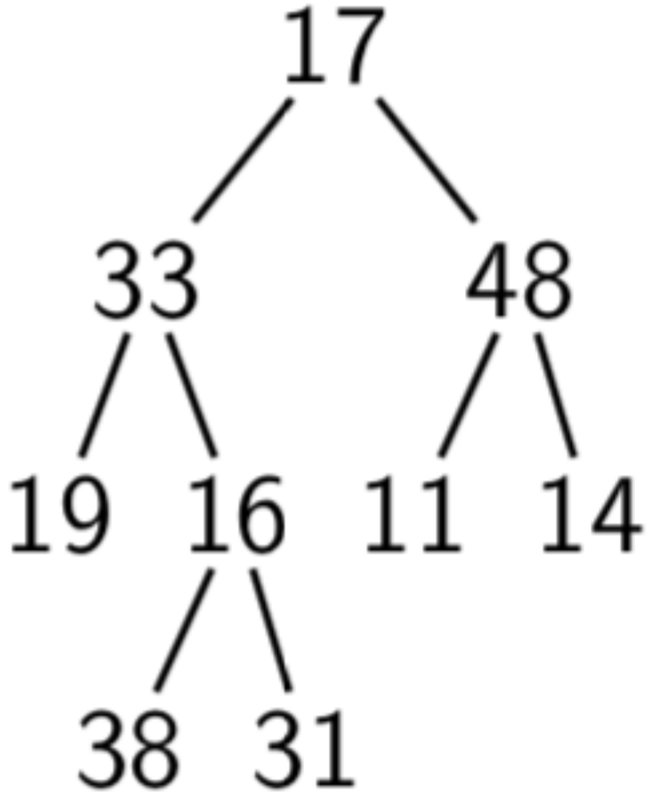
PostorderTraverse(38)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



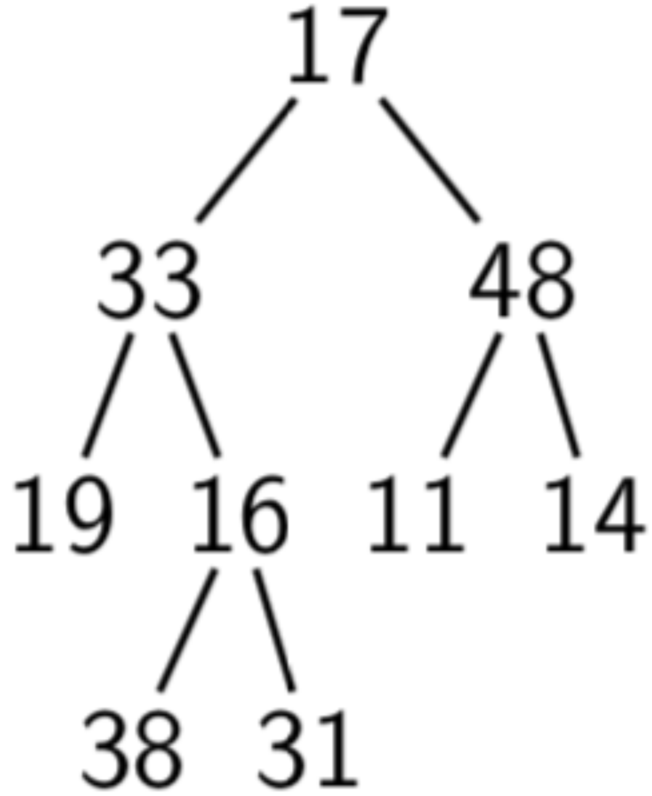
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



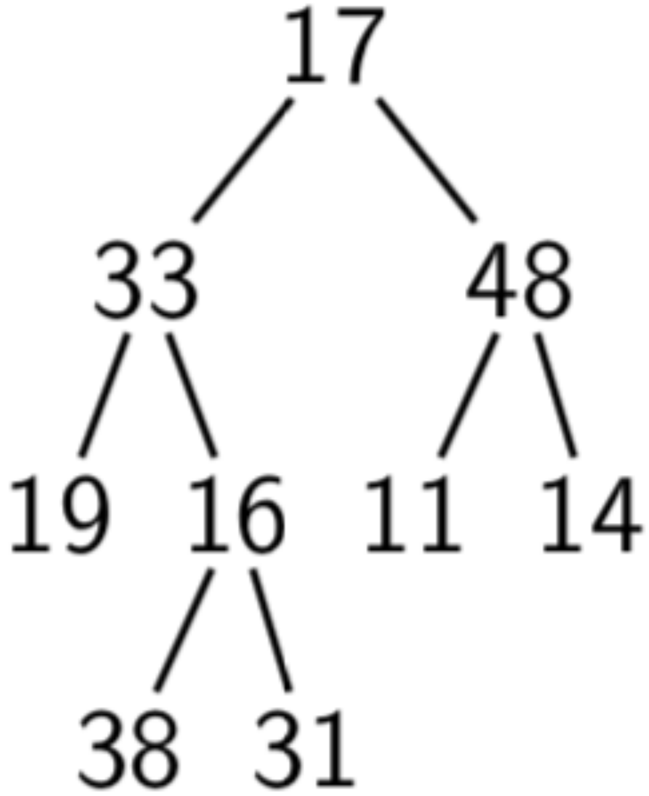
PostorderTraverse(31)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



PostorderTraverse(31)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

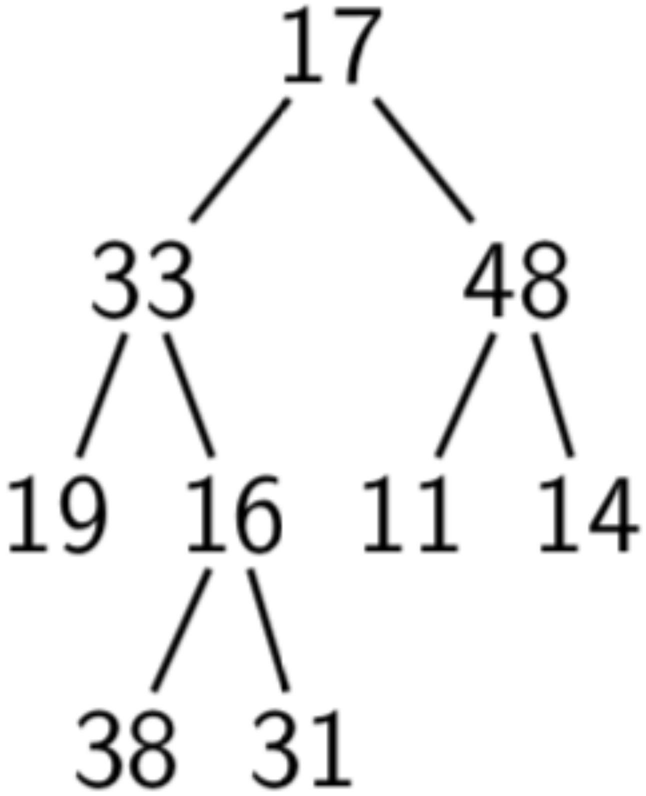
Call Stack

(…skipping the calls to PostorderTraverse(null)…)

Postorder Traversal

Visit order: 19 38 31

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



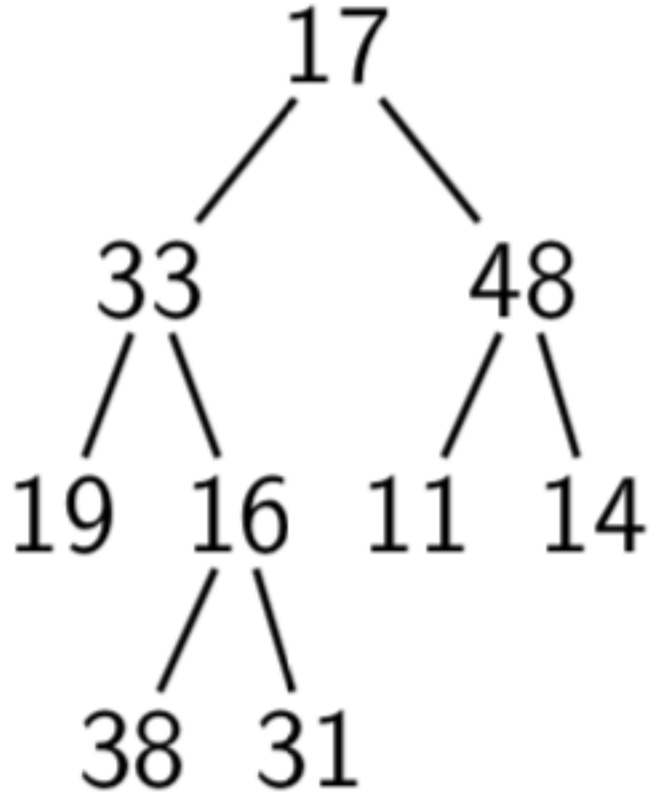
PostorderTraverse(31)
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



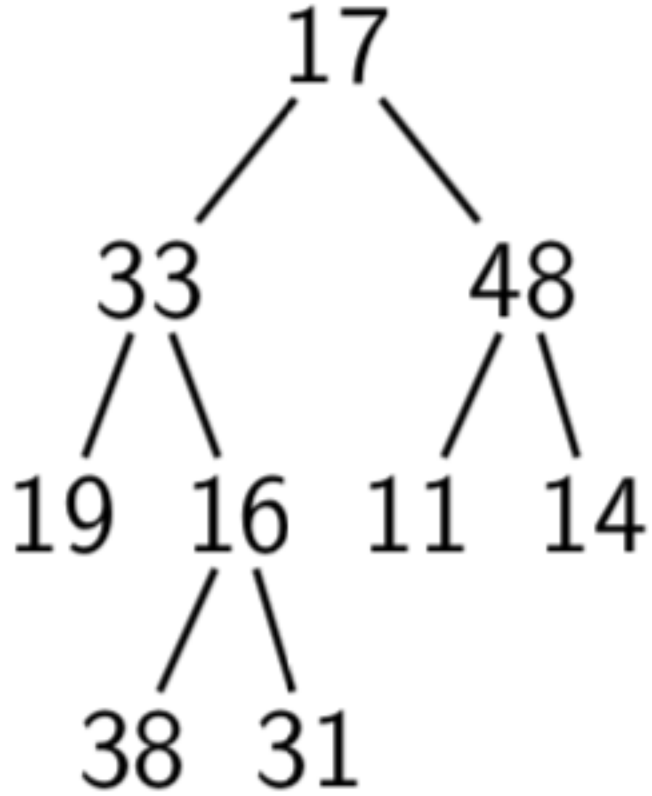
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



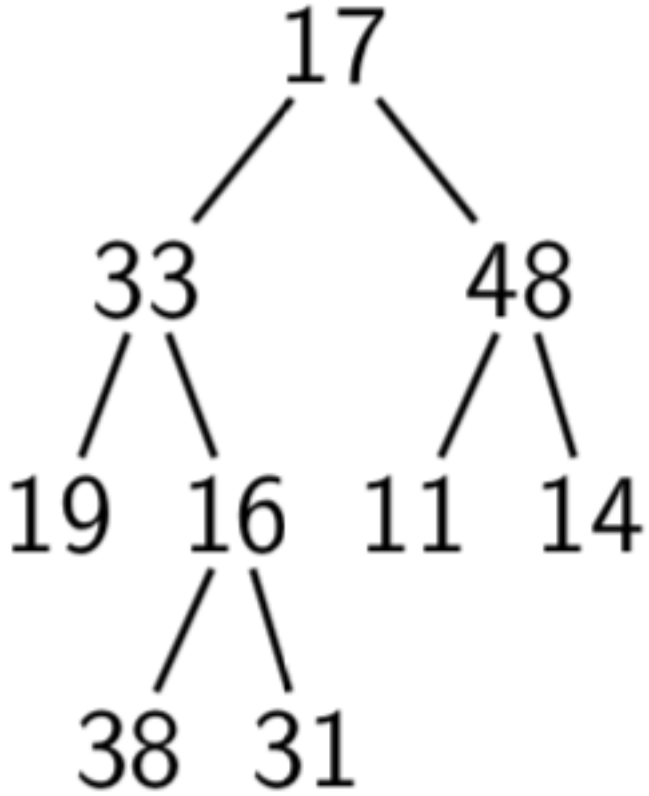
PostorderTraverse(16)
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



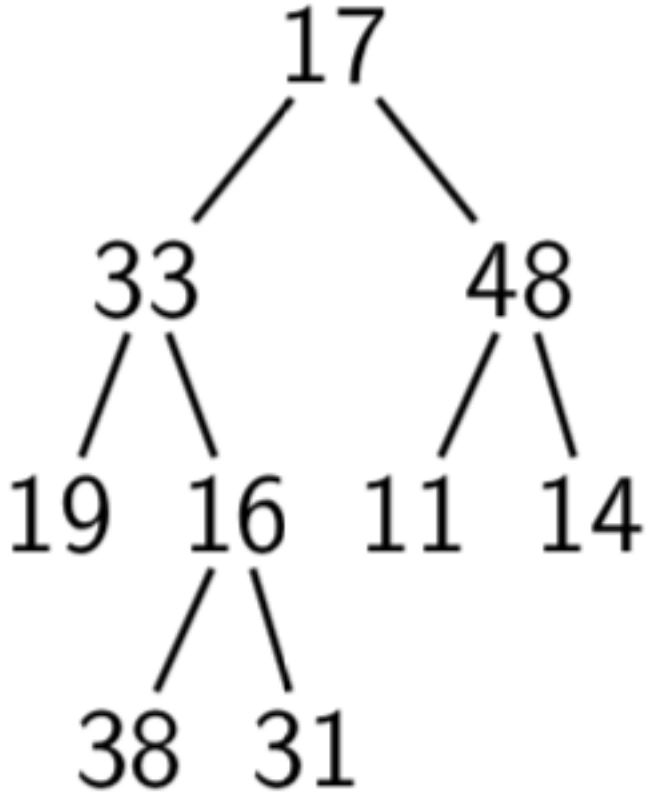
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



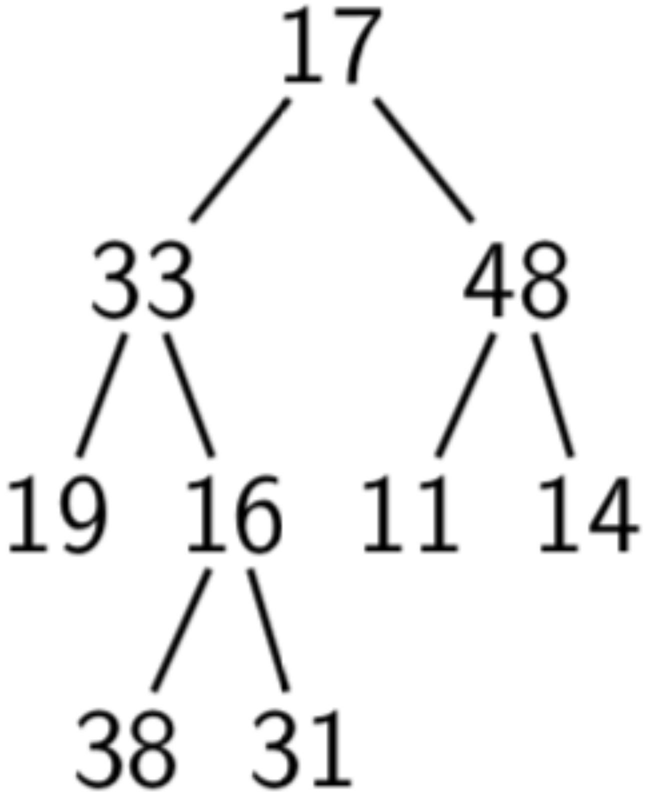
PostorderTraverse(33)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



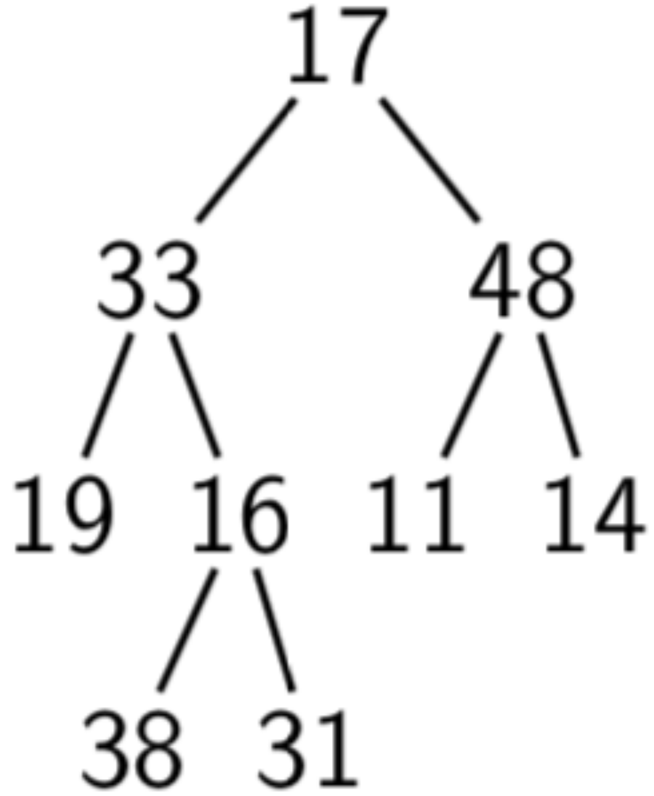
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



