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COMP90038 Algorithms and Complexity

Lecture 6: Recursion
(with thanks to Harald Søndergaard)

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Recursion

- We've already seen some examples
- A very natural approach when the data structure is recursive (e.g. lists, trees)
- But also examples of naturally recursive array processing algorithms
- Next week we'll express **depth first graph traversal** recursively (the natural way); later we'll meet other examples of recursion too

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
  return $\textsc{Fac}(n-1) * n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

n: 0

result: 120

Iterative version normally preferred since it is constant space

Example: Fibonacci Numbers

- To generate the *n*th number of sequence: 1 1 2 3 5 8 13 21 34 55 …

function $\textsc{Fib}(n)$
    if $n = 0$ **then**
        return 1
    if $n = 1$ **then**
        return 1
    return $\textsc{Fib}(n-1) + \textsc{Fib}(n-2)$

Follows the mathematical definition of Fibonacci numbers very closely.

Easy to understand

But performs lots of redundant computation

Basic operation: addition

Complexity is **exponential** in *n*

Fibonacci Again

- Of course we only need to remember the latest two items. Recursive version: left; iterative version: right

function $\text{FIB}(n, a, b)$
    if $n = 0$ **then**
        return a
    return $\text{FIB}(n - 1, a + b, a)$

Initial call: Fib(n, 1, 0)

Time complexity of both solutions is linear in n

function $\text{FIB}(n)$
    $a \leftarrow 1$
    $b \leftarrow 0$
    while $n > 0$ **do**
        $t \leftarrow a$
        $a \leftarrow a + b$
        $b \leftarrow t$
        $n \leftarrow n - 1$
    return a

(There is a cleverer, still recursive, way which is $O(\log n)$.)

Tracing Recursive Fibonacci

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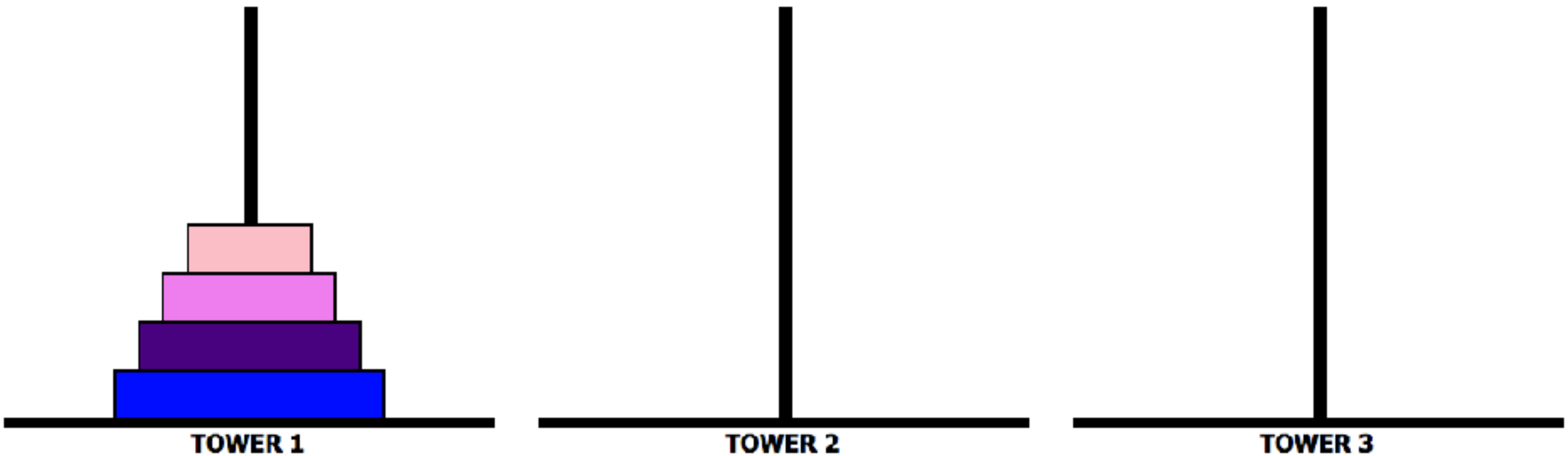
function $\textsc{Fib}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\textsc{Fib}(n-1, a+b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)
= Fib(3,2,1)
= Fib(2,3,2)
= Fib(1,5,3)
= Fib(0,8,5)   = 8

