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COMP90038
Algorithms and Complexity

Lecture 11: Sorting with Divide-and-Conquer
(with thanks to Harald Søndergaard)

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Divide and Conquer

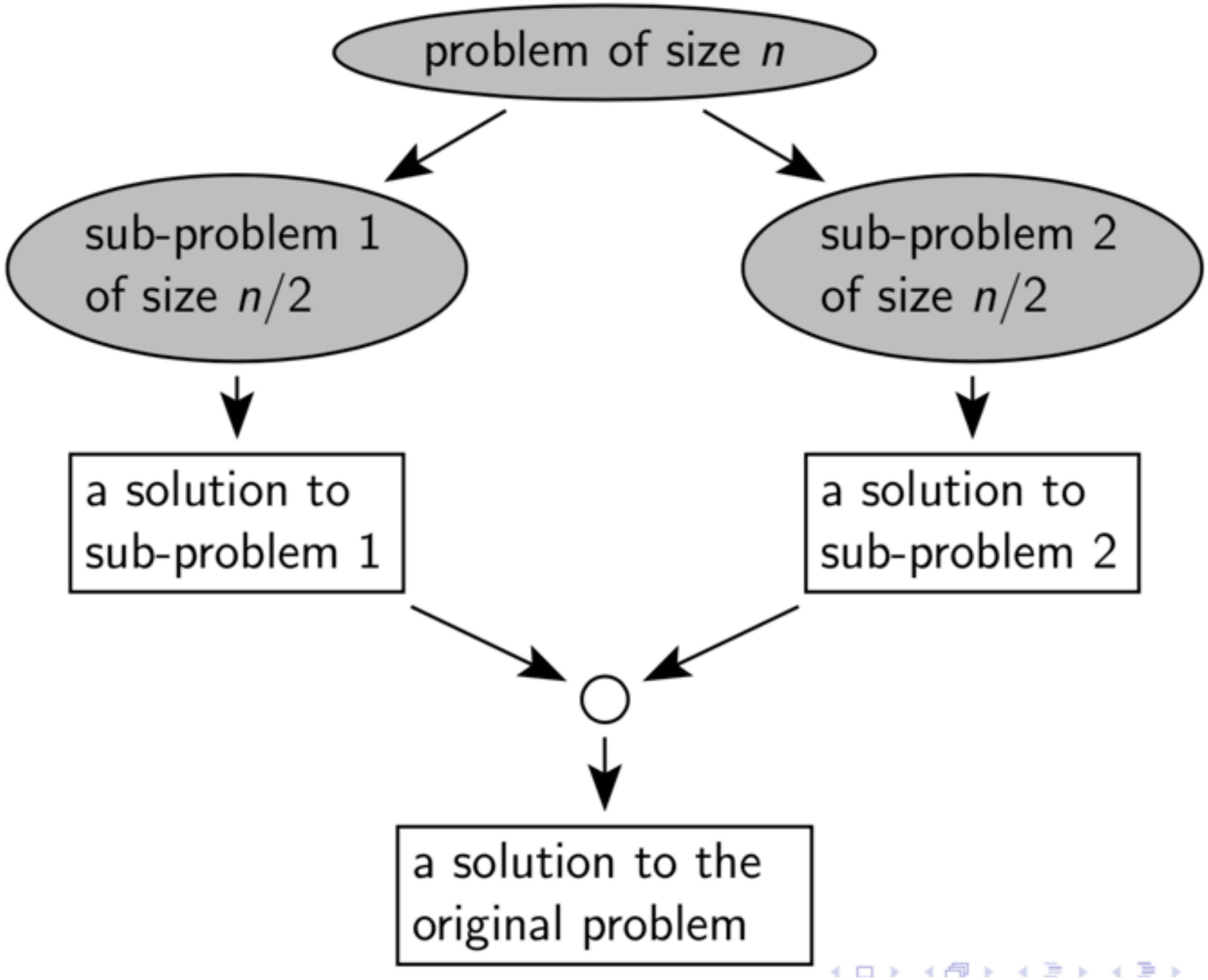
- We earlier studied recursion as a powerful problem solving technique.

- The **divide-and-conquer** strategy tries to make the most of this idea:

  1. Divide the given problem instance into smaller instances.
  2. Solve the smaller instances recursively.
  3. Combine the smaller solutions to solve the original instance.

- This works best when the smaller instances can be made to be of equal (or near-equal) size.

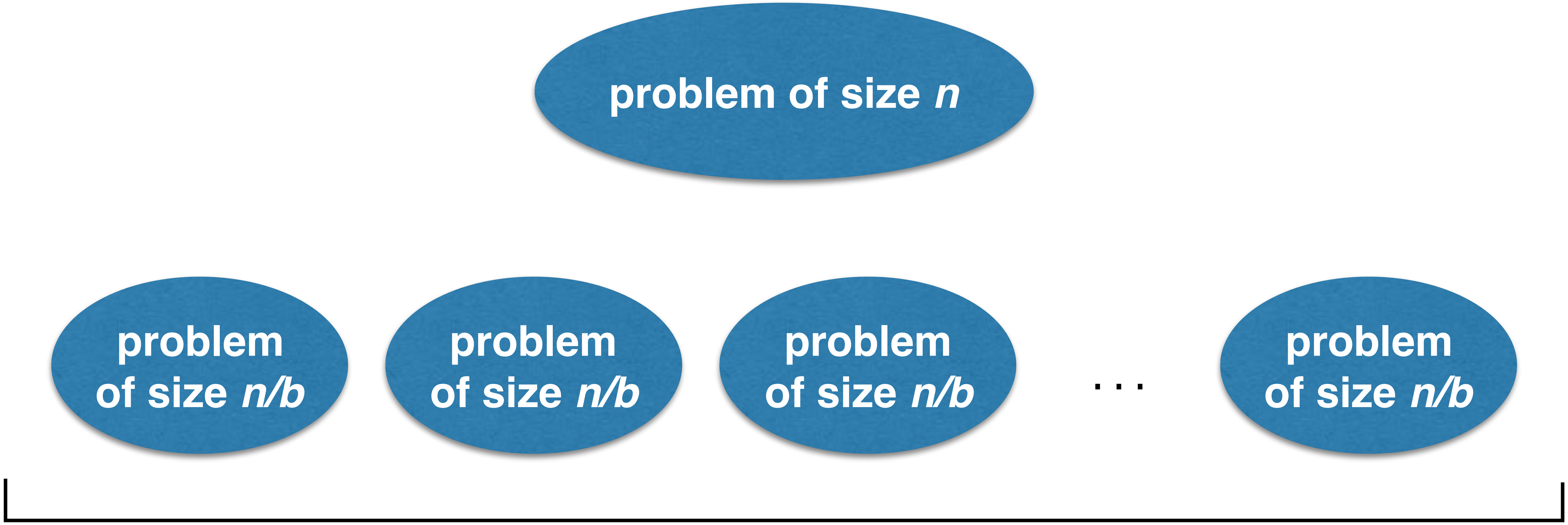
Split-Solve-and-Join Approach


problem of size n
sub-problem 1 of size $n/2$
sub-problem 2 of size $n/2$
a solution to sub-problem 1
a solution to sub-problem 2
a solution to the original problem

Divide and Conquer Algorithms

- We will discuss:
  - The Master Theorem
  - Mergesort
  - Quicksort
  - Tree traversal
  - Closest Pair revisited

Divide-and-Conqer General Case



problem of size *n*

problem of size *n/b*

problem of size *n/b*

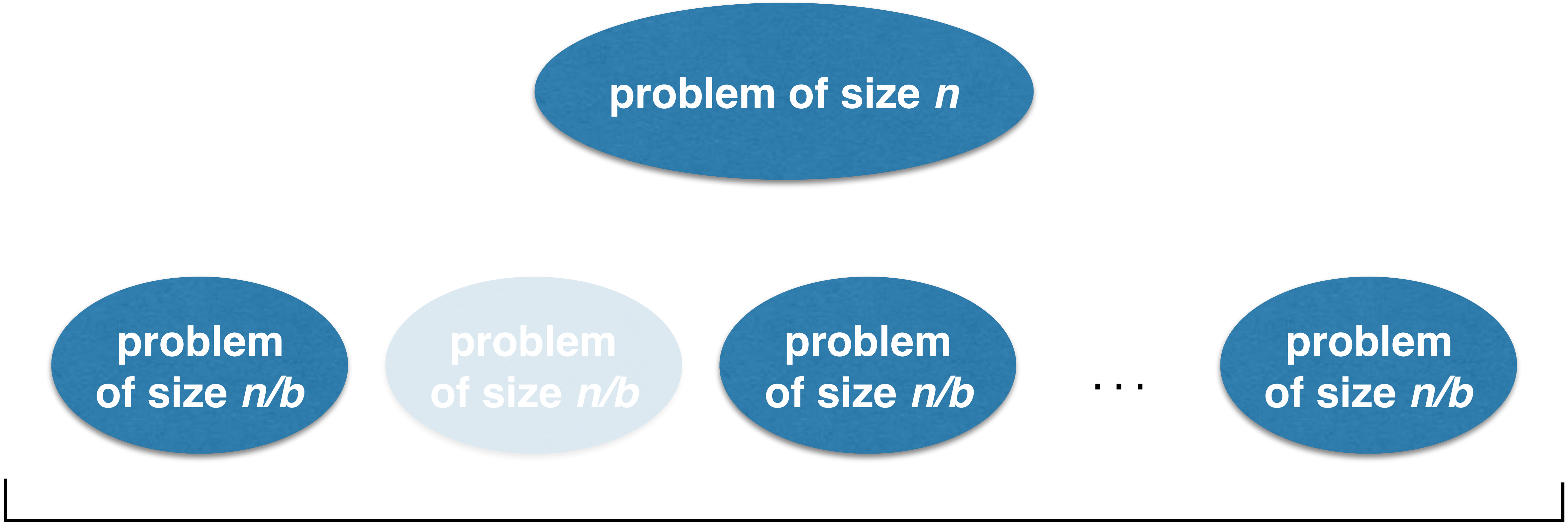
problem of size *n/b*

. . .

problem of size *n/b*

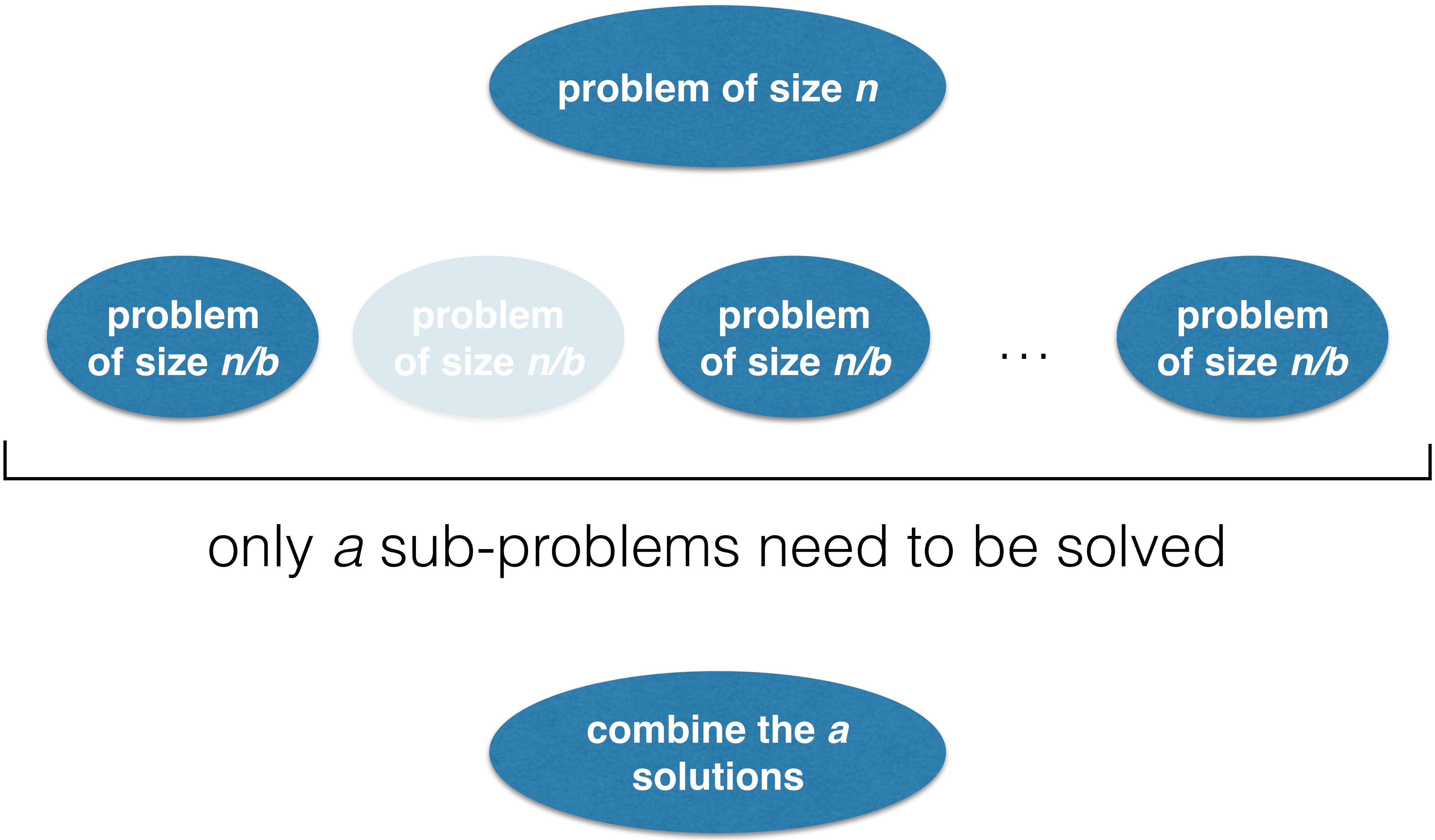
b sub-problems

Divide-and-Conqer General Case



only *a* sub-problems need to be solved

Divide-and-Conqer General Case



problem of size *n*

problem of size *n/b*

problem of size *n/b*

problem of size *n/b*

. . .

problem of size *n/b*

only *a* sub-problems need to be solved

combine the *a* solutions

Divide-and-Conquer Recurrences

- What is the time required to solve a problem of size *n* by divide-and-conquer?

- For the general case, assume we split the problem into *b* instances (each of size *n*/*b*), of which *a* need to be solved:

  $$T(n) = aT(n/b) + f(n)$$

  where *f*(*n*) expresses the time spent on dividing a problem into *b* sub-problems and combining the *a* results.

- (A very common case is $T(n) = 2T(n/2) + n$.)

- How to find closed forms for these recurrences?

The Master Theorem

- (A proof is in Levitin's Appendix B.)
- For integer constants $a \geq 1$ and $b > 1$, and function f with $\underline{f(n) \in \Theta(n^d)}$, $\underline{d \geq 0}$, the recurrence

$$T(n) = aT(n/b) + f(n)$$

(with T(1) = c) has solutions, and

$$\boxed{T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}}$$

- Note that we also allow a to be greater than b.

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$

a = 2, b = 2, d = 1

$a = b^d$

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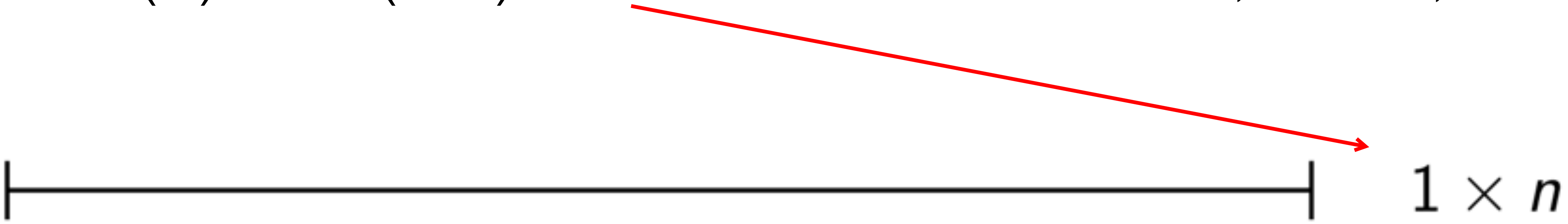
So, by the Master Theorem, $T(n) \in \Theta(n \log n)$

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

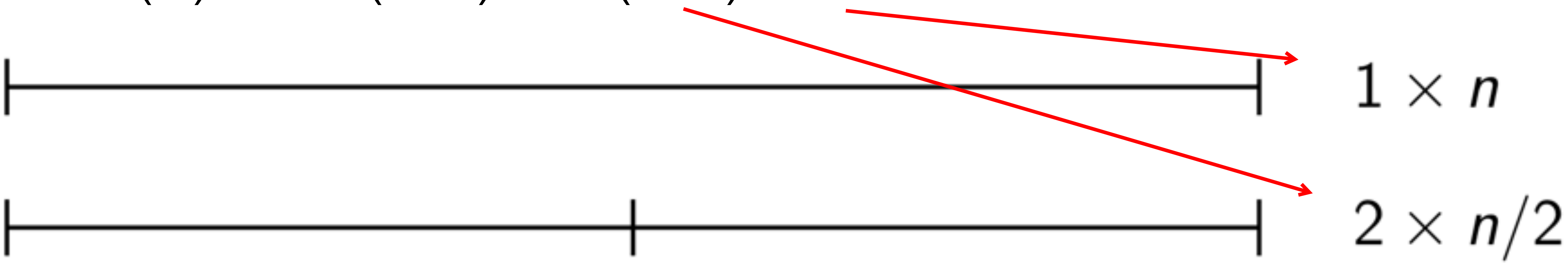
$T(n) = 2(2T(n/4) + (n/2)) + n$

$1 \times n$

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

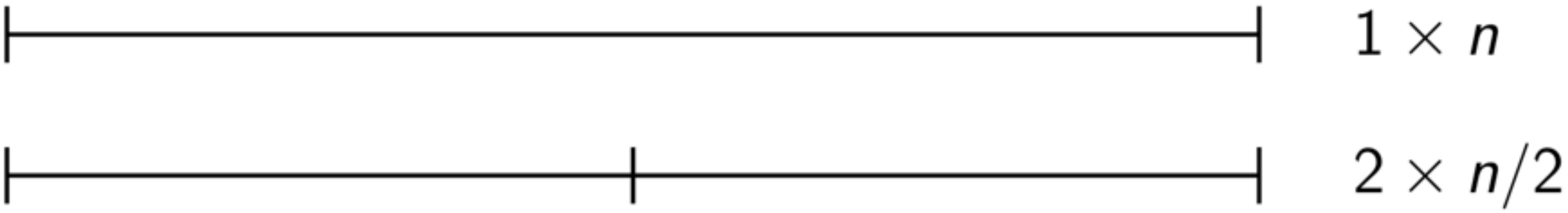
$T(n) = 4T(n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

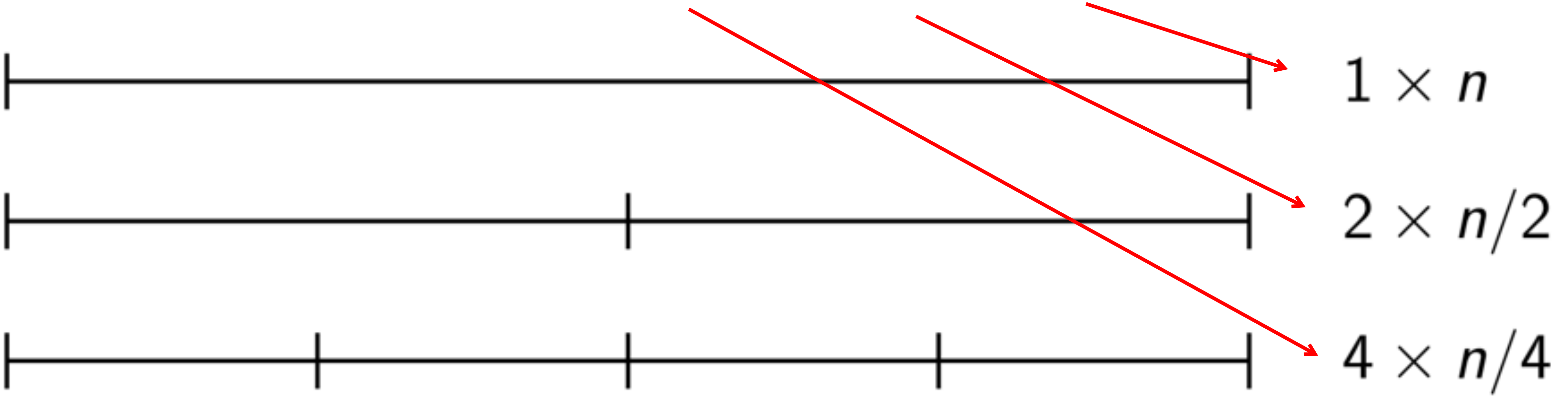
$T(n) = 4(2T(n/8) + n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

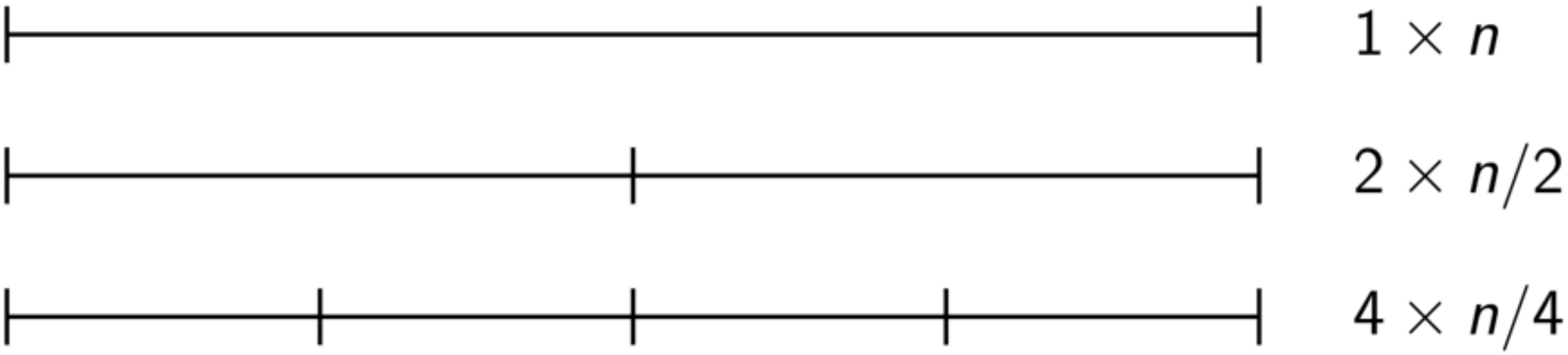
$T(n) = 8T(n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

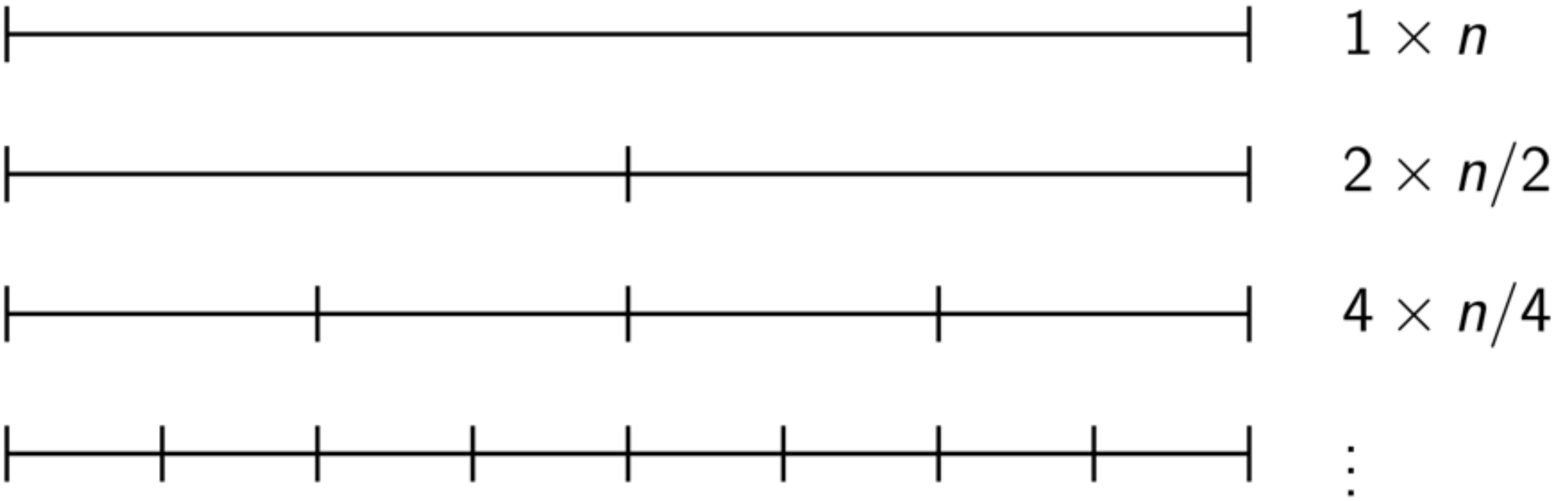
$T(n) = 8(2T(n/16) + n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

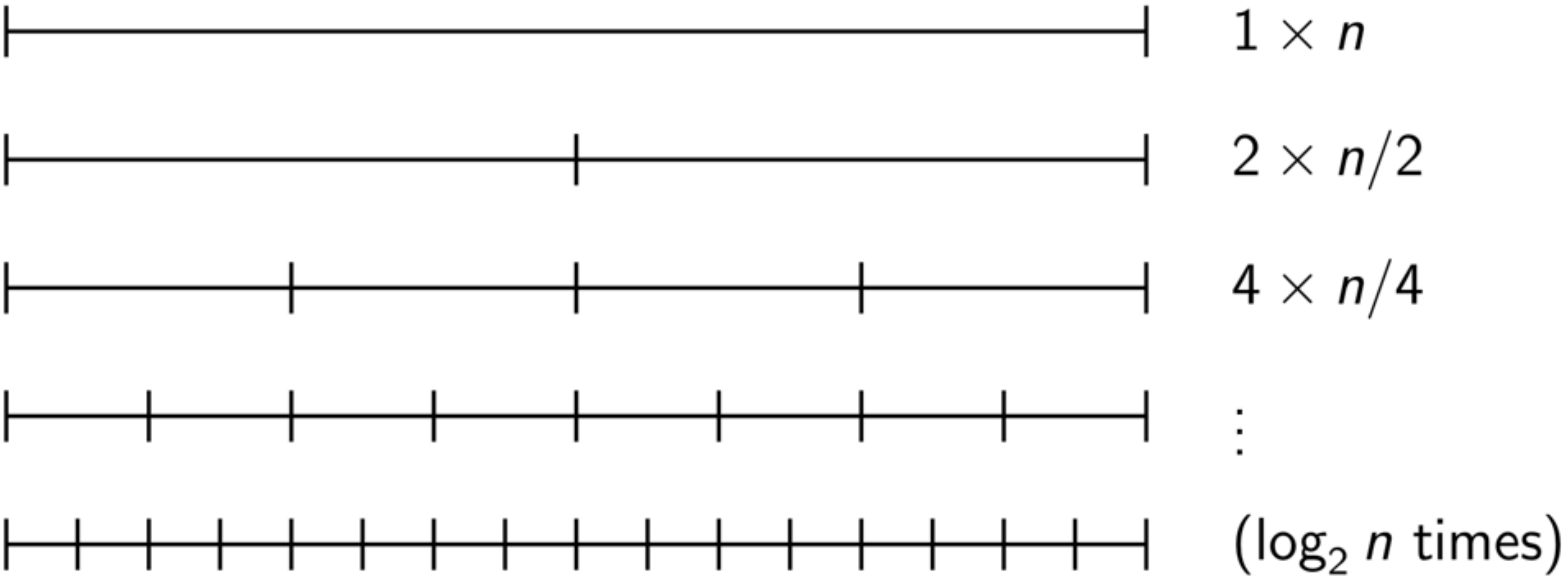
$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

$T(n) = 16T(n/16) + 8(n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

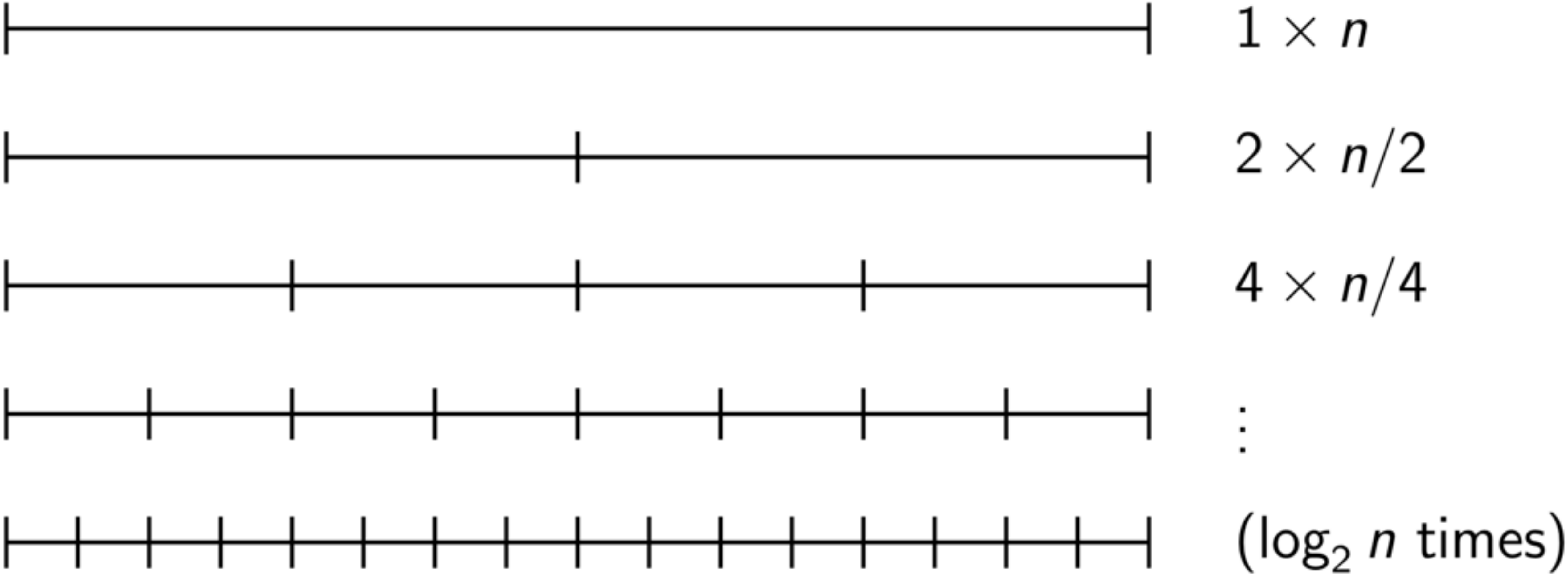
$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