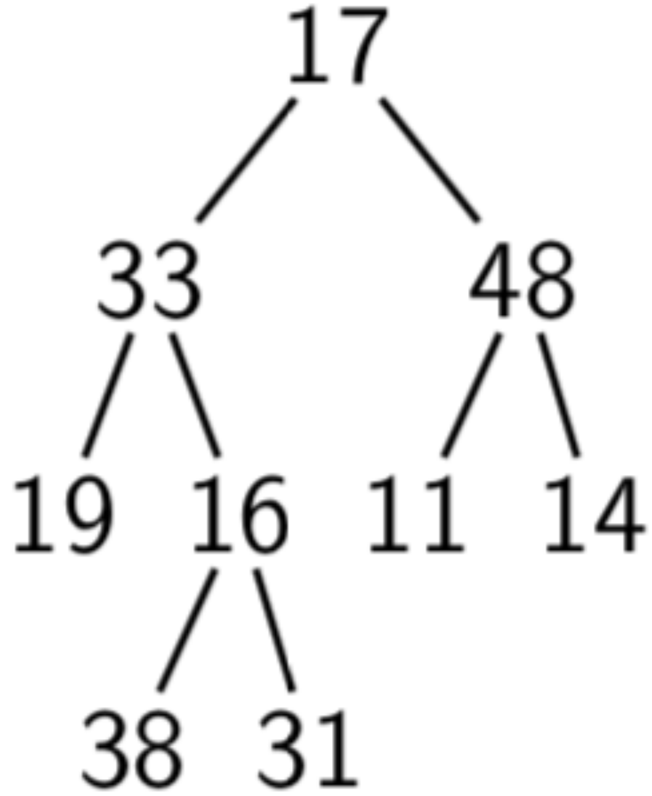
PostorderTraverse(48)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



PostorderTraverse(11)

PostorderTraverse(48)

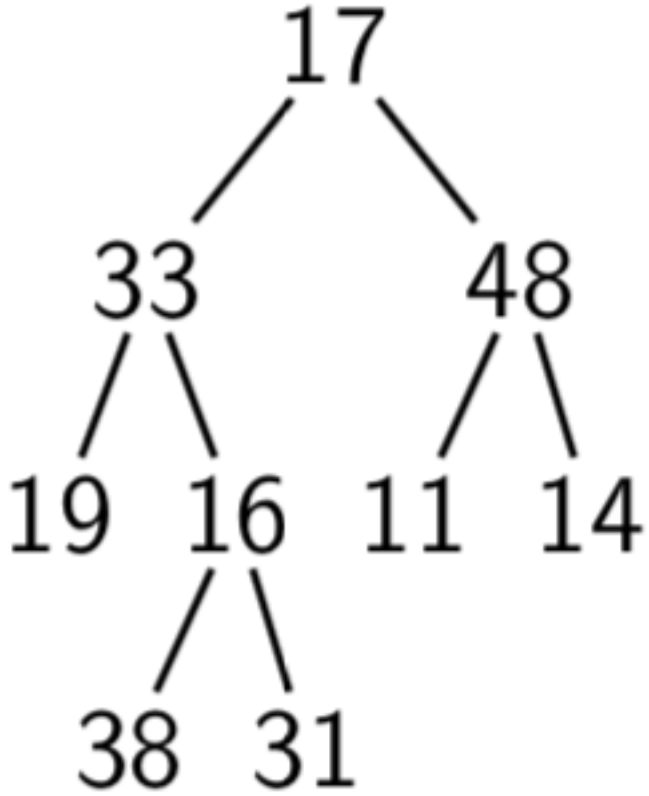
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



PostorderTraverse(11)

PostorderTraverse(48)

PostorderTraverse(17)

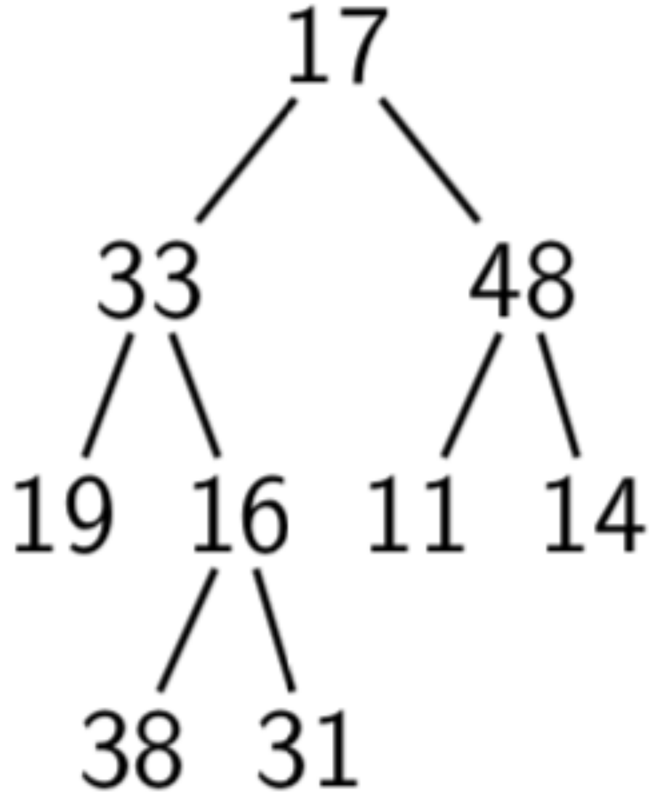
Call Stack

(…skipping the calls to PostorderTraverse(null)…)

Postorder Traversal

Visit order: 19 38 31 16 33 11

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



PostorderTraverse(11)

PostorderTraverse(48)

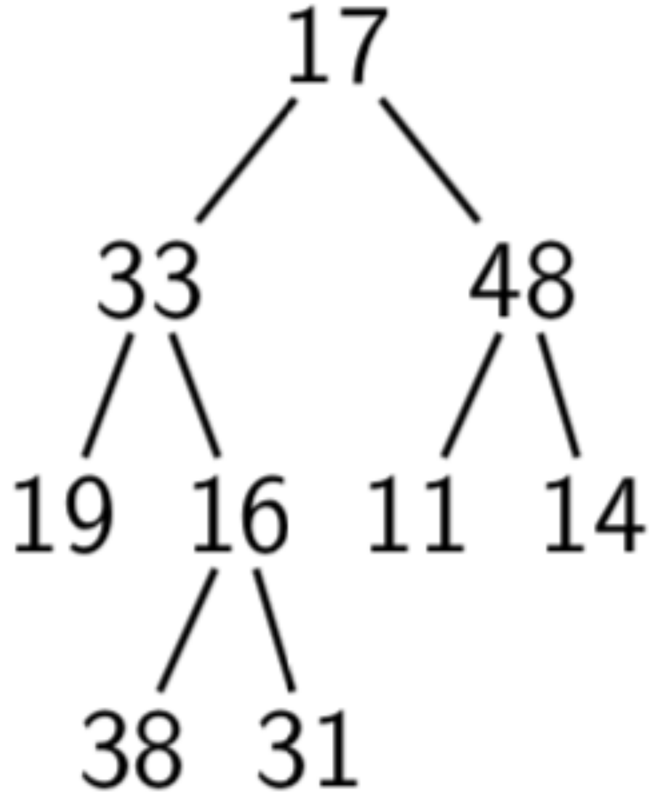
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



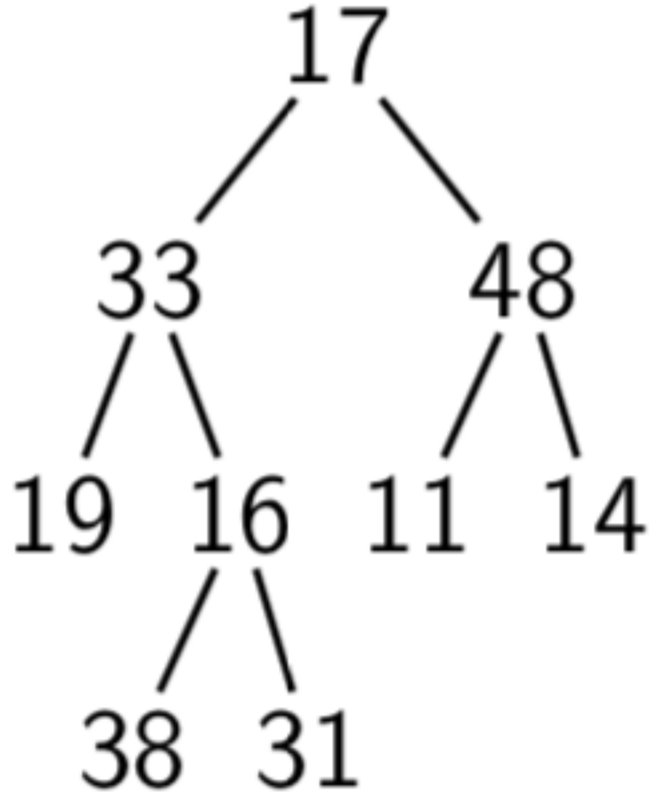
PostorderTraverse(48)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



PostorderTraverse(14)

PostorderTraverse(48)

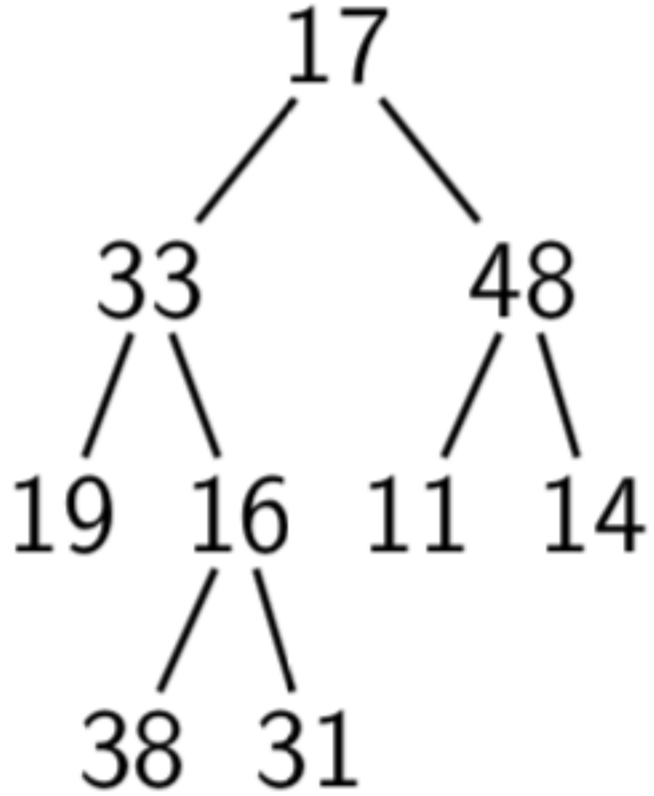
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



PostorderTraverse(14)

PostorderTraverse(48)

PostorderTraverse(17)

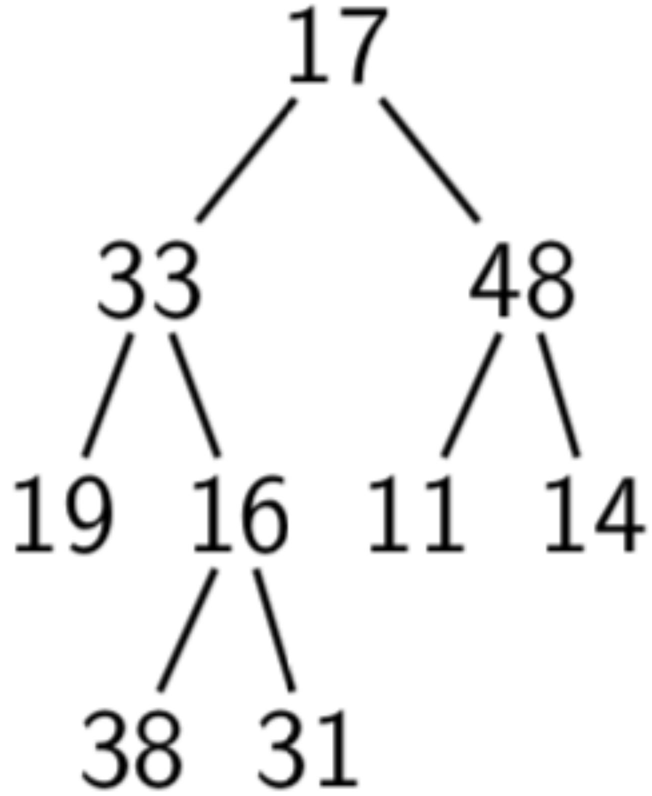
Call Stack

(…skipping the calls to PostorderTraverse(null)…)

Postorder Traversal

Visit order: 19 38 31 16 33 11 14

```
procedure PostorderTraverse(T)
    if T ≠ null then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



PostorderTraverse(14)

PostorderTraverse(48)

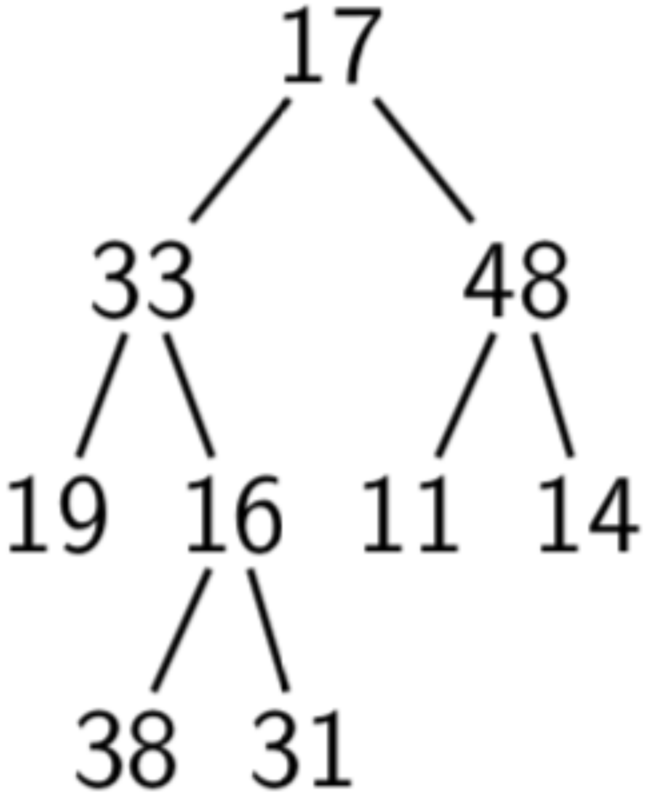
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11 14

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



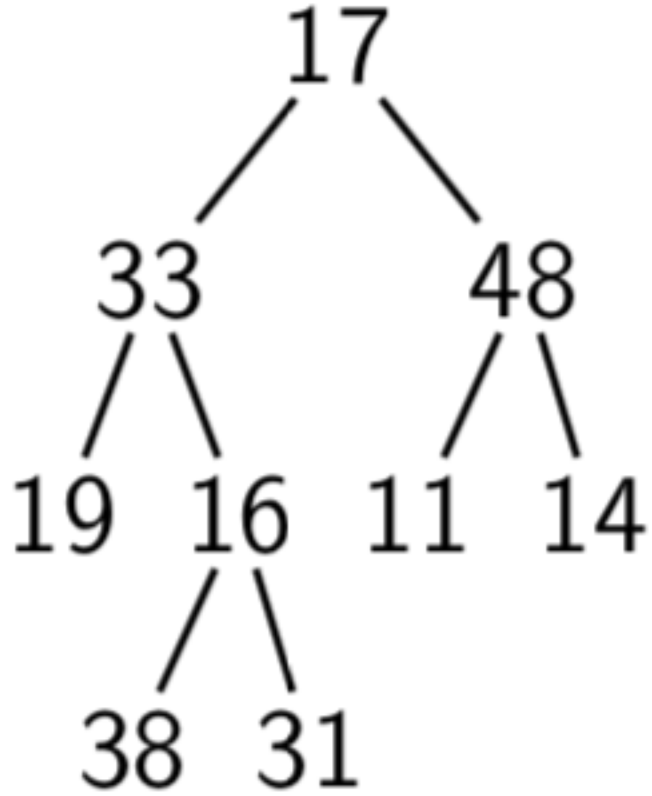
PostorderTraverse(48)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11 14 48