Fibonacci Sequence:
1 1 2 3 5 8 …

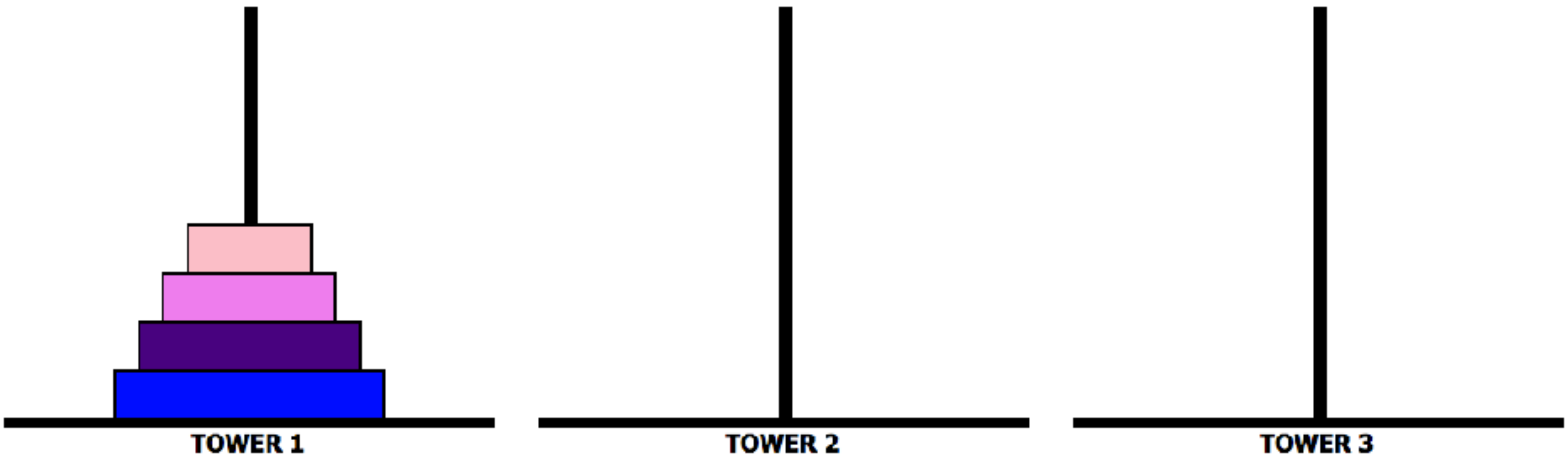
Tower of Hanoi



Init Aux Fin

Move *n* disks from *Init* to *Fin*. A larger disk can never be placed on top of a smaller one.