$T(n) \in \Theta(n \log n)$



Master Theorem: Example 2

$T(n) = 4T(n/4) + n$

a = 4, b = 4, d = 1

a = b^d

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

So, by the Master Theorem, $T(n) \in \Theta(n \log n)$

Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ $\quad$ $a = 4,\ b = 4,\ d = 1$

Master Theorem: Example 2

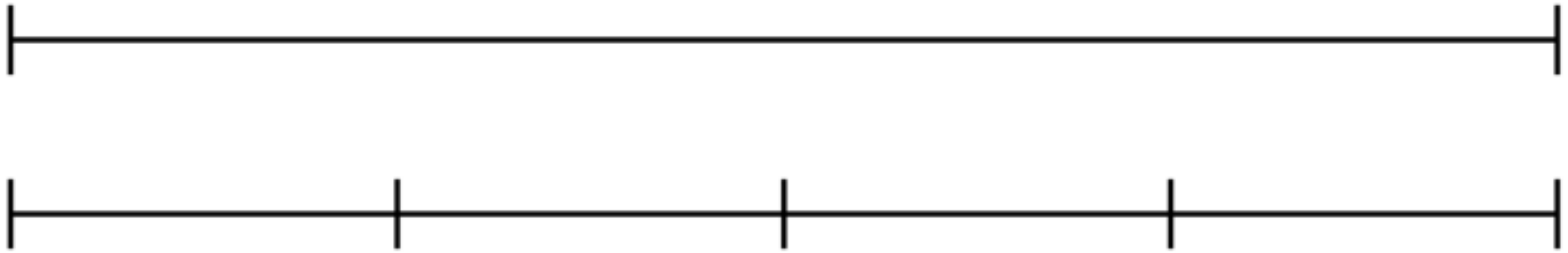
$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

$T(n) = 4(4T(n/16) + (n/4)) + n$

Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

$T(n) = 16T(n/16) + 4(n/4) + n$



Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

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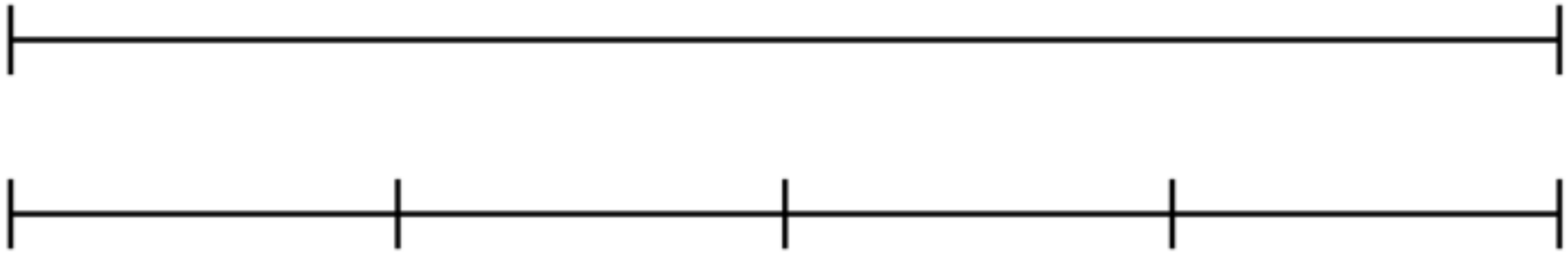
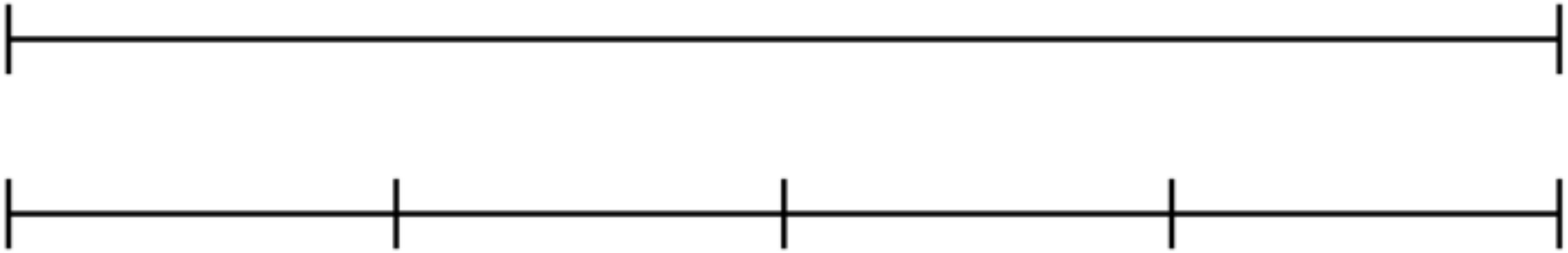


Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

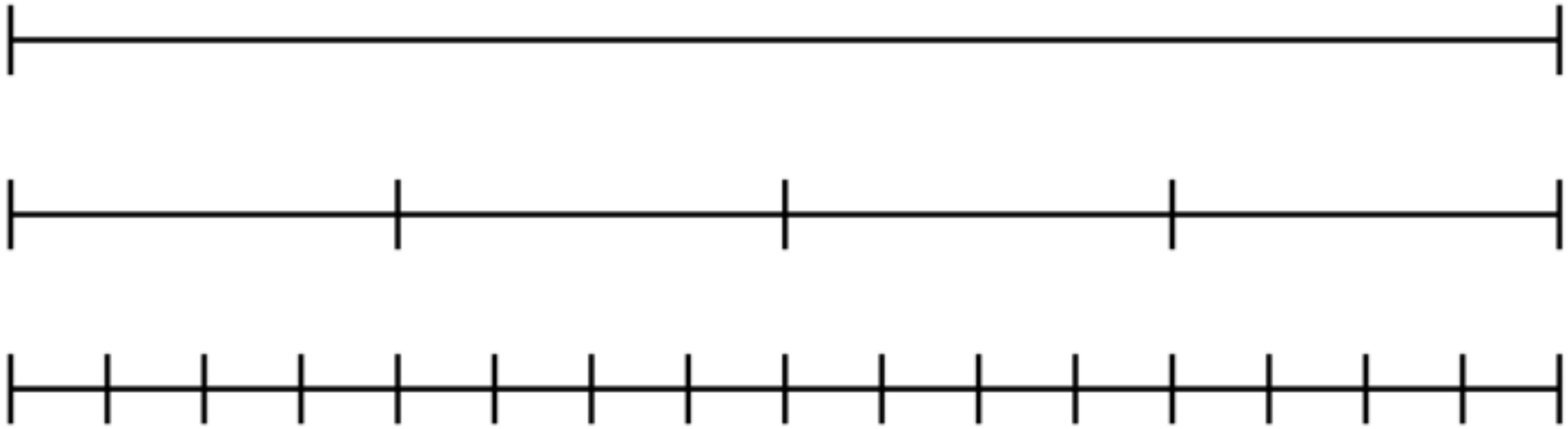
$T(n) = 16(4T(n/64) + n/16)\ 4(n/4) + n$



Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

$T(n) = 64T(n/64) + 16(n/16) + 4(n/4) + n$



Master Theorem: Example 2

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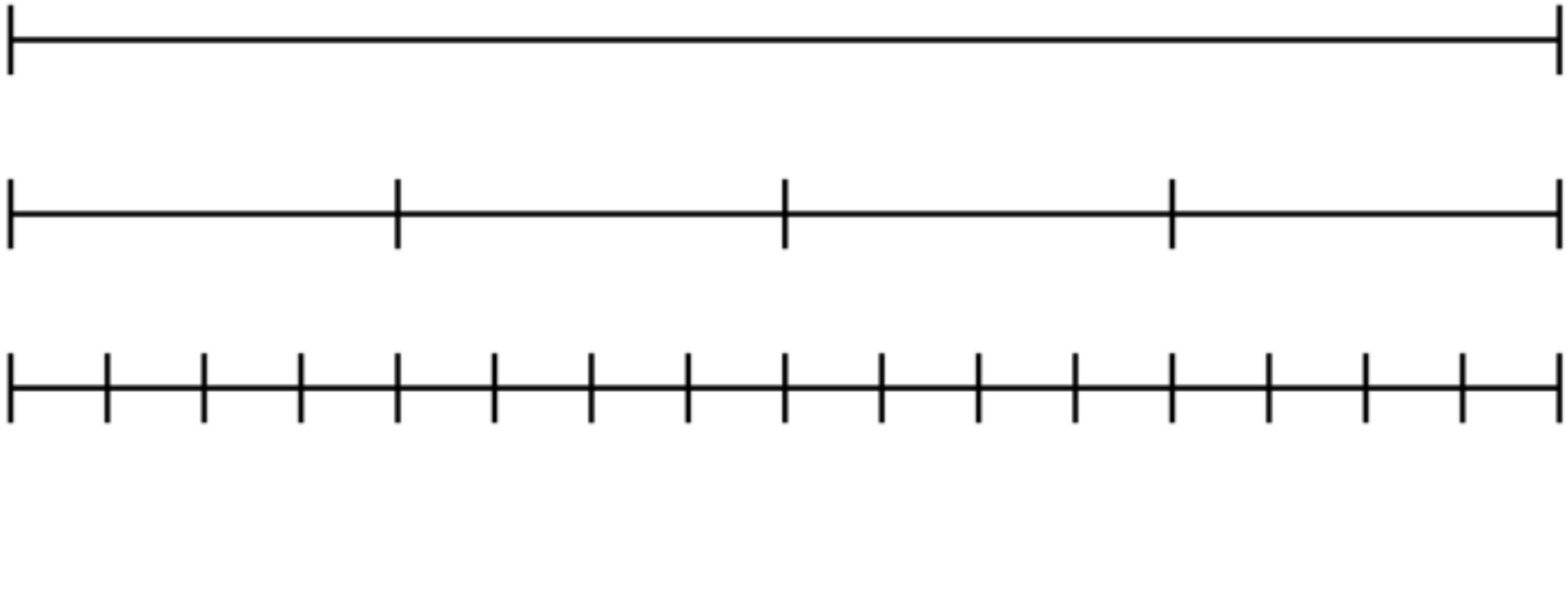

(log$_4$ n times)

Master Theorem: Example 2

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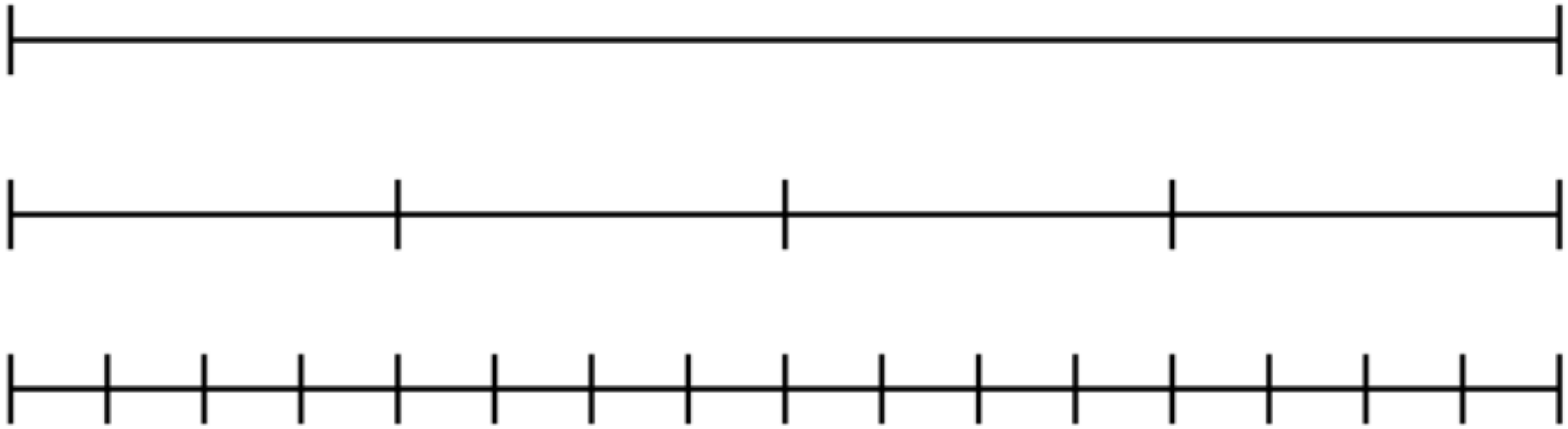

⋮

(log$_4$ n times)

Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

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$T(n) \in \Theta(n \log n)$

Master Theorem: Example 3

$T(n) = T(n/2) + n$ 		a = 1, b = 2, d = 1

Master Theorem: Example 3

$T(n) = T(n/2) + n$

a = 1, b = 2, d = 1

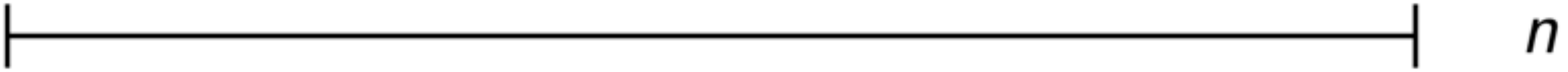
$a < b^d$

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

So, by the Master Theorem, $T(n) \in \Theta(n)$

Master Theorem: Example 3

$T(n) = T(n/2) + n$ a = 1, b = 2, d = 1



Master Theorem: Example 3

$T(n) = T(n/2) + n$ $\quad$ a = 1, b = 2, d = 1

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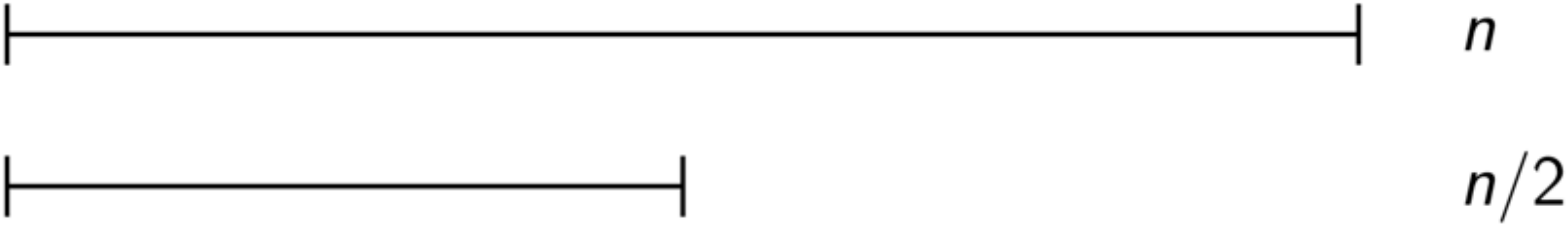
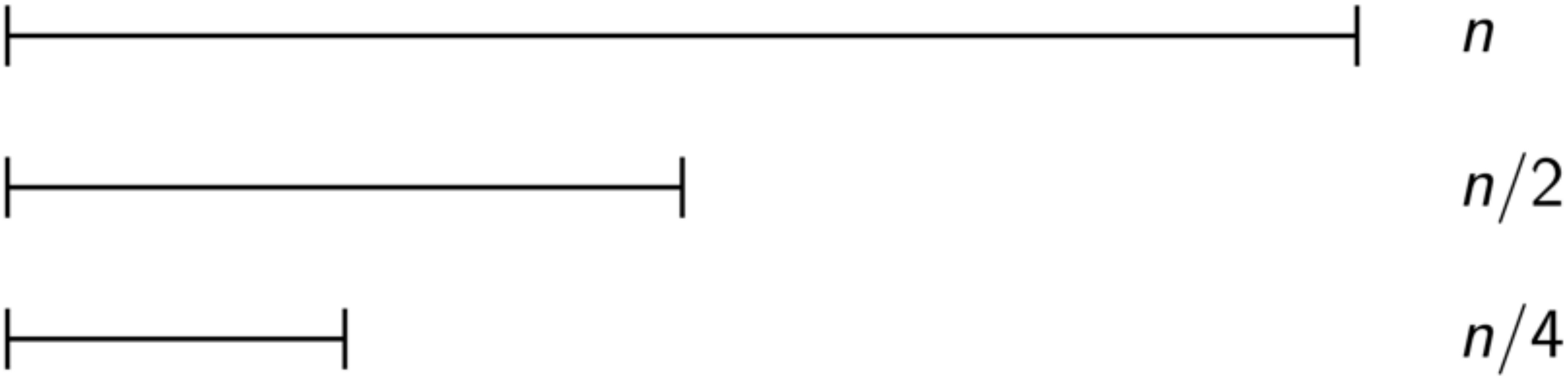


Master Theorem: Example 3

$T(n) = T(n/2) + n$ a = 1, b = 2, d = 1

$T(n) = T(n/8) + n/4 + n/2 + n$



Master Theorem: Example 3

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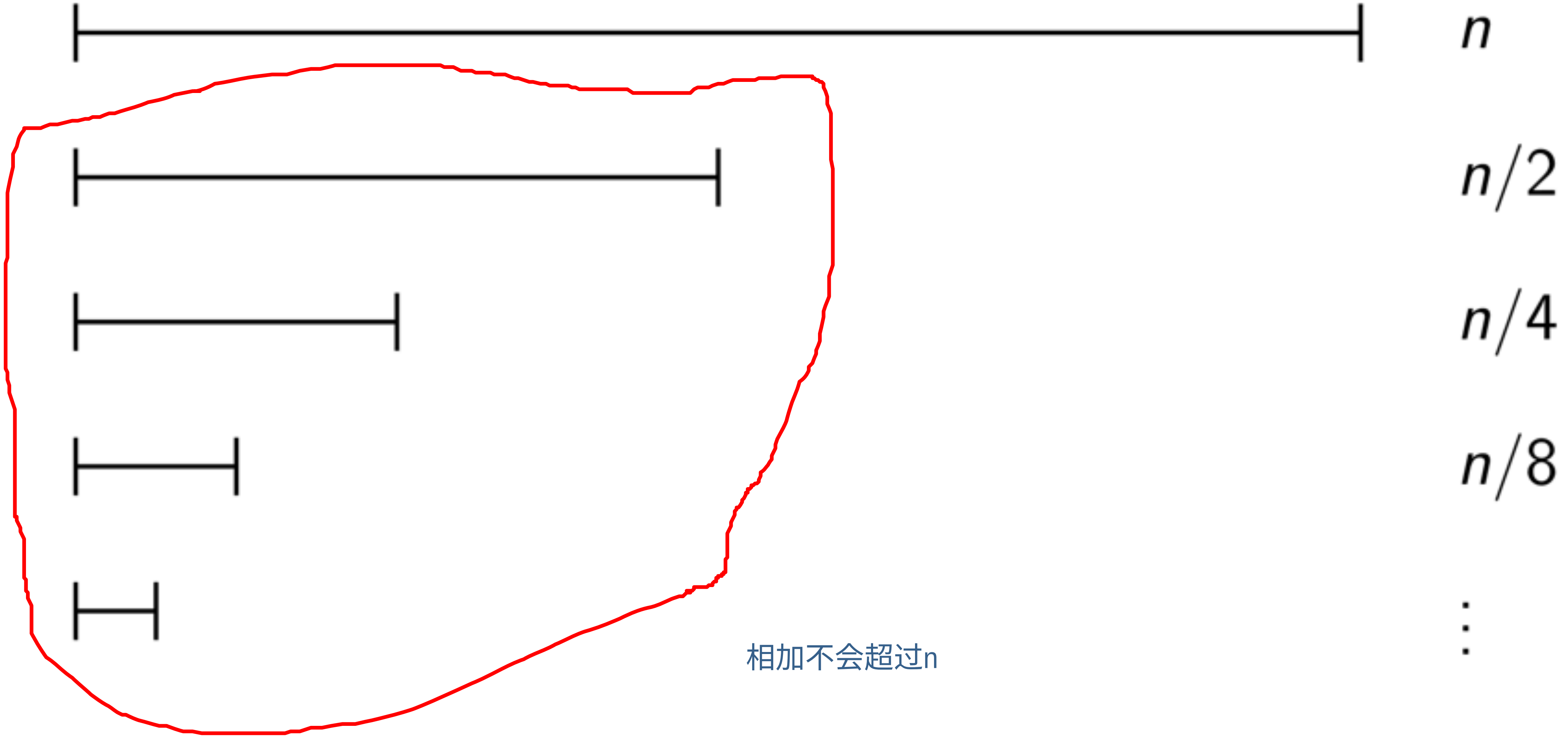


Master Theorem: Example 3

$T(n) = T(n/2) + n$ a = 1, b = 2, d = 1

$T(n) = T(n/8) + \mathrm{n}/4 + \mathrm{n}/2 + n$

$T(n) \in \Theta(\mathrm{n})$

Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$

$a = 2, b = 2, d = 2$

Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$

a = 2, b = 2, d = 2

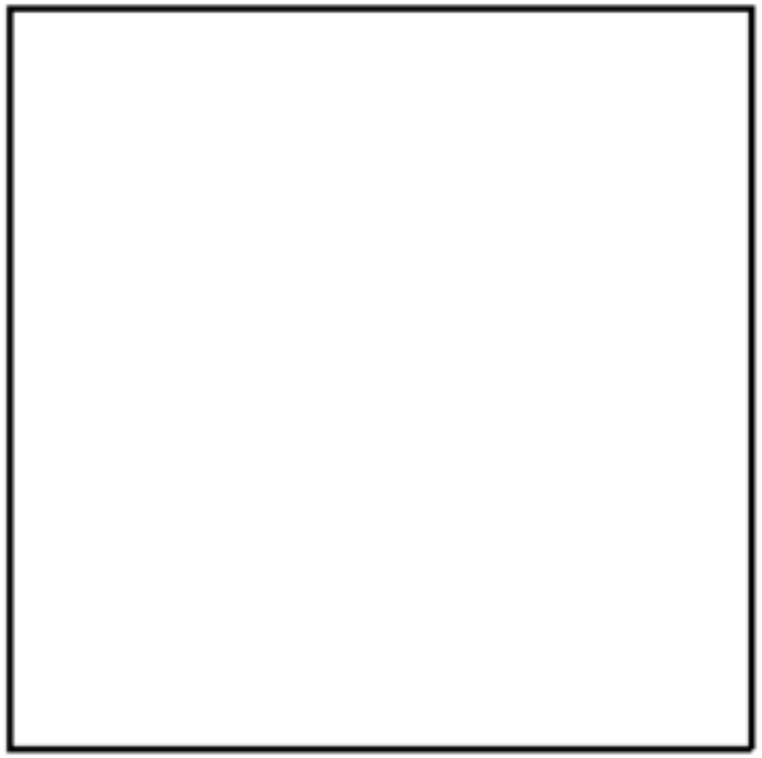
$a < b^d$

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

So, by the Master Theorem, $T(n) \in \Theta(n^2)$

Master Theorem: Example 4

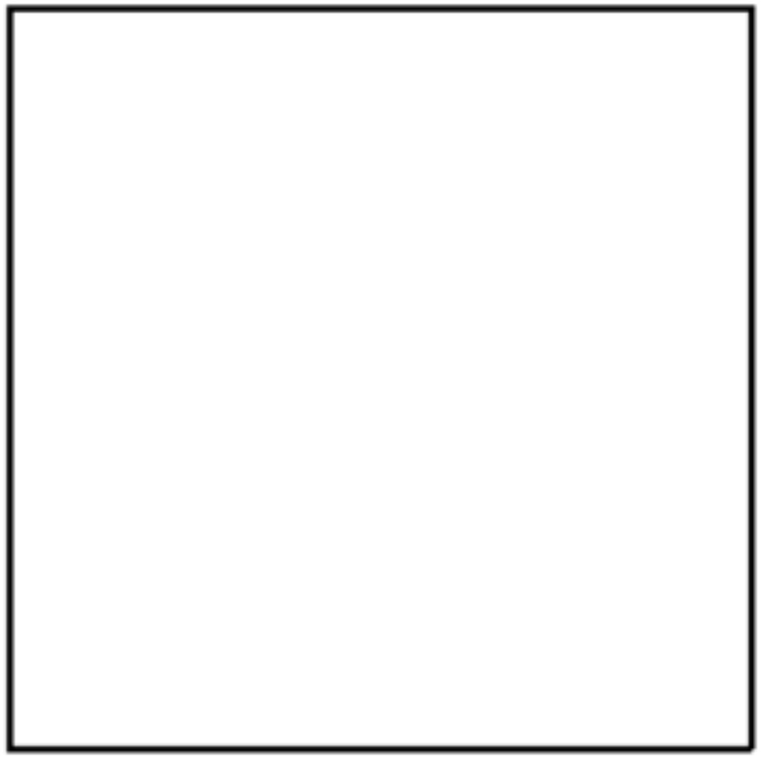
$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2



Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2

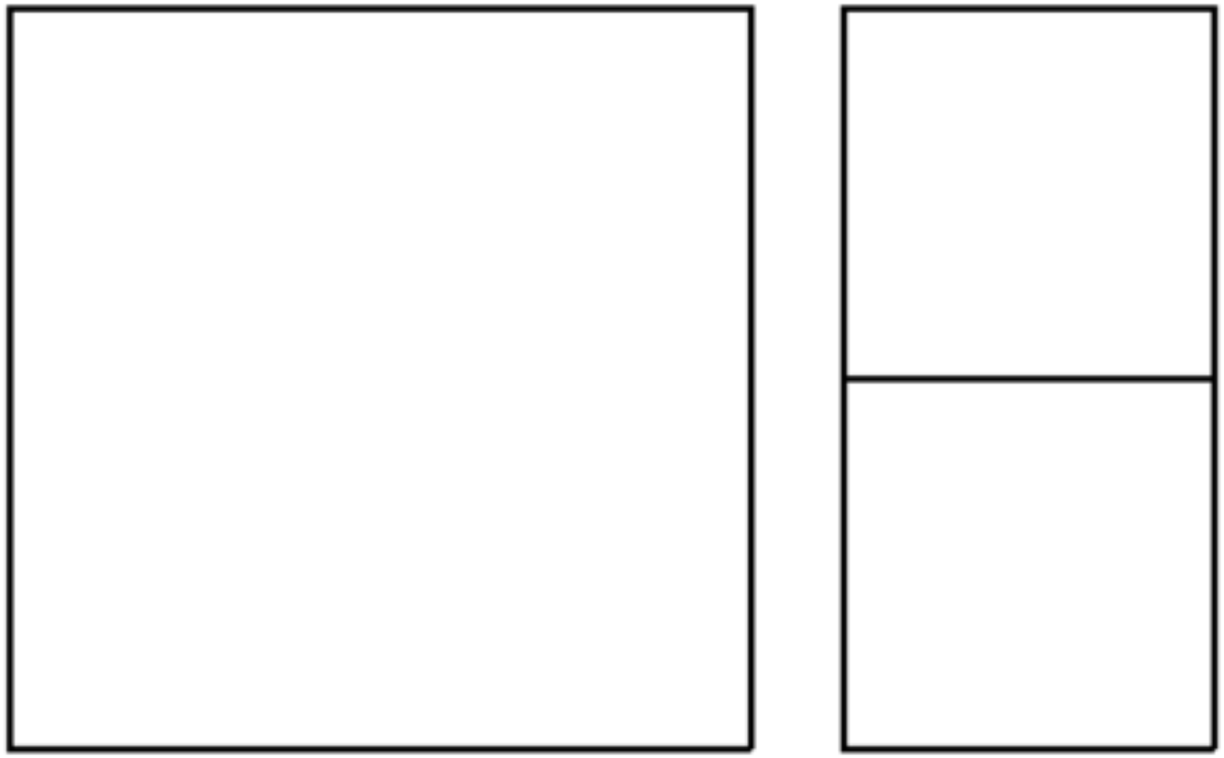
$T(n) = 2(2T(n/4) + (n/2)^2) + n^2$



Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2

$T(n) = 4T(n/4) + 2(\mathrm{n}/2)^2 + n^2$



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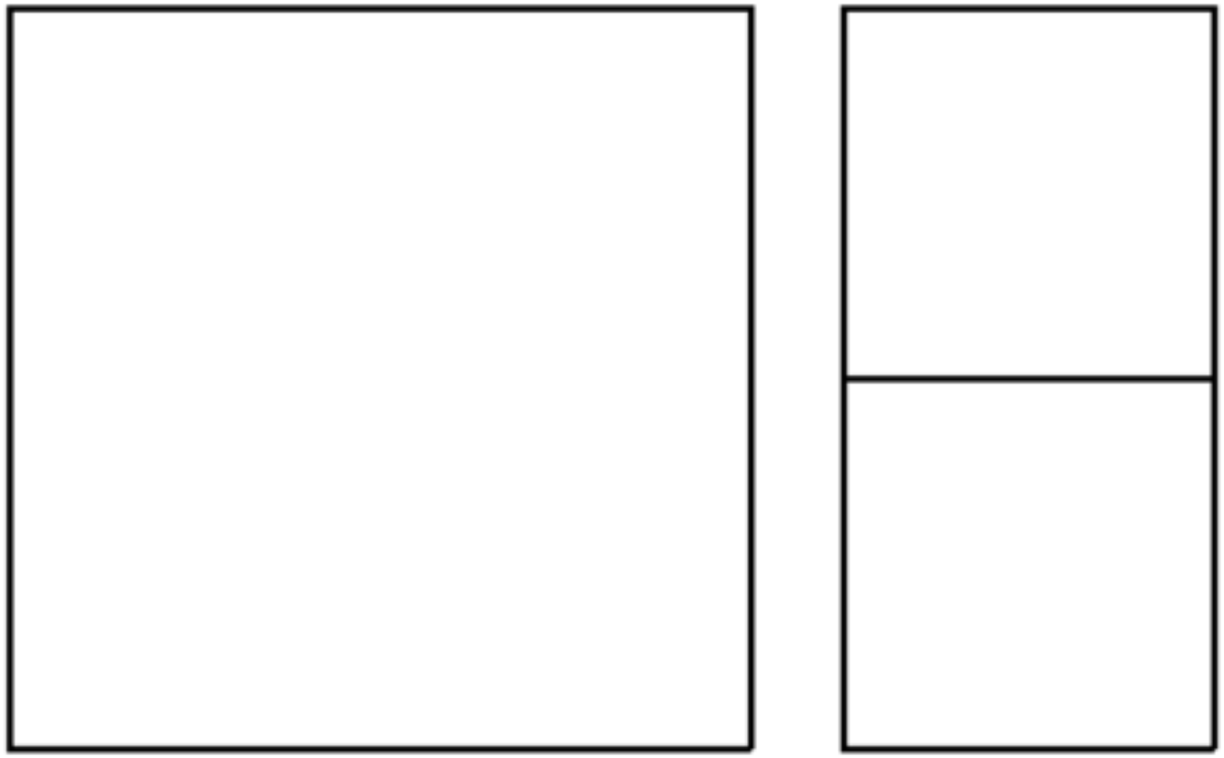
Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2