```
procedure PostorderTraverse($T$)
    if $T \neq null$ then
        PostorderTraverse($T.left$)
        PostorderTraverse($T.right$)
        visit $T.root$
```



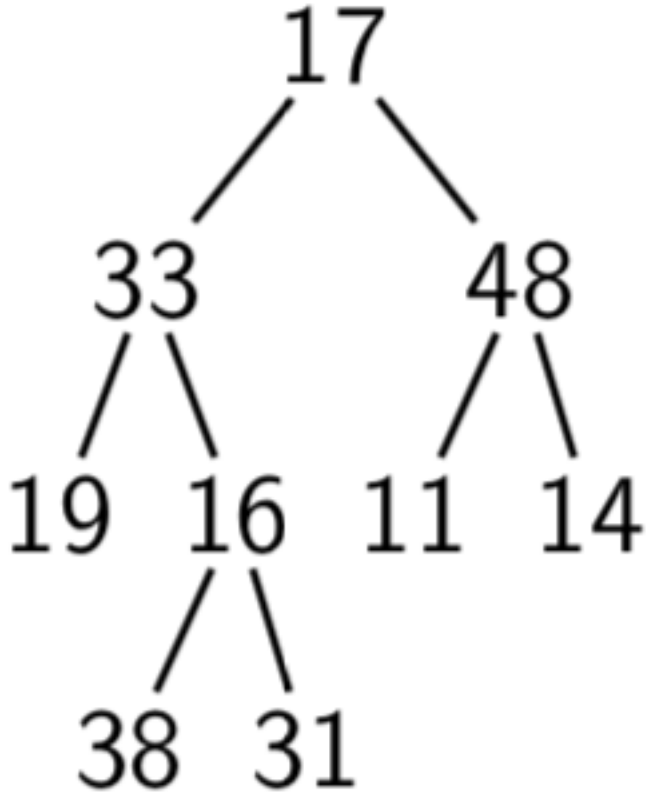
PostorderTraverse(48)
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11 14 48

```
procedure POSTORDERTRAVERSE($T$)
    if $T \neq null$ then
        POSTORDERTRAVERSE($T.left$)
        POSTORDERTRAVERSE($T.right$)
        visit $T.root$
```



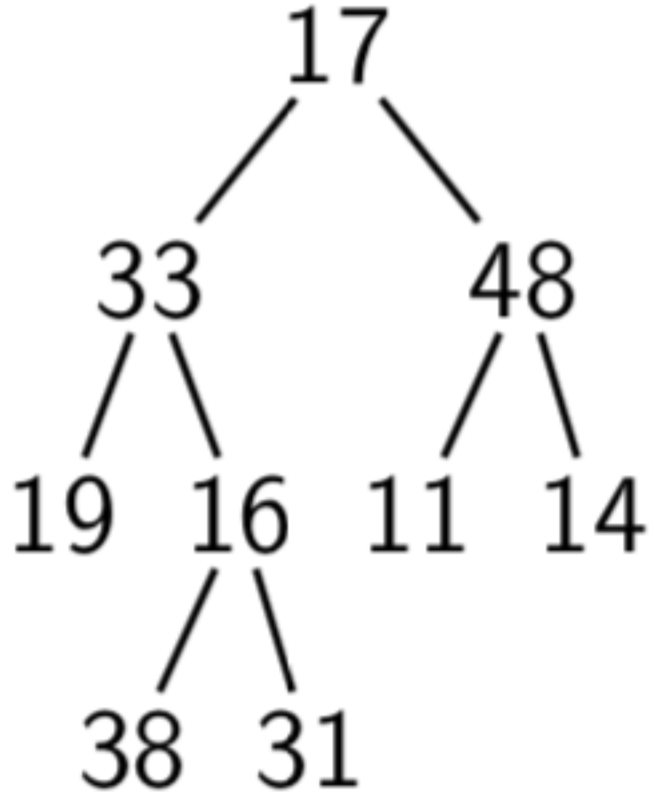
POSTORDERTRAVERSE(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11 14 48 17

```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



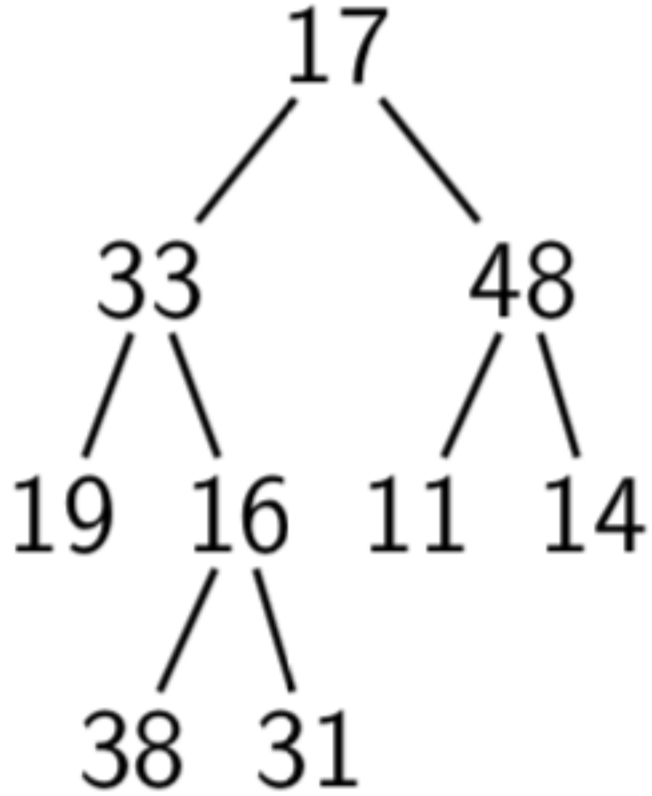
PostorderTraverse(17)

Call Stack

Postorder Traversal

Visit order: 19 38 31 16 33 11 14 48 17

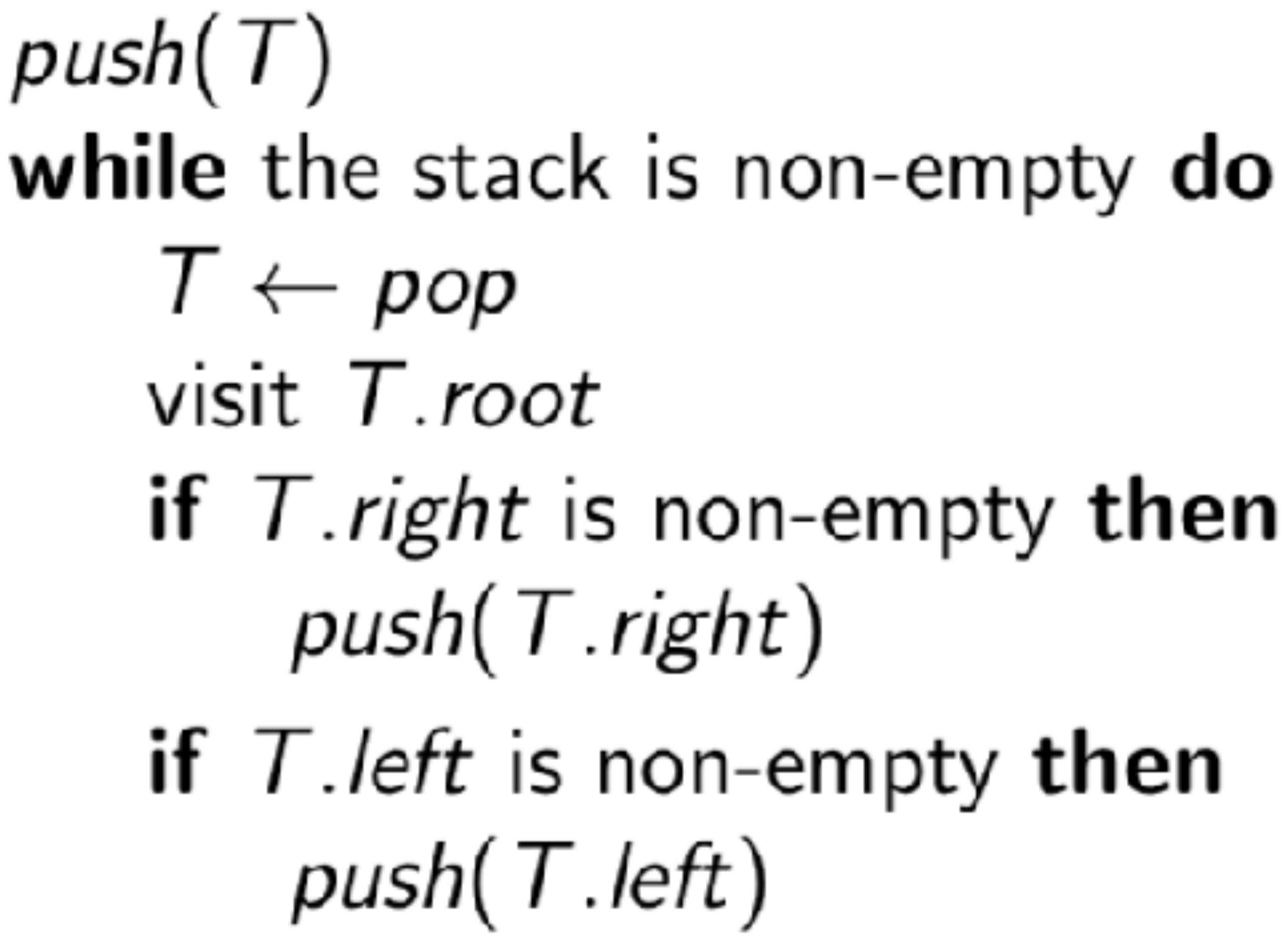
```
procedure PostorderTraverse(T)
    if $T \neq null$ then
        PostorderTraverse(T.left)
        PostorderTraverse(T.right)
        visit T.root
```



Call Stack

Preorder Traversal Using a Stack

- Explicitly maintain a stack of nodes



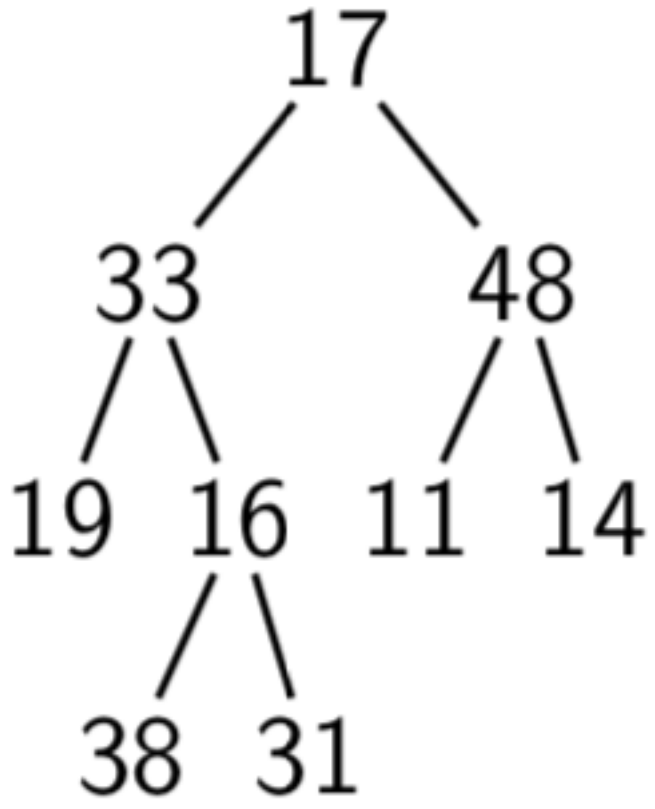
```
$push(T)$
while the stack is non-empty do
    $T \leftarrow pop$
    visit $T.root$
    if $T.right$ is non-empty then
        $push(T.right)$
    if $T.left$ is non-empty then
        $push(T.left)$
```

- In an implementation, the elements placed onto the stack would not be whole trees, but **pointers** to the corresponding internal nodes

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



Queue:

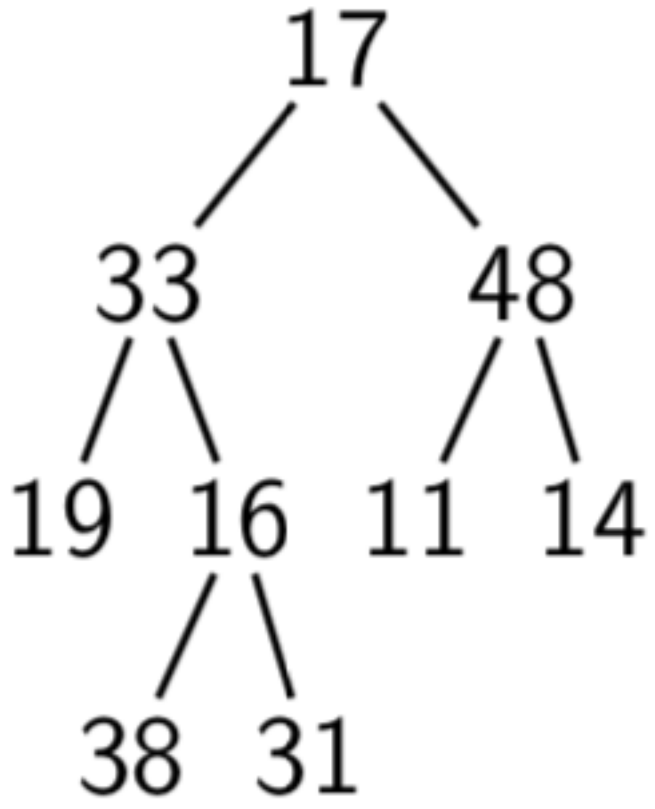
Traversal order:

breadth first

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



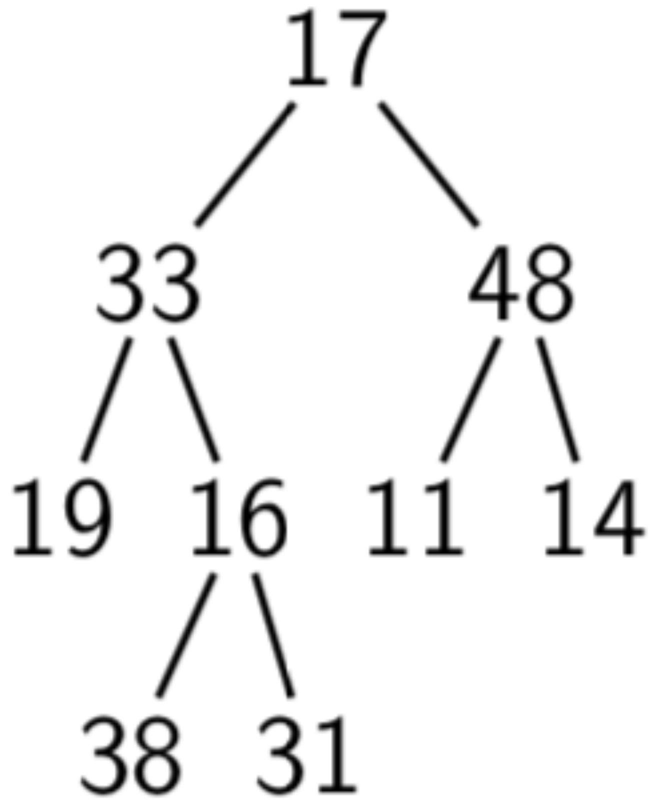
Queue: 17

Traversal order:

