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COMP90038
Algorithms and Complexity

Lecture 10: Decrease-and-Conquer-by-a-Factor
(with thanks to Harald Søndergaard)

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Decrease-and-Conquer

- **Last lecture:** to solve a problem of size n, try to express the solution in terms of a solution to the same problem of size n–1.

  decrease and conquer by a constant

- A simple example was sorting: To sort an array of length n, just:

  1. sort the first n – 1 items, then
  2. locate the cell A[j] that should hold the last item, right-shift all elements to its right, then place the last element in A[j].

- This led to an $O(n^2)$ algorithm called **insertion sort**. We can implement the idea either with recursion or iteration (we chose iteration).

Decrease-and-Conquer by-a-Factor

- We now look at better utilization of the approach, often leading to methods with logarithmic time behaviour or better!

- **Decrease-by-a-constant-factor** is exemplified by binary search.

- **Decrease-by-a-variable-factor** is exemplified by interpolation search.

- Let us look at these and other instances.

Binary Search

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- This is a well-known approach for searching for an element k in a sorted array.

- Start by comparing against the array's middle element A[m]. If A[m] = k we are done.

- If A[m] > k, search the sub-array up to A[m – 1] recursively.

- If A[m] < k, search the sub-array from A[m + 1] recursively.

$$\begin{array}{ccc} & = k? & \\ & \downarrow & \\ \underbrace{A[0] \ \cdots \ A[m-1]}_{\text{search here if } A[m] > k} & {\color{blue}A[m]} & \underbrace{A[m+1] \ \cdots \ A[n-1]}_{\text{search here if } A[m] < k} \end{array}$$

Binary Search

- We have already seen a recursive formulation in Lecture 4. Here is an iterative one.

```
function BinSearch(A[·], n, k)
    lo ← 0
    hi ← n − 1
    while lo ≤ hi do
        m ← ⌊(lo + hi)/2⌋
        if A[m] = k then
            return m
        if A[m] > k then
            hi ← m − 1
        else
            lo ← m + 1
    return −1
```

Example:
Binarv Search in Sorted Array

function $\textsc{BinSearch}(A[\cdot], n, k)$
    $lo \leftarrow 0$
    $hi \leftarrow n - 1$
    while $lo \leq hi$ **do**
        $m \leftarrow \lfloor (lo + hi)/2 \rfloor$
        if $A[m] = k$ **then**
            return m
        if $A[m] > k$ **then**
            $hi \leftarrow m - 1$
        else
            $lo \leftarrow m + 1$
    return -1

k: 41

lo: 0

hi: 6

m: 3

A:

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,7,41)

Example:
Binarv Search in Sorted Array

function BINSEARCH($A[\cdot], n, k$)
  $lo \leftarrow 0$
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k: 41

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k: 41

lo: 4

hi: 4

m: 4

A:

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

BinSearch(A,7,41)

Complexity of Binary Search

- Worst-case input to binary sarch:
  - When *k* is not in the array
- In that case, its complexity is given by the following recursive equation:

$$C(n) = \begin{cases} 1 & \text{if } n = 1 \\ C(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

- A closed form is: $C(n) = \lfloor \log_2 n \rfloor + 1$
- In the worst case, searching for k in an array of size 1,000,000 requires 20 comparisons.
- The average-case time complexity is also Θ(log n)

Russian Peasant Multiplication

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- A way of doing multiplication.

- For even *n*:

$$n \cdot m = \frac{n}{2} \cdot 2m$$

- For odd *n:*

$$n \cdot m = \frac{n-1}{2} \cdot 2m + m$$

- Thus, ~halve *n* repeatedly, until n = 1. Add up all odd values of *m*

n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	
1	5888	5888

= 7452

Finding the Median

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- Given an array, an important problem is how to find the **median**, that is, an array value which is no larger than half the elements and no smaller than half.