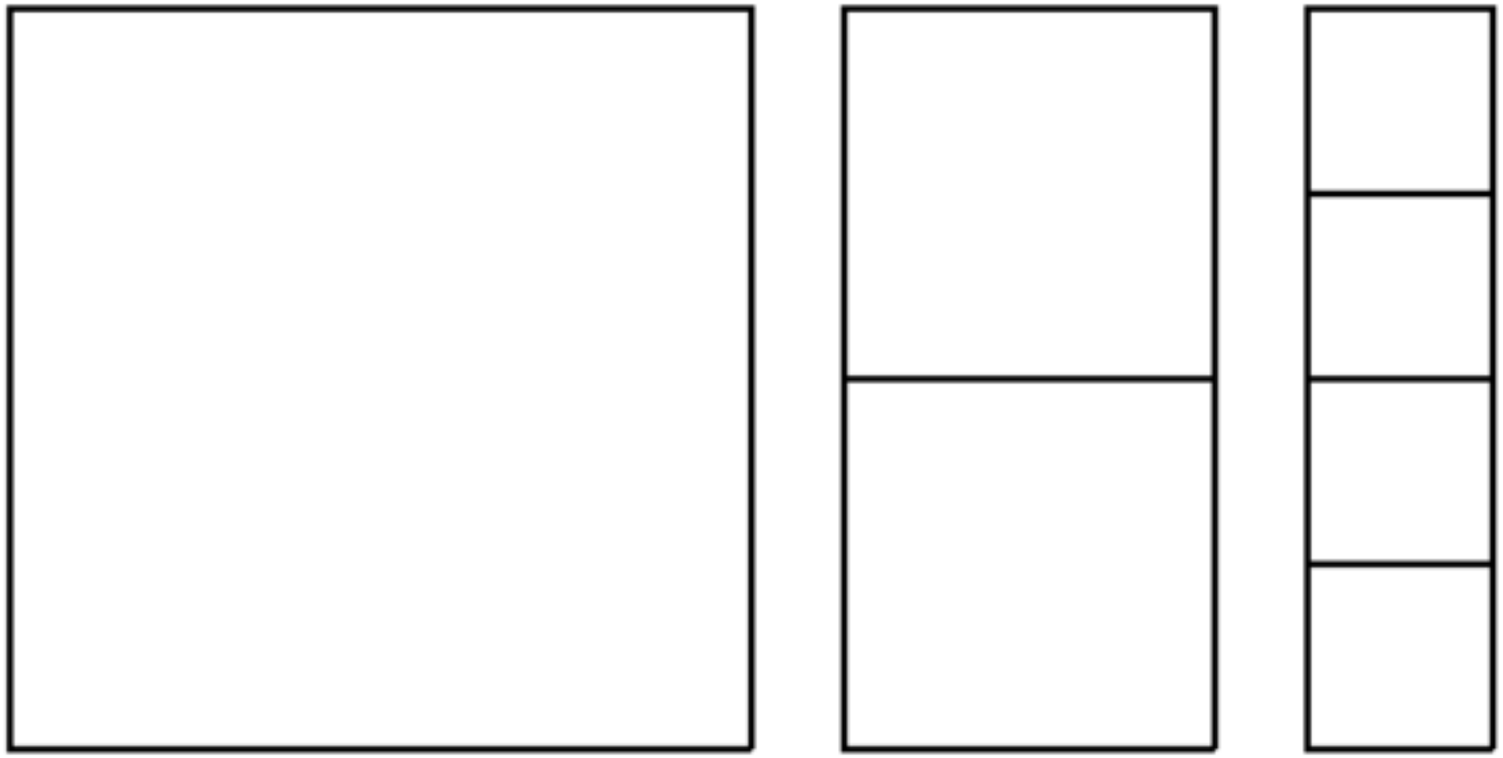
$T(n) = 4(2T(n/8) + (n/4)^2) + 2(n/2)^2 + n^2$



Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2

$T(n) = 8T(n/8) + 4(n/4)^2 + 2(n/2)^2 + n^2$



Master Theorem: Example 4

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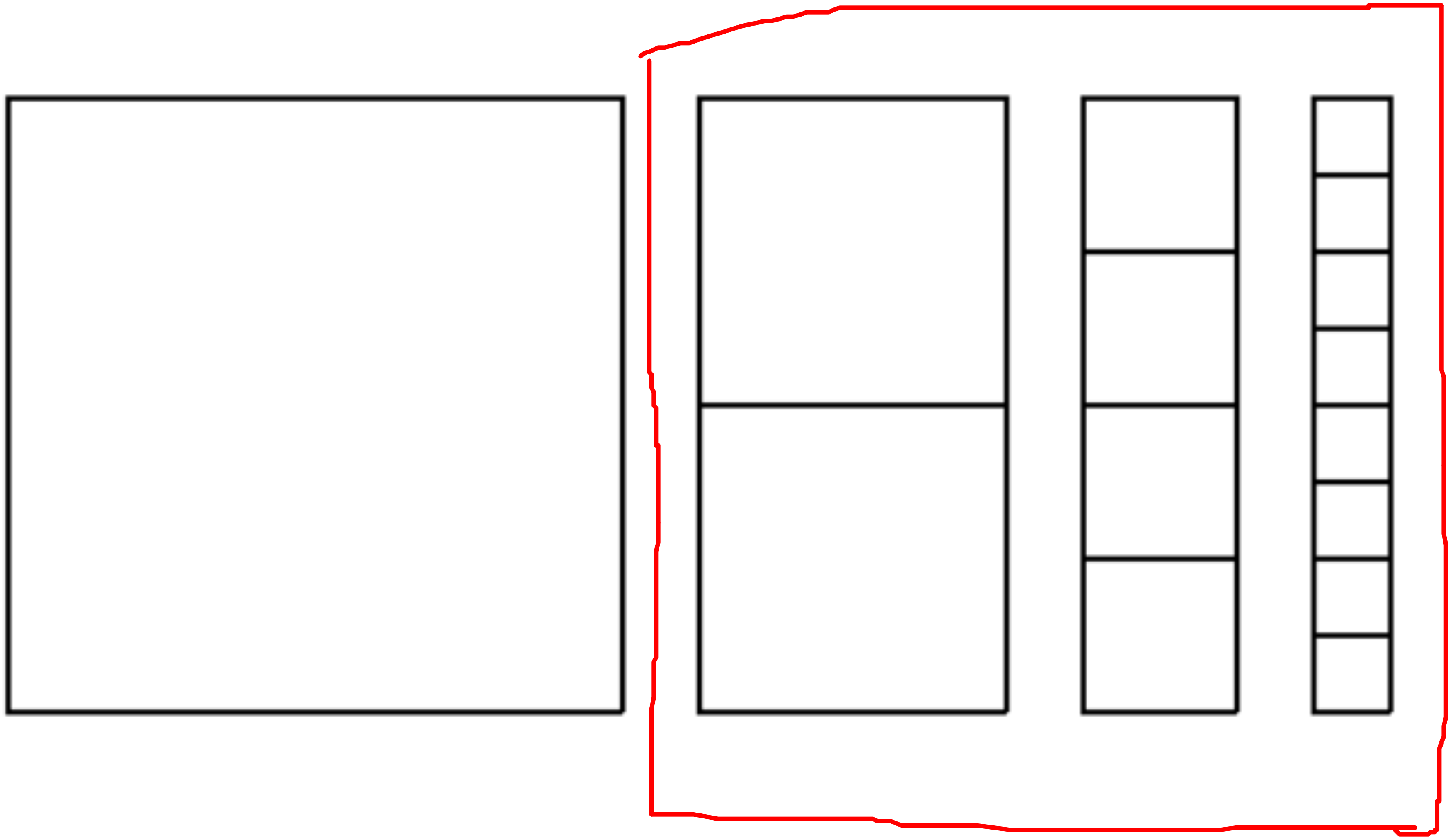


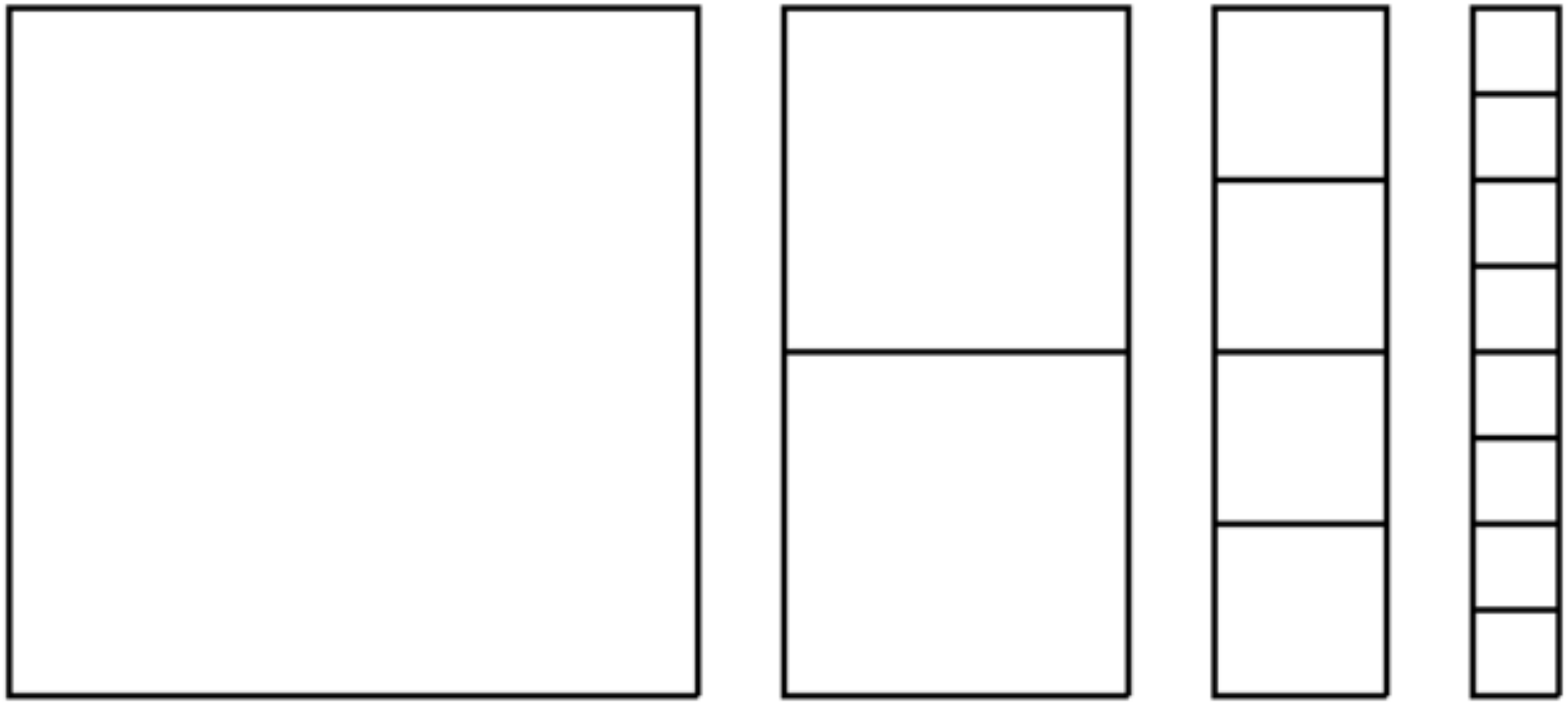
相加不会超过n^2

Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$ a = 2, b = 2, d = 2

$T(n) = 8T(n/8) + 4(n/4)^2 + 2(n/2)^2 + n^2$

$T(n) \in \Theta(n^2)$



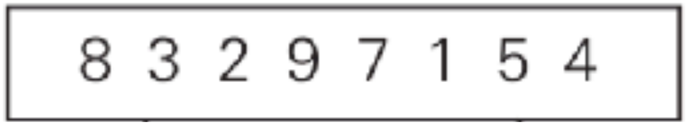
Mergesort

- Perhaps the most obvious application of divide-and-conquer:

- To sort an array (or a list), cut it into two halves, sort each half, and merge the two results.

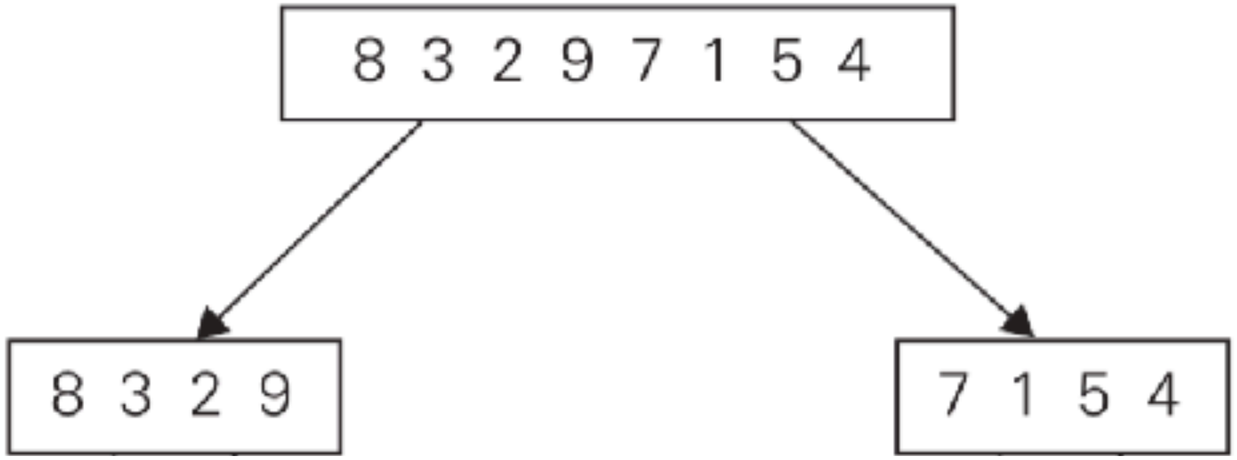
```
procedure MERGESORT($A[\cdot], n$)                     ▷ Sort $A[0]..A[n-1]$
    if $n > 1$ then
        for $i \leftarrow 0$ to $\lfloor n/2 \rfloor - 1$ do          ▷ Copy left half of $A$ to $B$
            $B[i] \leftarrow A[i]$
        for $i \leftarrow 0$ to $\lceil n/2 \rceil - 1$ do          ▷ Copy right half of $A$ to $C$
            $C[i] \leftarrow A[\lfloor n/2 \rfloor + i]$
        MERGESORT($B, \lfloor n/2 \rfloor$)                          ▷ Sort $B$
        MERGESORT($C, \lceil n/2 \rceil$)                          ▷ Sort $C$
        MERGE($B, \lfloor n/2 \rfloor, C, \lceil n/2 \rceil, A$)        ▷ Merge $B$ and $C$ into $A$
```