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject(T)
while the queue is non-empty do
    T ← eject
    visit T.root
    if T.left is non-empty then
        inject(T.left)
    if T.right is non-empty then
        inject(T.right)
```



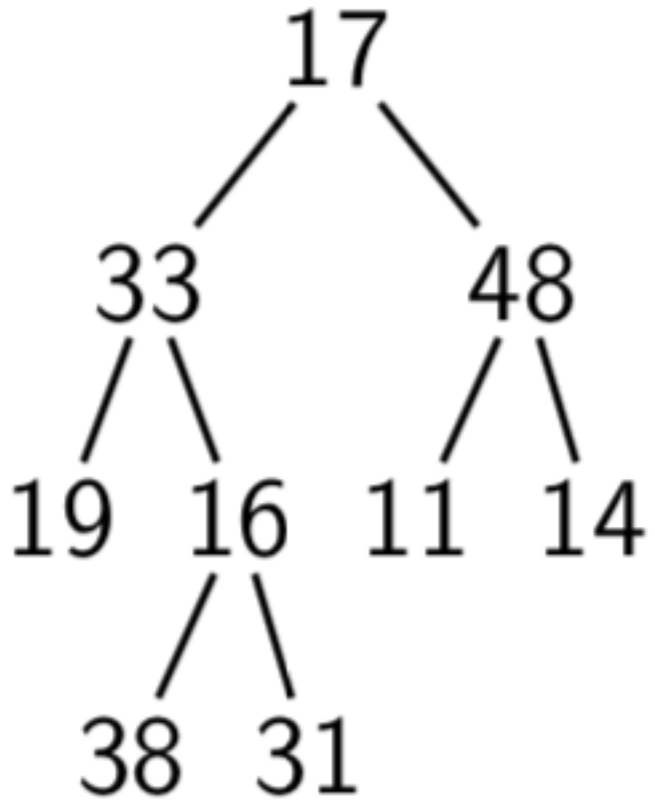
Queue:

Traversal order:

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject($T$)
**while** the queue is non-empty **do**
    $T \leftarrow eject$
    visit $T.root$
    **if** $T.left$ is non-empty **then**
        inject($T.left$)
    **if** $T.right$ is non-empty **then**
        inject($T.right$)
```



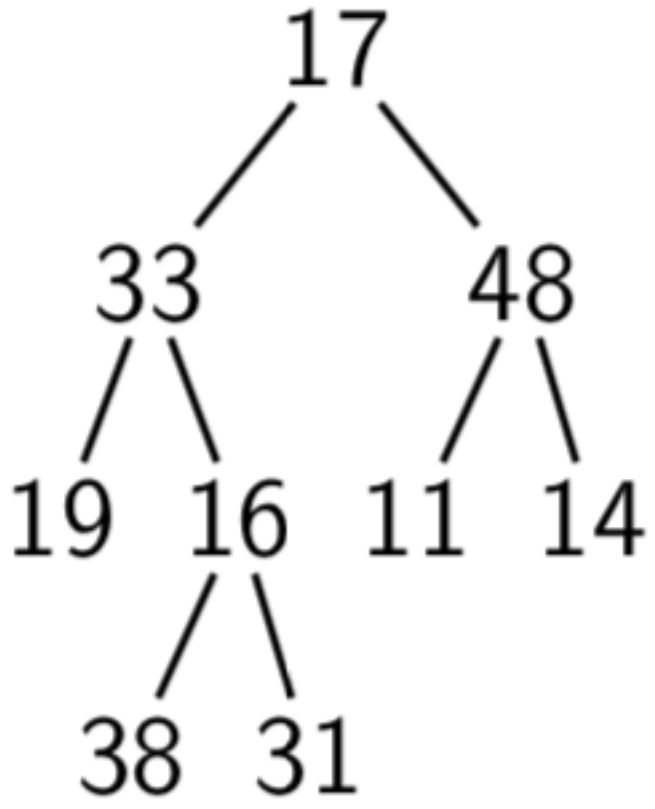
Queue:

Traversal order: 17

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject($T$)
while the queue is non-empty do
    $T \leftarrow$ eject
    visit $T$.root
    if $T$.left is non-empty then
        inject($T$.left)
    if $T$.right is non-empty then
        inject($T$.right)
```



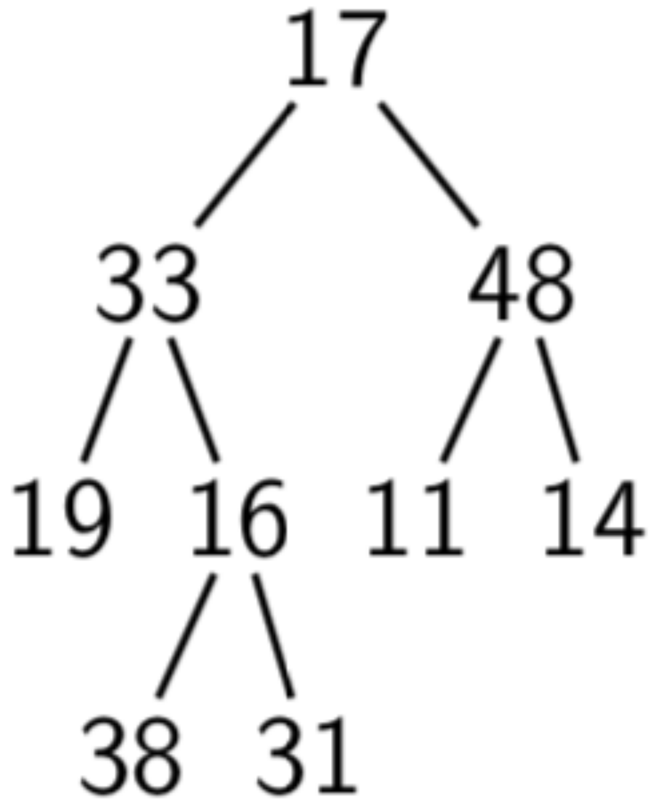
Queue: 33

Traversal order: 17

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject(T)
while the queue is non-empty do
    T ← eject
    visit T.root
    if T.left is non-empty then
        inject(T.left)
    if T.right is non-empty then
        inject(T.right)
```



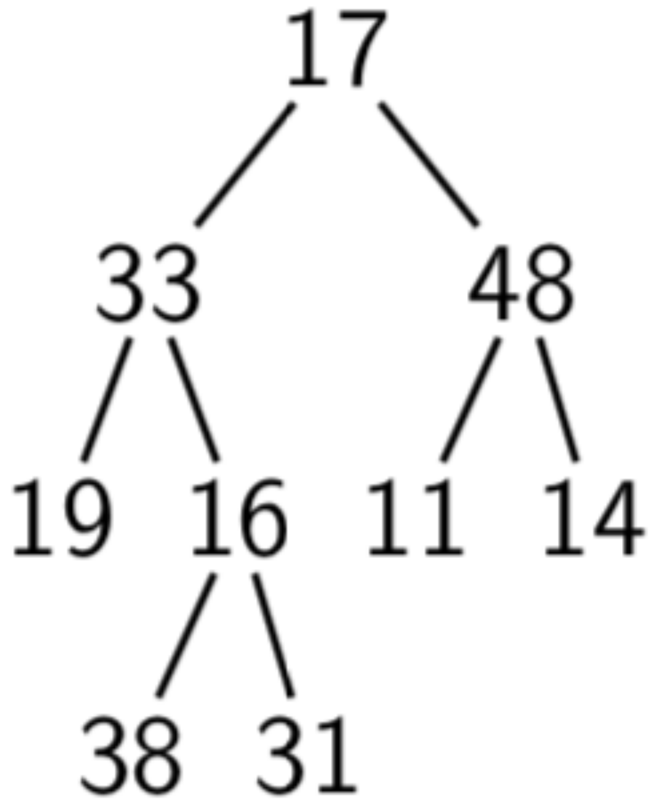
Queue: 33 48

Traversal order: 17

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



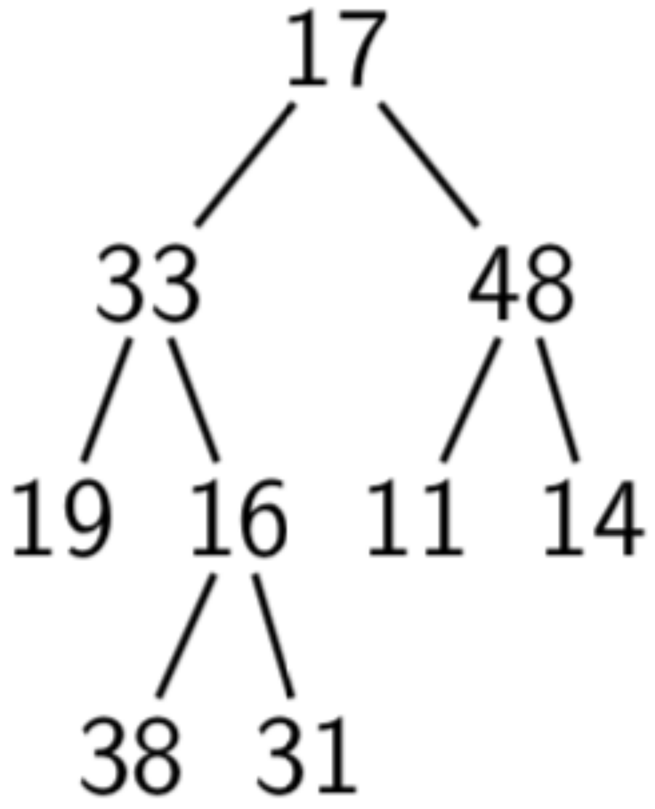
Queue: 48

Traversal order: 17

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject($T$)
while the queue is non-empty do
    $T \leftarrow$ eject
    visit $T.root$
    if $T.left$ is non-empty then
        inject($T.left$)
    if $T.right$ is non-empty then
        inject($T.right$)
```



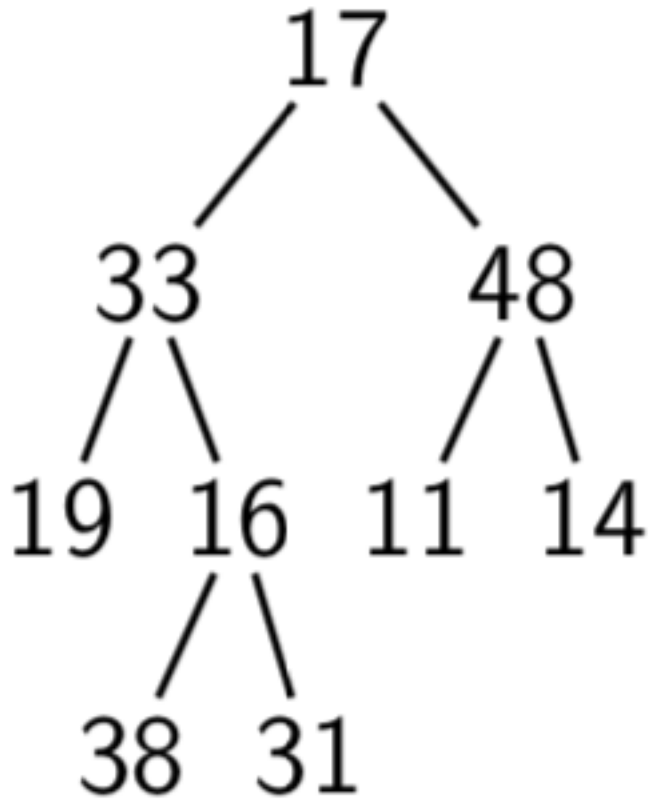
Queue: 48

Traversal order: 17 33

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject(T)
while the queue is non-empty do
    T ← eject
    visit T.root
    if T.left is non-empty then
        inject(T.left)
    if T.right is non-empty then
        inject(T.right)
```



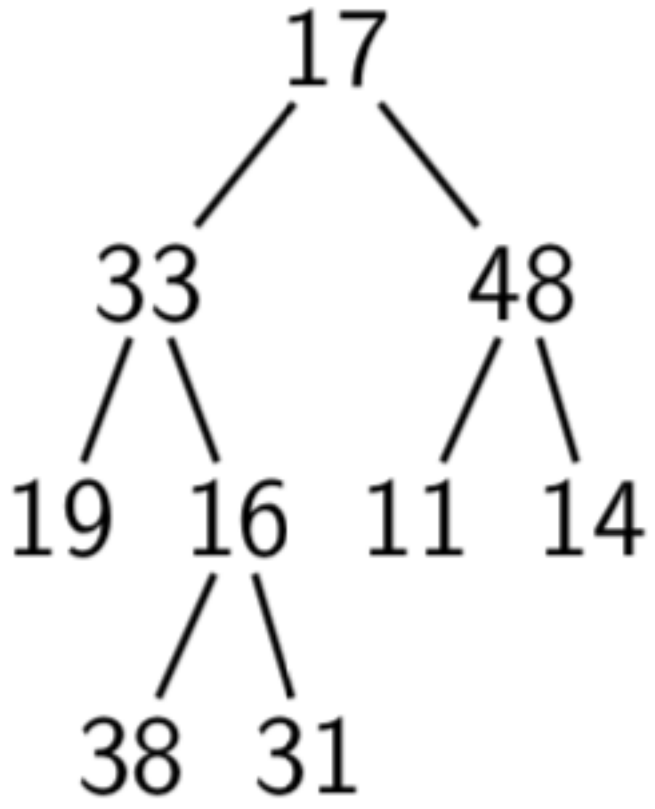
Queue: 48 19

Traversal order: 17 33

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject($T$)
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        inject($T.left$)
    if $T.right$ is non-empty then
        inject($T.right$)
```



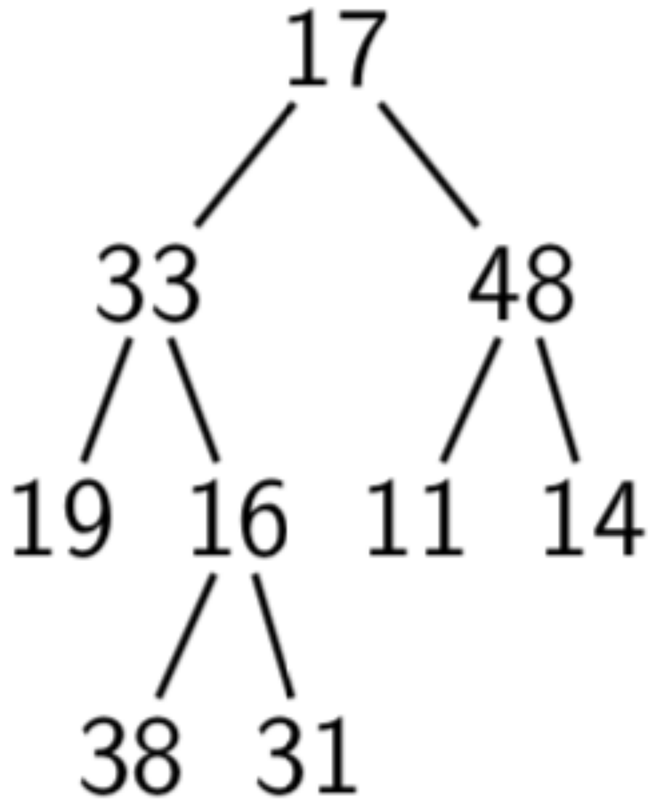
Queue: 48 19 16

Traversal order: 17 33

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



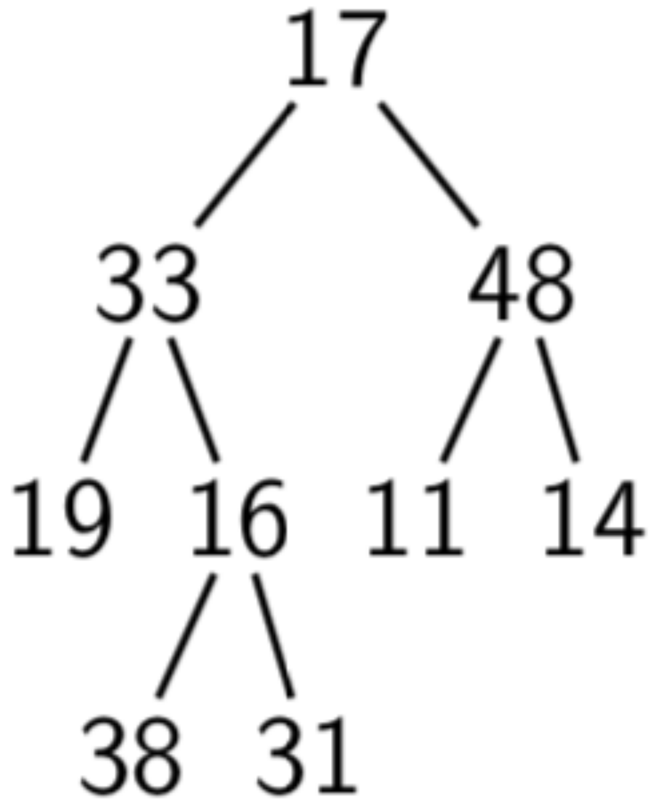
Queue: 19 16

Traversal order: 17 33

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject(T)
while the queue is non-empty do
    T ← eject
    visit T.root
    if T.left is non-empty then
        inject(T.left)
    if T.right is non-empty then
        inject(T.right)
```