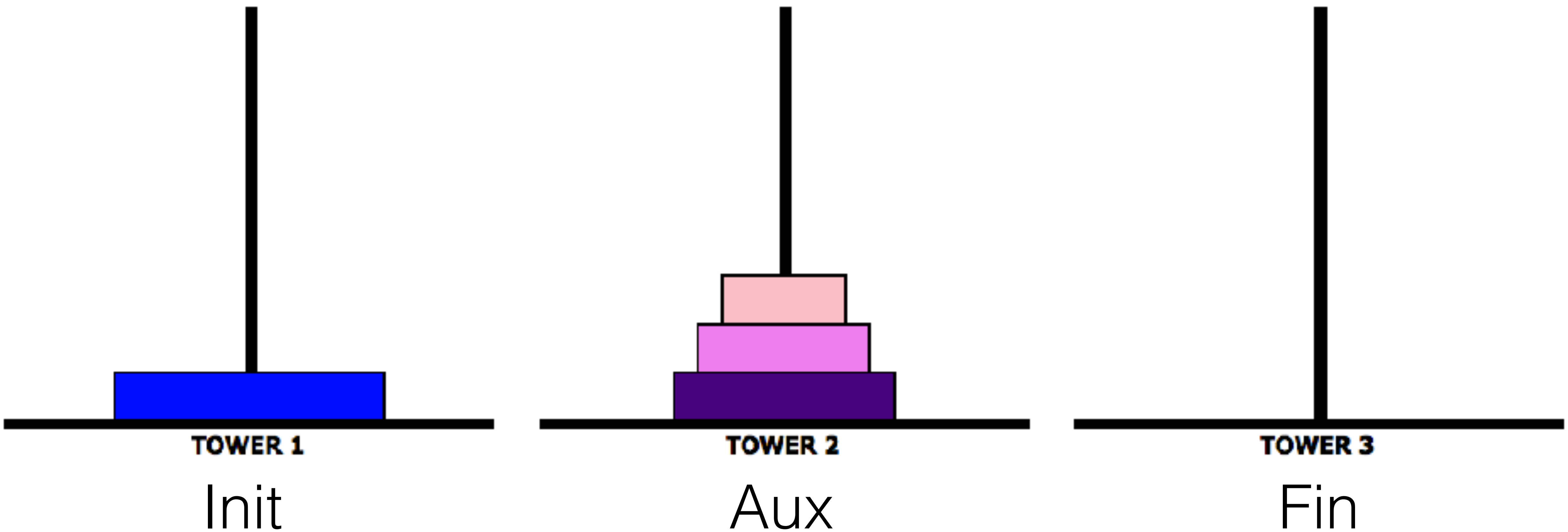
Tower of Hanoi: Recursive Solution



Init Aux Fin

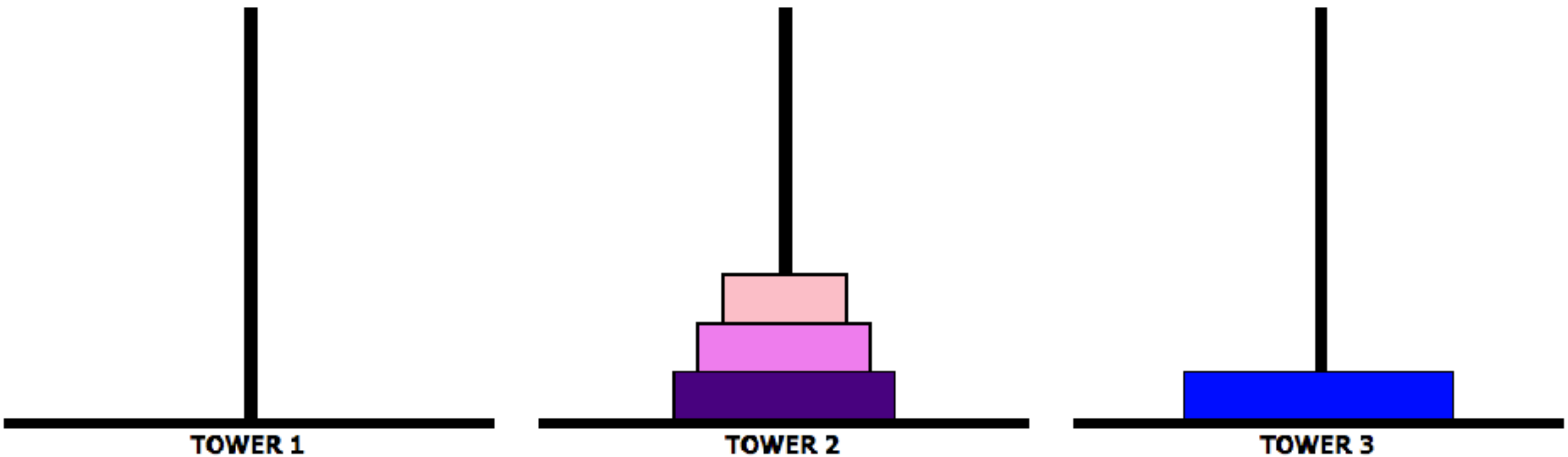
Move *n-1* disks from Init to Aux.