A:

9	23	3	41	22	8	46
0	1	2	3	4	5	6

- More generally, we would like to solve the problem of finding the kth smallest element. (e.g. when k=3)

A:

9	23	3	41	22	8	46
0	1	2	3	4	5	6

- If the array is sorted, the solution is straight-forward, so one approach is to start by sorting (as we'll soon see, this can be done in time O (n log n)).

A:

3	8	9	22	23	41	46
0	1	2	3	4	5	6

- However, sorting the array seems like overkill.

A Detour via Partitioning

- Partitioning an array around some pivot element p means reorganizing the array so that all elements to the left of p are no greater than p, while those to the right are no smaller.

A:

0	1	2	3	4	5	6
9	23	3	41	22	8	46

Partitioning around the pivot 9

A:

0	1	2	3	4	5	6
3	8	9	23	22	41	46

Lomuto Partitioning

function LomutoPartition($A[\cdot], lo, hi$)

$\quad p \leftarrow A[lo]$

$\quad s \leftarrow lo$

$\quad$ **for** $i \leftarrow lo + 1$ **to** hi **do**

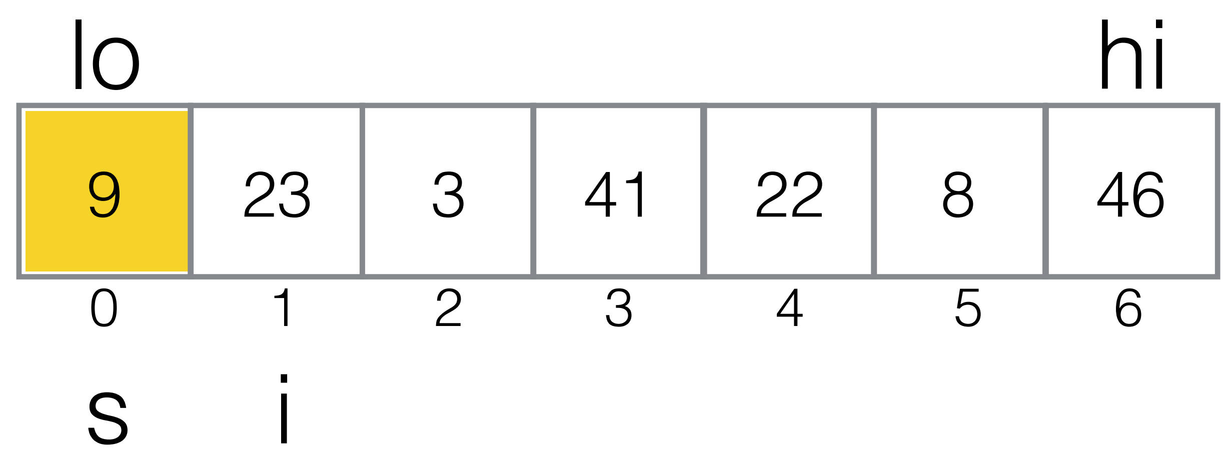
$\quad\quad$ **if** $A[i] < p$ **then**

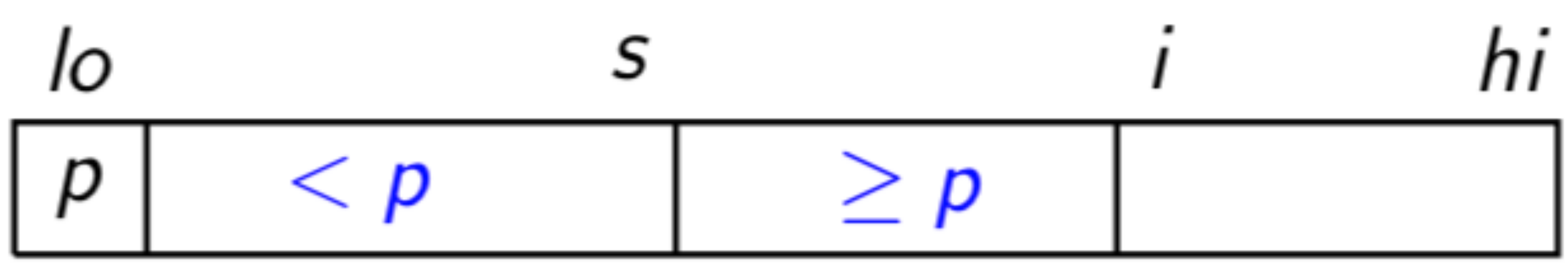
$\quad\quad\quad s \leftarrow s + 1$

$\quad\quad\quad swap(A[s], A[i])$

$\quad swap(A[lo], A[s])$

$\quad$ **return** s





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    swap(A[lo], A[s])
    return s
```

lo						hi
9	23	3	41	22	8	46
0	1	2	3	4	5	6
s		i				

lo	s	i	hi
p	$< p$	$\geq p$	

Lomuto Partitioning

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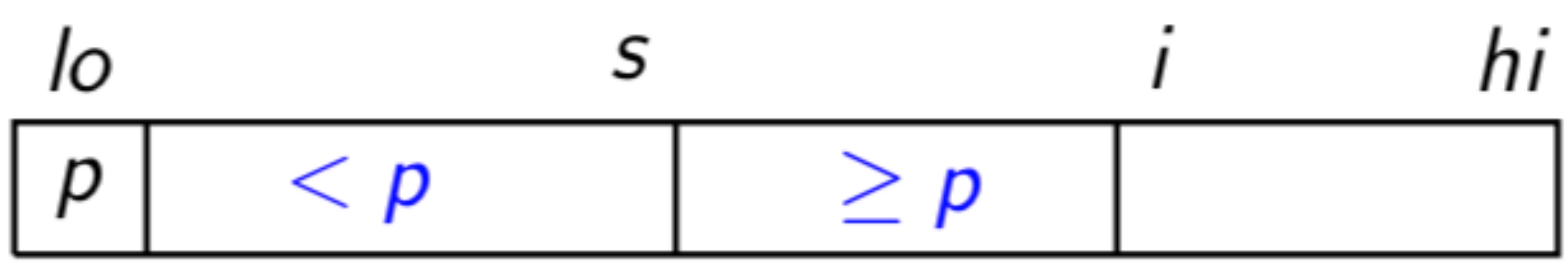

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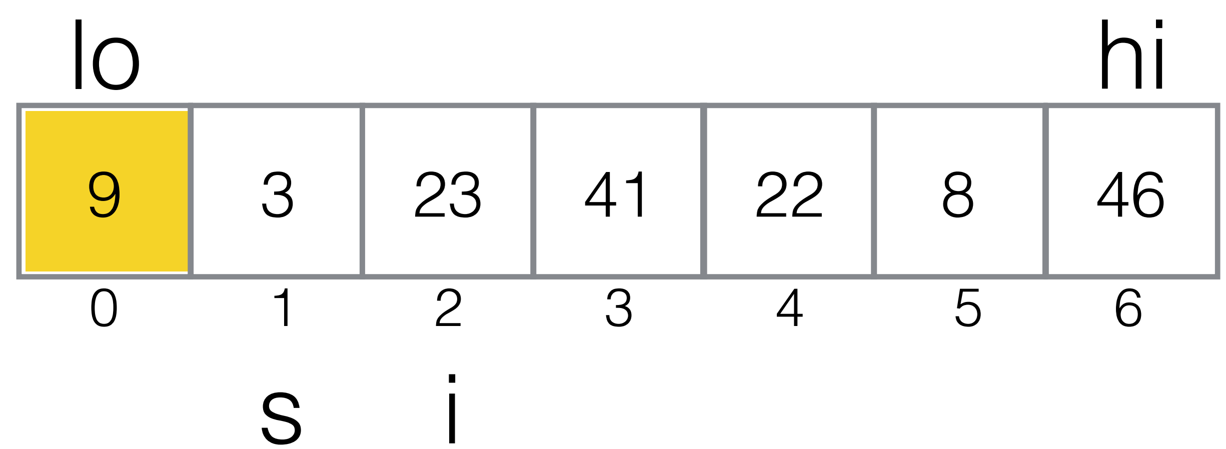

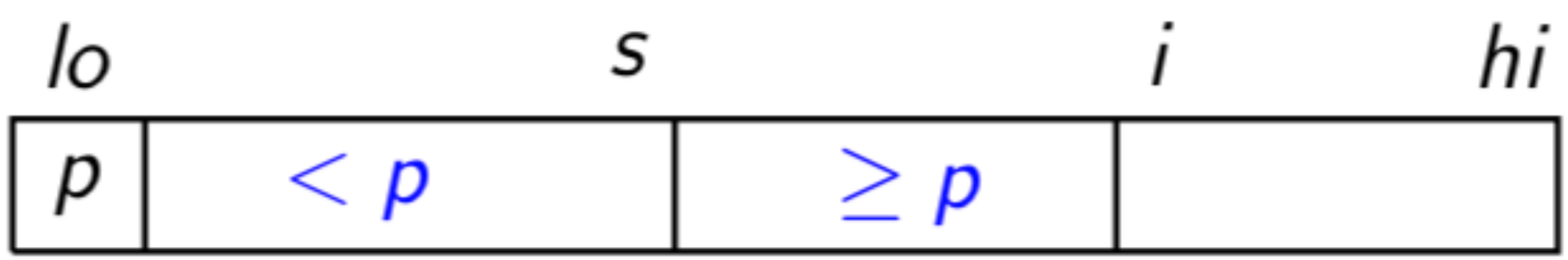