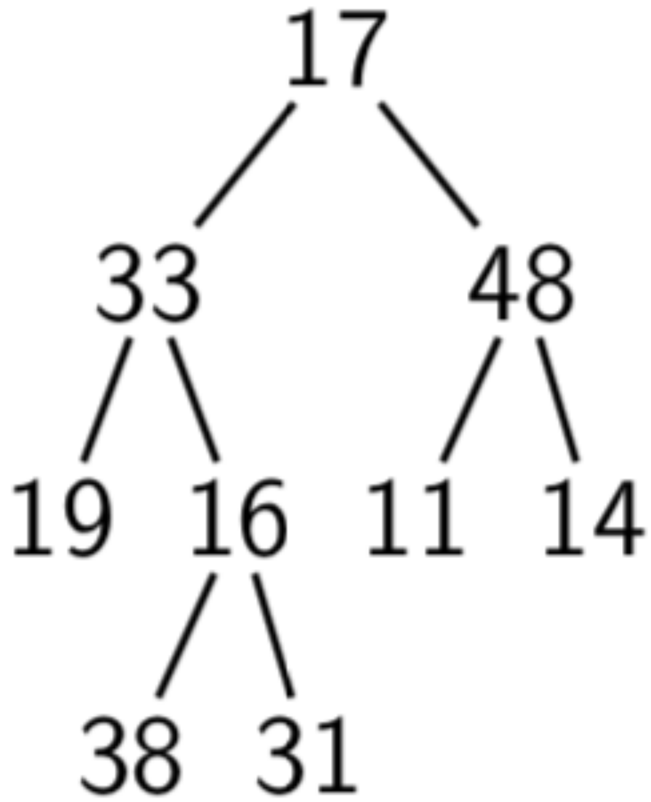
Queue: 19 16

Traversal order: 17 33 48

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject(T)
while the queue is non-empty do
    $T \leftarrow eject$
    visit T.root
    if T.left is non-empty then
        inject(T.left)
    if T.right is non-empty then
        inject(T.right)
```



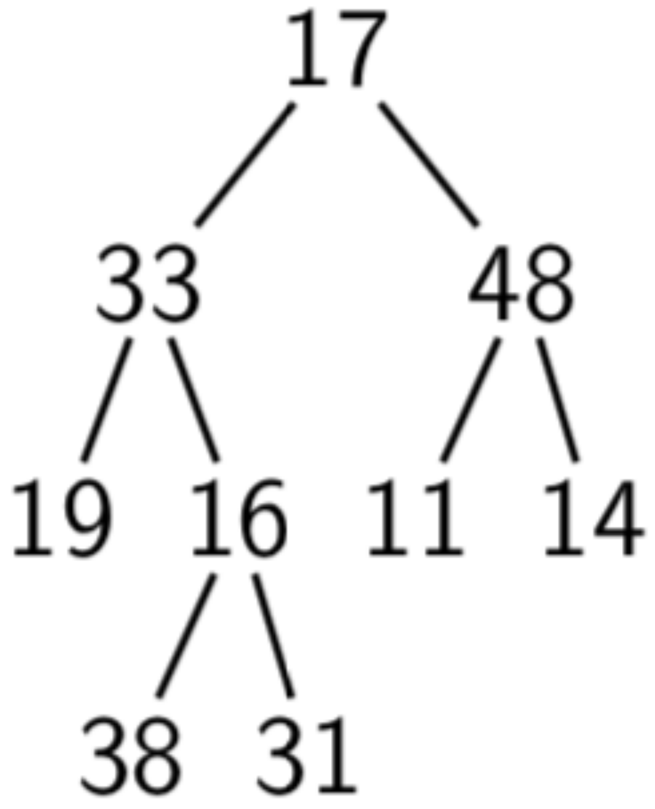
Queue: 19 16 11

Traversal order: 17 33 48

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject($T$)
while the queue is non-empty do
    $T \leftarrow$ eject
    visit $T.root$
    if $T.left$ is non-empty then
        inject($T.left$)
    if $T.right$ is non-empty then
        inject($T.right$)
```



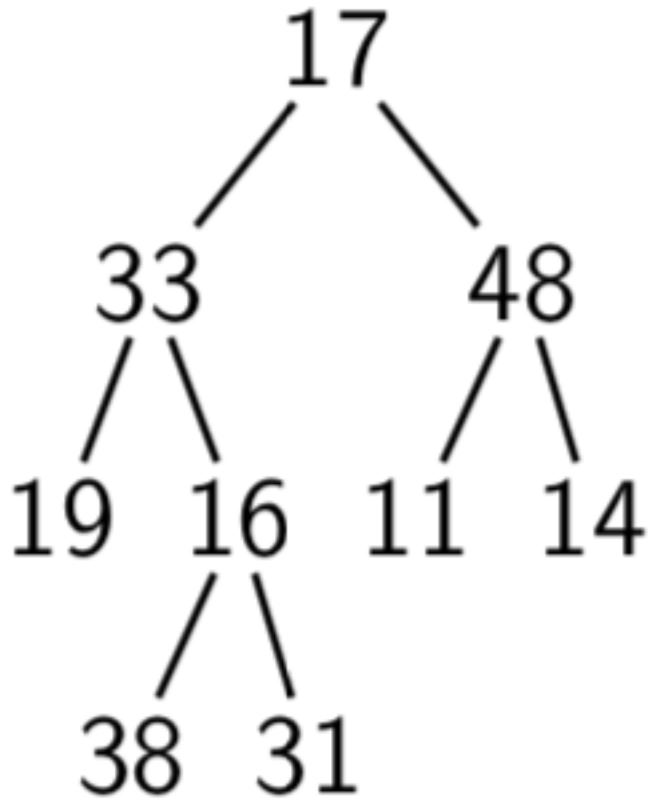
Queue: 19 16 11 14

Traversal order: 17 33 48

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



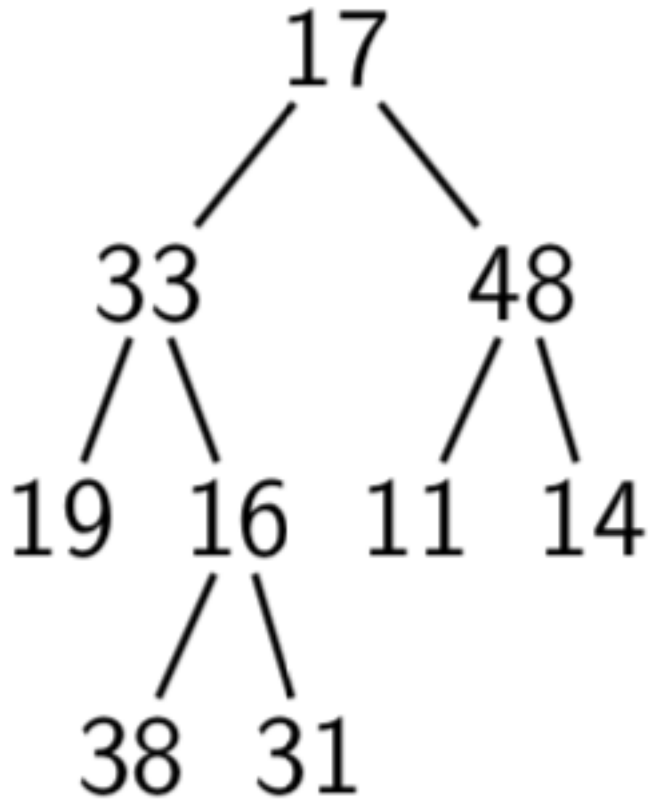
Queue: 16 11 14

Traversal order: 17 33 48

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



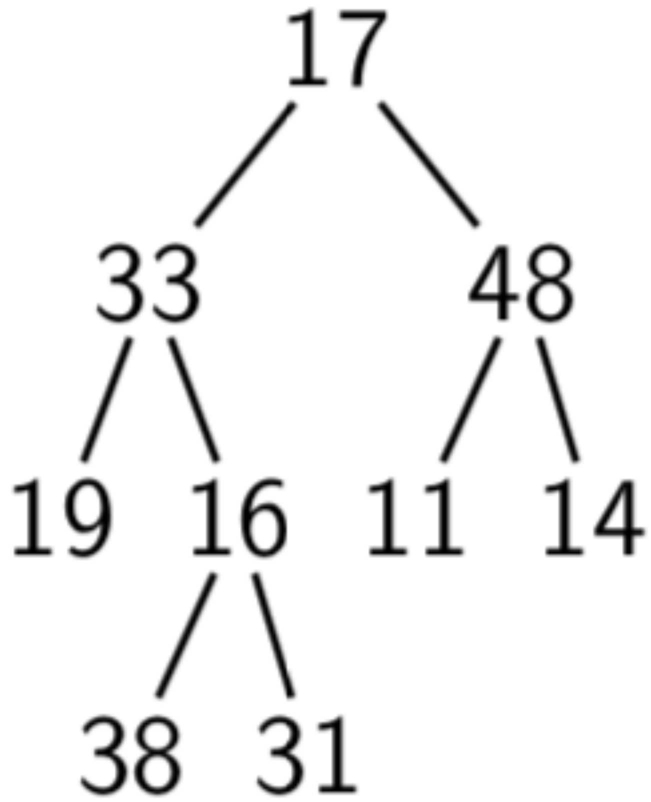
Queue: 16 11 14

Traversal order: 17 33 48 19

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



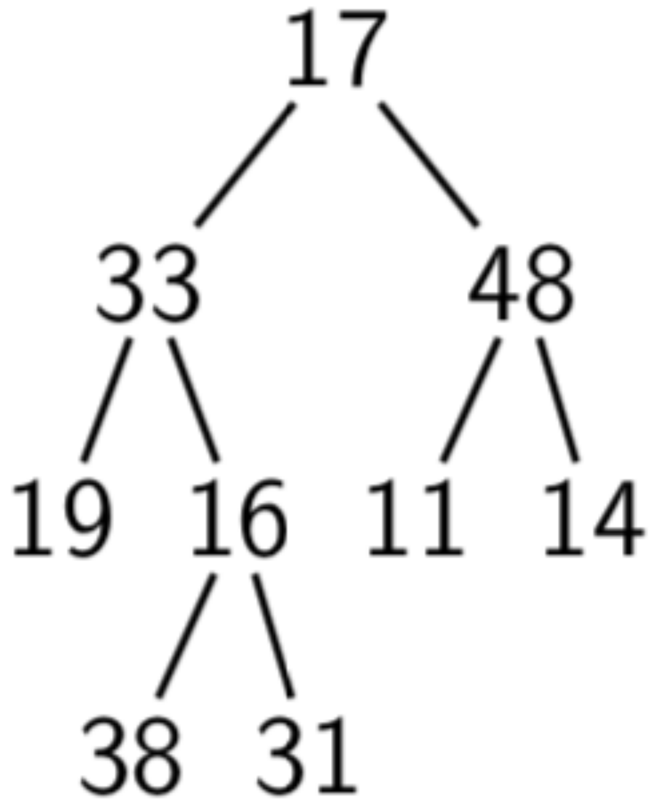
Queue: 11 14

Traversal order: 17 33 48 19

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
$inject(T)$
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        $inject(T.left)$
    if $T.right$ is non-empty then
        $inject(T.right)$
```



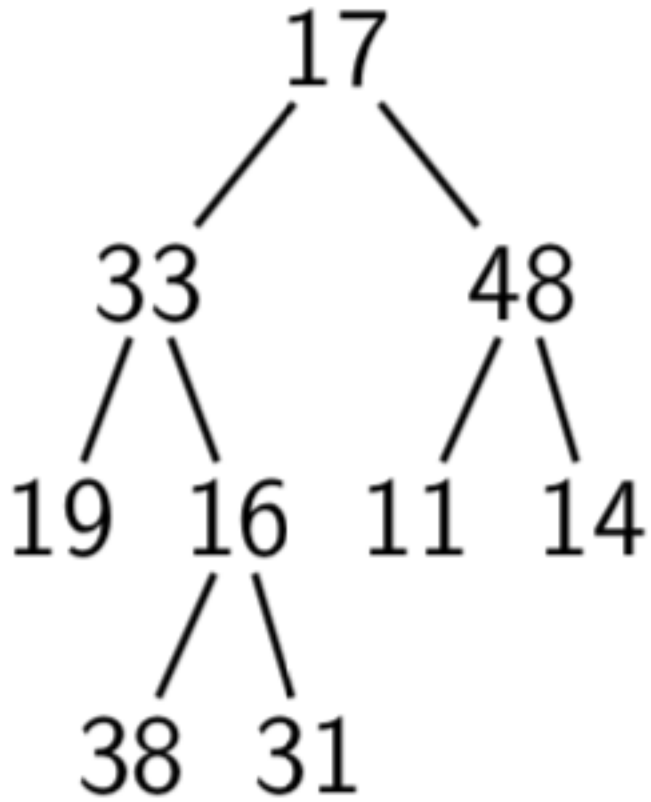
Queue: 11 14

Traversal order: 17 33 48 19 16

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