Tower of Hanoi: Recursive Solution



Move *n-1* disks from Init to Aux.
Then move the *n*th disk to Fin.

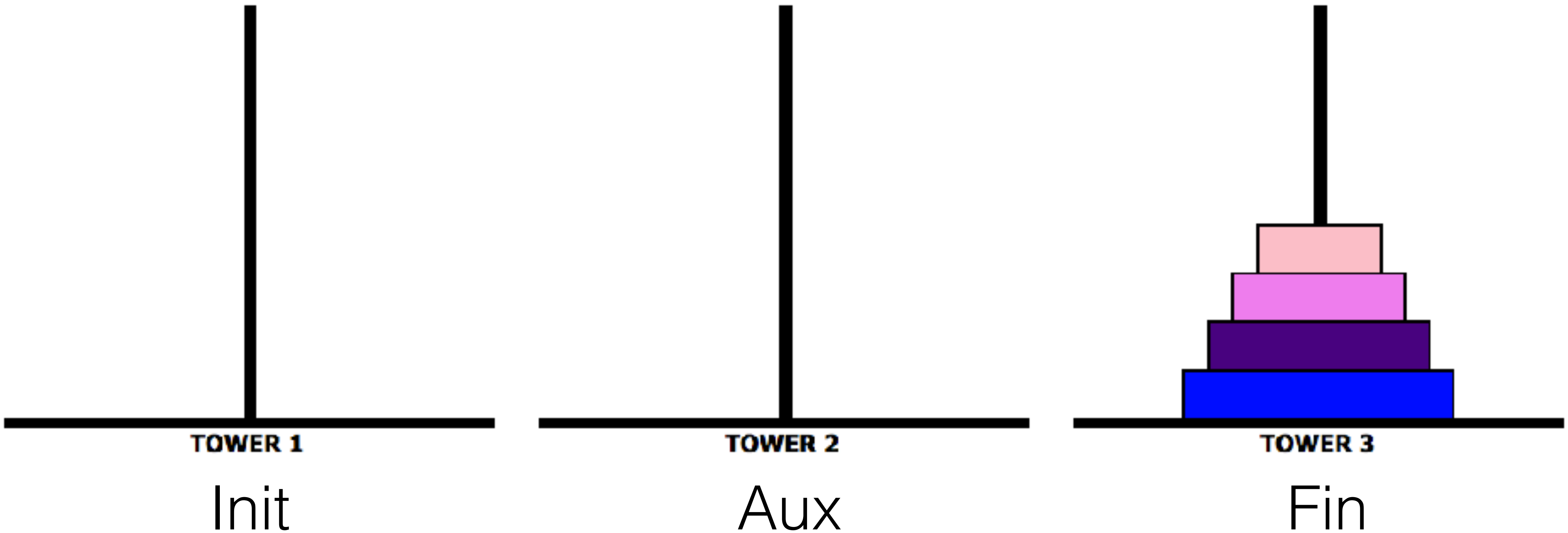
Tower of Hanoi: Recursive Solution



Init Aux Fin

Move *n-1* disks from Init to Aux.
Then move the *n*th disk to Fin.
Then move the *n-1* disks from Aux to Fin.