Mergesort

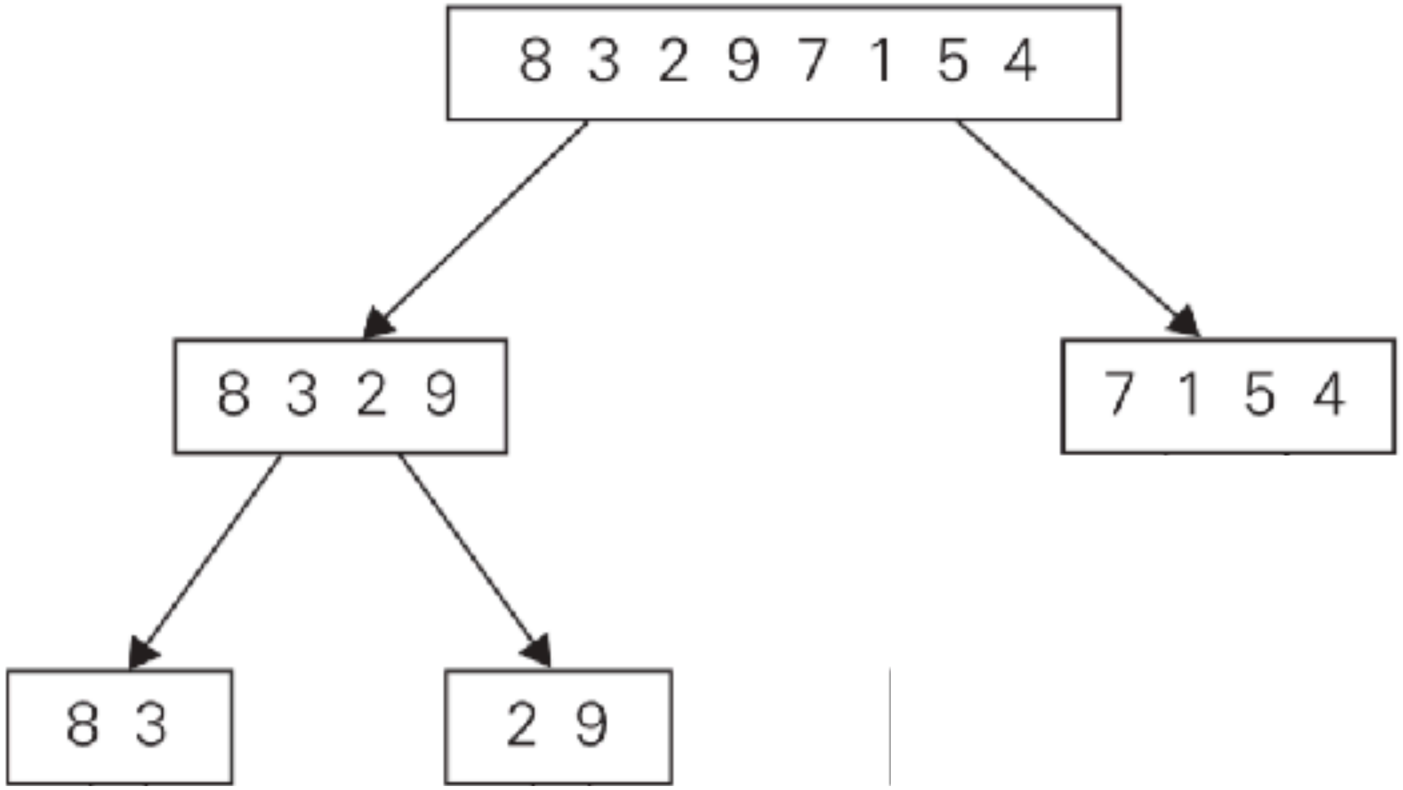


8 3 2 9 7 1 5 4

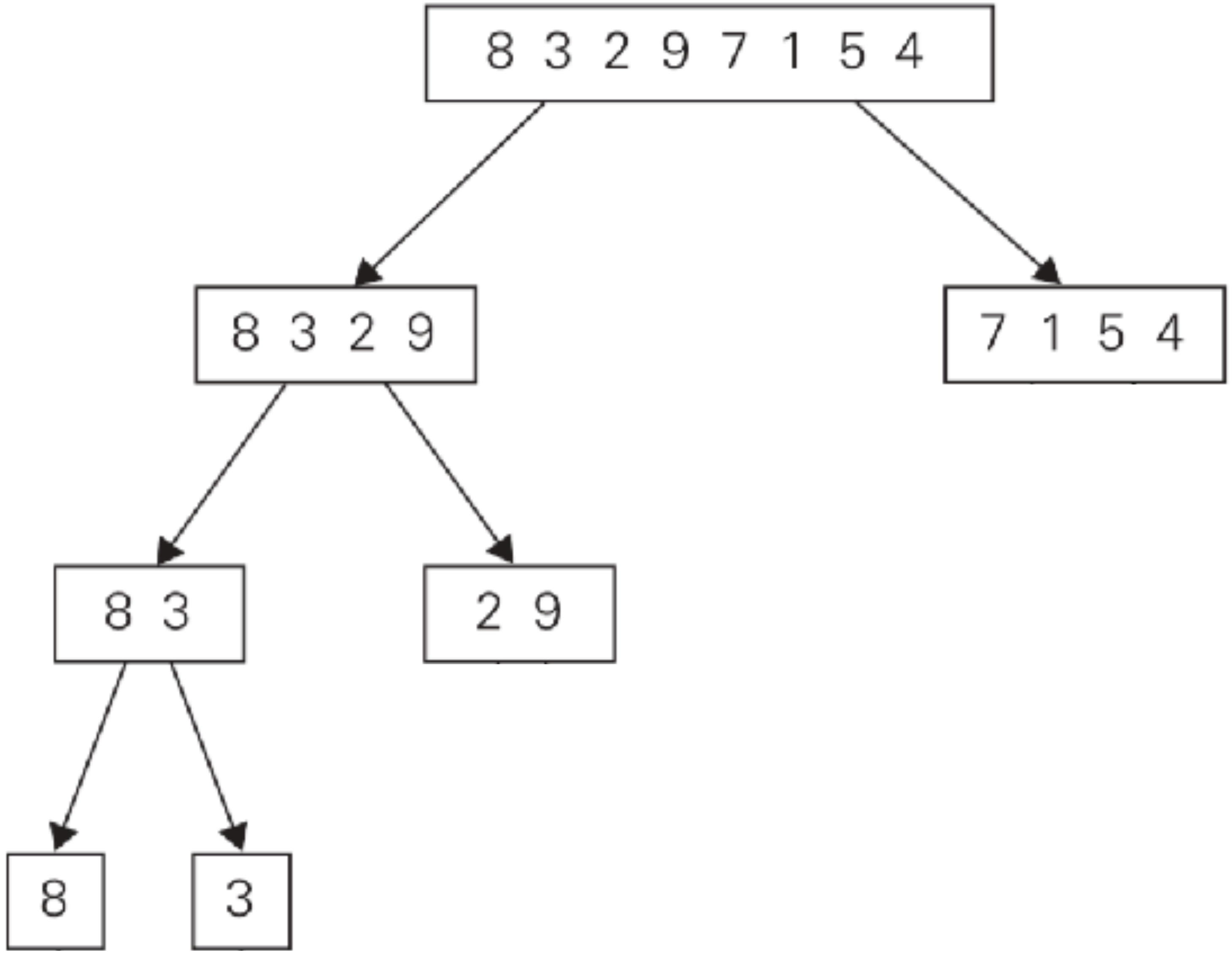
Mergesort



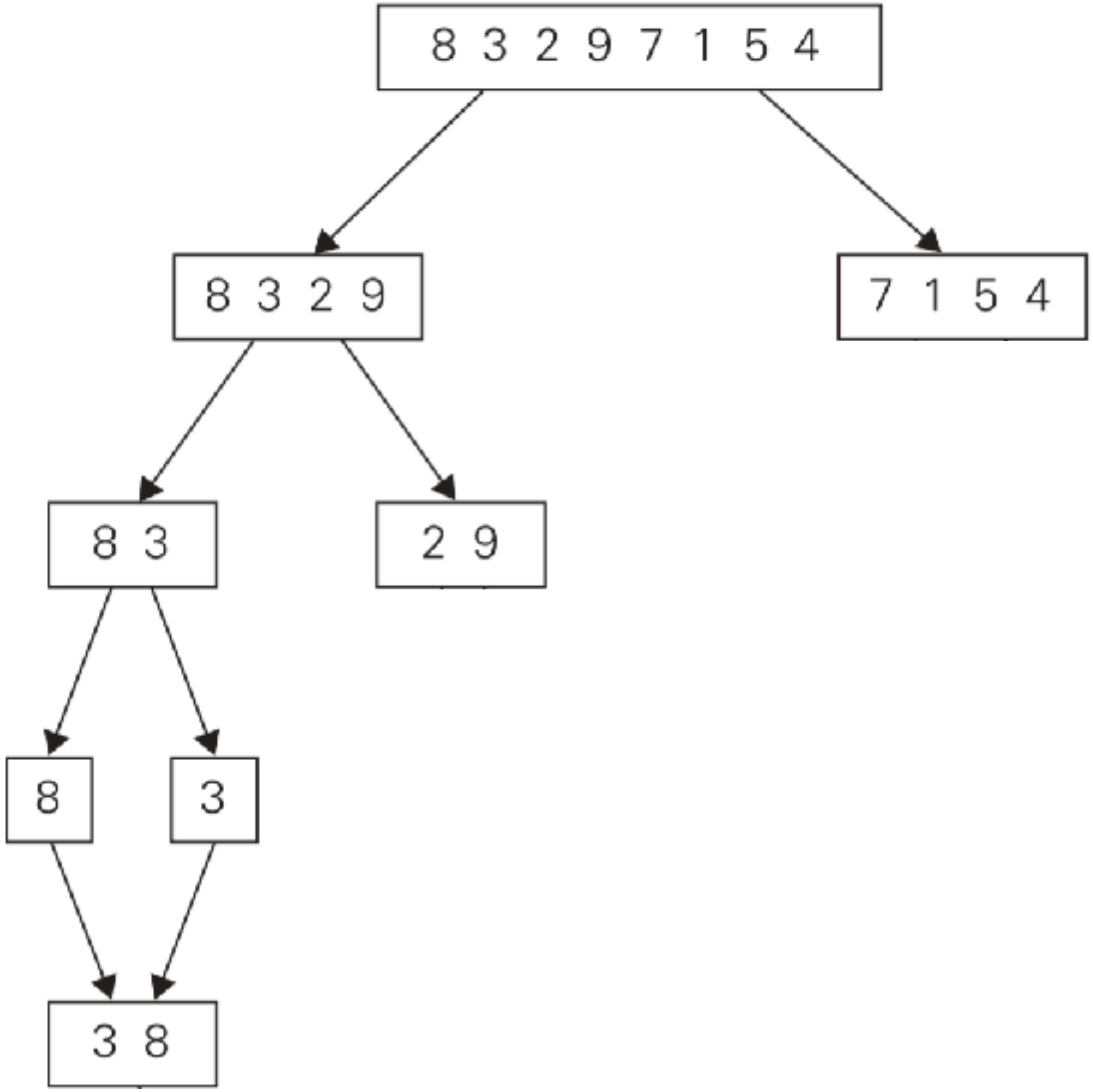
Mergesort



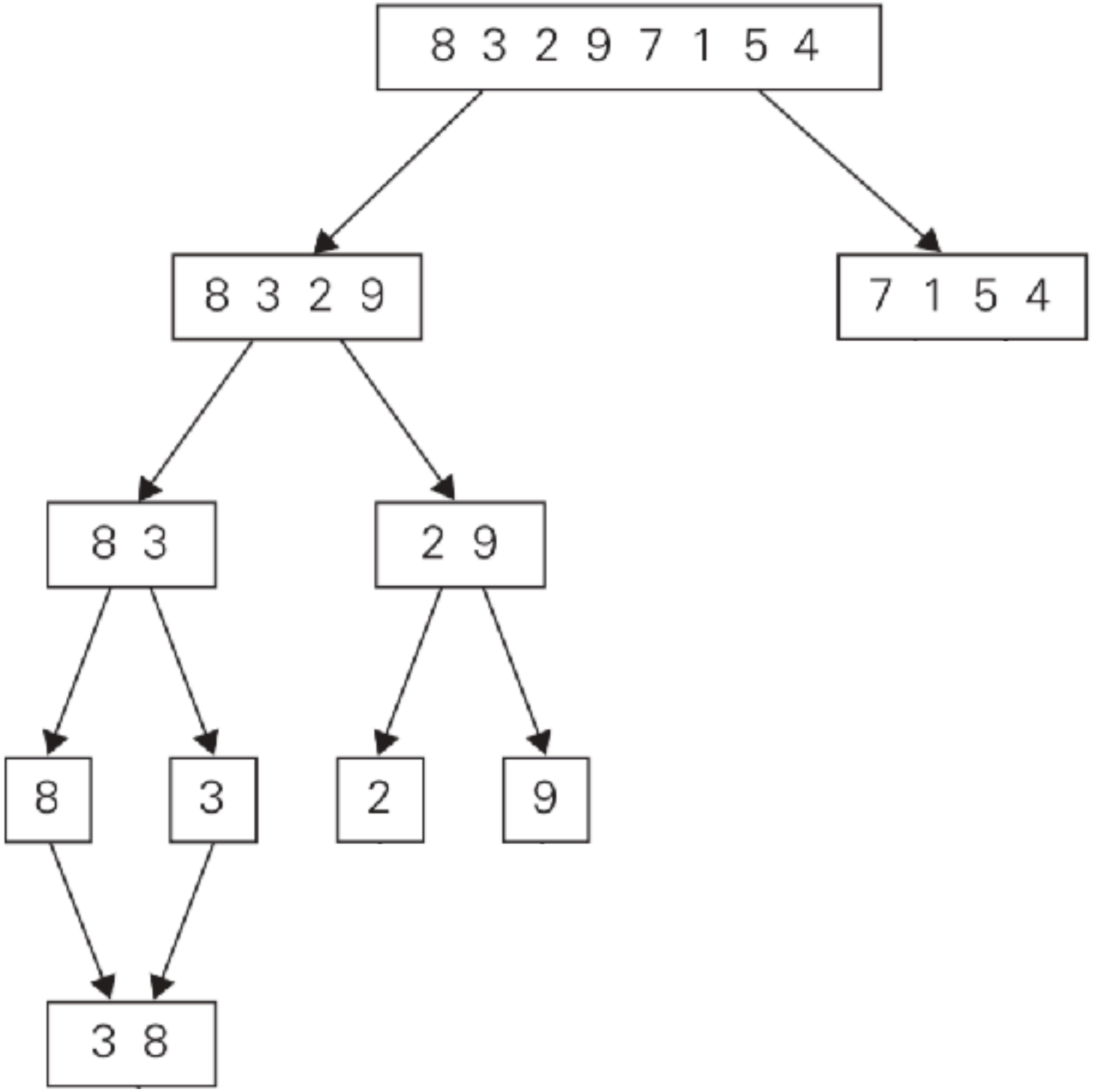
Mergesort



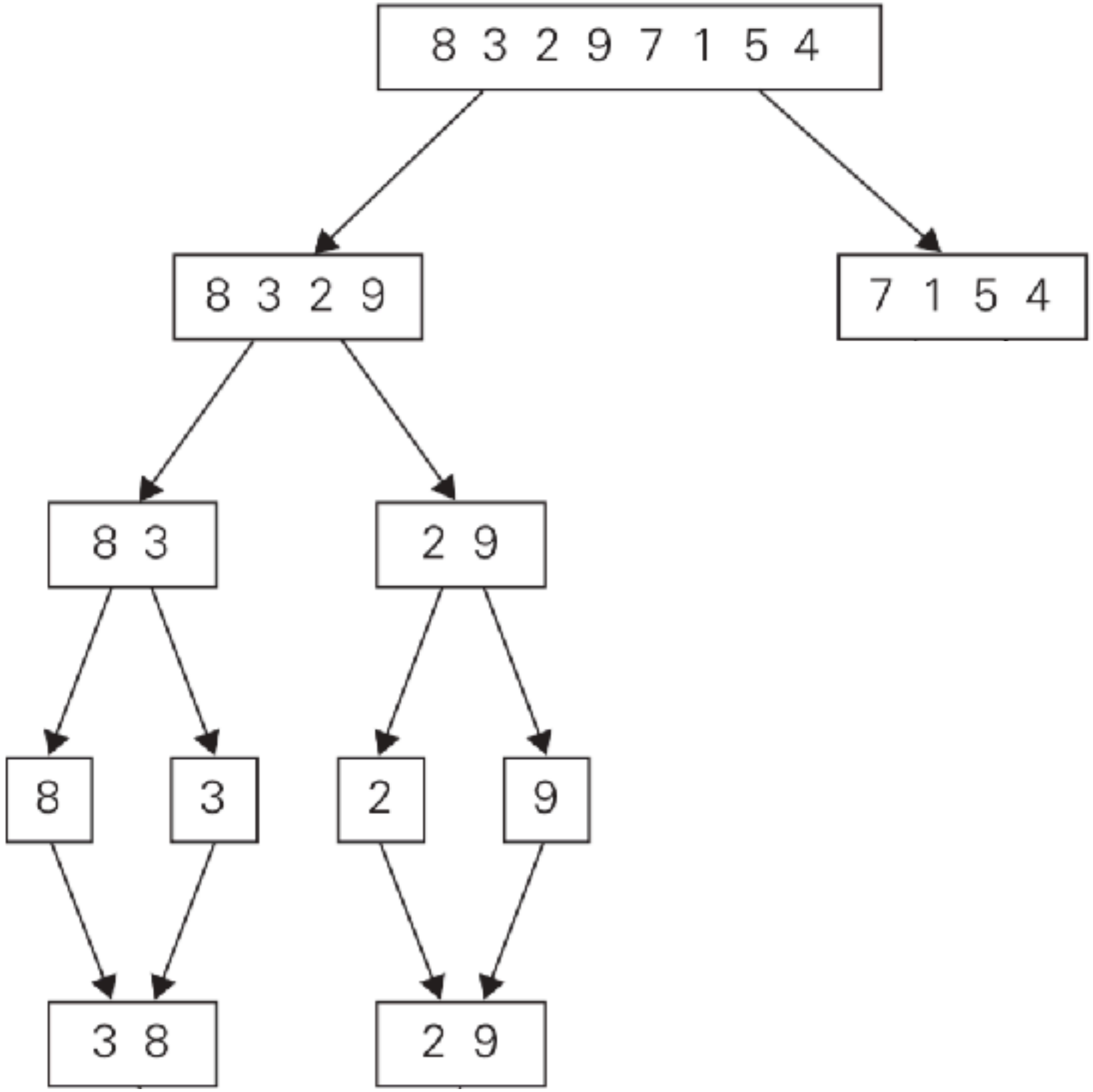
Mergesort



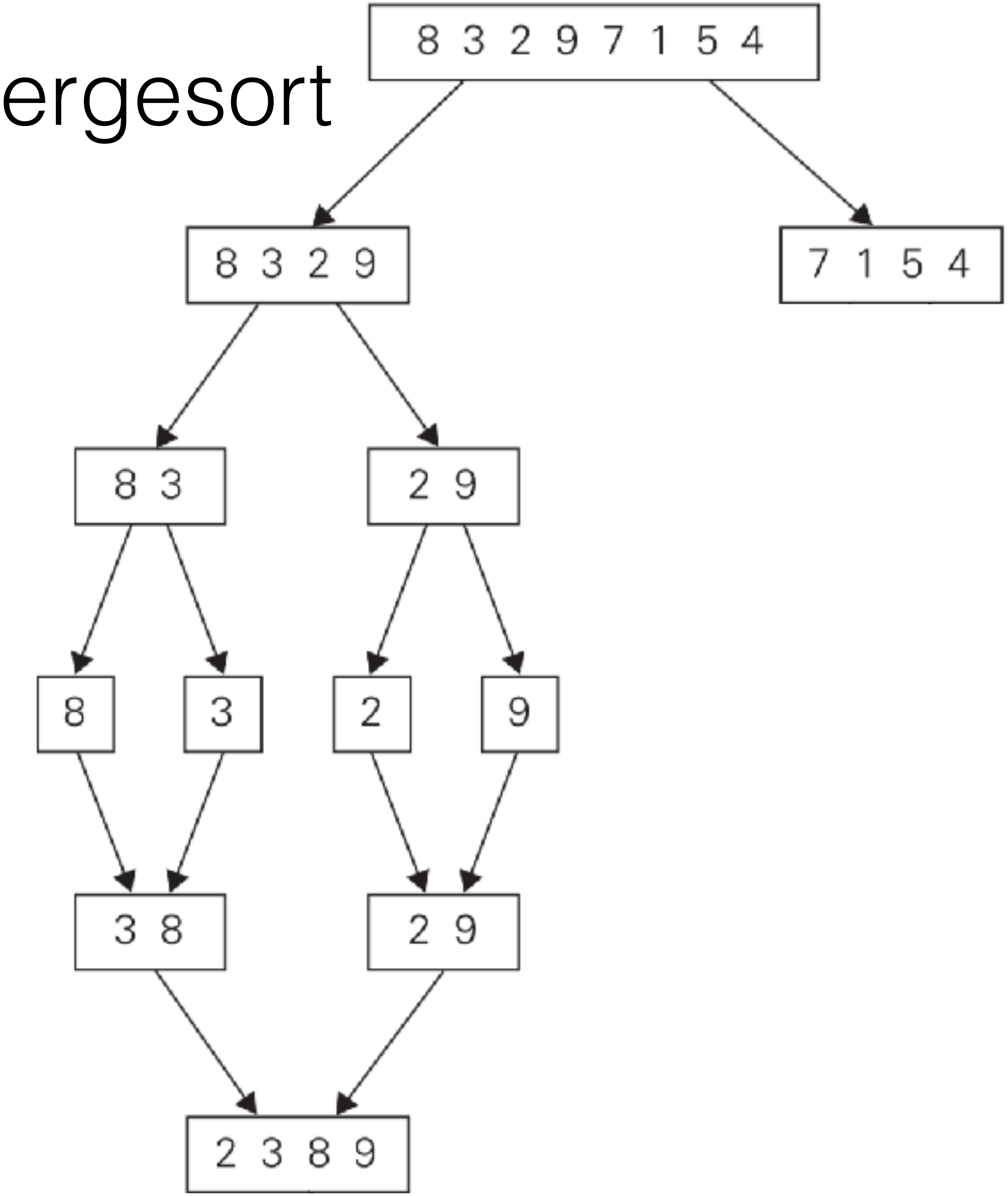
Mergesort



Mergesort



Mergesort



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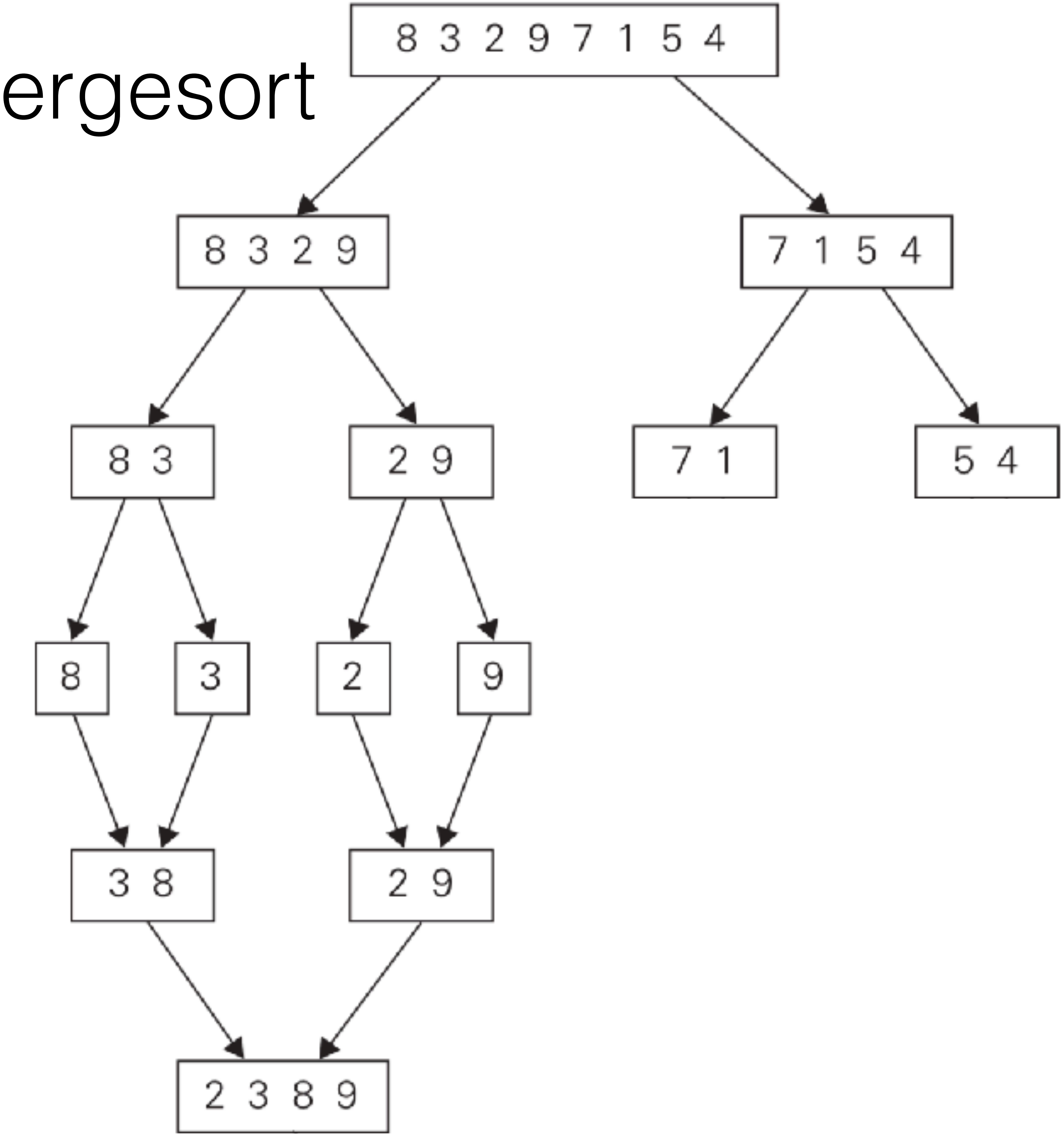
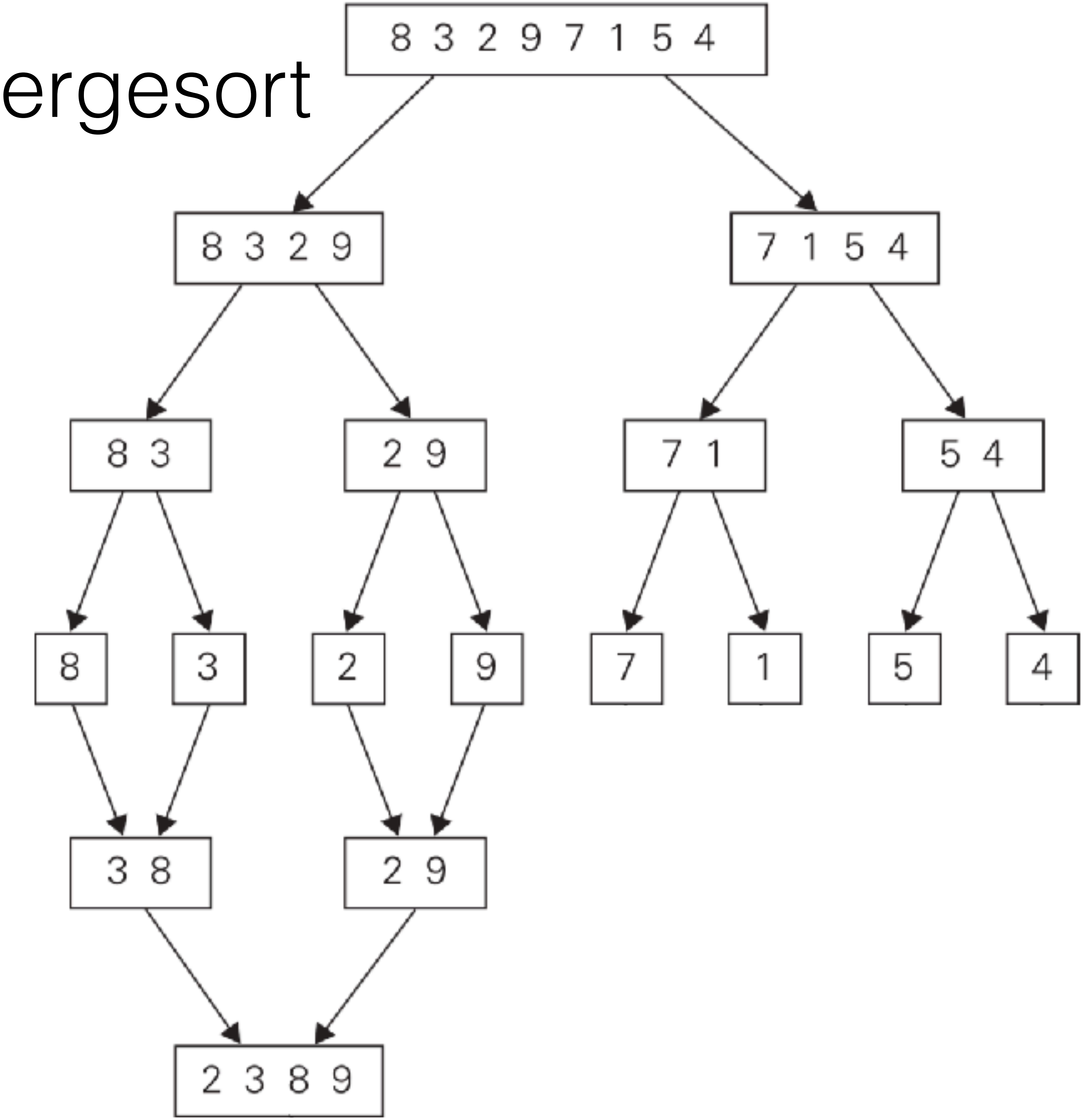
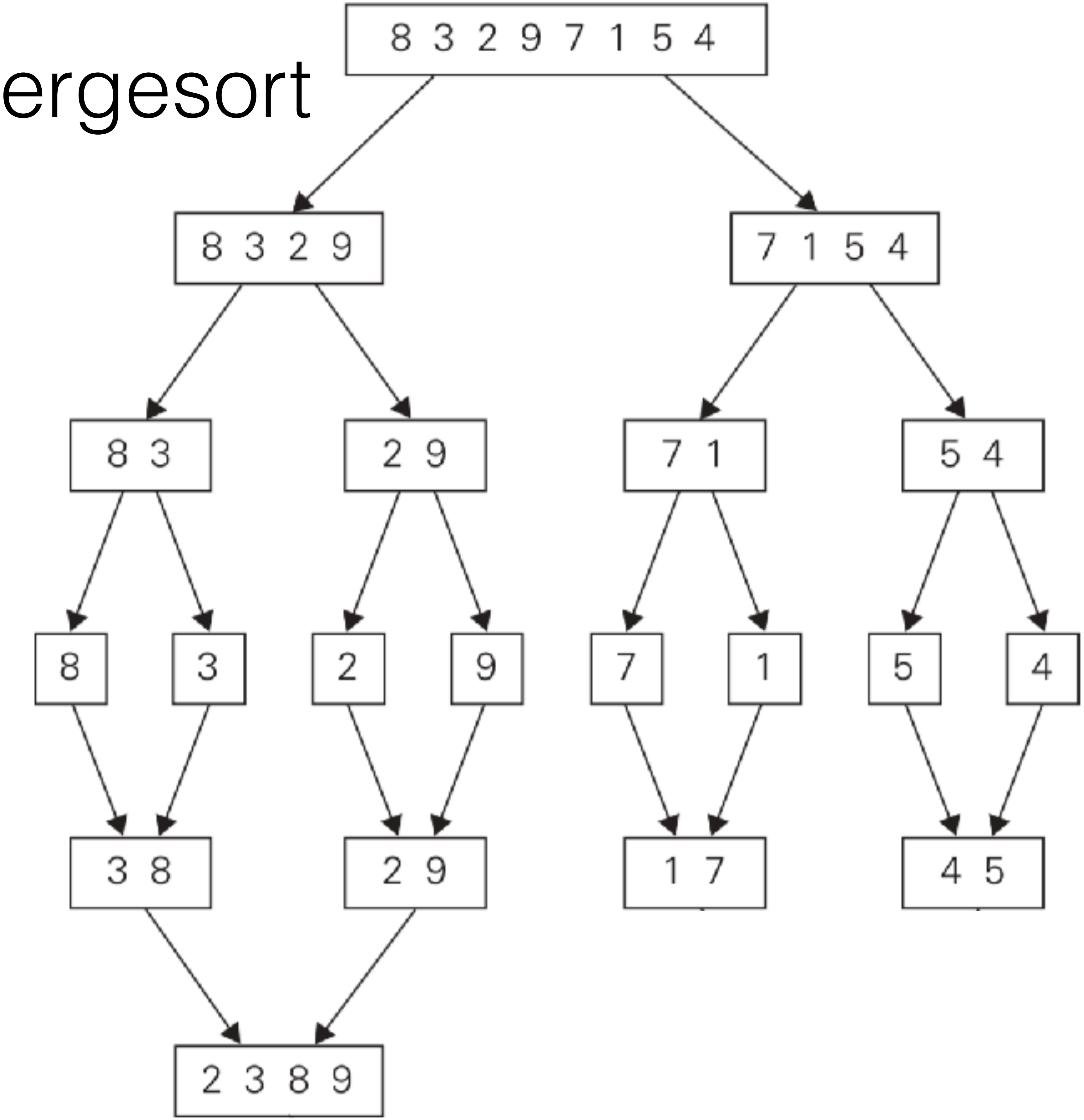
Mergesort



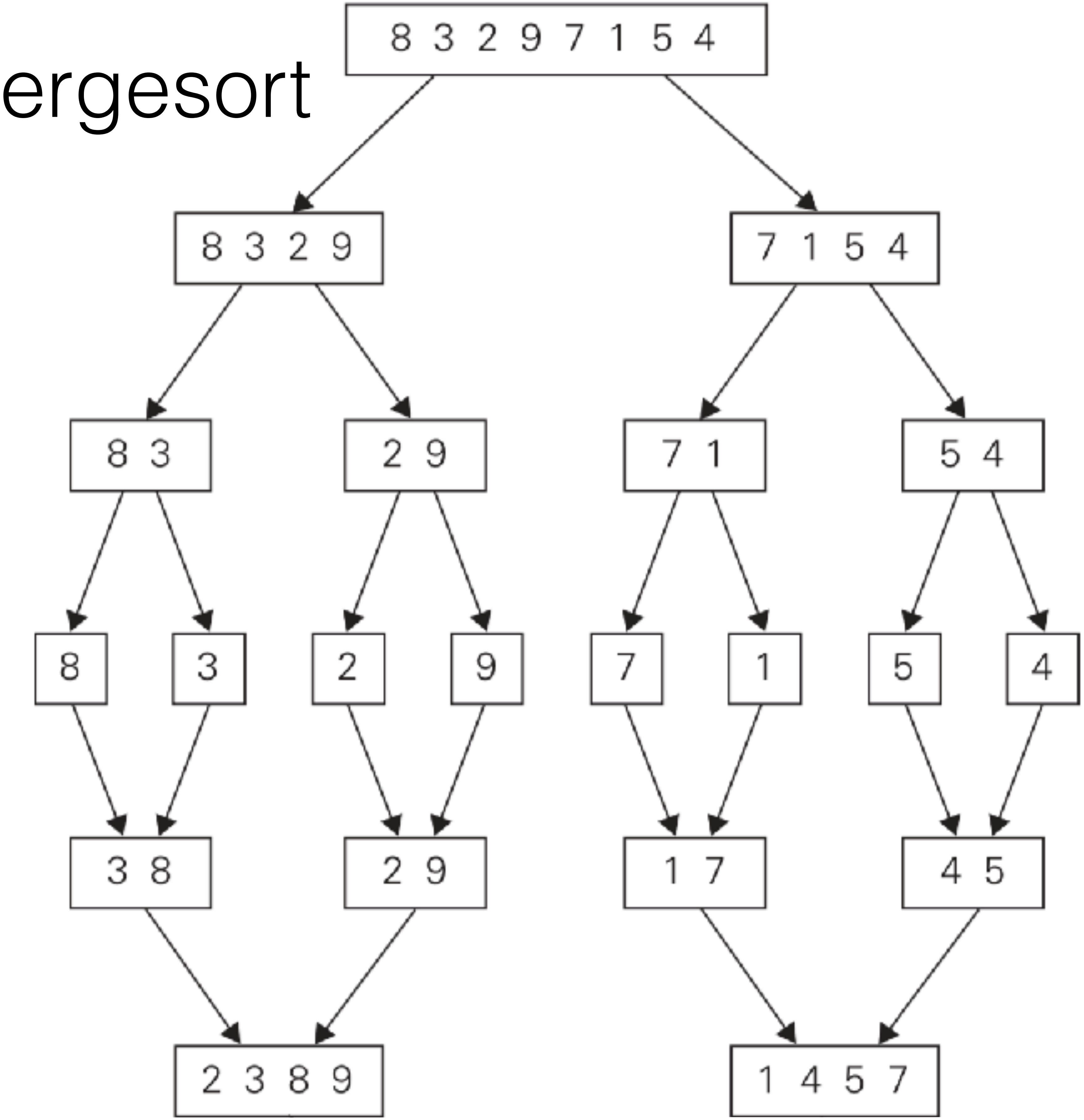
Mergesort



Mergesort



Mergesort



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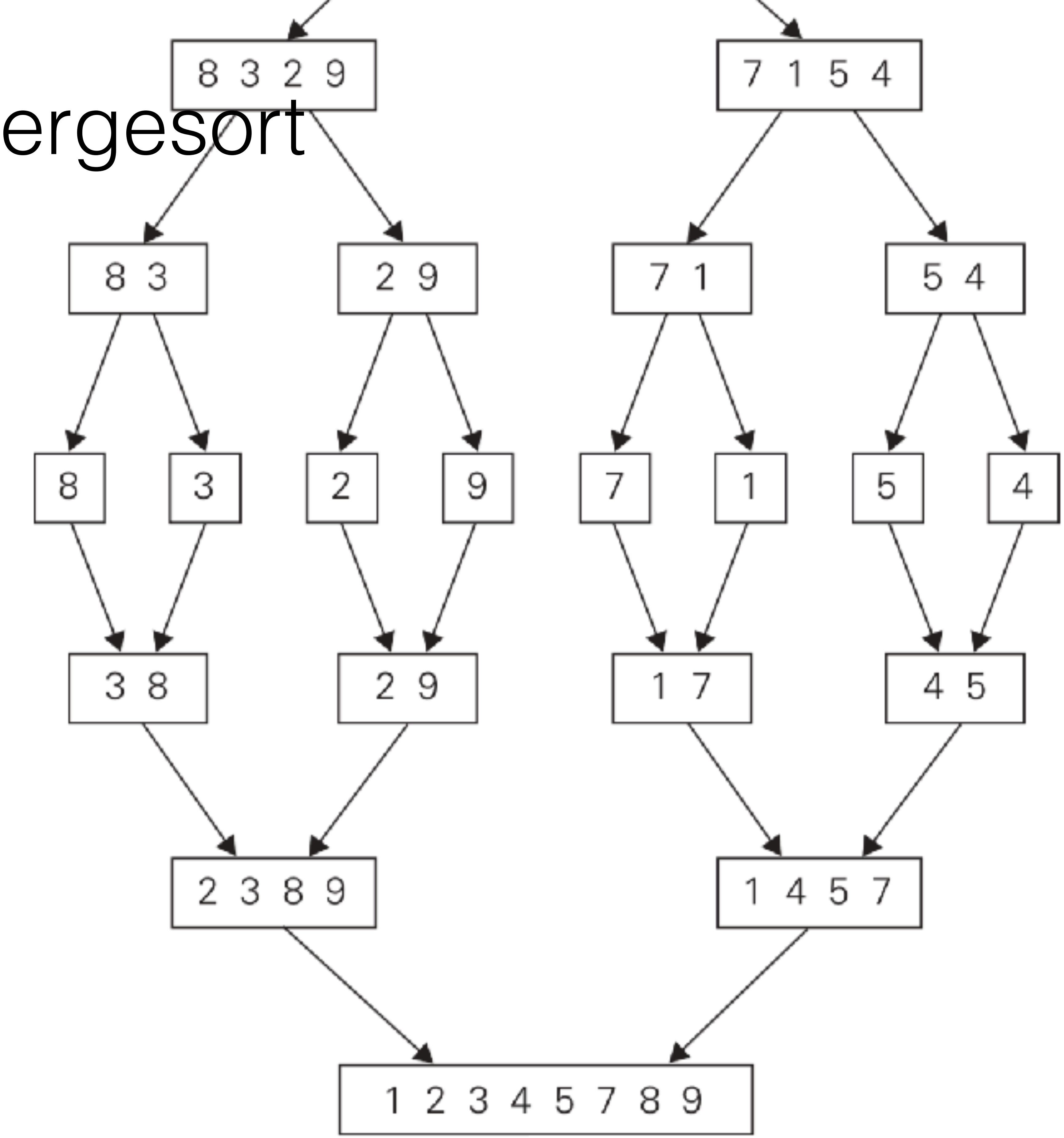
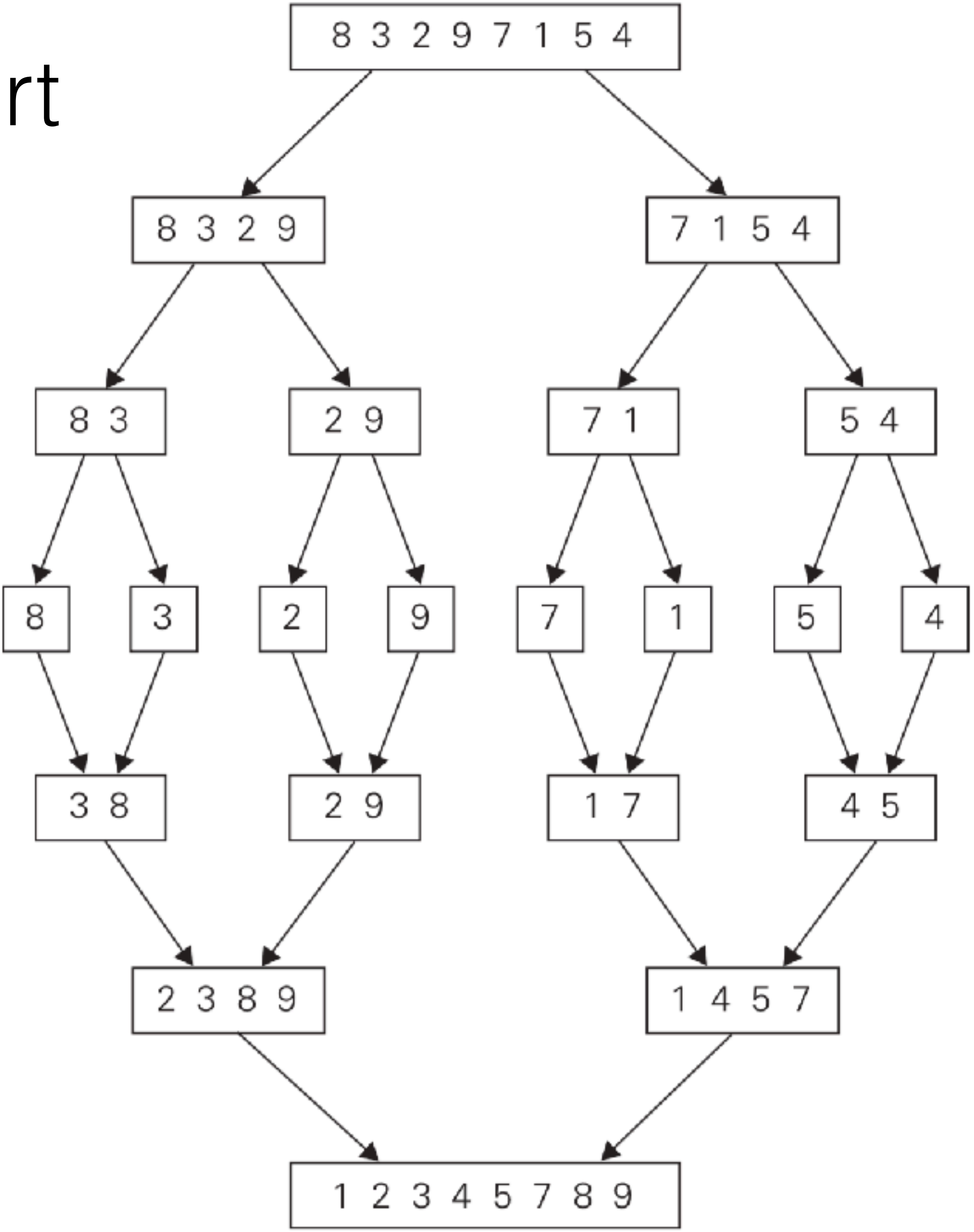
Mergesort