```
inject(T)
while the queue is non-empty do
    $T \leftarrow eject$
    visit $T.root$
    if $T.left$ is non-empty then
        inject($T.left$)
    if $T.right$ is non-empty then
        inject($T.right$)
```



Queue: 11 14 38

Traversal order: 17 33 48 19 16

Level-Order Traversal Using a Queue

- Replace the stack with a **queue**

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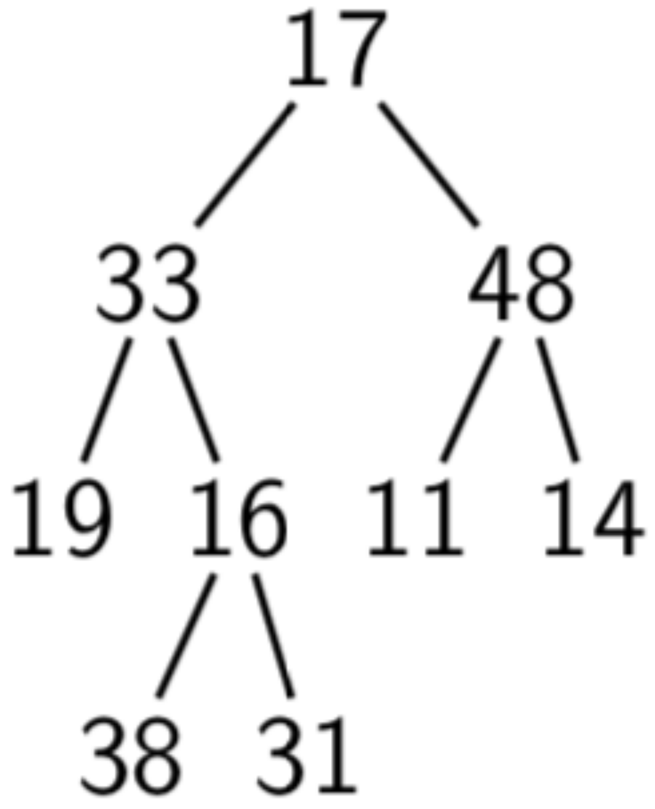

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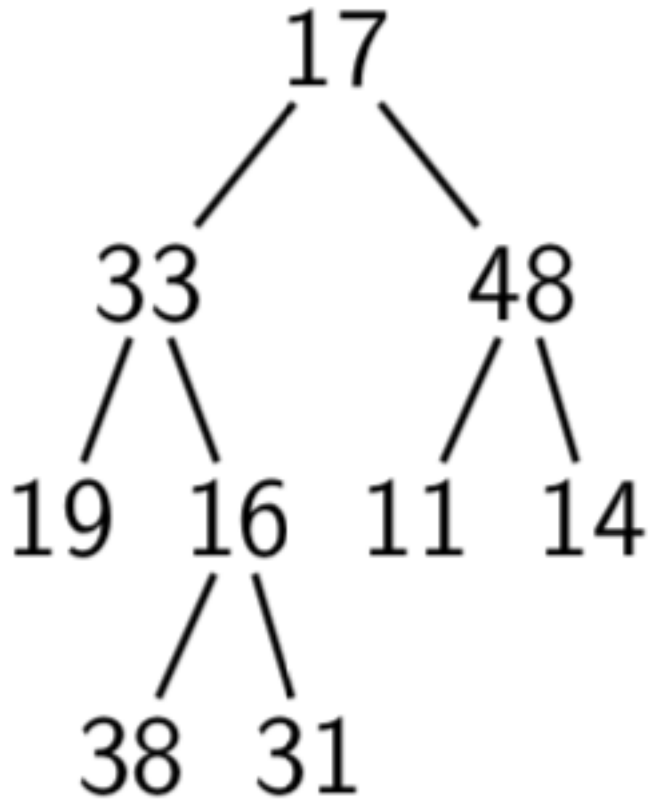

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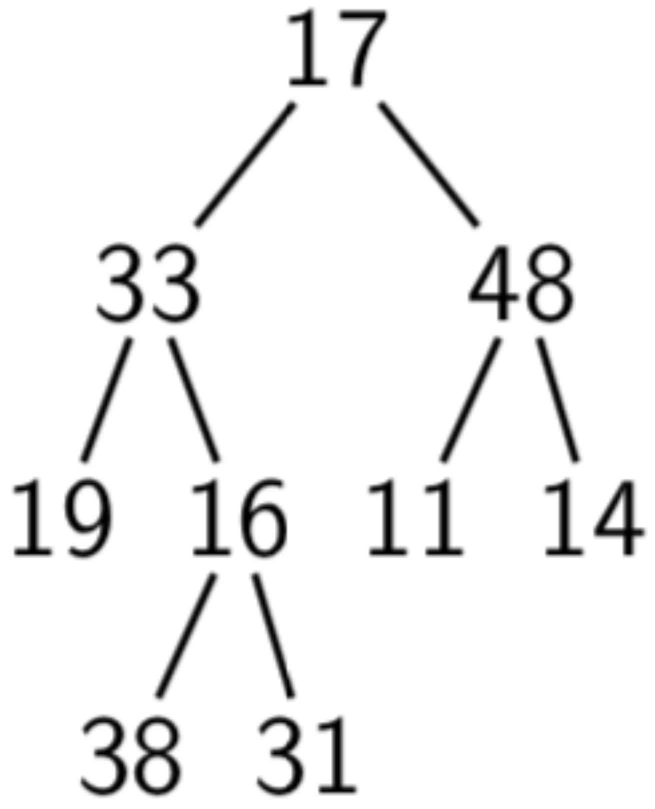

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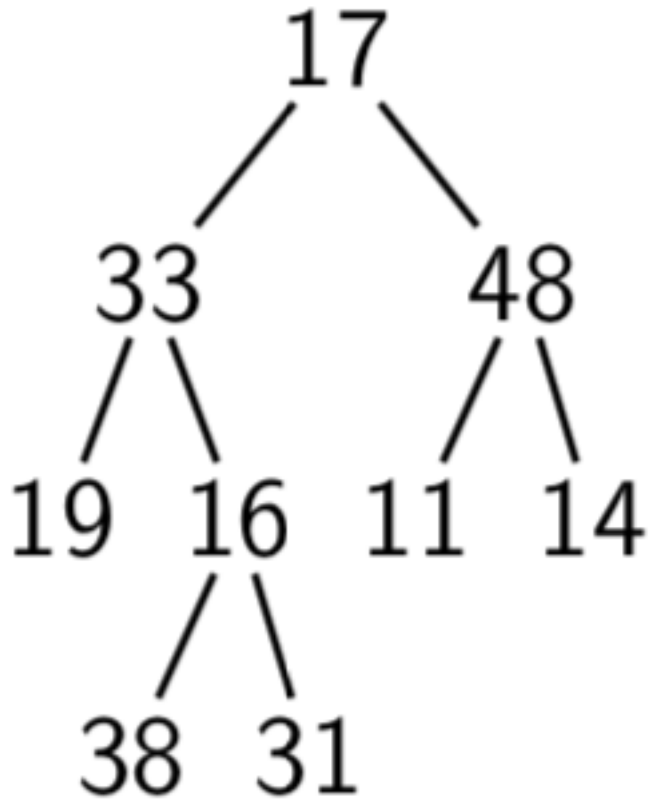

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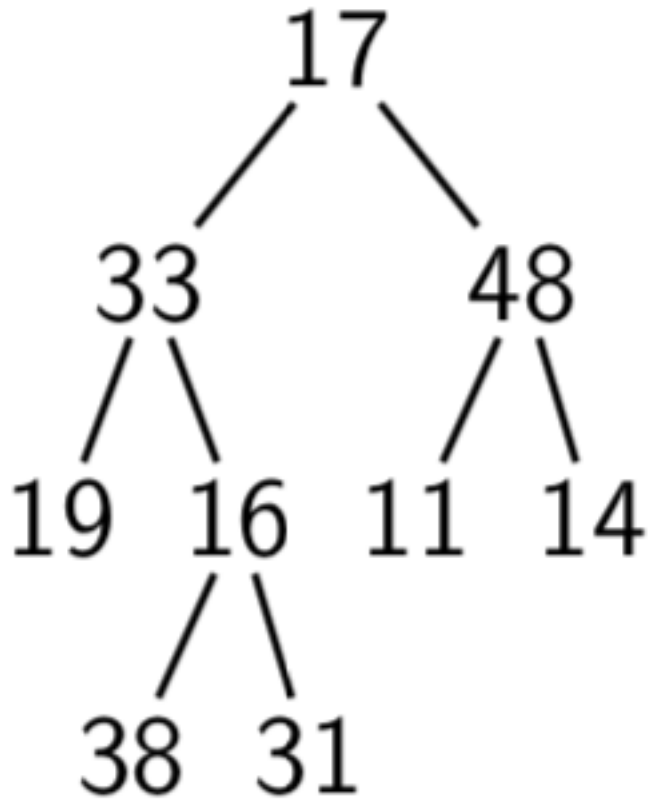

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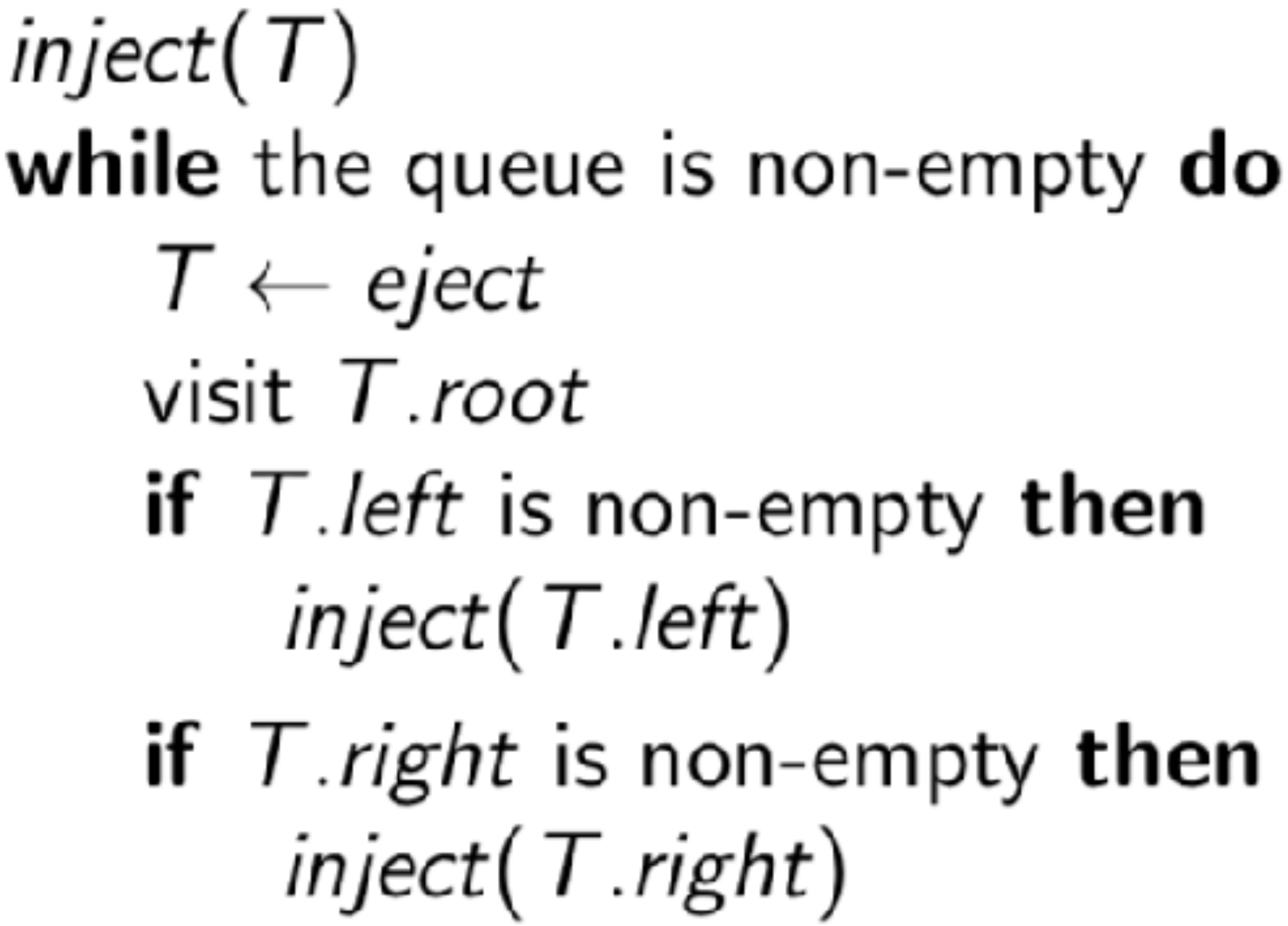

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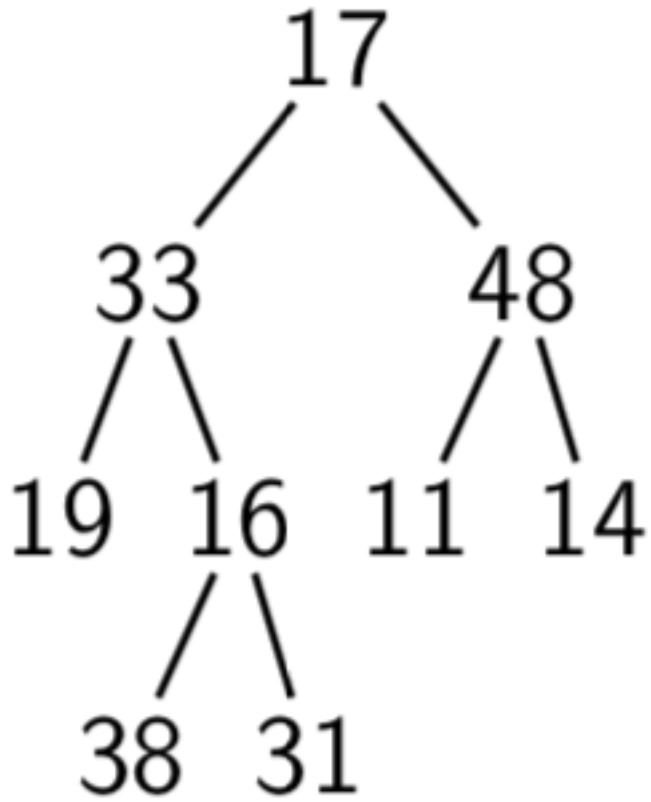

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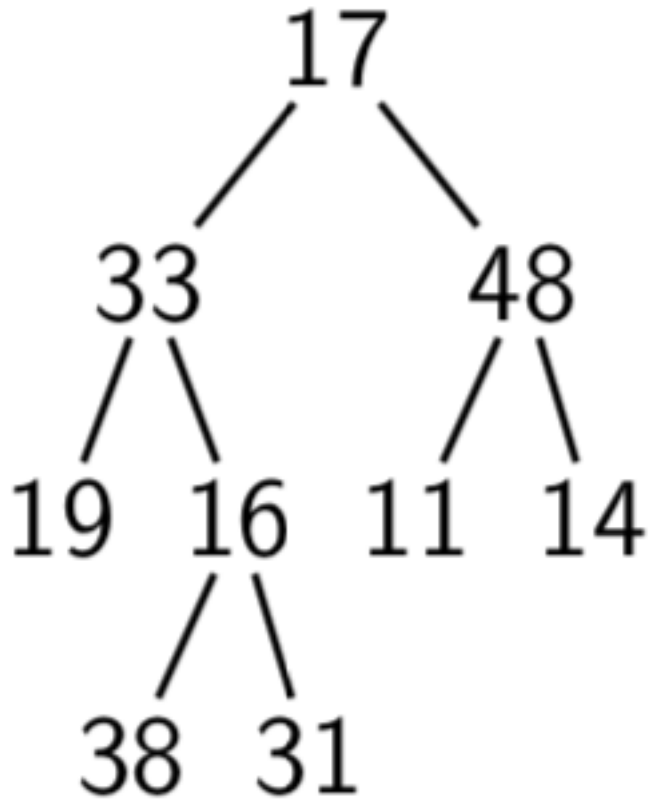

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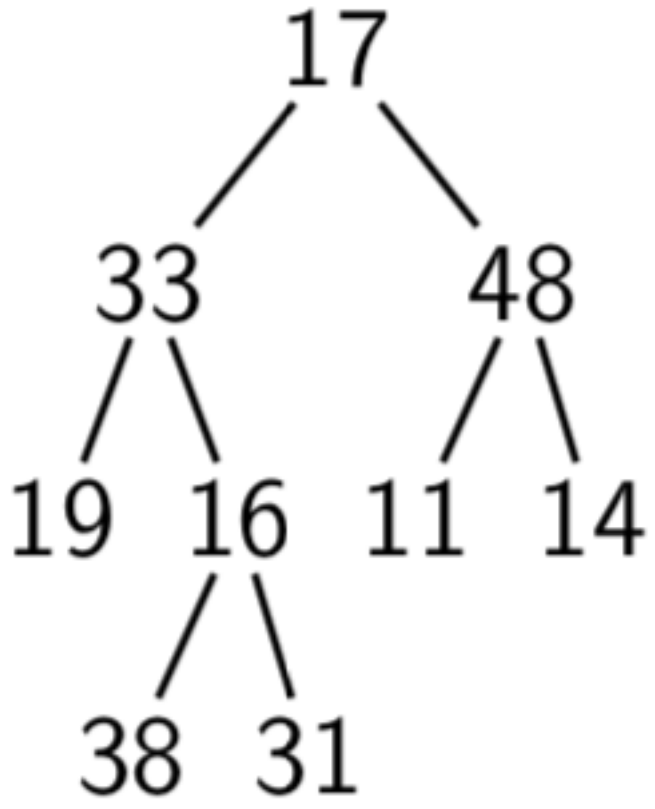

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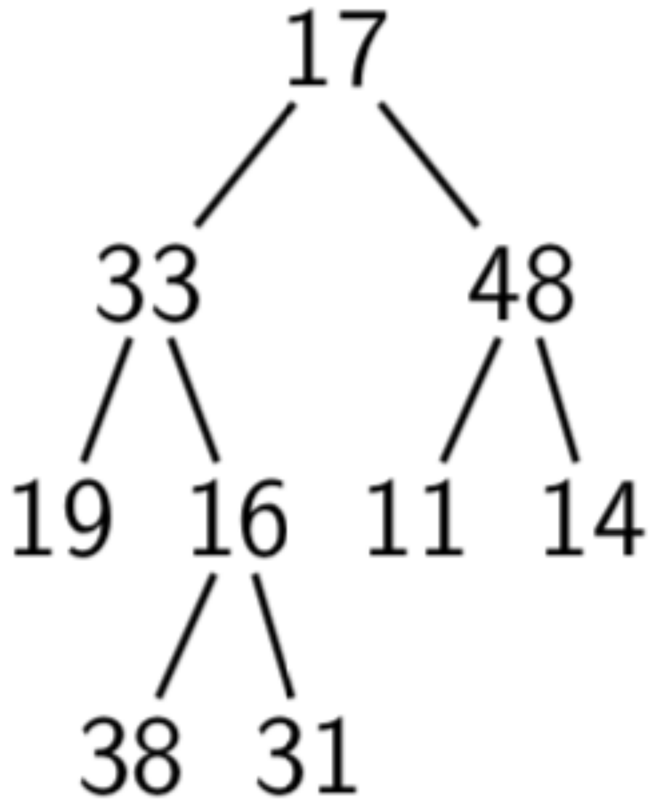

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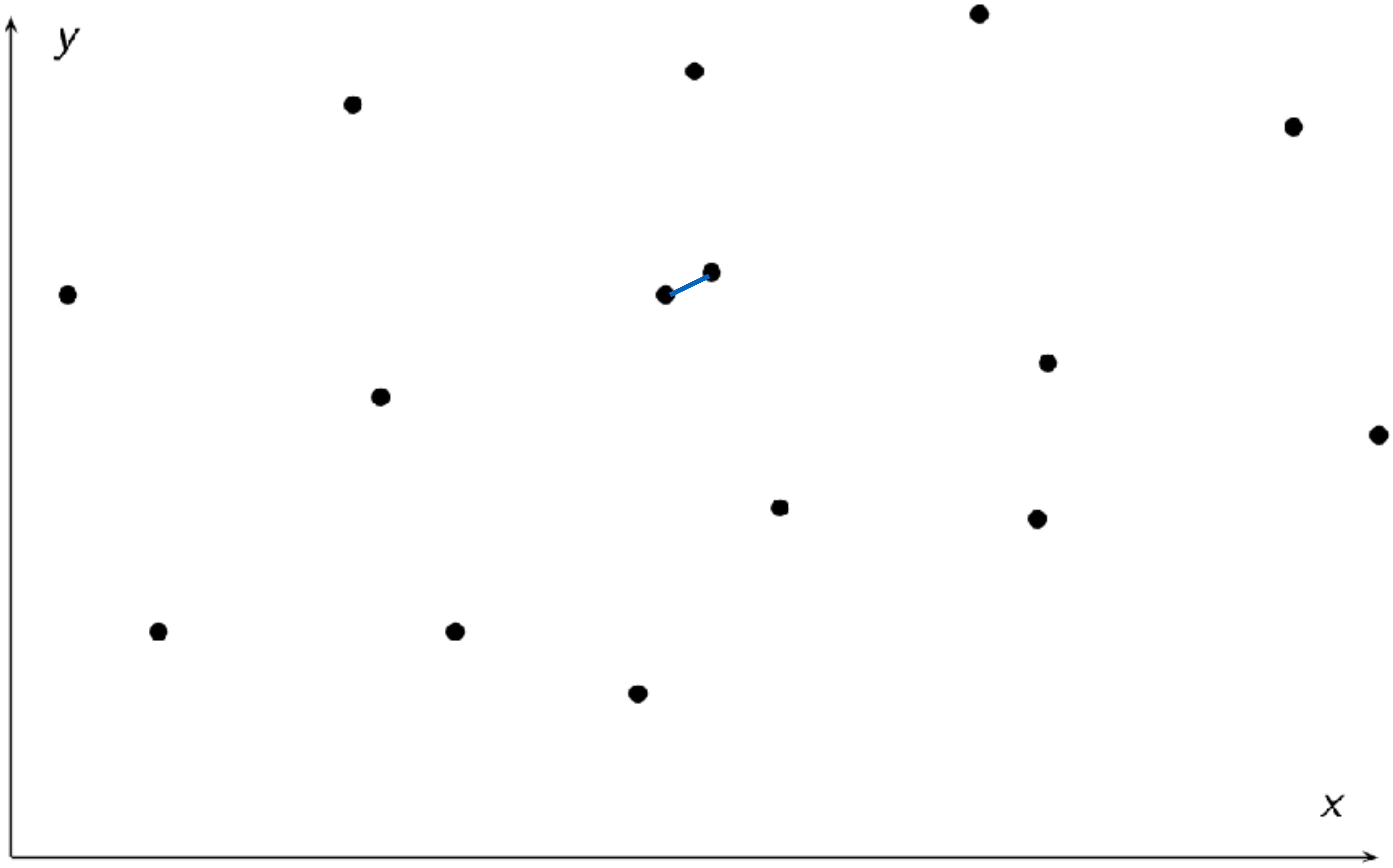