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lo	s	i	hi
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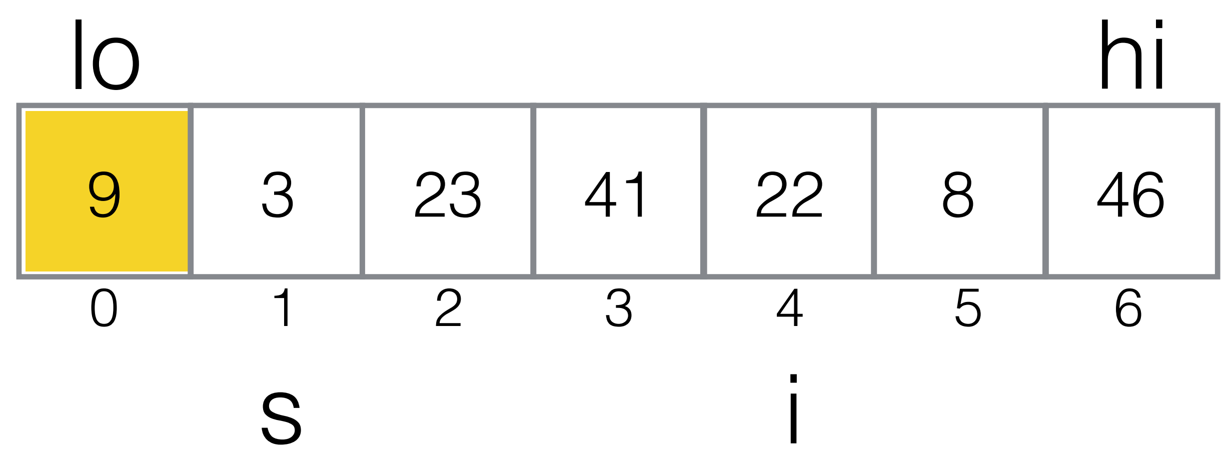

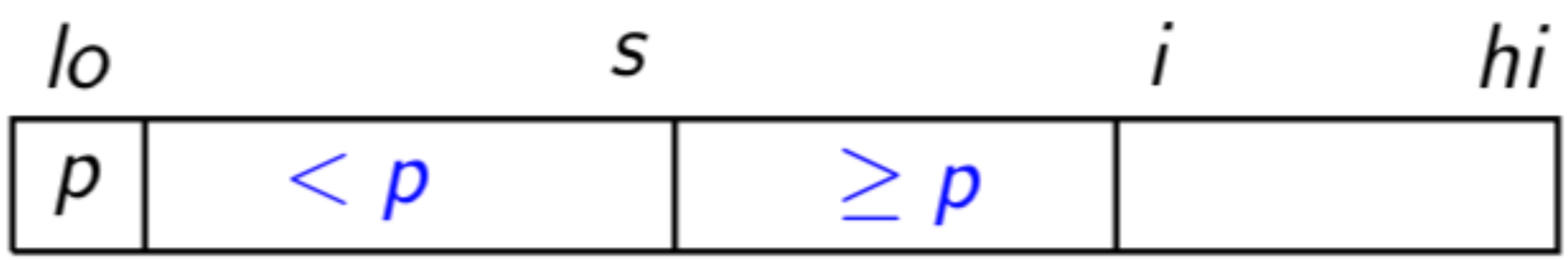

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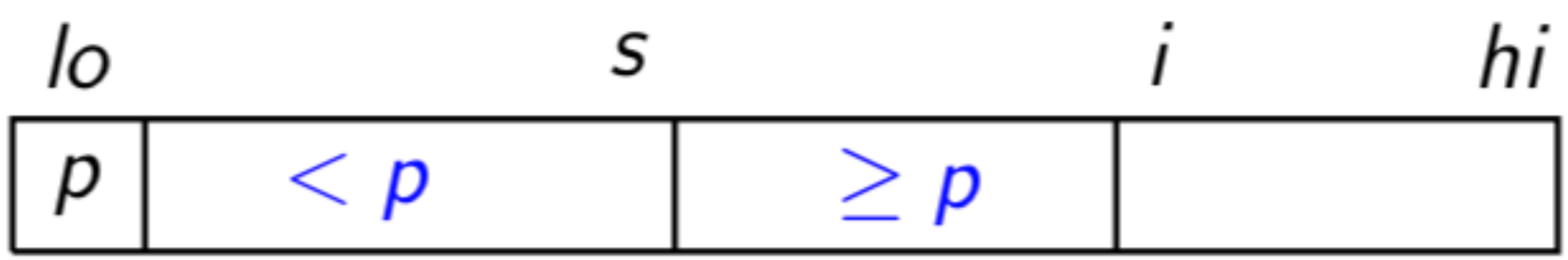

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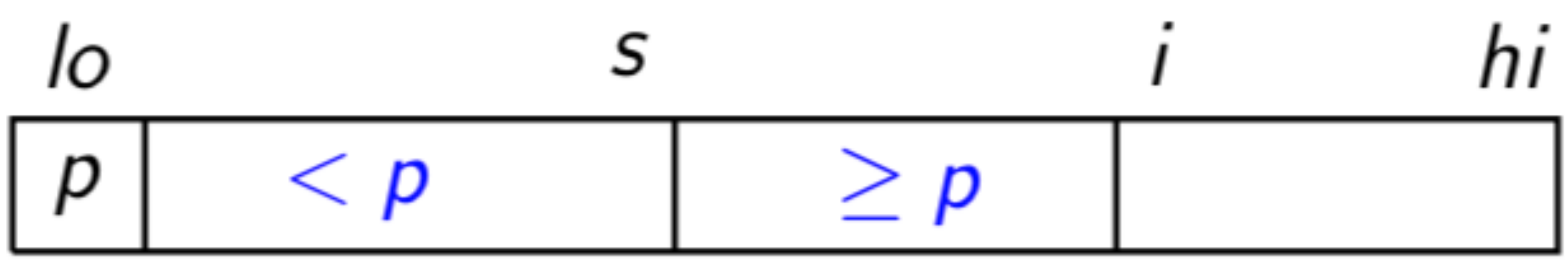

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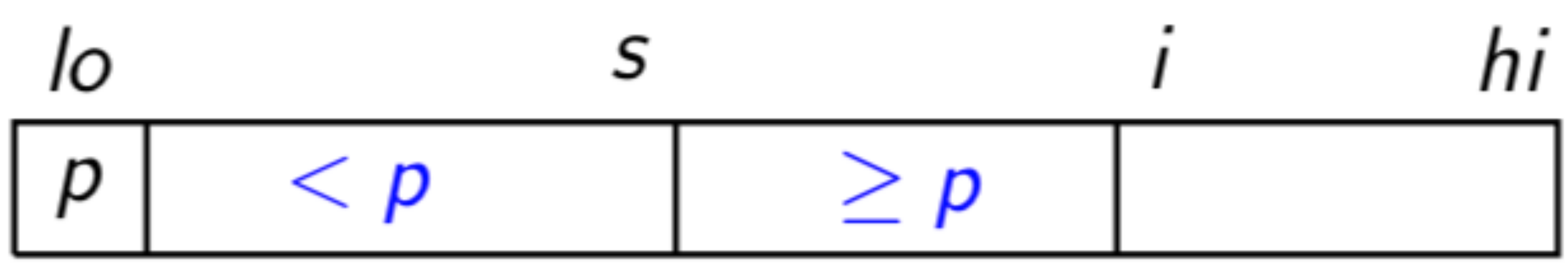

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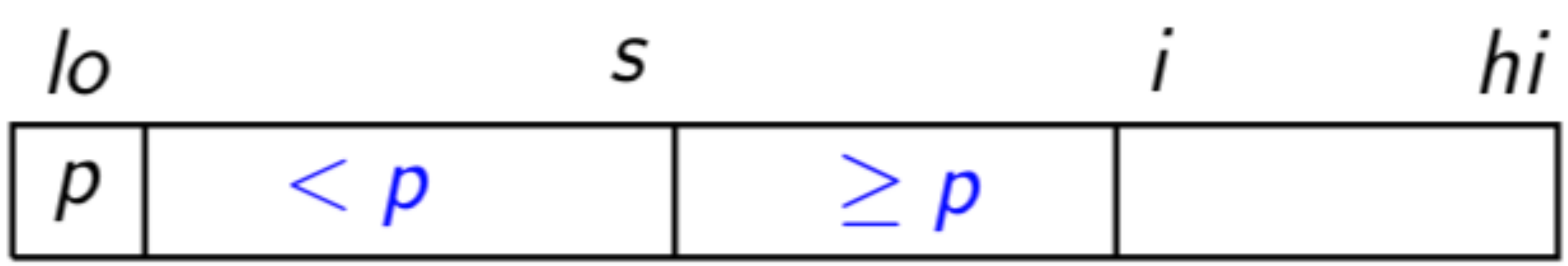

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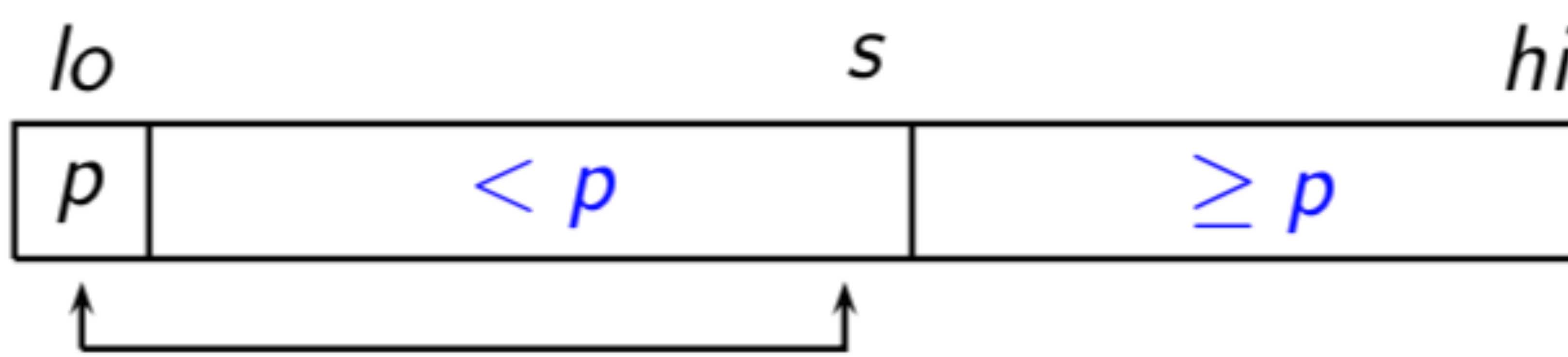

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lo						hi
8	3	9	41	22	23	46
0	1	2	3	4	5	6
		s				

lo	s	hi
$< p$	p	$\geq p$

Finding the *k*th-smallest Element