Mergesort



8 3 2 9 7 1 5 4

8 3 2 9   7 1 5 4

8 3   2 9   7 1   5 4

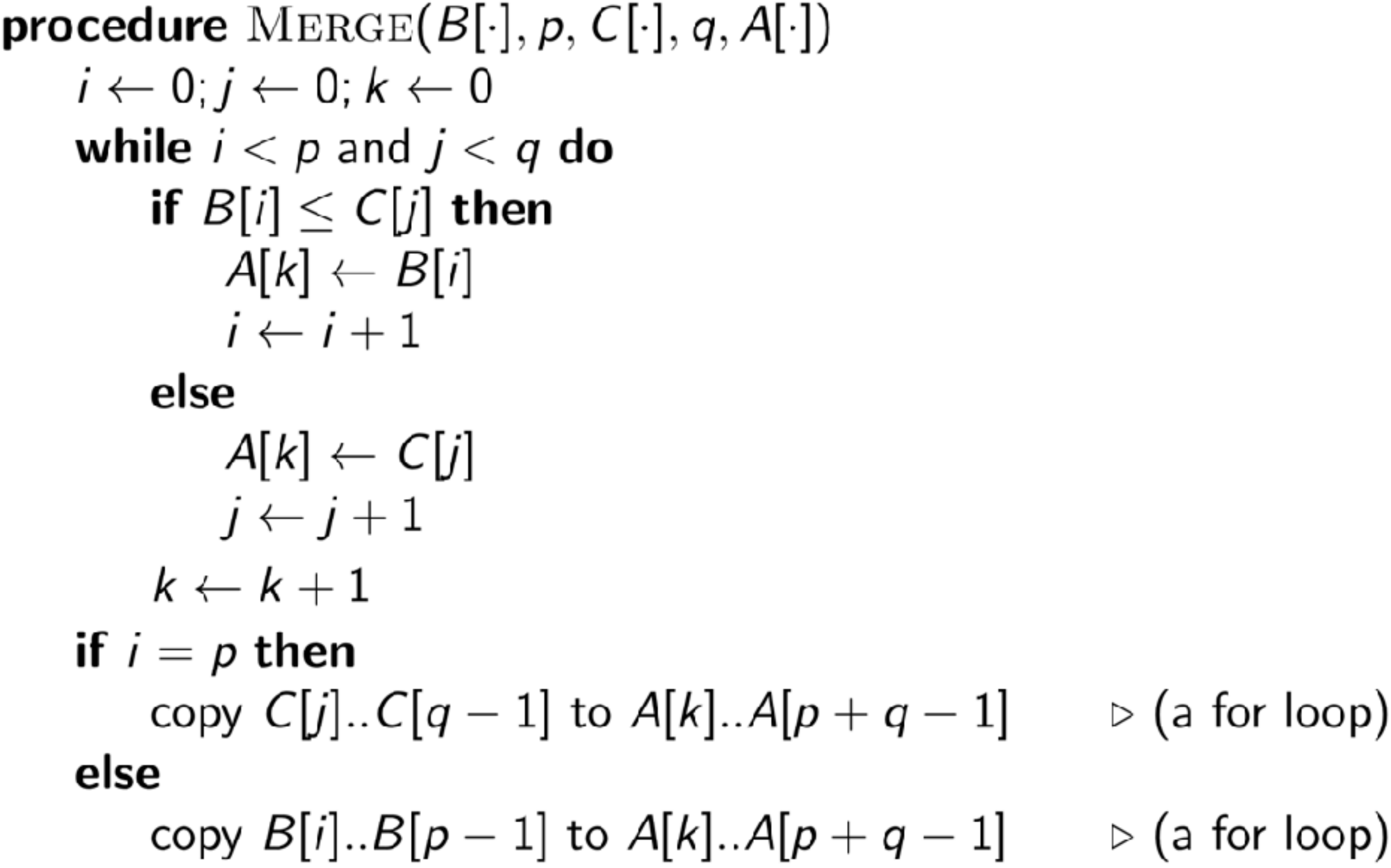
8   3   2   9   7   1   5   4

3 8   2 9   1 7   4 5

2 3 8 9   1 4 5 7

1 2 3 4 5 7 8 9

Mergesort: Merging Arrays



```
procedure MERGE($B[\cdot], p, C[\cdot], q, A[\cdot]$)
    $i \leftarrow 0; j \leftarrow 0; k \leftarrow 0$
    while $i < p$ and $j < q$ do
        if $B[i] \leq C[j]$ then
            $A[k] \leftarrow B[i]$
            $i \leftarrow i + 1$
        else
            $A[k] \leftarrow C[j]$
            $j \leftarrow j + 1$
        $k \leftarrow k + 1$
    if $i = p$ then
        copy $C[j]..C[q-1]$ to $A[k]..A[p+q-1]$        $\triangleright$ (a for loop)
    else
        copy $B[i]..B[p-1]$ to $A[k]..A[p+q-1]$        $\triangleright$ (a for loop)
```

Mergesort: Merging Arrays

B:

2	3	8	9
0	1	2	3
i			

C:

1	4	5	7
0	1	2	3
j			

p: 4

q: 4

A:

0	1	2	3	4	5	6	7
k							

Mergesort: Merging Arrays

B:	2	3	8	9
	0	1	2	3
	i			

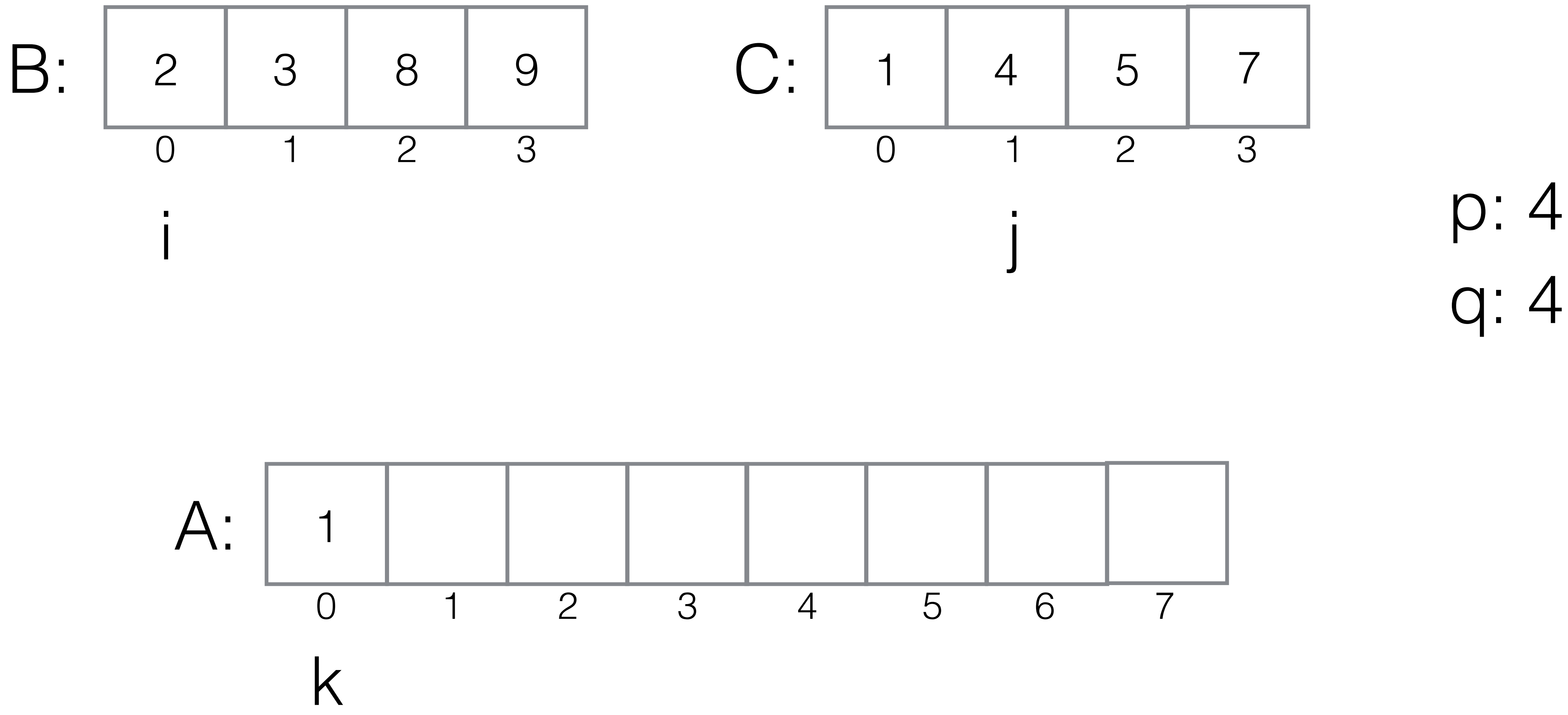
C:	1	4	5	7
	0	1	2	3
	j			

p: 4

q: 4

A:	1							
	0	1	2	3	4	5	6	7
	k							

Mergesort: Merging Arrays



B:

0	1	2	3
2	3	8	9
i			

C:

0	1	2	3
1	4	5	7
	j		

p: 4

q: 4

A:

0	1	2	3	4	5	6	7
1							
k							

Mergesort: Merging Arrays

B:	2	3	8	9
	0	1	2	3
	i			

C:	1	4	5	7
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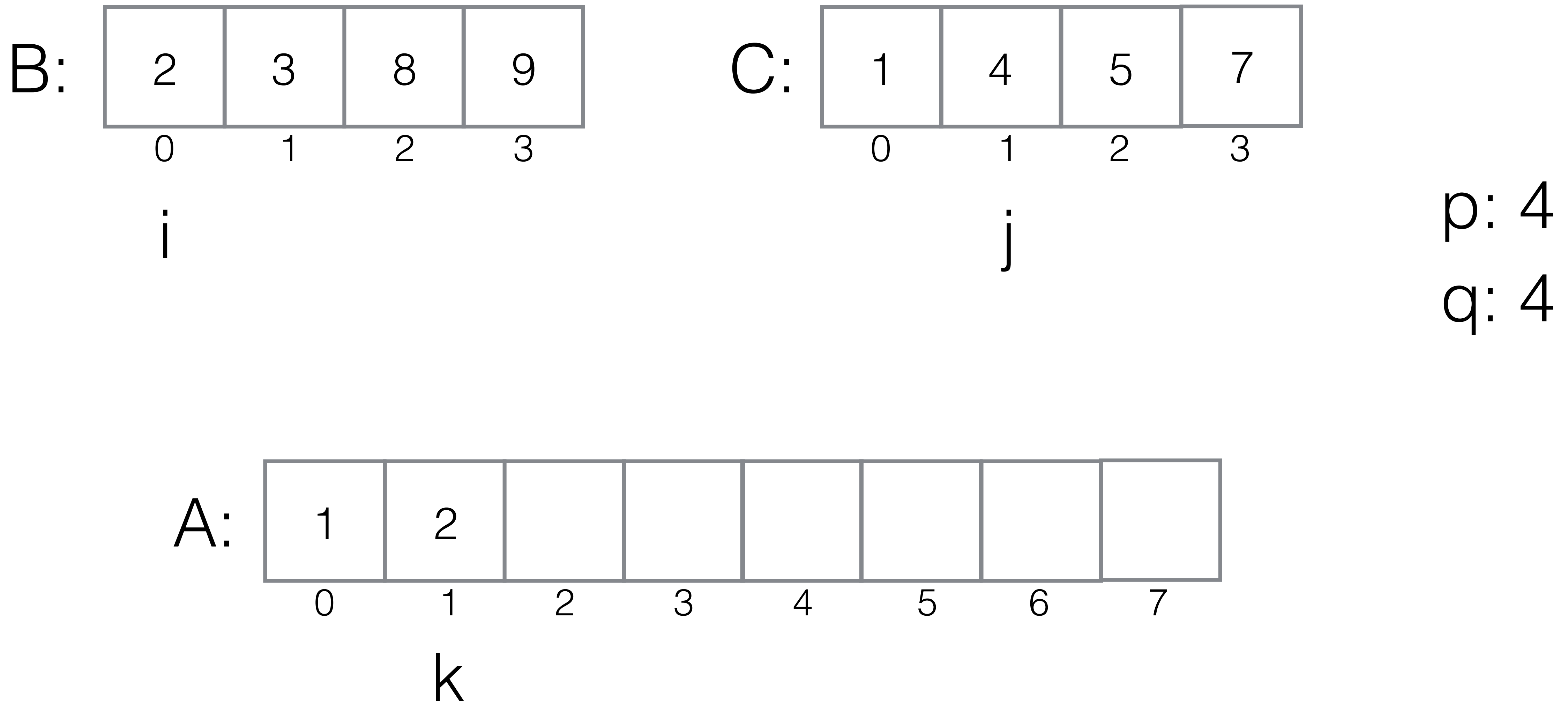
		j		

p: 4

q: 4

A:	1							
	0	1	2	3	4	5	6	7
		k						

Mergesort: Merging Arrays



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1	4	5	7
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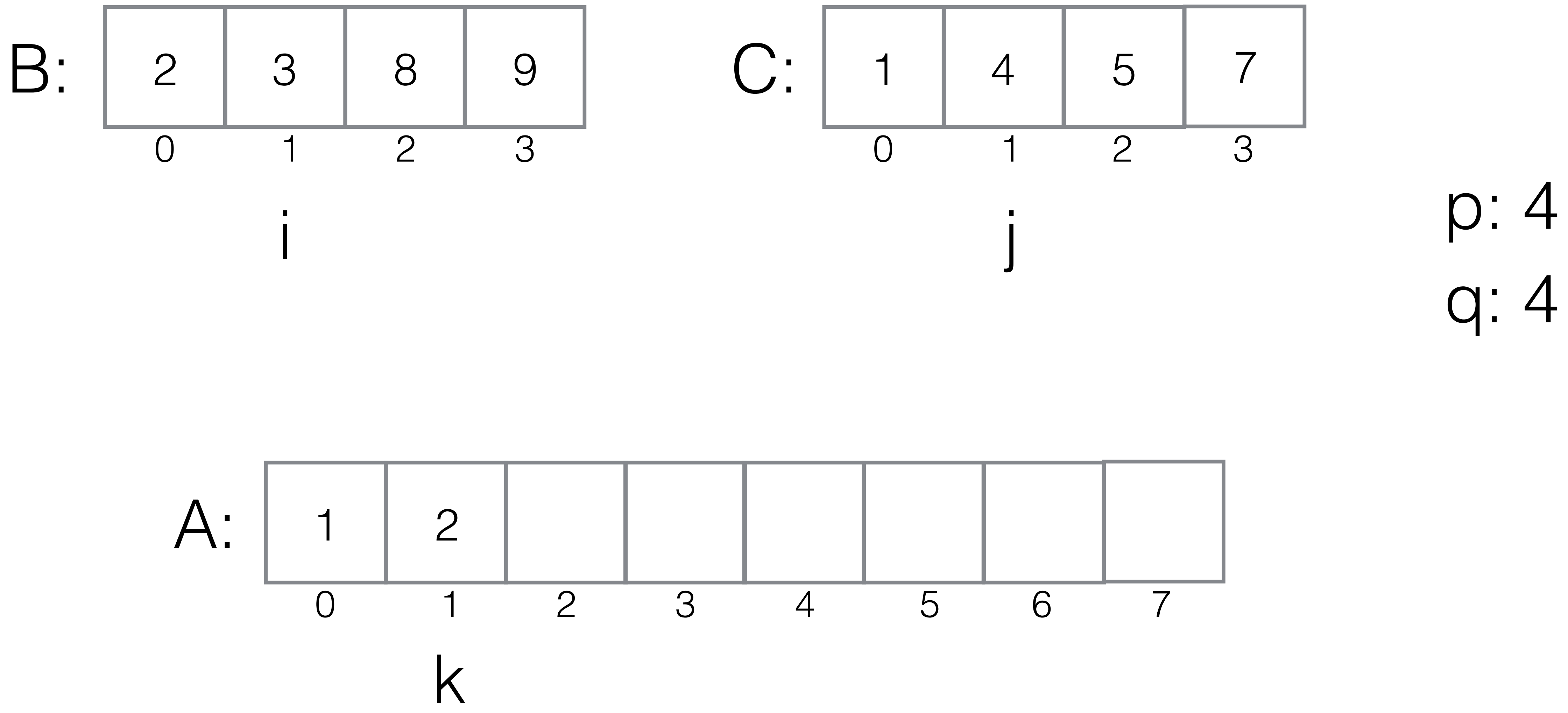
Mergesort: Merging Arrays



B:	2	3	8	9
	0	1	2	3
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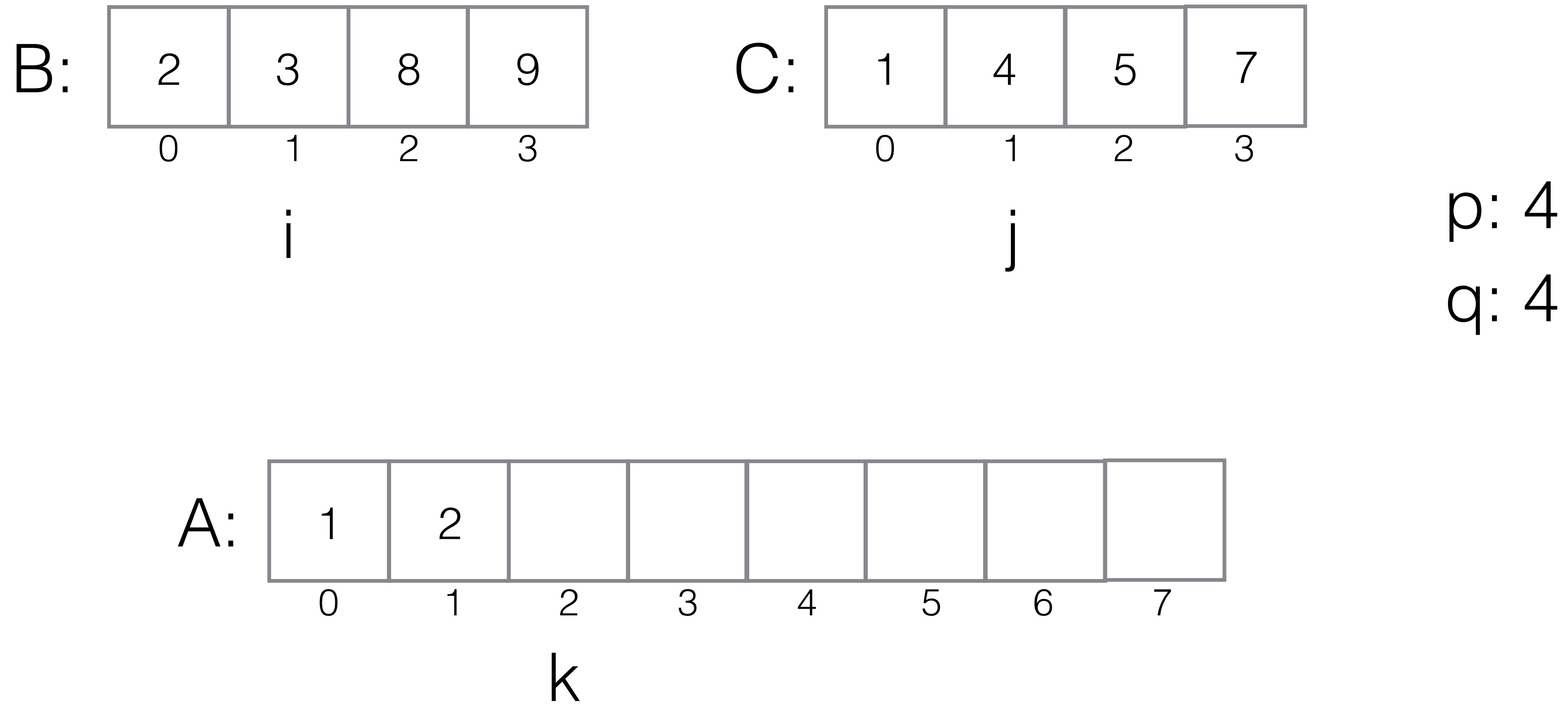
Mergesort: Merging Arrays



B:				
	2	3	8	9
	0	1	2	3
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		j		

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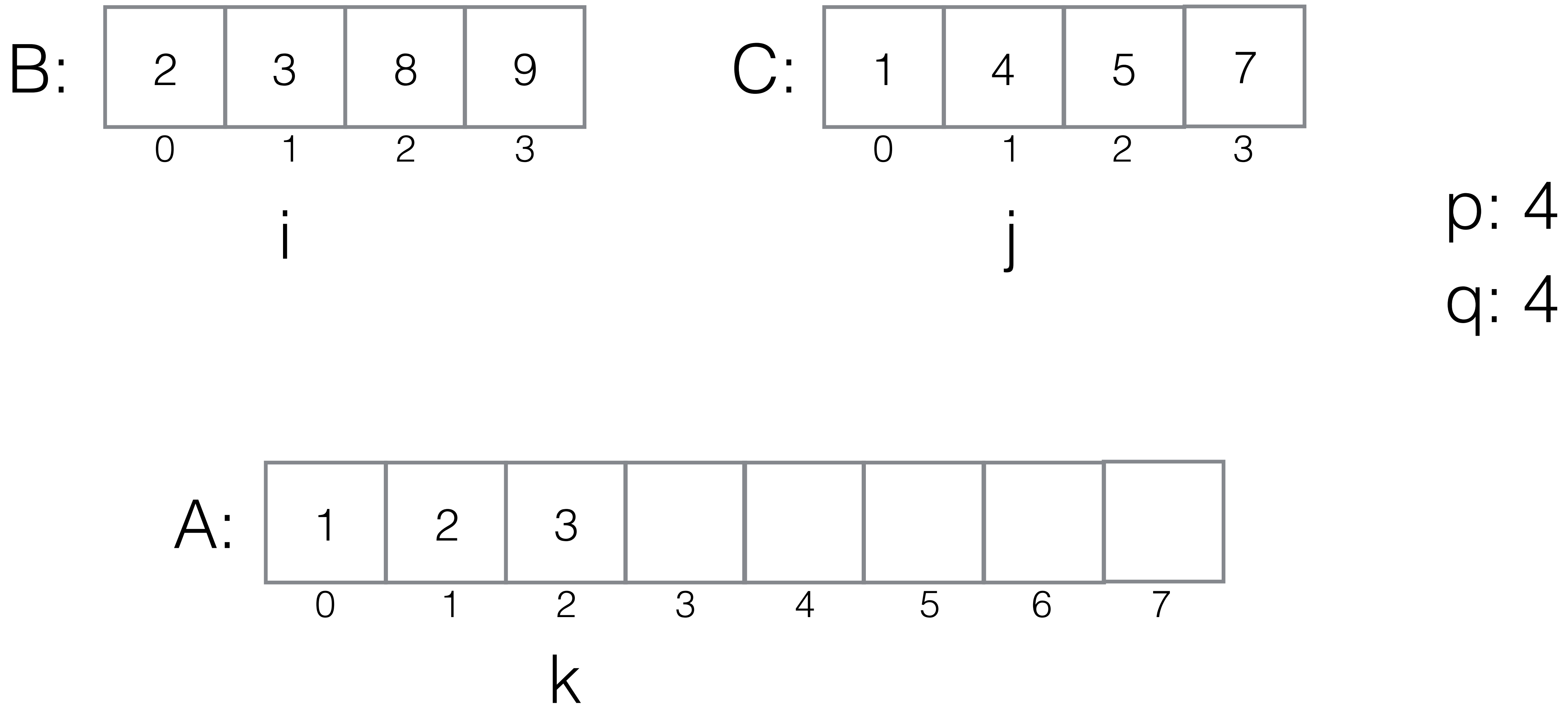
Mergesort: Merging Arrays



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	0	1	2	3
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C:	1	4	5	7
	0	1	2	3
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q: 4

A:	1	2	3					
	0	1	2	3	4	5	6	7
			k					

Mergesort: Merging Arrays

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B:				
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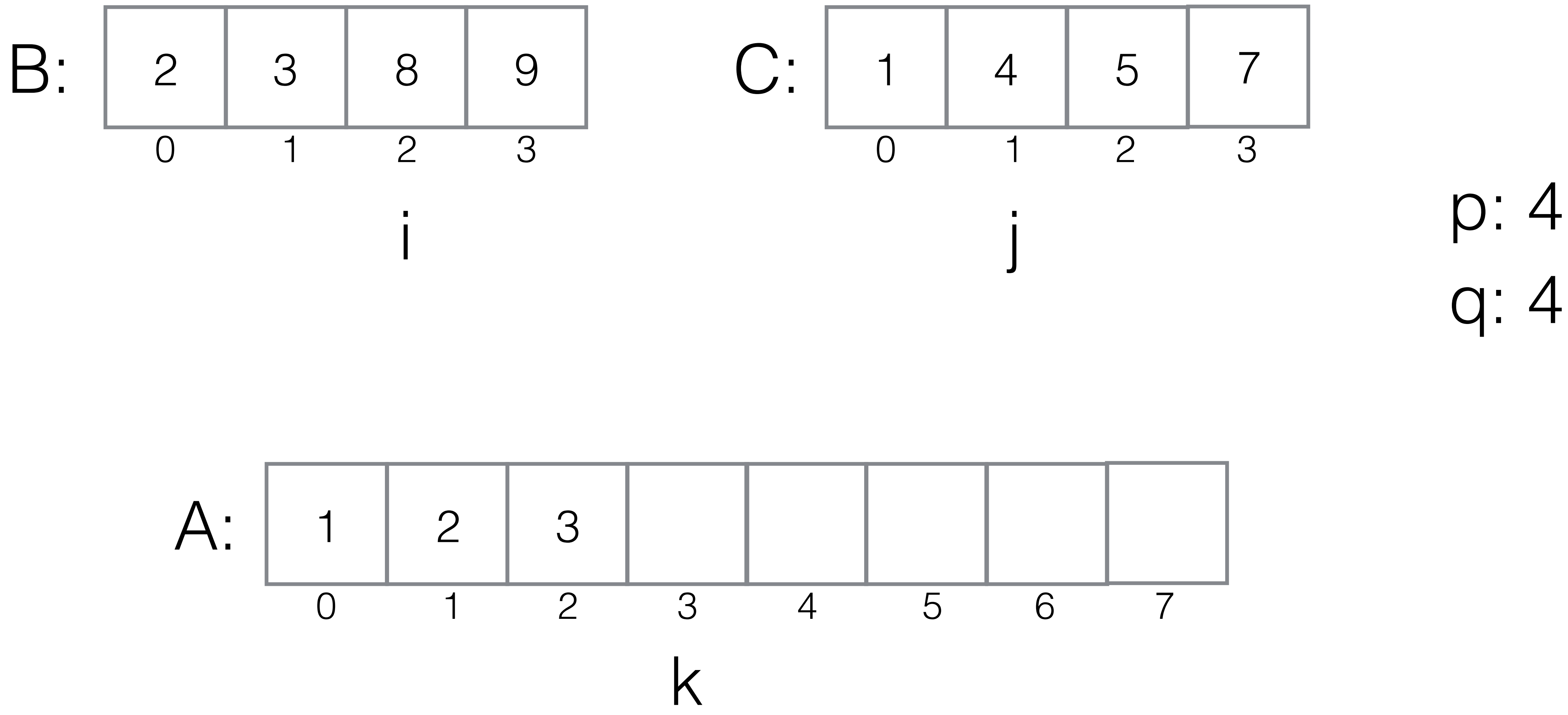
Mergesort: Merging Arrays



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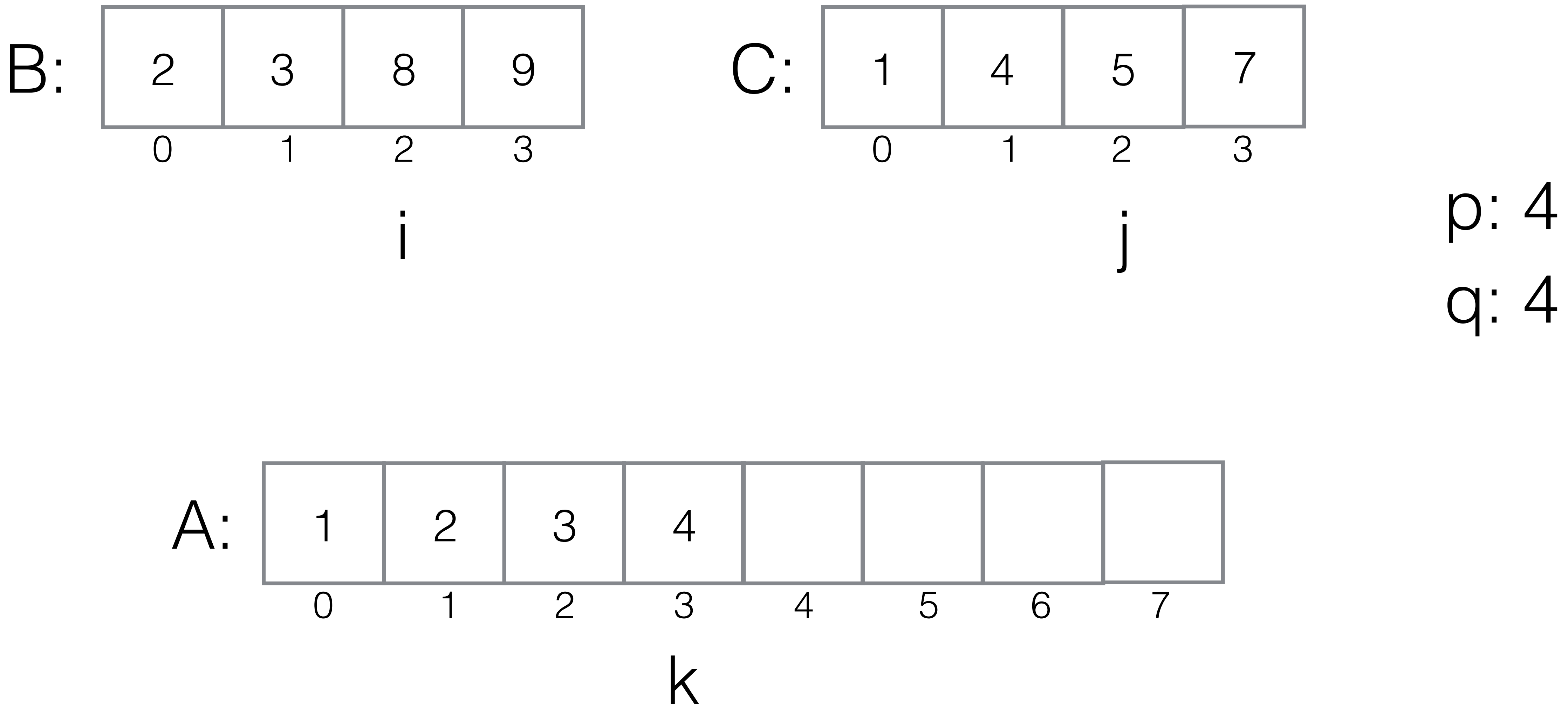