Closest Pair Problem (2D) Revisited (see Lecture 5)



Closest Pair Problem Revisited

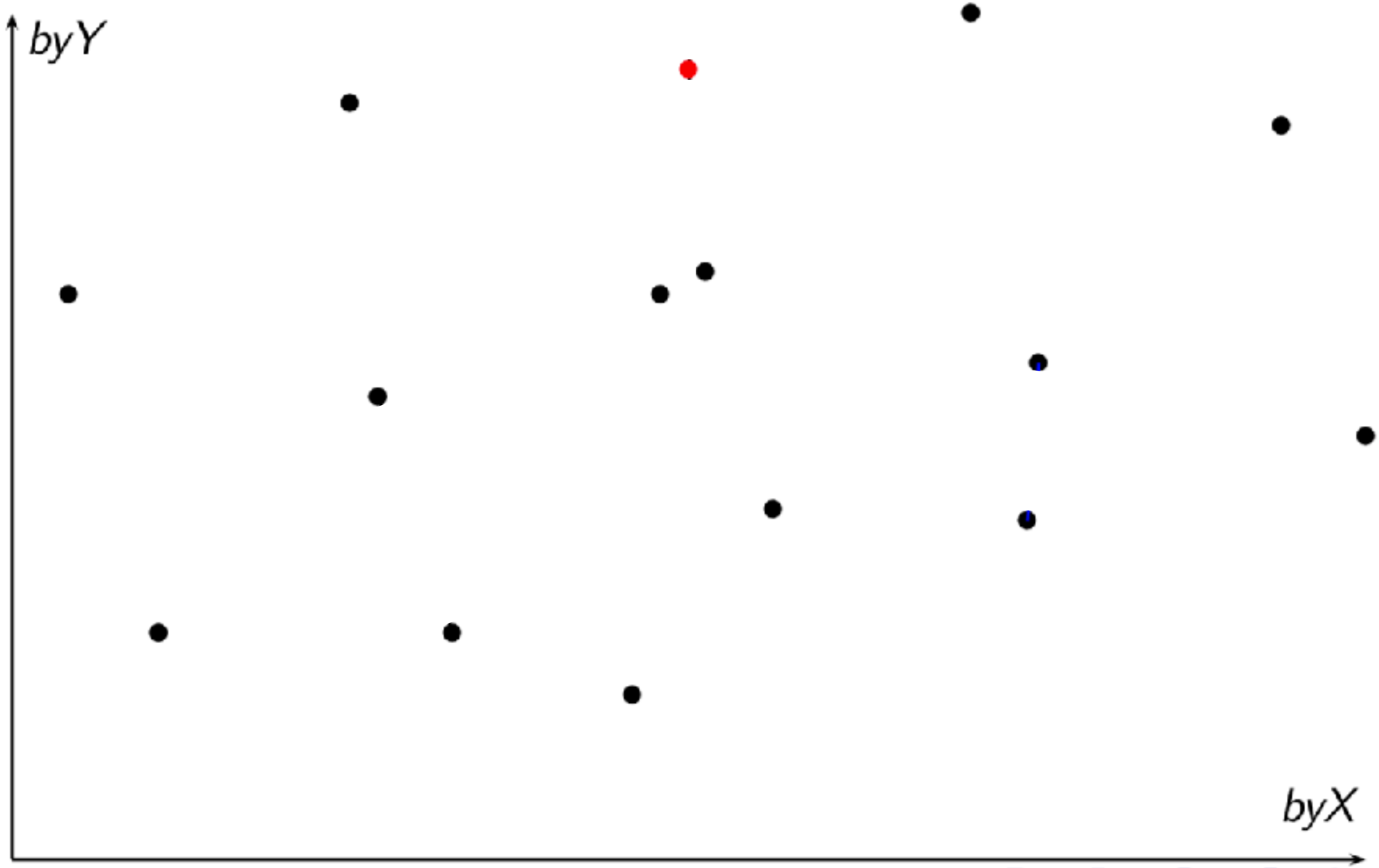
- In Lecture 5 we gave a brute-force algorithm for the closest pair problem: Given n points in the Cartesian plane, find a pair with minimal distance.

- The brute-force method had complexity $\Theta(n^2)$. We can use divide-and-conquer to do better, namely $\Theta(n \log n)$.

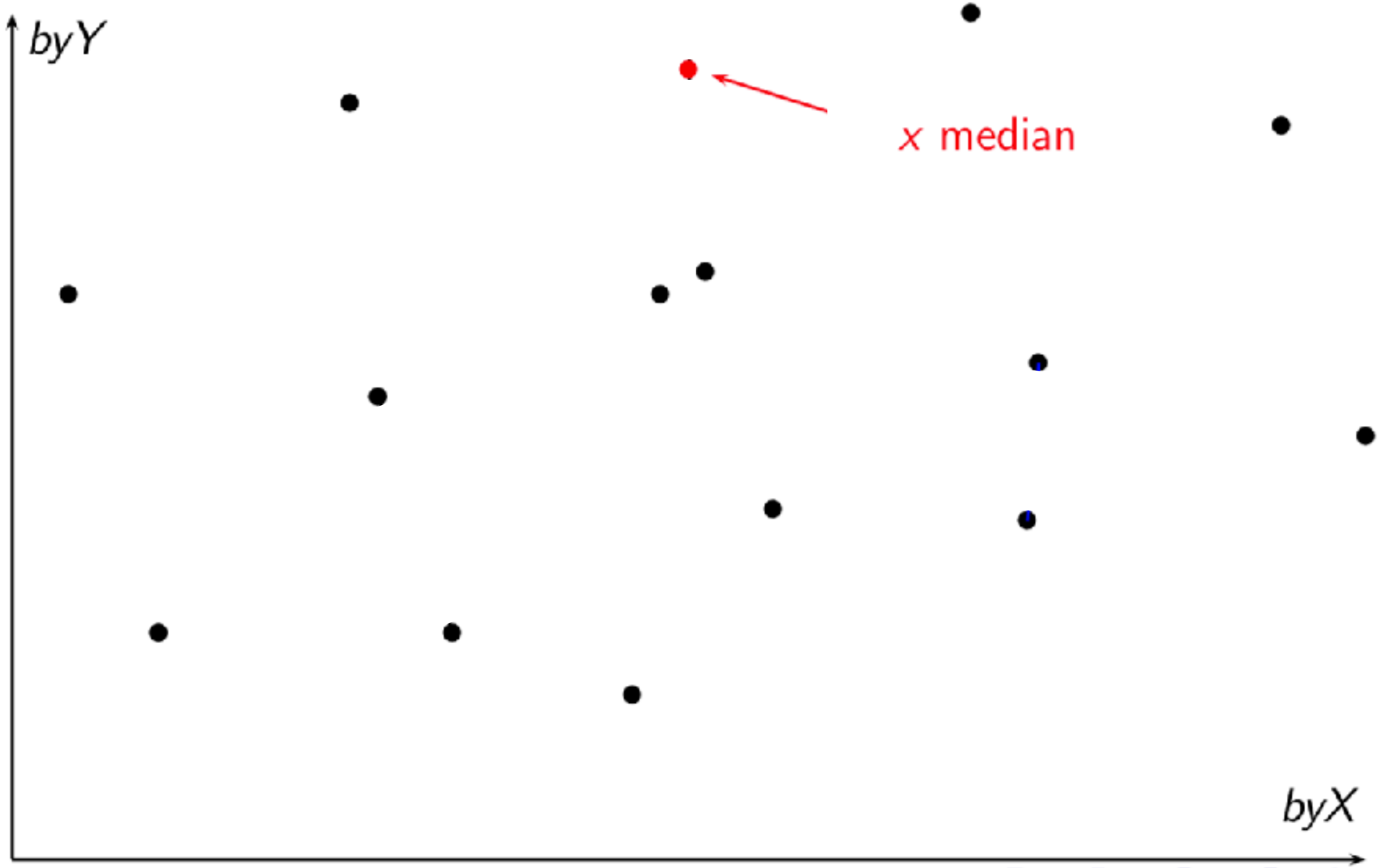
- First, sort the points by x value and store the result in array **byX**. Also sort the points by y value and store the result in array **byY**.

- Now we can identify the x median, and recursively process the set P_L of points with lower x values, as well as the set P_R with higher x values.

Closest Pair Problem Revisited



Closest Pair Problem Revisited



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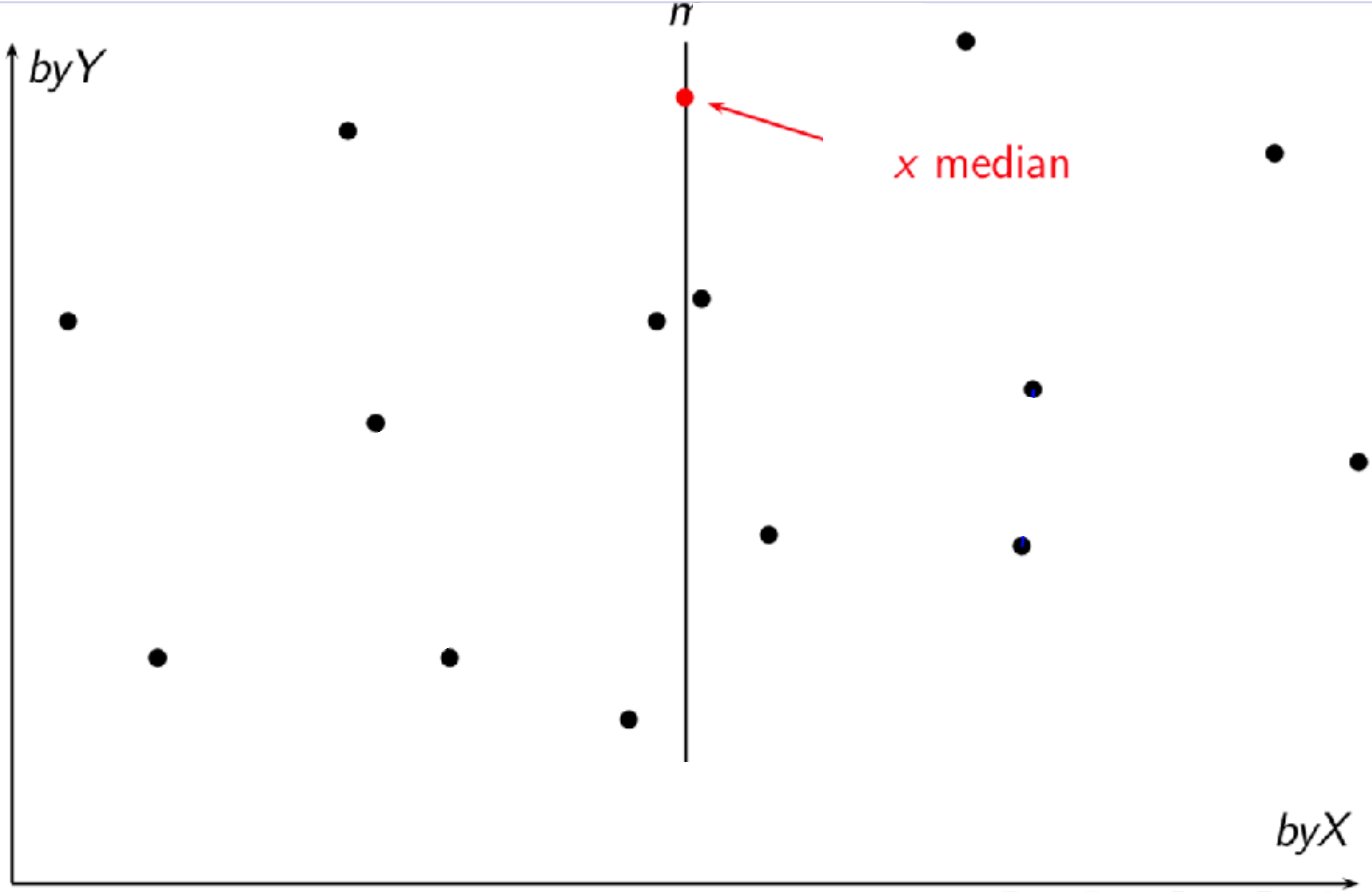
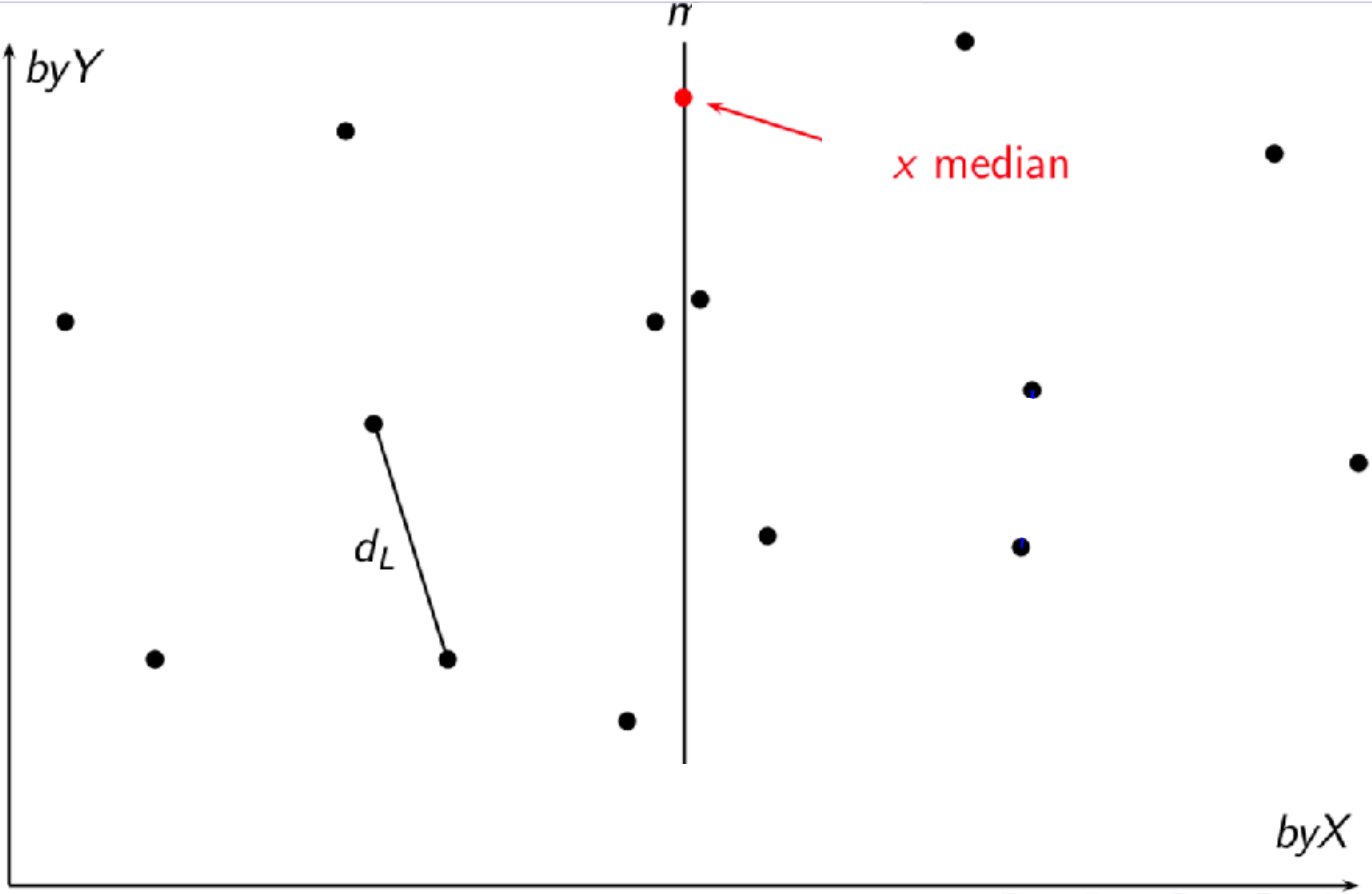


Closest Pair Problem Revisited



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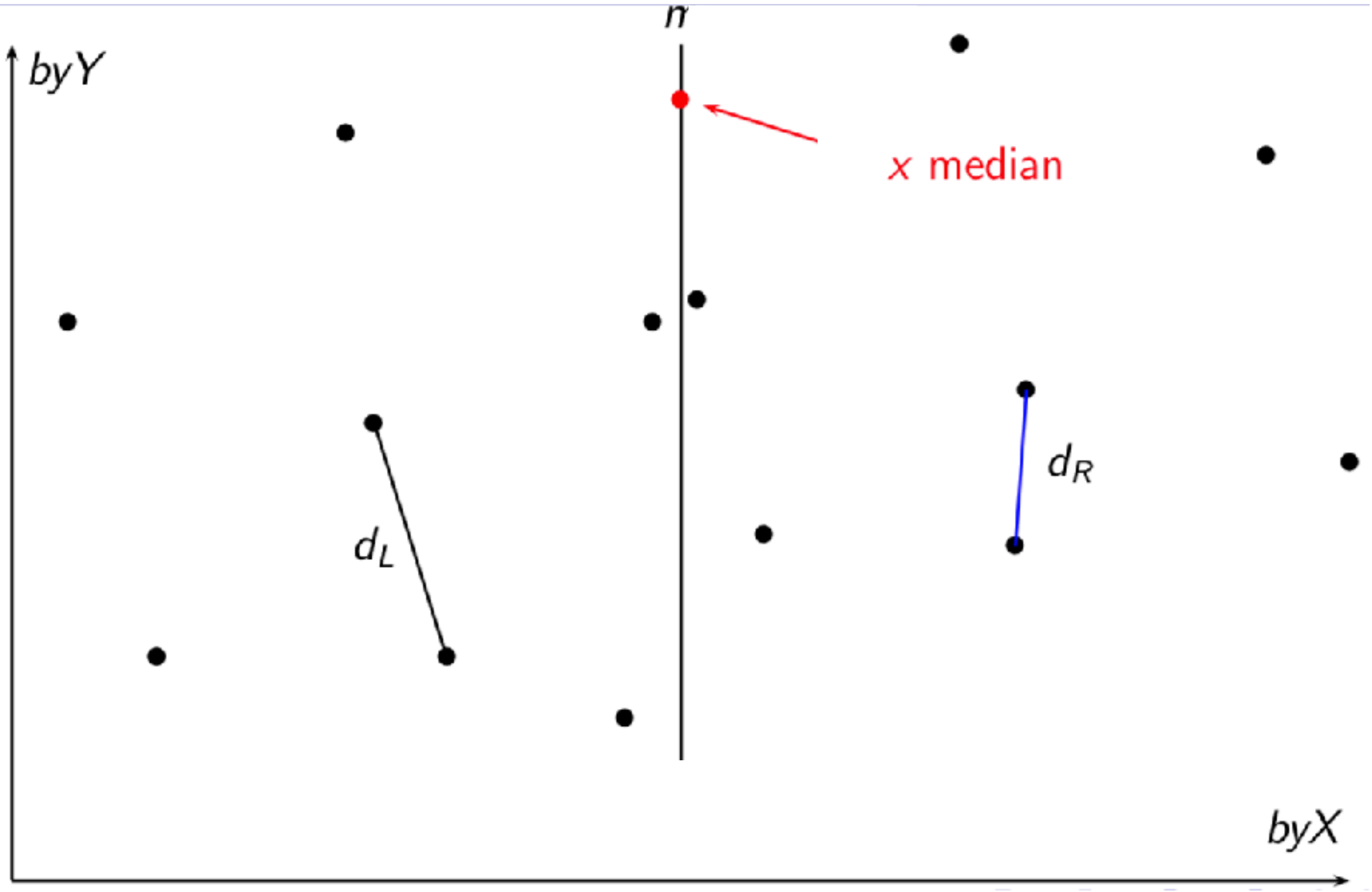

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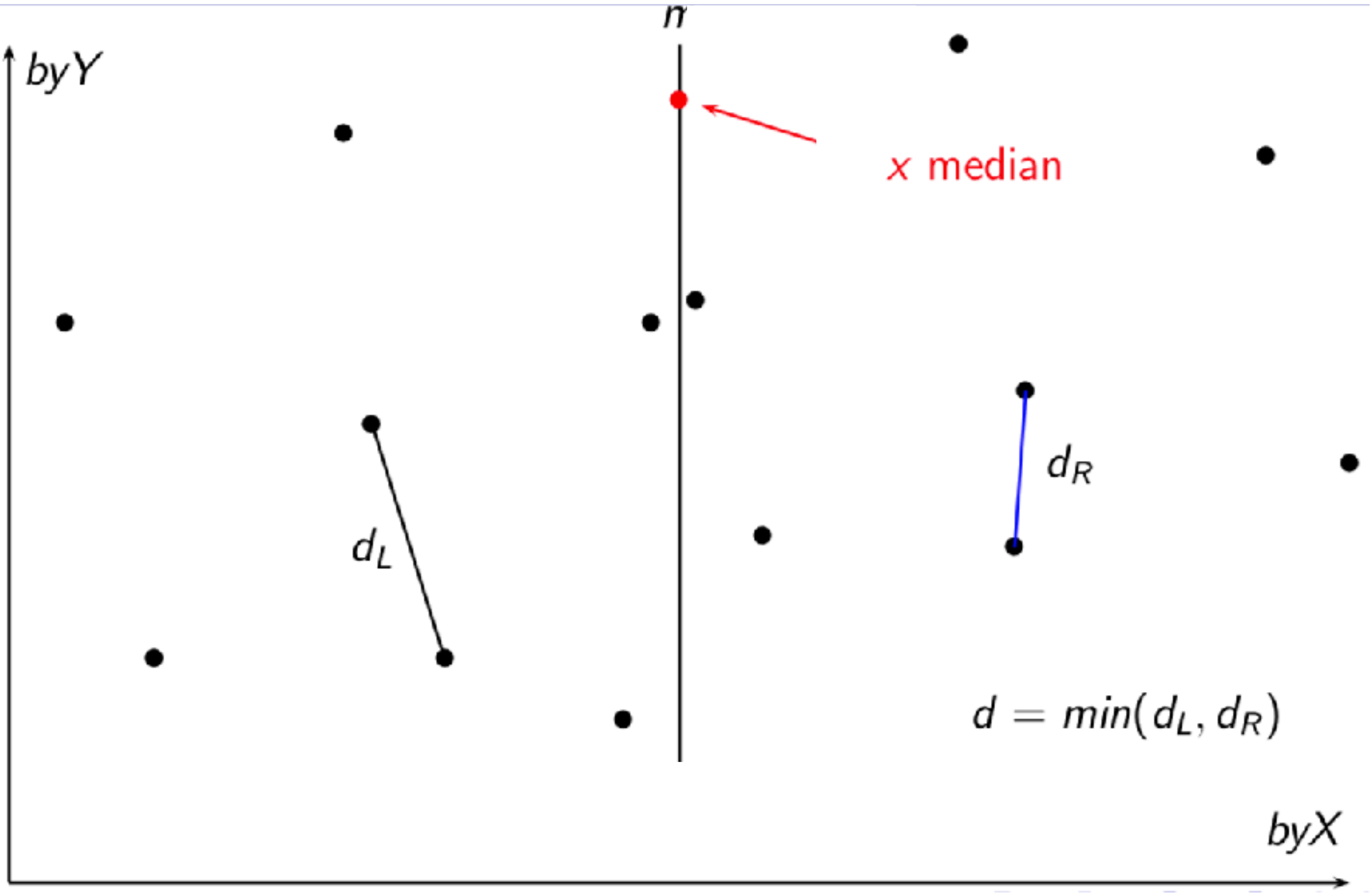

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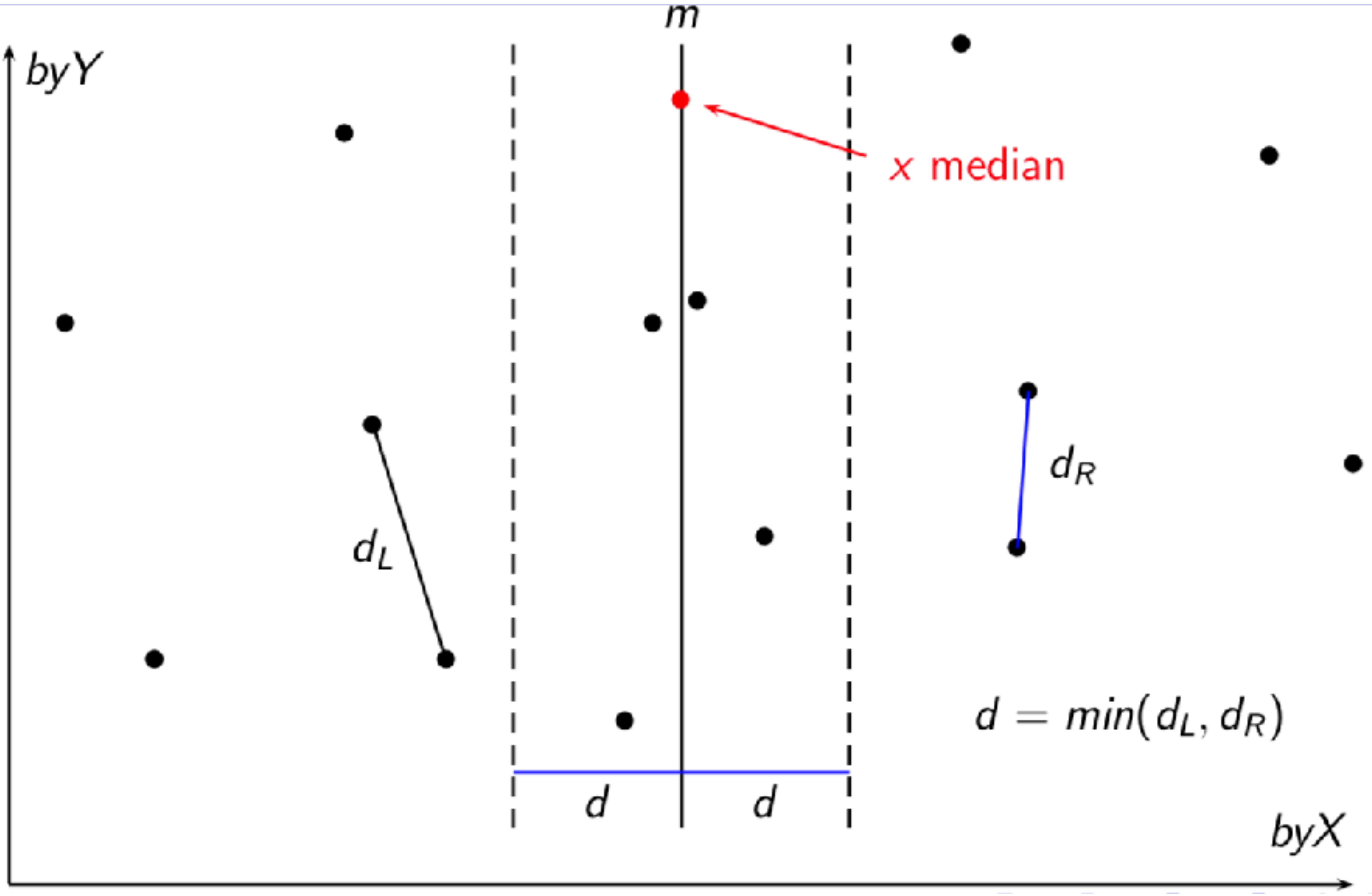


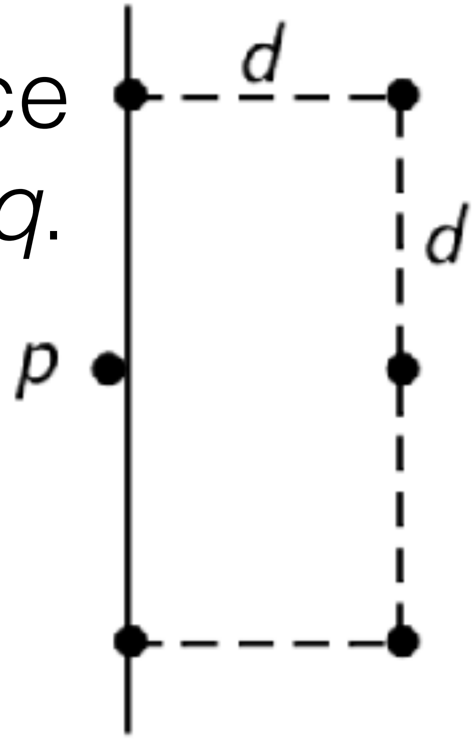