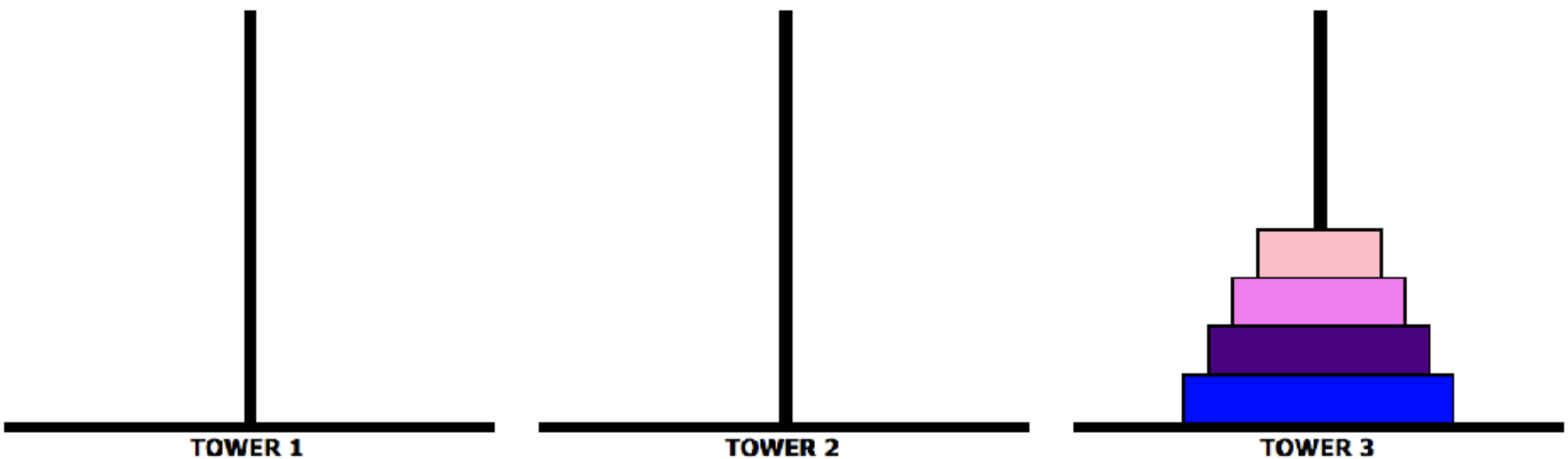
Tower of Hanoi: Recursive Solution



Move *n-1* disks from Init to Aux.
Then move the *n*th disk to Fin.
Then move the *n-1* disks from Aux to Fin.

Tower Of Hanoi: Recursive Algorithm

function HANOI(n, *init*, *aux*, *fin*)
  if $n > 0$ **then**
    HANOI($n - 1$, *init*, *fin*, *aux*)
    Move one disk from *init* to *fin*
    HANOI($n - 1$, *aux*, *init*, *fin*)



Init Aux Fin

Tracing Tower of Hanoi Recursive Algorithm

http://vornlocher.de/tower.html

n disks needs minimum no. of moves: 2^n - 1

A Challenge:
Coin Change Problem

- There are 6 different kinds of Australian coin
- In cents, their values are: 5, 10, 20, 50, 100, 200
- In how many different ways can I produce a handful of coins adding up to $4?
- This is not an easy problem!
- Key to solving it is to find a way to break it down into simpler sub-problems

Coin Change Problem: Decomposition

Coin Change Problem: Decomposition

The number of ways of making $4 is therefore:

1 x the number of ways of making $2

+

the number of ways of making $4 without using a $2 coin

Coin Change Problem: Partial Algorithm

function WAYS(amount, denominations)

// … base cases ….

d ← selectLargest(denominations)

return WAYS(amount – d, denominations) +

WAYS(amount, denominations \ {d})

For example:

Ways(400, {5,10,20,50,100,200}) =

Ways(200, {5,10,20,50,100,200}) +

Ways(400, {5,10,20,50,100})

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)
- Consider each of these separately.
  - amount base cases:
    - amount = 0:
    - amount < 0:
    - denominations = ∅ (and amount > 0):

Coin Change Problem: Full Recursive Algorithm

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function WAYS(amount, denominations)
    if amount = 0 **then**
        return 1
    if amount < 0 **then**
        return 0
    if denominations = ∅ **then**
        return 0
    d ← selectLargest(denominations)
    return WAYS(amount − d, denominations) +
                WAYS(amount, denominations \ {d})

Initial call: WAYS(amount, {5, 10, 20, 50, 100, 200}).

Recursive Solution and its Complexity

- Although our recursive algorithm is short and elegant, it is not the most efficient way of solving the problem.

- Its running time grows **exponentially** as you grow the input amount.

- More efficient solutions can be developed using **memoing** or **dynamic programming**—more about that later (around Week 10).

Next Time…

- Graphs, trees, graph traversal and allied algorithms.