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0	1	2	3	4	5	6	7
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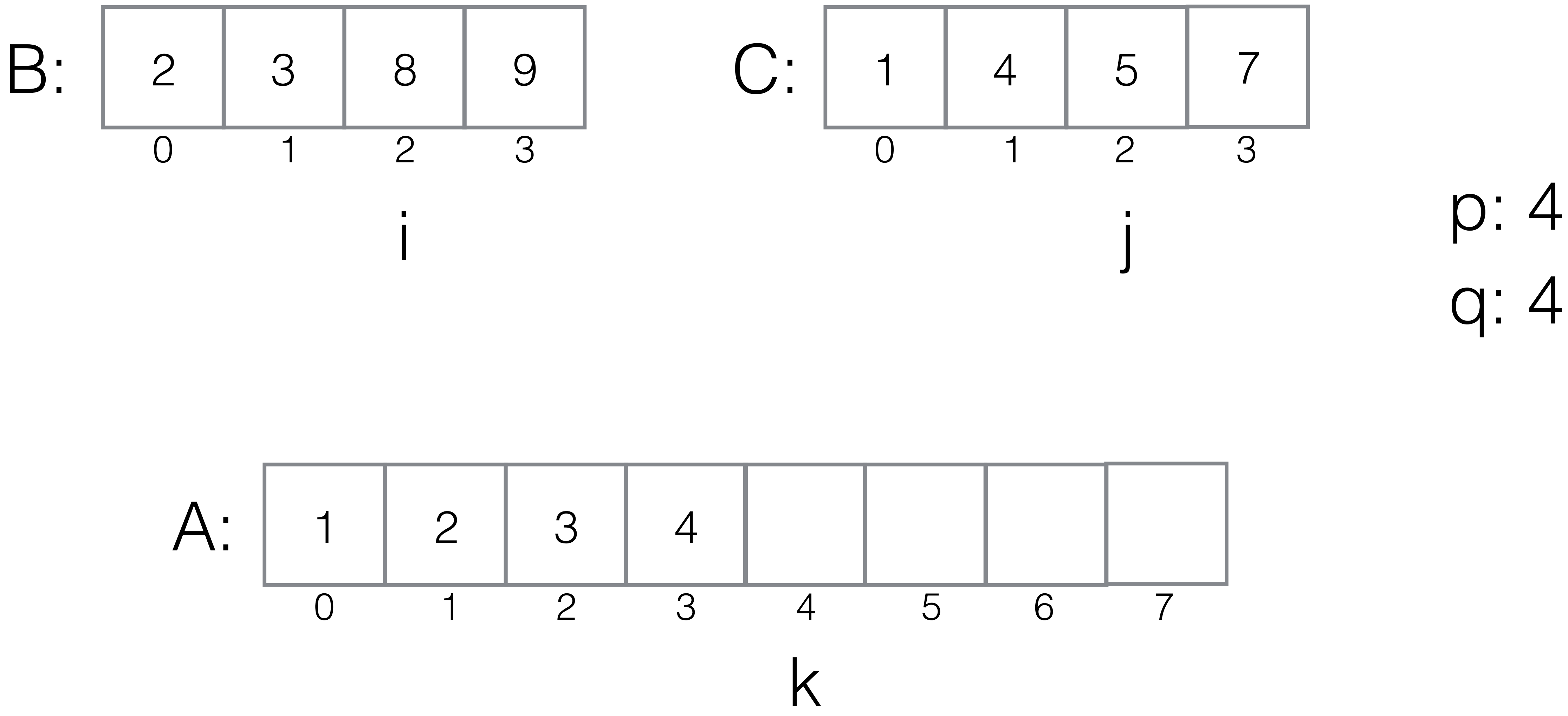
Mergesort: Merging Arrays



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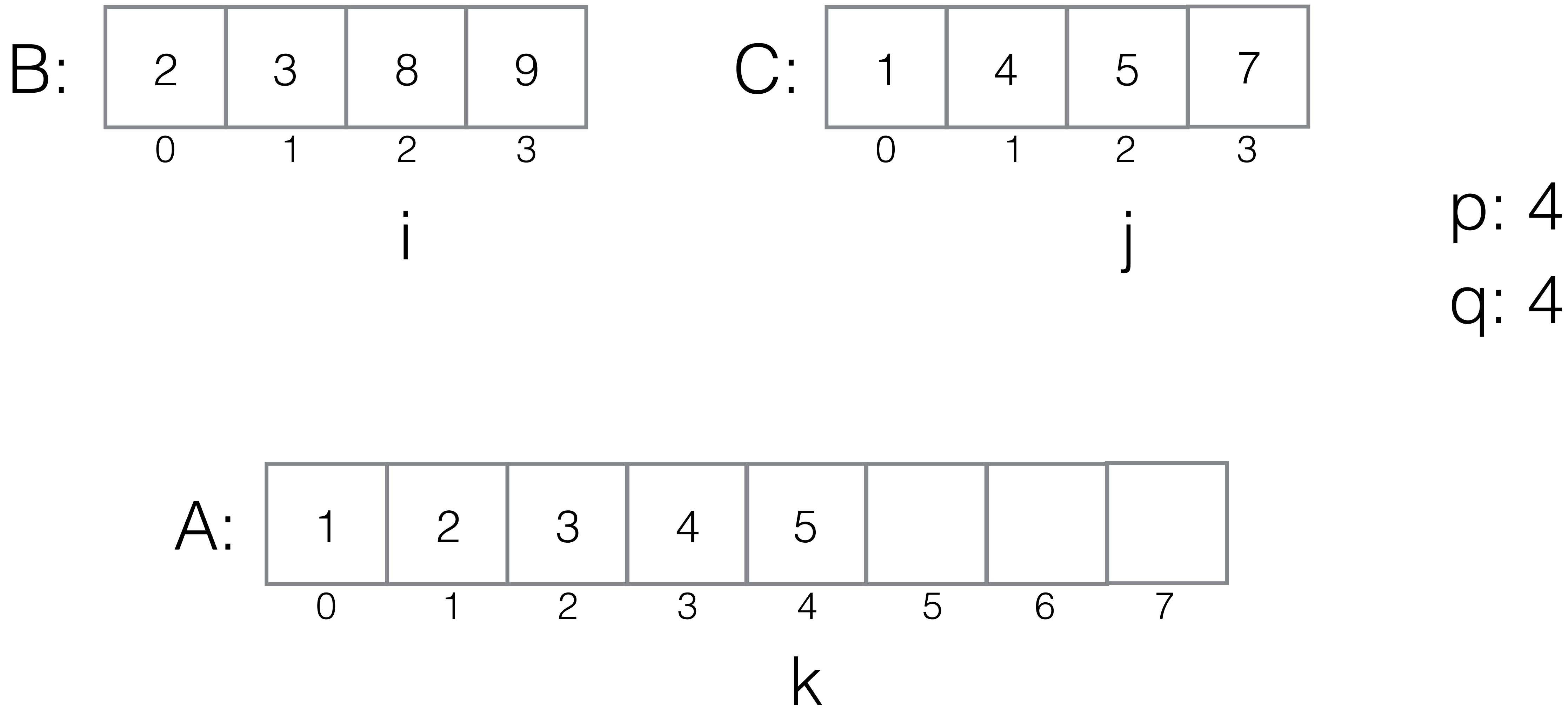

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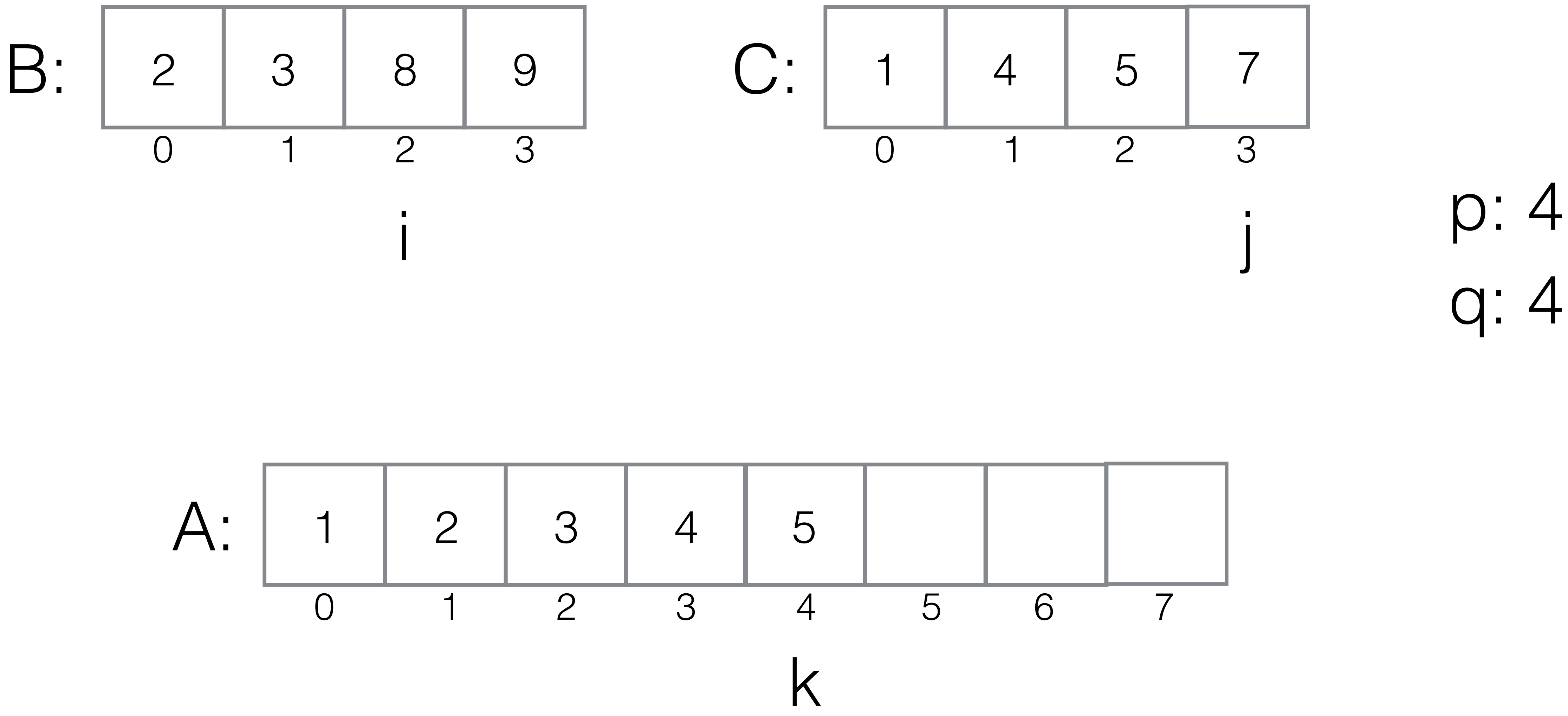

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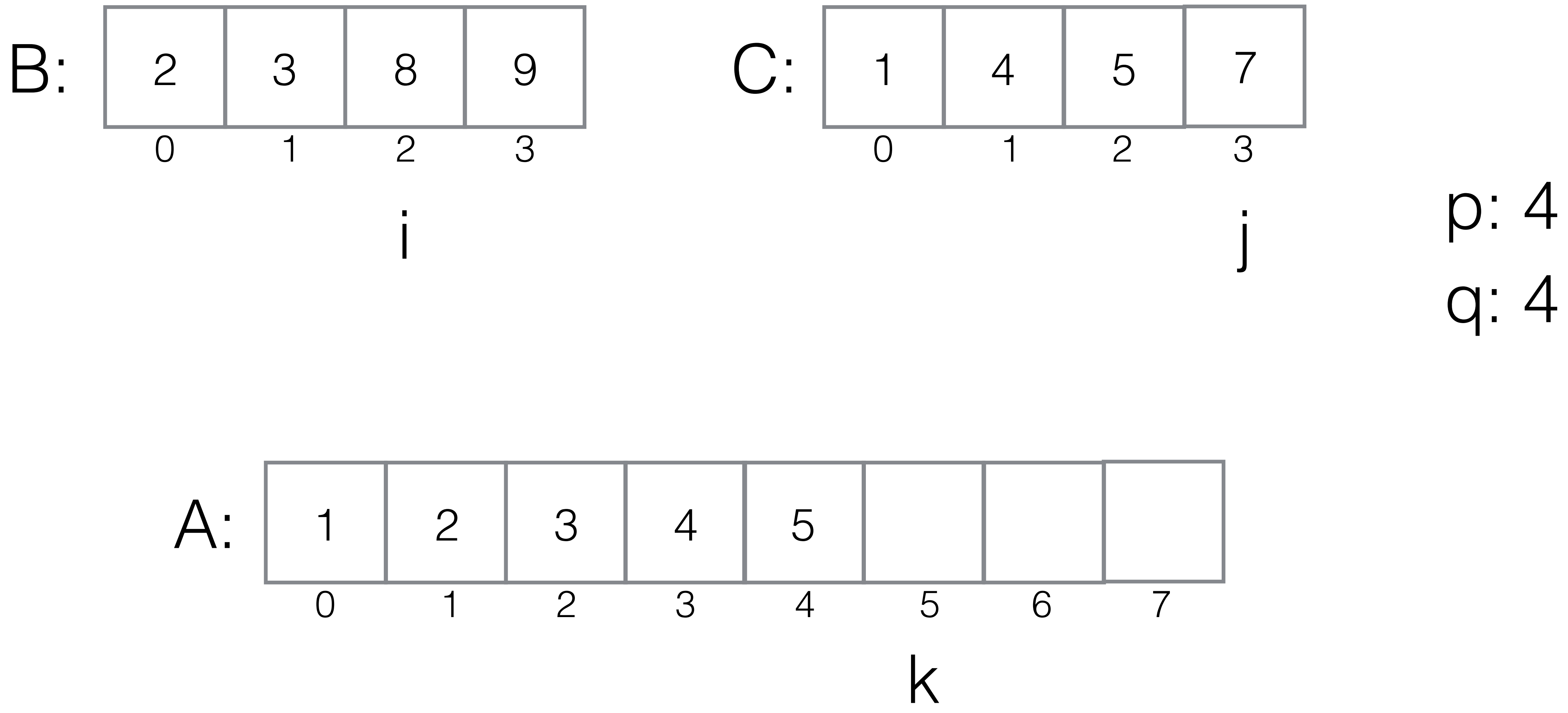

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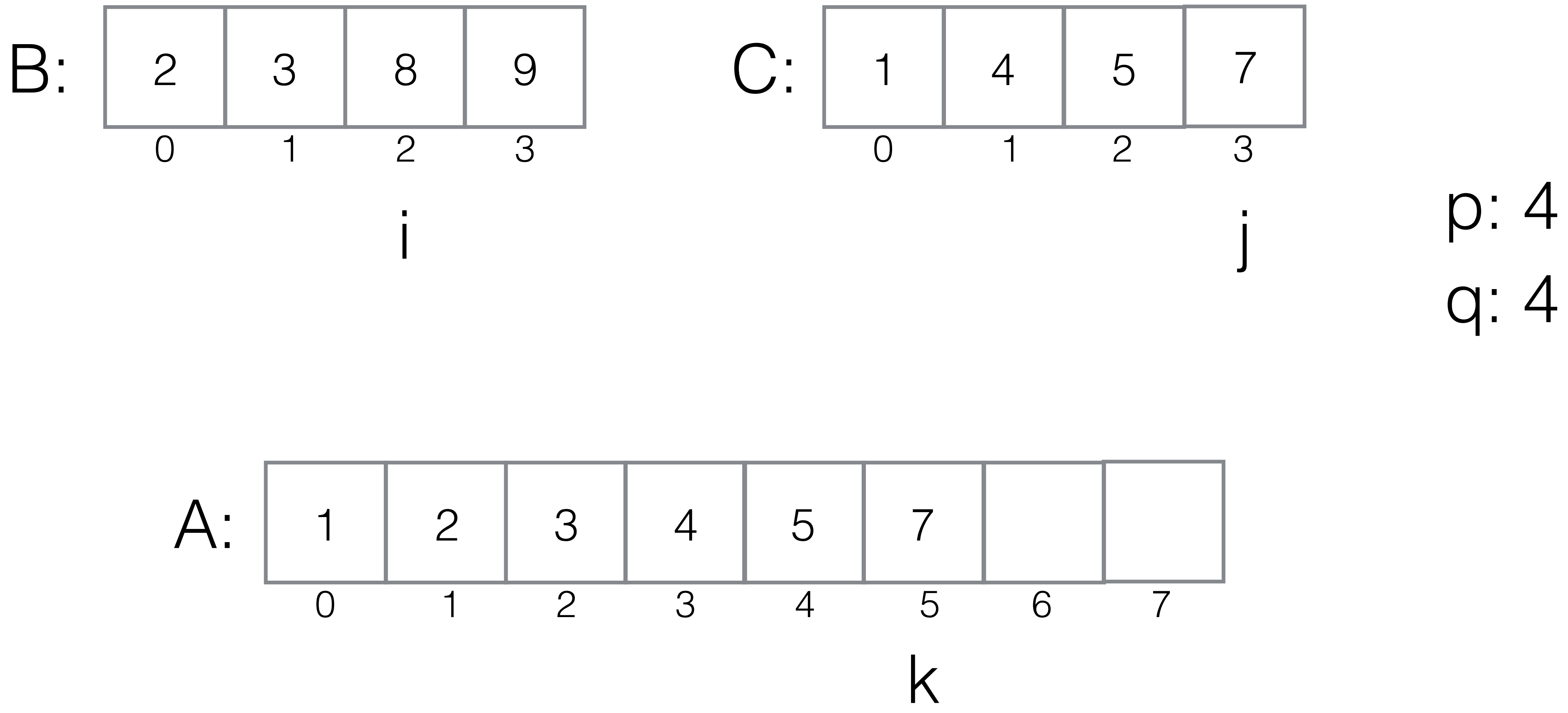

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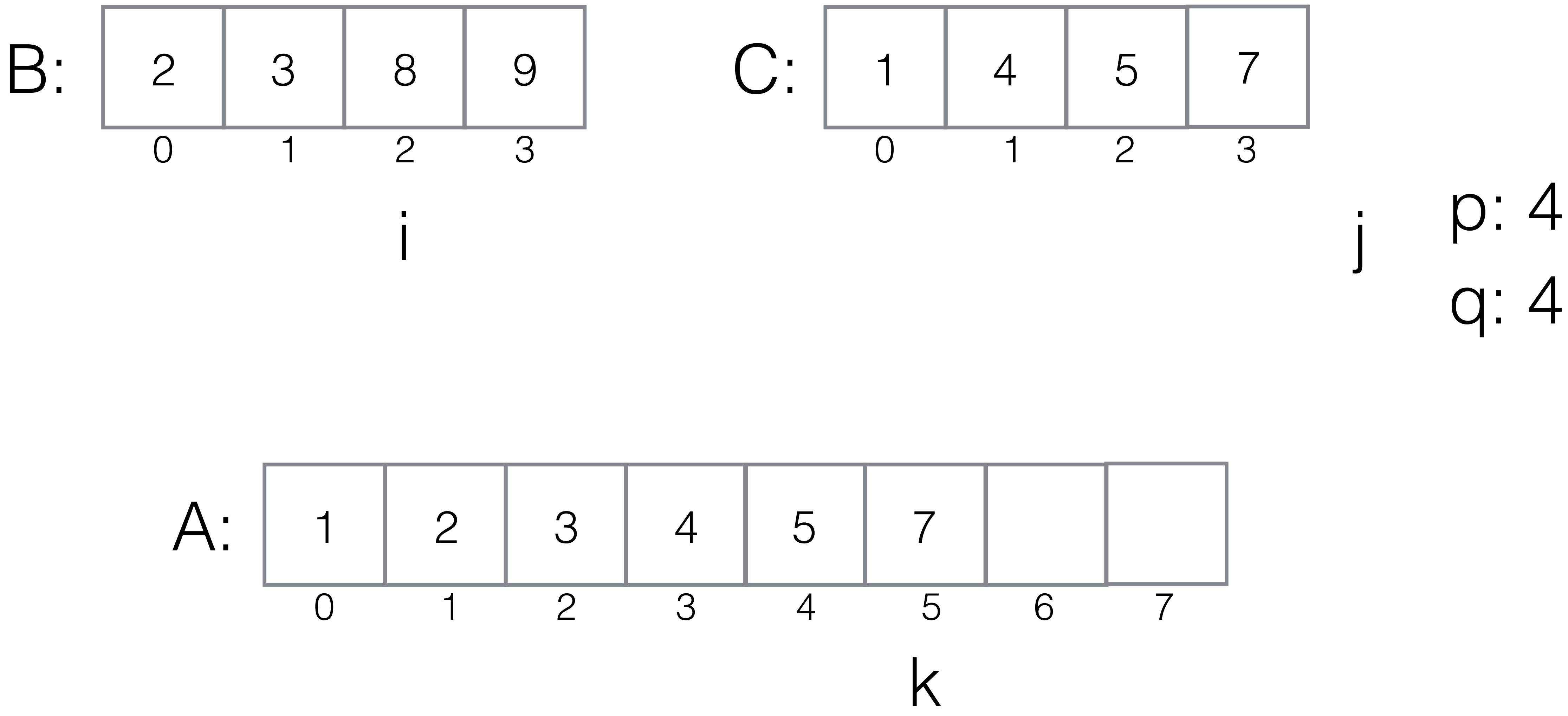
Mergesort: Merging Arrays



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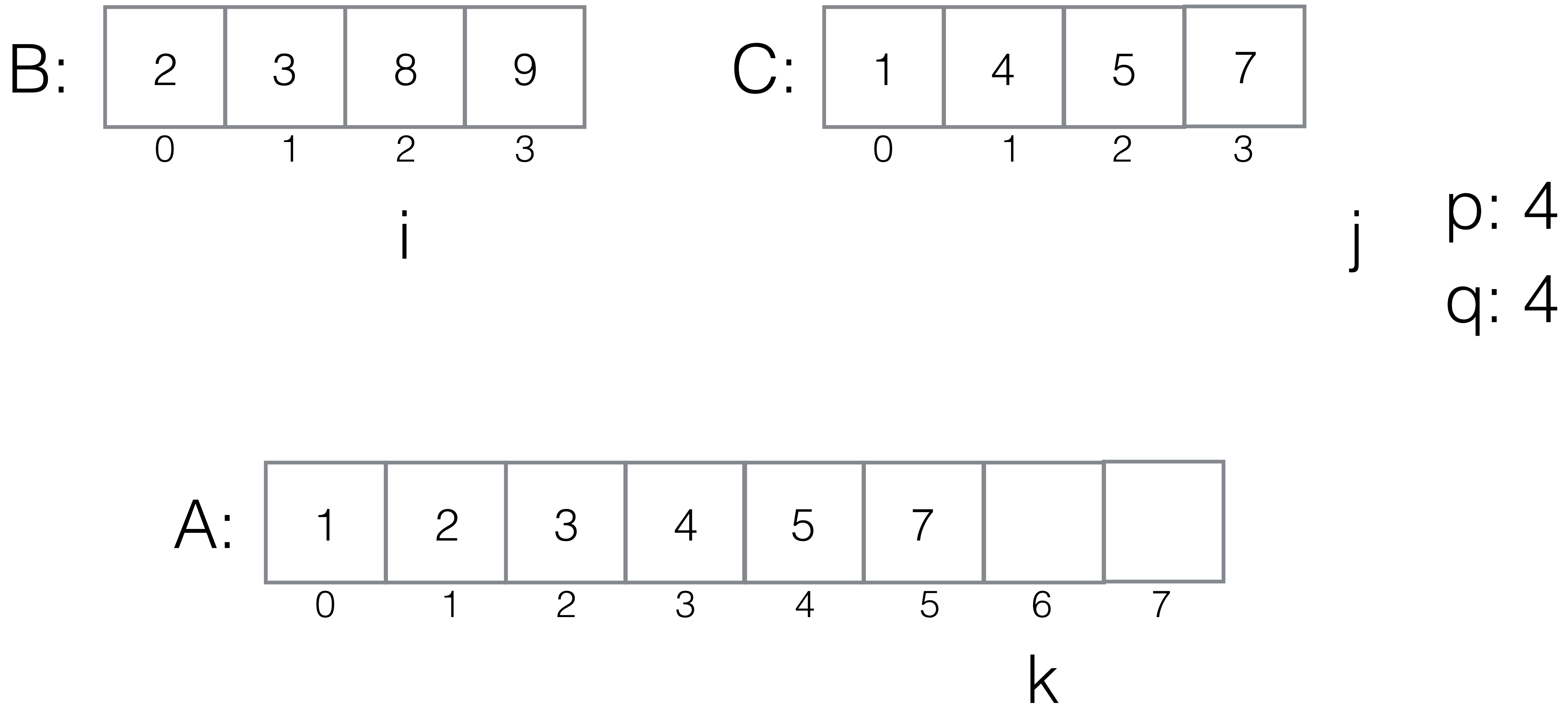
Mergesort: Merging Arrays