function $\textsc{QuickSelect}(A[\cdot], lo, hi, k)$
  $s \leftarrow \textsc{LomutoPartition}(A, lo, hi)$
  if $s - lo = k - 1$ **then**
    return $A[s]$
  else
    if $s - lo > k - 1$ **then**
      $\textsc{QuickSelect}(A, lo, s - 1, k)$
    else
      $\textsc{QuickSelect}(A, s + 1, hi, (k - 1) - (s - lo))$

lo						hi
8	3	9	41	22	23	46
0	1	2	3	4	5	6
		S				

Example: Find the Sixth Smallest Element

function $\text{QUICKSELECT}(A[\cdot], lo, hi, k)$
 $s \leftarrow \text{LOMUTOPARTITION}(A, lo, hi)$
 if $s - lo = k - 1$ **then**
  return $A[s]$
 else
  if $s - lo > k - 1$ **then**
   $\text{QUICKSELECT}(A, lo, s - 1, k)$
  else
   $\text{QUICKSELECT}(A, s + 1, hi, (k - 1) - (s - lo))$

k: 6

lo						hi
9	23	3	41	22	8	46
0	1	2	3	4	5	6

Example: Find the ~~Fifth~~ Sixth Smallest Element

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k: 3

			lo			hi
8	3	9	41	22	23	46
0	1	2	3	4	5	6

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					s	

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  return s

5

			lo			hi
8	3	9	23	22	41	46
0	1	2	3	4	5	6
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Example: Find the ~~Fifth~~ Sixth Smallest Element

function $\textsc{QuickSelect}(A[\cdot], lo, hi, k)$

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k: 3

			lo			hi
8	3	9	23	22	41	46
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					s	

Example: Find the ~~Fifth~~ Sixth Smallest Element

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      QUICKSELECT($A, lo, s - 1, k$)
    else
      QUICKSELECT($A, s + 1, hi, (k - 1) - (s - lo)$)

k: 3

returns 41!

			lo			hi
8	3	9	23	22	41	46