Closest Pair Problem Revisited

- The recursive calls will identify d_L, the shortest distance for pairs in P_L, and d_R, the shortest distance for pairs in P_R.
- Let m be the x median and let $d = \min(d_L, d_R)$. This d is a candidate for the smallest distance.
- But d may not be the global minimum—there could be some close pair whose points are on opposite sides of the median line $x = m$.
- For candidates that may improve on d we only need to look at those in the band $m - d \le x \le m + d$.
- So pick out, from array byY, each point p with x-coordinate between $m-d$ and $m+d$, and keep these in array S.
- For each point in S, consider just its “close” neighbours in S.

Closest Pair Problem Revisited

- The following calculates the smallest distance and leaves the (square of the) result in *minsq*.

- It can be shown that the while loop can execute **at most 5 times** for each *i* value—see diagram.



```
$minsq \leftarrow d^2$
copy all points of $byY$ with $|x - m| < d$ to array $S$
$k \leftarrow |S|$
for $i \leftarrow 0$ to $k - 2$ do
    $j \leftarrow i + 1$
    while $j \leq k - 1$ and $(S[j].y - S[i].y)^2 < minsq$ do
        $minsq \leftarrow min(minsq, (S[j].x - S[i].x)^2 + (S[j].y - S[i].y)^2)$
        $j \leftarrow j + 1$
```

You’re Learning Heaps!

- Next up: Priority queues, heaps and heapsort.