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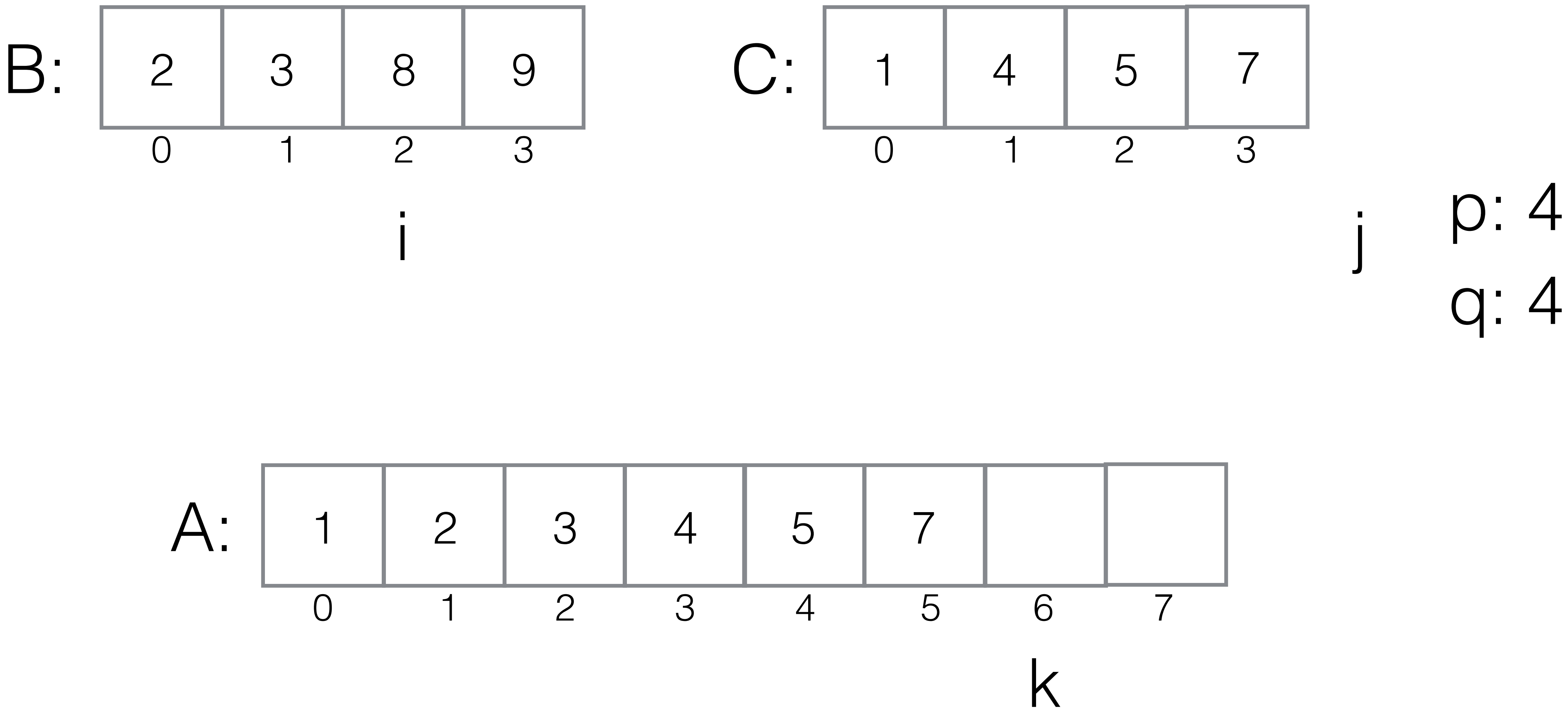
```

Mergesort: Merging Arrays



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1	4	5	7
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j

p: 4

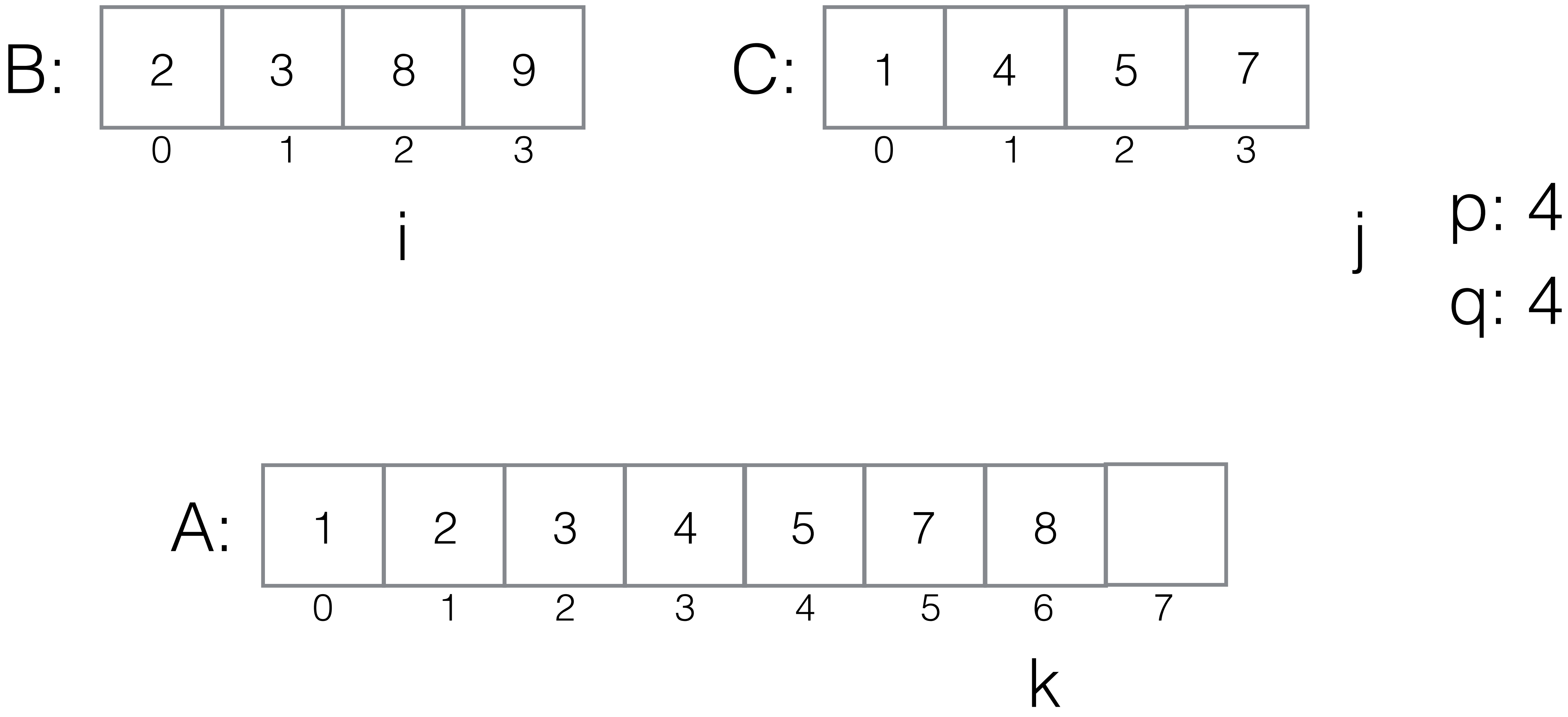
q: 4

A:

1	2	3	4	5	7		
0	1	2	3	4	5	6	7

k

Mergesort: Merging Arrays



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1	4	5	7
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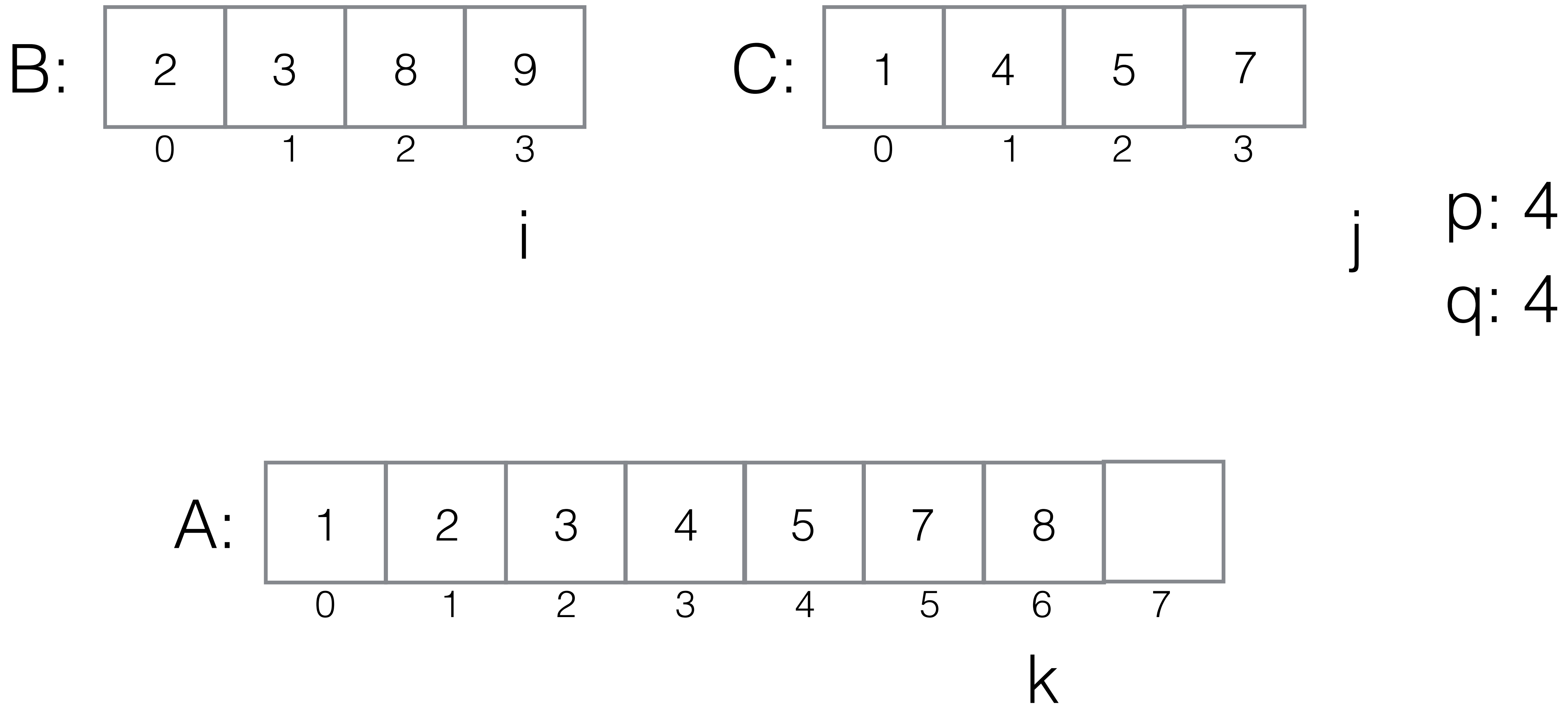
k

Mergesort: Merging Arrays



B:

2	3	8	9
0	1	2	3
			i

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1	4	5	7	
0	1	2	3	
				j

p: 4

q: 4

A:

1	2	3	4	5	7	8	
0	1	2	3	4	5	6	7
						k	

Mergesort: Merging Arrays

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	0	1	2	3
				i

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	0	1	2	3	
					j

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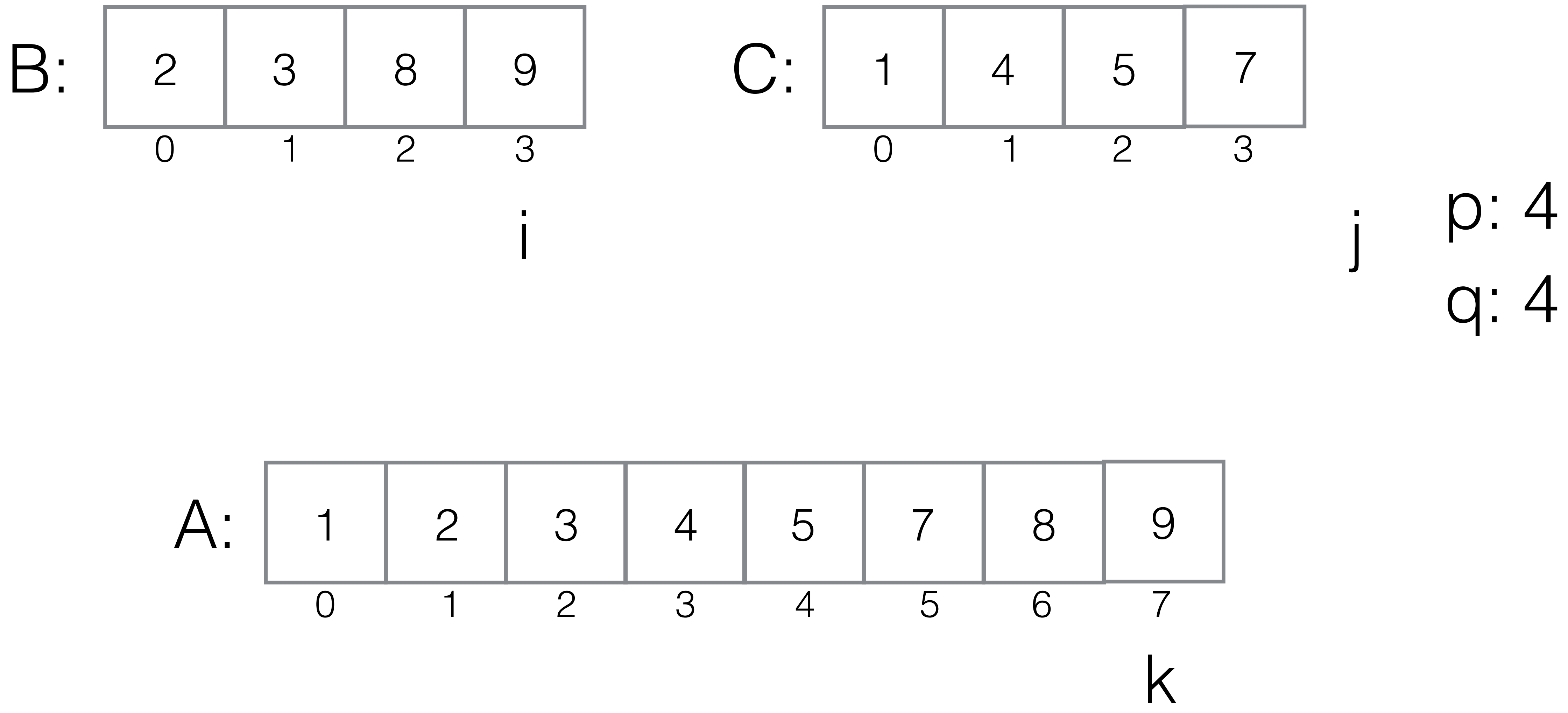
	0	1	2	3	4	5	6	7
								k

Mergesort: Merging Arrays



B:	2	3	8	9
	0	1	2	3

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	0	1	2	3

i

j

p: 4

q: 4

A:	1	2	3	4	5	7	8	9
	0	1	2	3	4	5	6	7

k

Mergesort: Analysis

- How many comparisons will MERGE need to make in the worst case, when given arrays of size ⌊n/2⌋ and ⌈n/2⌉ ?

C(n) = 2C(n/2) + C_merge(n)

C_worst(n) = 2C(n/2) + n-1

- If the largest and second-largest elements are in different arrays, then n – 1 comparisons. Hence the cost equation for Mergesort is

$$C(n) = \begin{cases} 0 & \text{if } n < 2 \\ 2C(n/2) + n - 1 & \text{otherwise} \end{cases}$$

- By the Master Theorem, C(n) ∈ Θ(n log n).

Mergesort: Properties

- For large n, the number of comparisons made tends to be around 75% of the worst-case scenario.

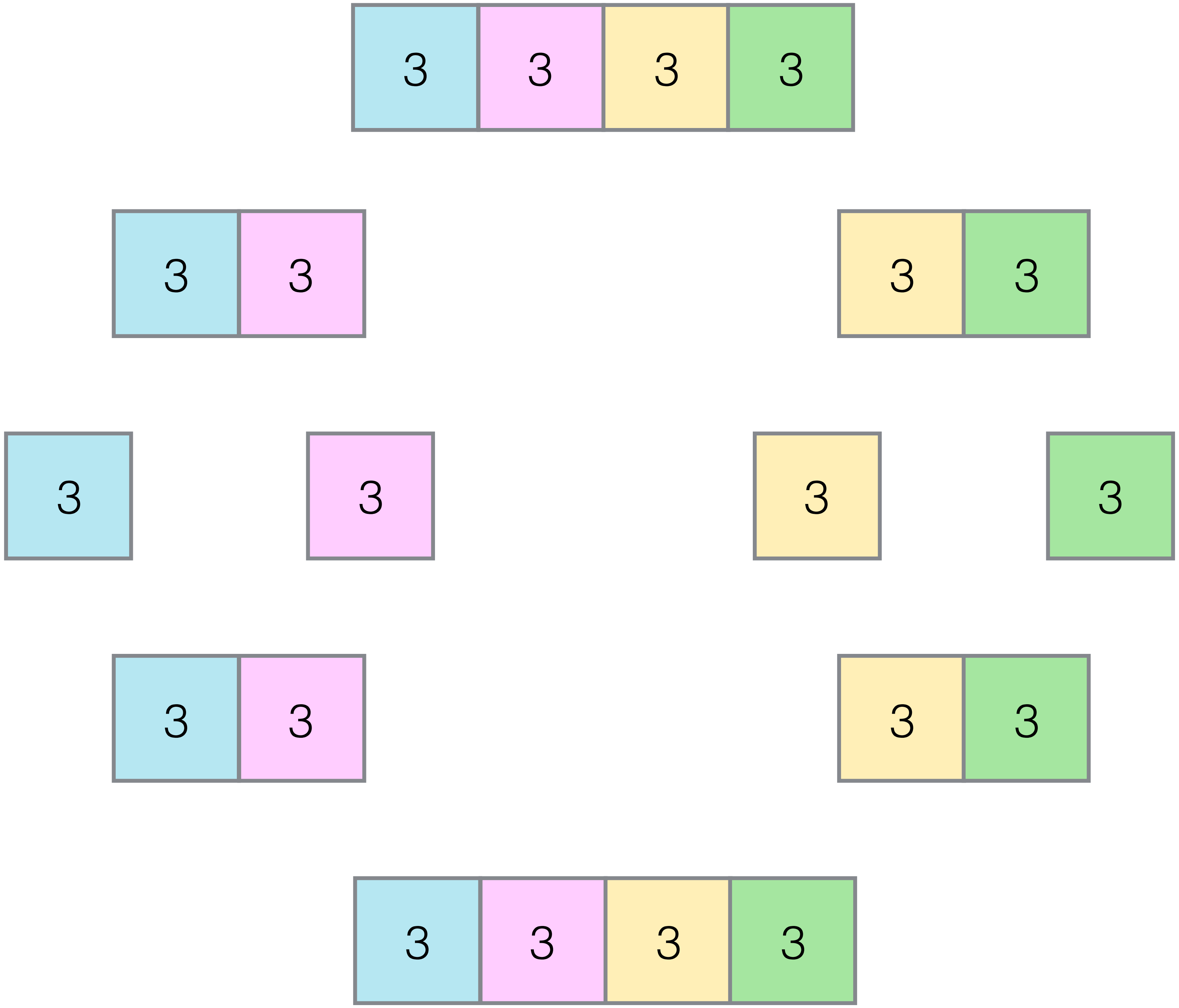
- Is mergesort stable? ?

- Is mergesort in-place? no

- If comparisons are fast, mergesort ranks between quicksort and heapsort (covered next week) for time, assuming random data.

- Mergesort is the method of choice for linked lists and for very large collections of data.

Mergesort: Stability



Mergesort: Properties

- For large n, the number of comparisons made tends to be around 75% of the worst-case scenario.

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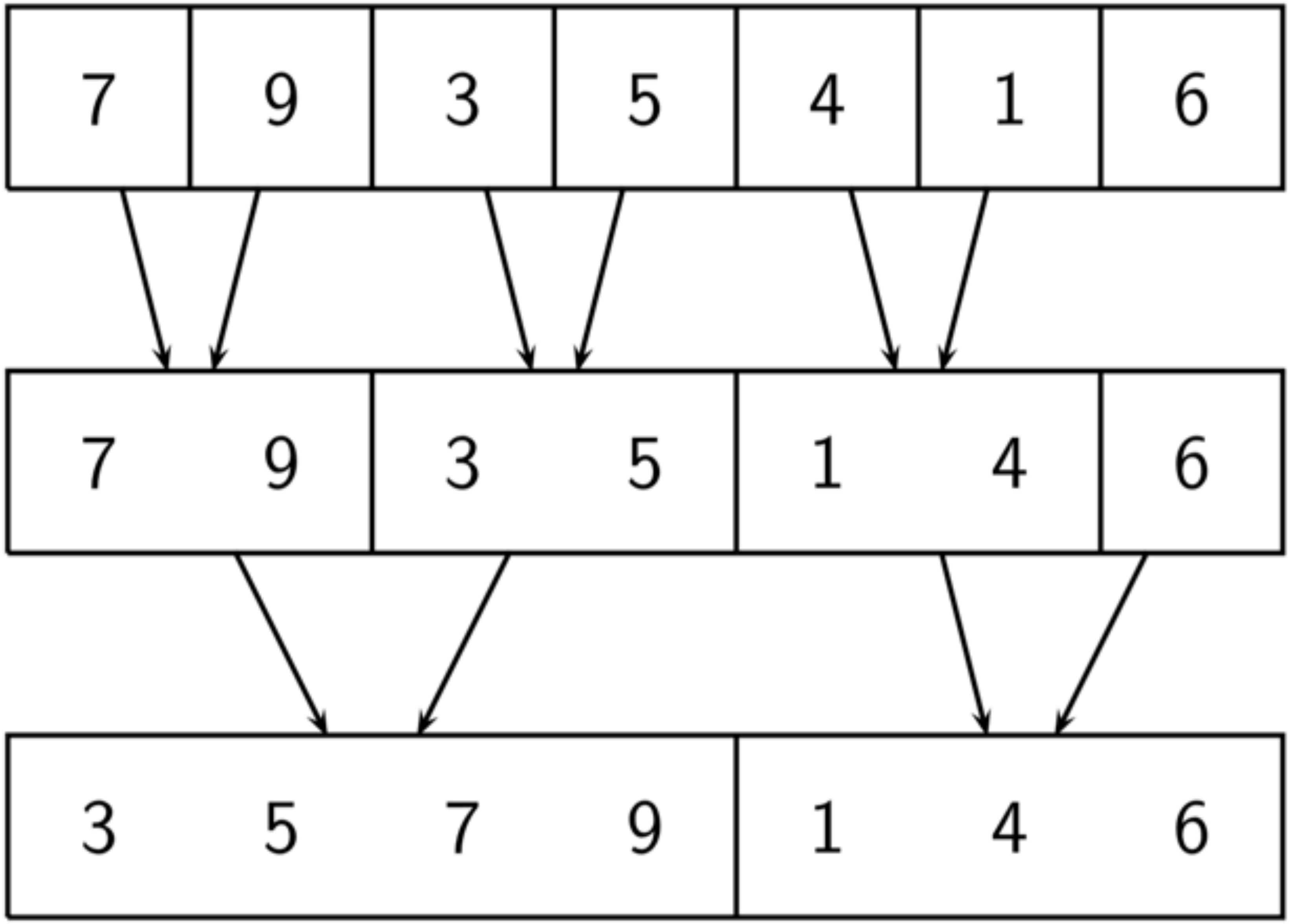
- Mergesort is the method of choice for linked lists and for very large collections of data.

Bottom-Up Mergesort

- An alternative way of doing mergesort:
- Generate **runs** of length 2, then of length 4, and so on:



7 | 9 | 3 | 5 | 4 | 1 | 6

7 9 | 3 5 | 1 4 | 6

3 5 7 9 | 1 4 6

Quicksort

- Quicksort takes a divide-and-conquer approach that is different to mergesort's.

- It uses the **partitioning** idea from QuickSelect, picking a pivot element, and partitioning the array around that, so as to obtain this situation: pivot: for now, we use the simplest strategy of selecting subarray's first element

$$\underbrace{A[0]\ldots A[s-1]}_{\text{all are } \le A[s]} \quad A[s] \quad \underbrace{A[s+1]\ldots A[n-1]}_{\text{all are } \ge A[s]}$$

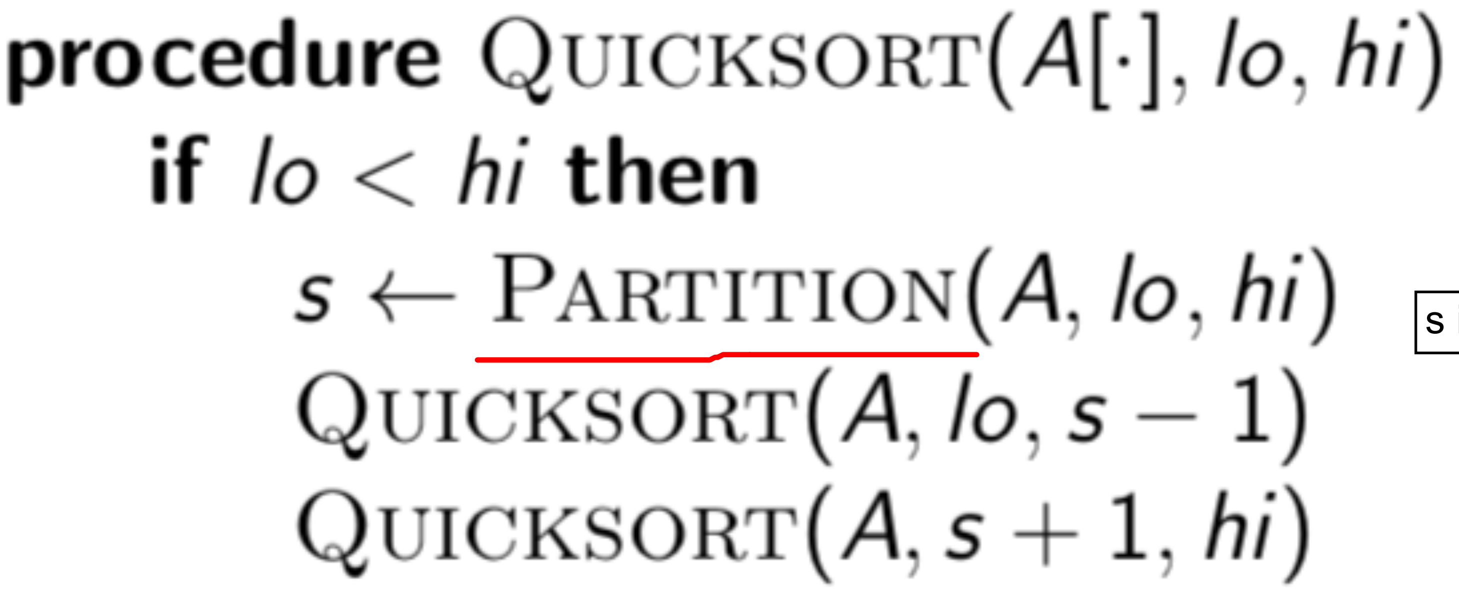
- The element A[s] will be in its final position (it is the (s + 1)th smallest element).

- All that then needs to be done is to sort the segment to the left, recursively, as well as the segment to the right.

Quicksort

- Very short and elegant:

indices



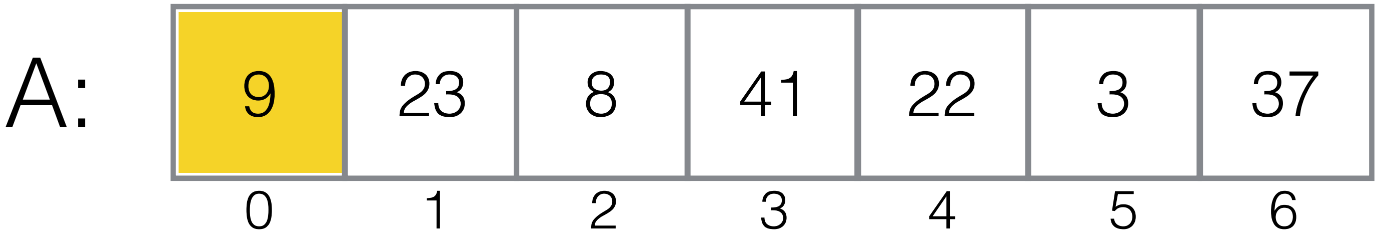
```
procedure $\text{QUICKSORT}(A[\cdot], lo, hi)$
    if $lo < hi$ then
        $s \leftarrow \underline{\text{PARTITION}}(A, lo, hi)$
        $\text{QUICKSORT}(A, lo, s - 1)$
        $\text{QUICKSORT}(A, s + 1, hi)$
```

s is a split position

- Initial call: Quicksort(A, 0, n – 1).

Quicksort: Example

```
procedure $\textsc{Quicksort}(A[\cdot], lo, hi)$
    if $lo < hi$ then
        $s \leftarrow \textsc{Partition}(A, lo, hi)$
        $\textsc{Quicksort}(A, lo, s - 1)$
        $\textsc{Quicksort}(A, s + 1, hi)$
```



A:	0	1	2	3	4	5	6
	9	23	8	41	22	3	37

Quicksort: Example

```
procedure QUICKSORT($A[\cdot]$, $lo$, $hi$)
    if $lo < hi$ then
        $s \leftarrow$ PARTITION($A$, $lo$, $hi$)
        QUICKSORT($A$, $lo$, $s - 1$)
        QUICKSORT($A$, $s + 1$, $hi$)
```

A:	8	3	9	41	22	23	37
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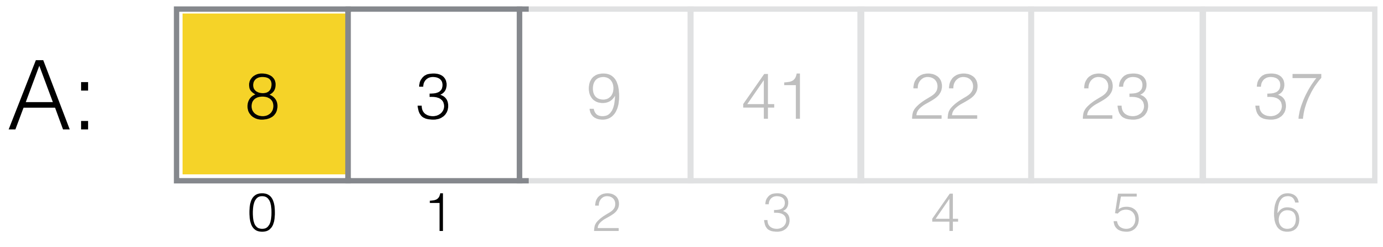

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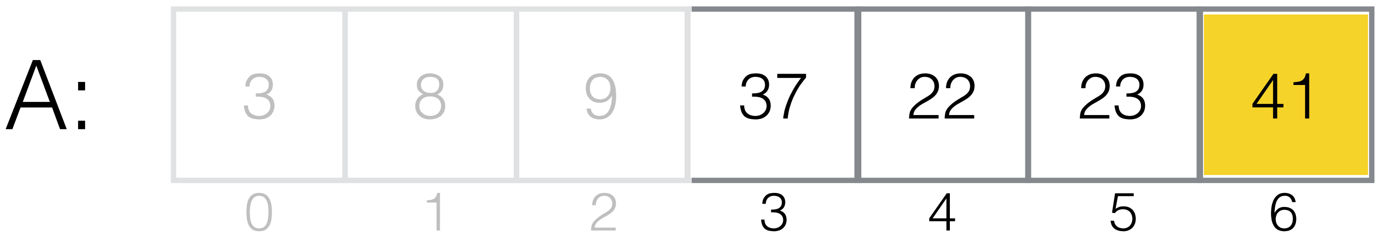

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        $s \leftarrow$ PARTITION($A$, $lo$, $hi$)
        QUICKSORT($A$, $lo$, $s - 1$)
        QUICKSORT($A$, $s + 1$, $hi$)
```
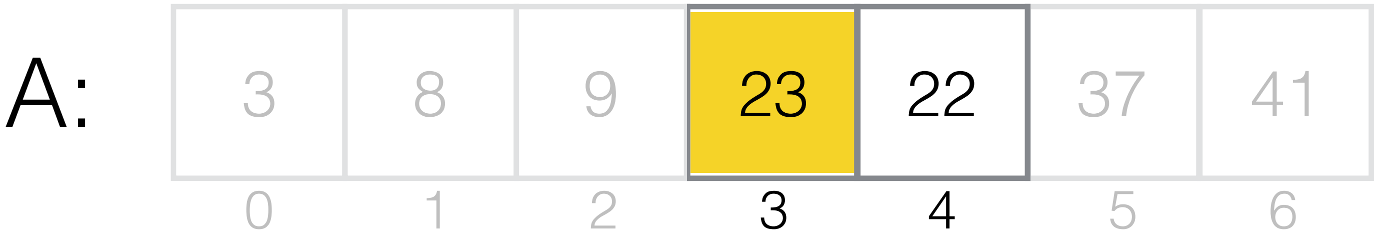
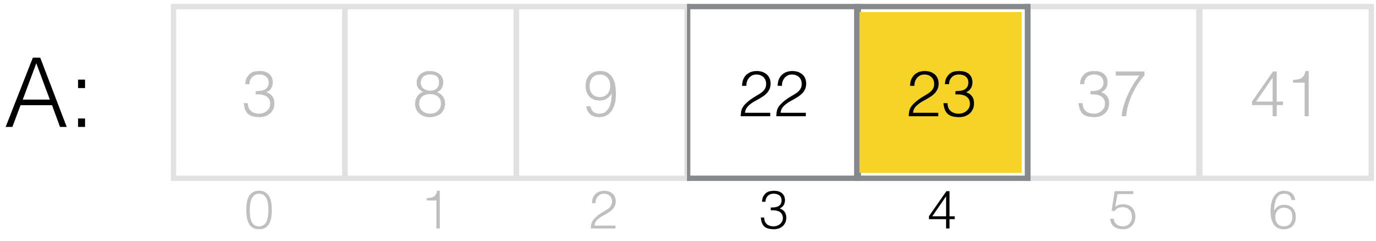