0	1	2	3	4	5	6
					s	

QuickSelect Complexity

- **Worst case** complexity for QuickSelect is quadratic,

- **Average-case** complexity is linear.

best case: 1 partition   O(n)
worst case: (n-1) + (n-2) + ... + 1   O(n^2) 每次减少一个数
avg case: n + n/2 + n/4 + ... + 1   O(n) pivot 每次分一半

Interpolation Search

- If the elements of a sorted array are distributed reasonably evenly, we can do better than binary search!
- Think about how you search for an entry in the telephone directory: If you look for ‘Zobel’, you make a rough estimate of where to do the first probe—very close to the end of the directory.
- This is the idea in interpolation search.
- When searching for k in the array segment A[lo] to A[hi], take into account where k is, relative to A[lo] and A[hi].

Interpolation Search

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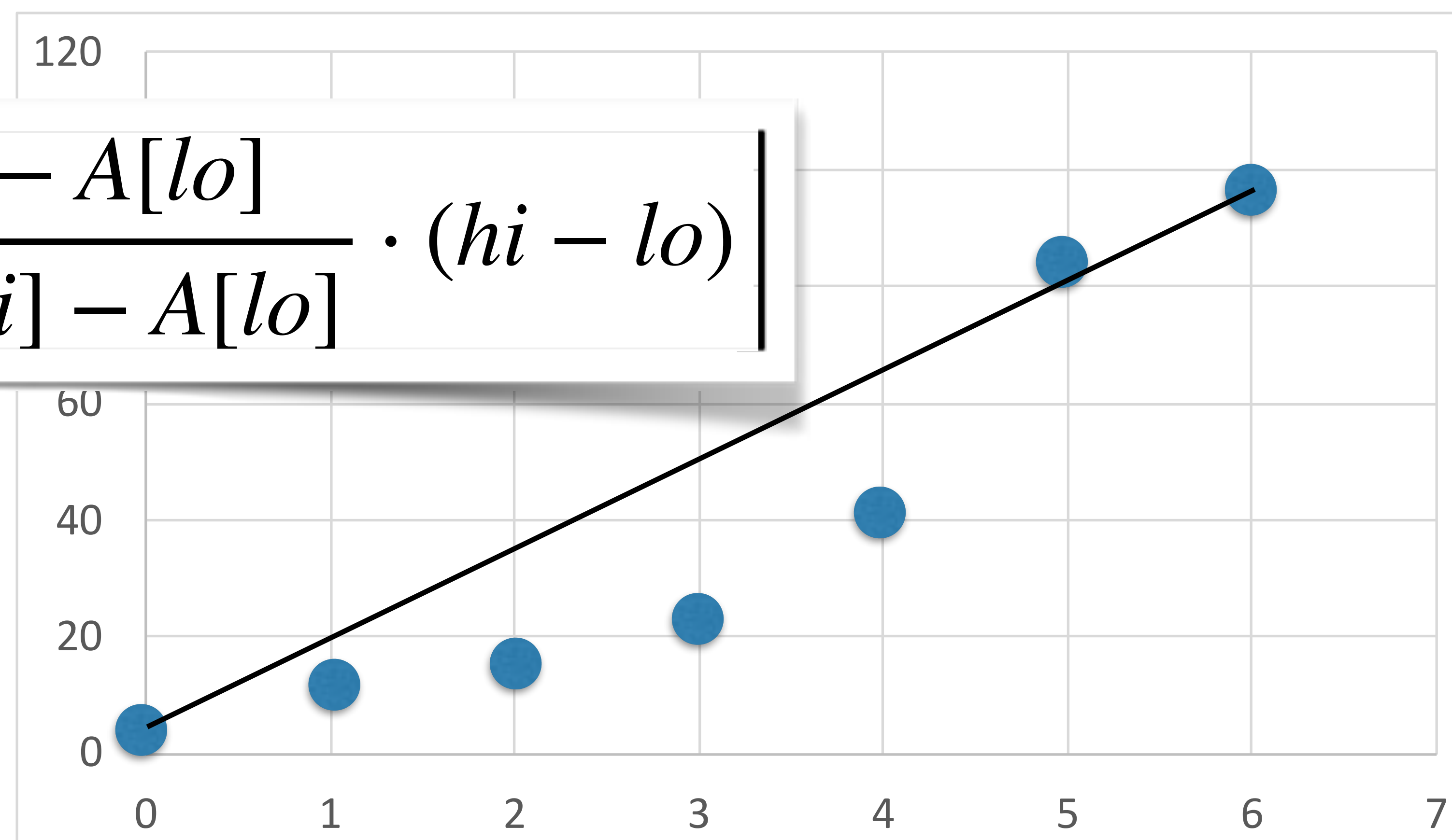
A:						
4	9	13	22	41	83	96
0	1	2	3	4	5	6

Suppose we are searching for k = 83

$$mid = A[lo] + \left\lfloor \frac{k - A[lo]}{A[hi] - A[lo]} \cdot (hi - lo) \right\rfloor$$



Interpolation Search

- Instead of computing the mid-point *m* as in binary search:

$$m \leftarrow \lfloor (lo + hi)/2 \rfloor$$

we instead perform linear interpolation between the points (lo,A[lo]) and (hi,A[hi]). That is, we use:

$$\boxed{m \leftarrow lo + \left\lfloor \frac{k - A[lo]}{A[hi] - A[lo]}(hi - lo) \right\rfloor}$$

- Interpolation search has average complexity O(log log n)

- It is the right choice for large arrays when elements are **uniformly distributed**

Next Week

- Learn to divide and conquer!
- Read Levitin Chapter 5, but skip 5.4.