A:

3	8	9	23	22	37	41
0	1	2	3	4	5	6

Quicksort: Example

```
procedure Quicksort($A[\cdot]$, $lo$, $hi$)
    if $lo < hi$ then
        $s \leftarrow$ Partition($A$, $lo$, $hi$)
        Quicksort($A$, $lo$, $s - 1$)
        Quicksort($A$, $s + 1$, $hi$)
```



A:

3	8	9	22	23	37	41
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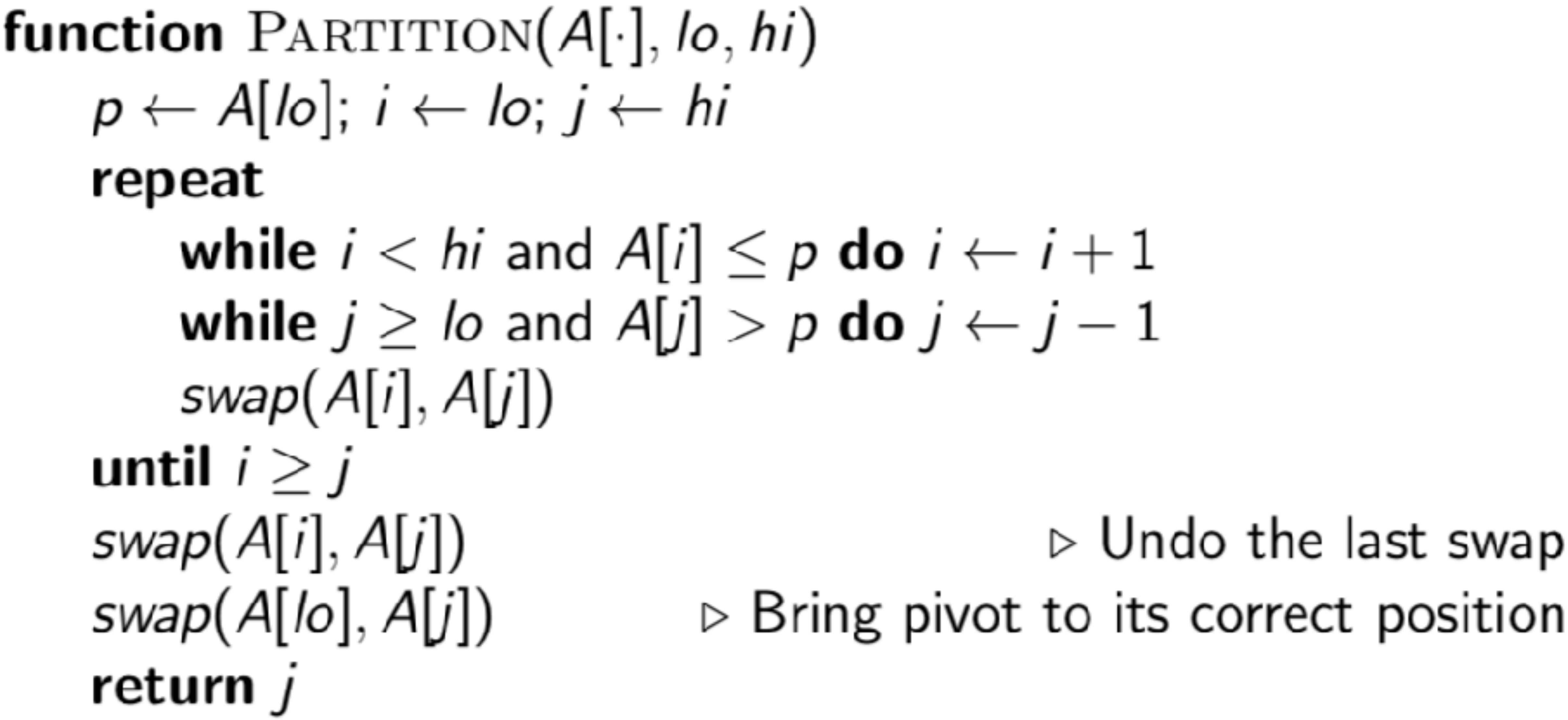
Quicksort: Example

```
procedure QUICKSORT(A[·], lo, hi)
    if $lo < hi$ then
        $s \leftarrow$ PARTITION(A, lo, hi)
        QUICKSORT(A, lo, $s - 1$)
        QUICKSORT(A, $s + 1$, hi)
```

A:	3	8	9	22	23	37	41
	0	1	2	3	4	5	6

Hoare Partitioning

- The standard way of doing partitioning in Quicksort



```
function PARTITION($A[\cdot], lo, hi$)
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi$
    repeat
        while $i < hi$ and $A[i] \leq p$ do $i \leftarrow i + 1$
        while $j \geq lo$ and $A[j] > p$ do $j \leftarrow j - 1$
        $swap(A[i], A[j])$
    until $i \geq j$
    $swap(A[i], A[j])$                    ▷ Undo the last swap
    $swap(A[lo], A[j])$        ▷ Bring pivot to its correct position
    return $j$
```

Hoare Partitioning

```
function PARTITION($A[\cdot], lo, hi$)
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi$
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        while $i < hi$ and $A[i] \le p$ do $i \leftarrow i + 1$
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    return $j$
```

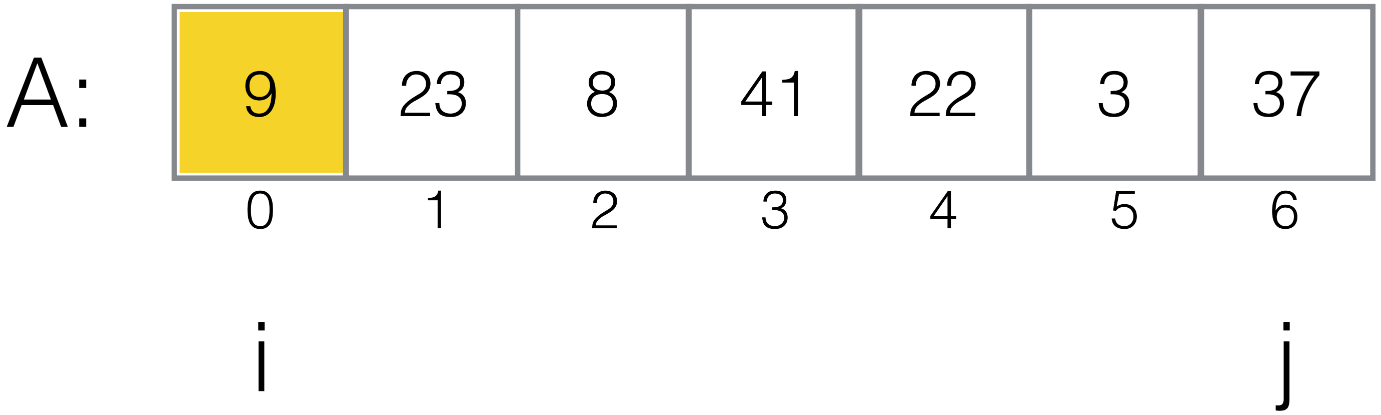

A:	9	23	8	41	22	3	37
	0	1	2	3	4	5	6
	i						j

p: 9

Hoare Partitioning

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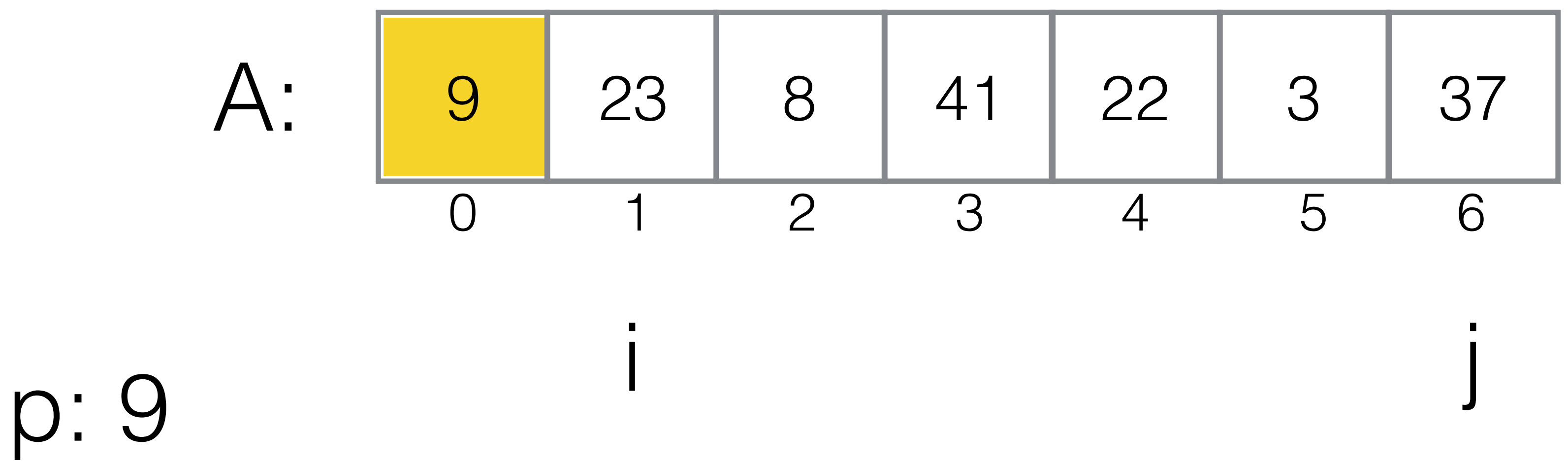

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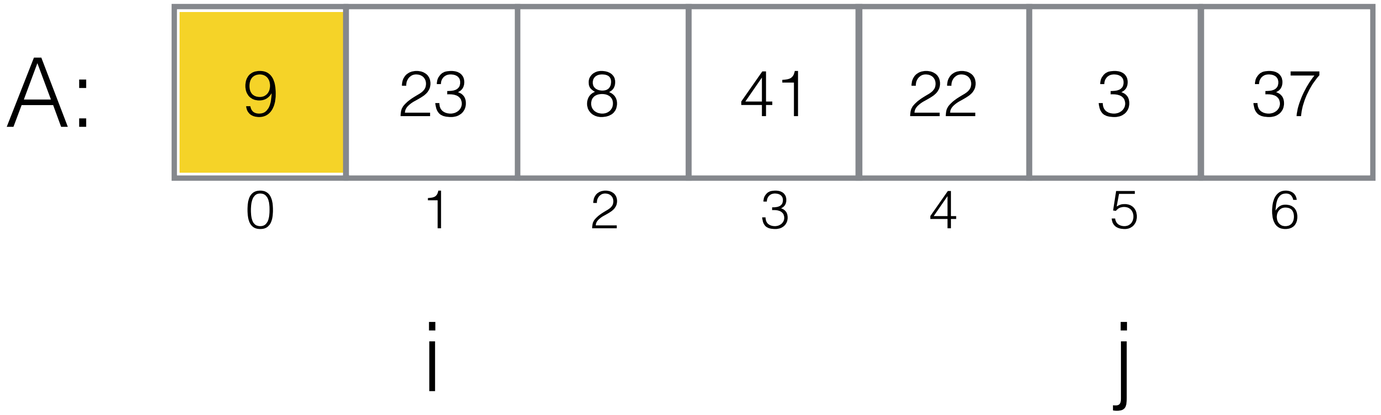

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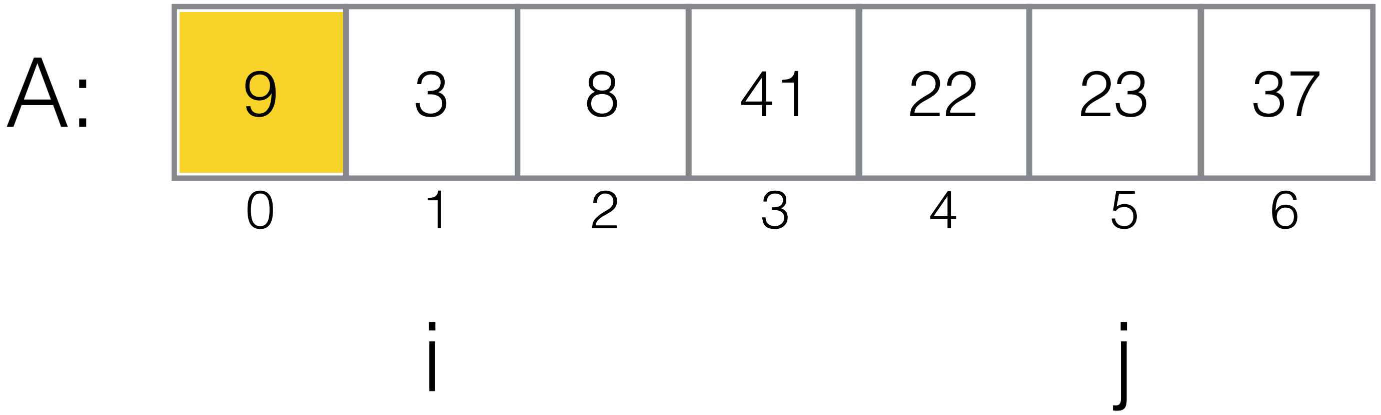

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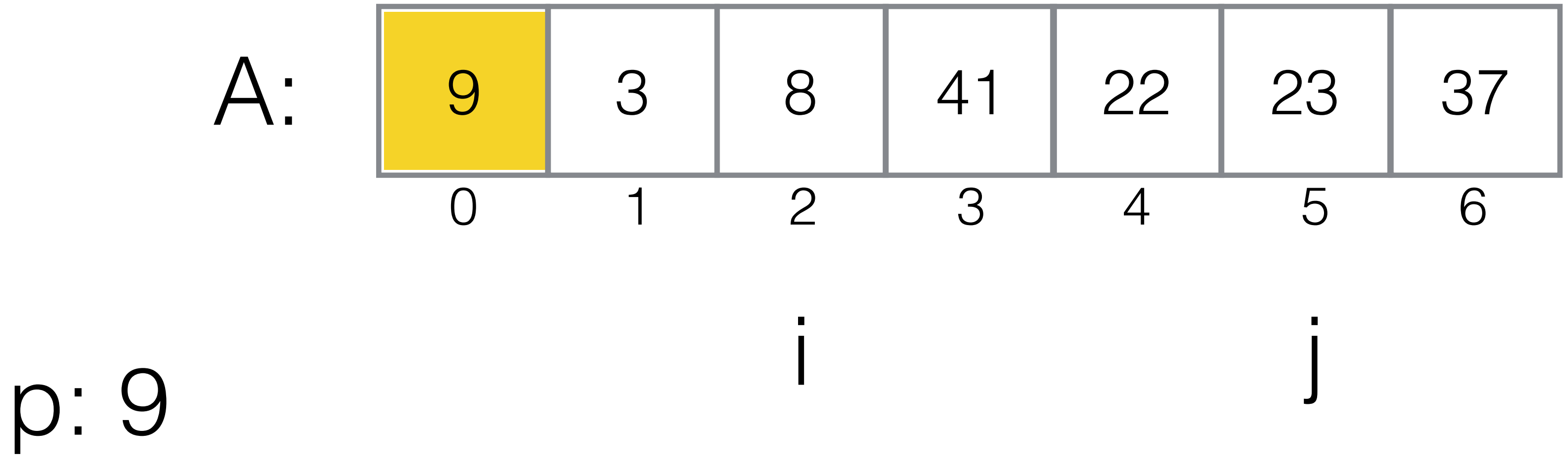

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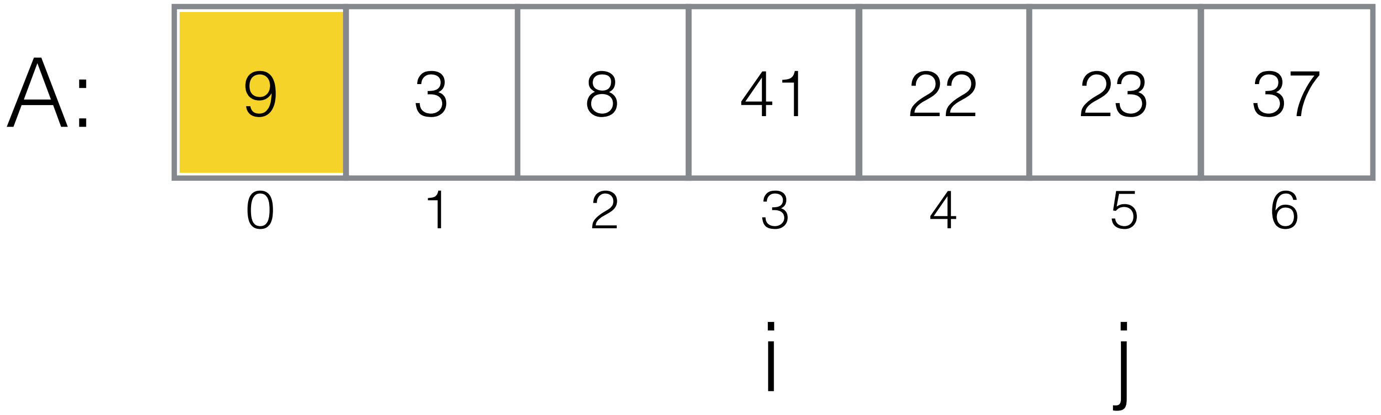

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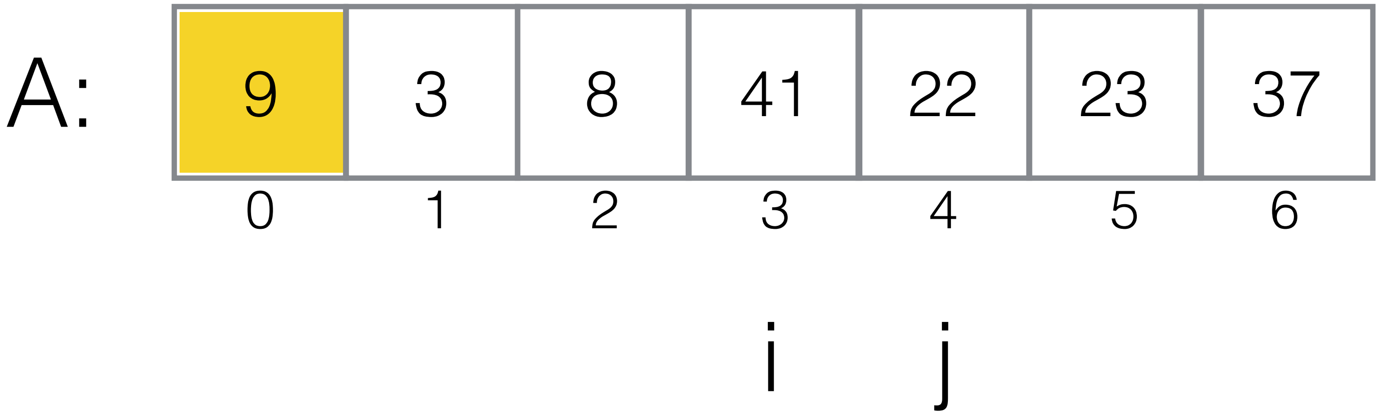

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				i	j		

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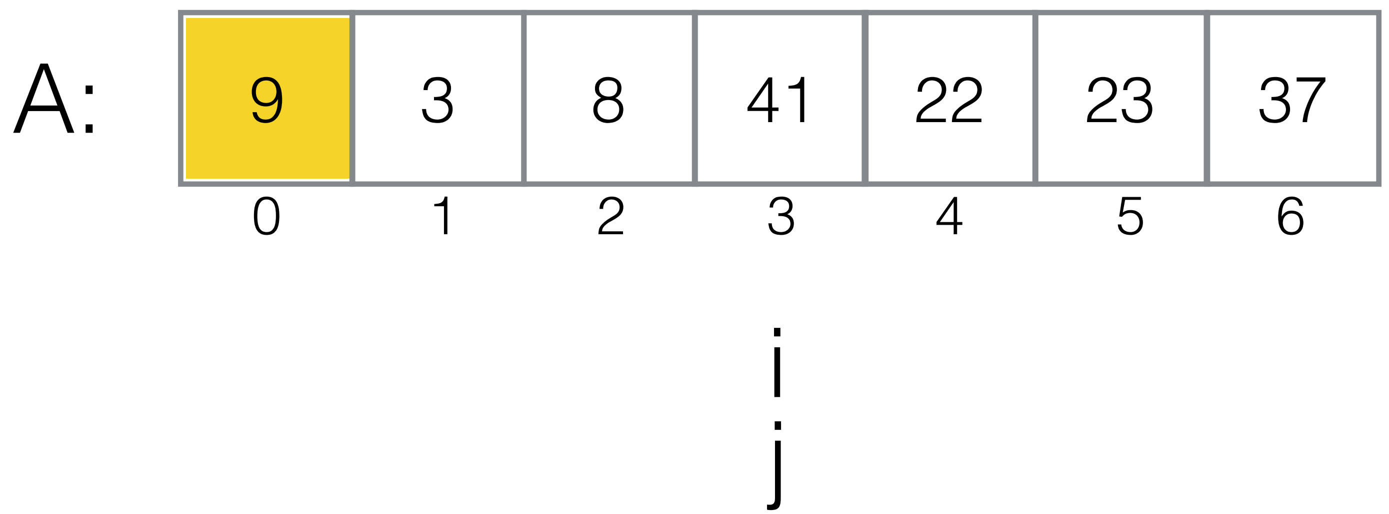

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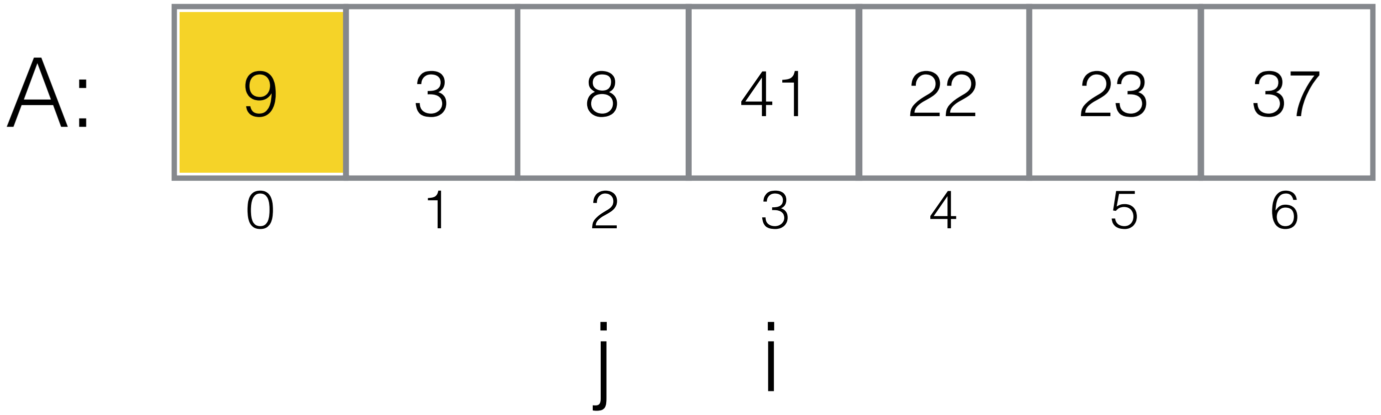

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		j	i			

p: 9

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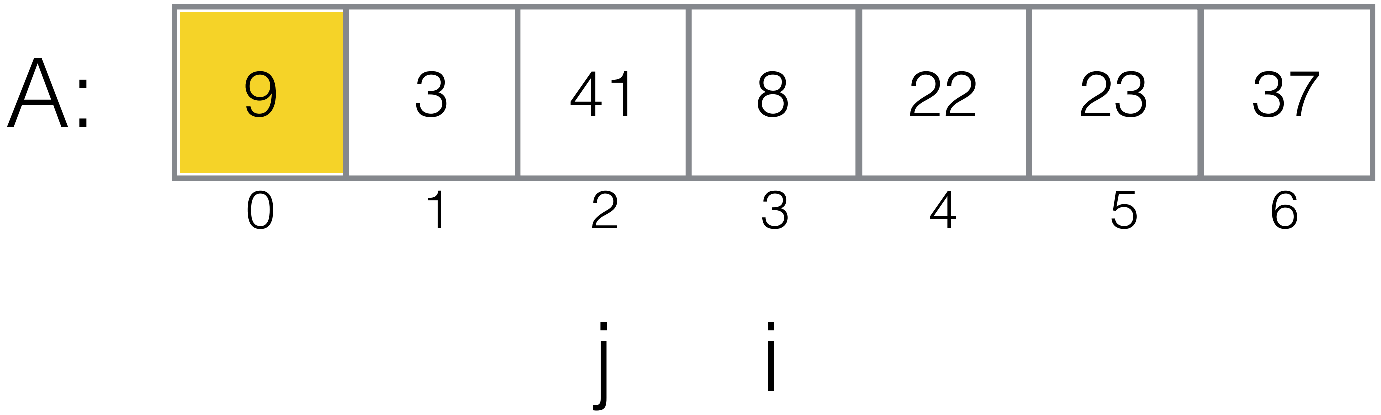

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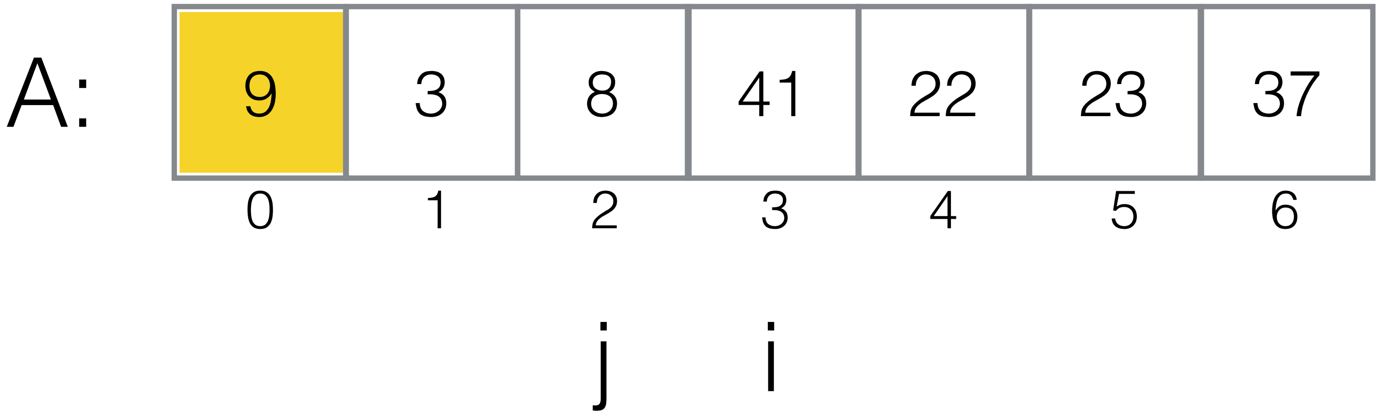
    return $j$
```



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p: 9

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```
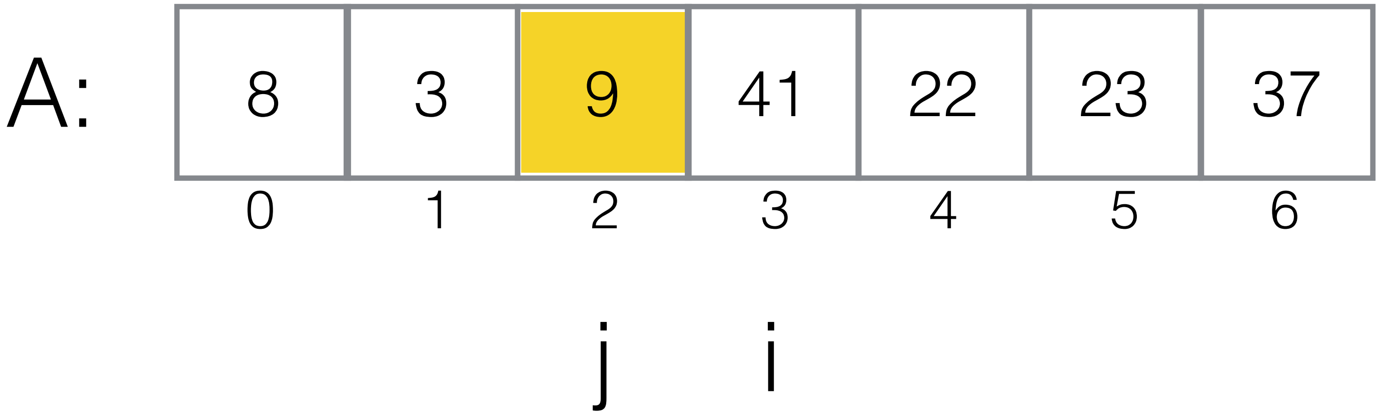



A:

8	3	9	41	22	23	37
0	1	2	3	4	5	6
		j	i			

p: 9

Quicksort Analysis:
Best Case Analysis

- The best case happens when the pivot is the median; that results in two sub-tasks of equal size.

$$C_{best}(n) = \begin{cases} 0 & \text{if } n < 2 \\ 2C_{best}(n/2) + n & \text{otherwise} \end{cases}$$

The ‘n’ is for the n key comparisons performed by Partition.

- By the Master Theorem, $C_{best}(n) \in \Theta(n \log n)$, just as for mergesort, so quicksort’s best case is (asymptotically) no better than mergesort’s worst case.

Quicksort Worst Case

A:

Quicksort Analysis: Worst Case Analysis

- The worst case happens if the array is already sorted.

- In that case, we don't really have divide-and-conquer, because each recursive call deals with a problem size that has only been decremented by 1:

$$C_{worst}(n) = \begin{cases} 0 & \text{if } n < 2 \\ C_{worst}(n-1) + n & \text{otherwise} \end{cases}$$

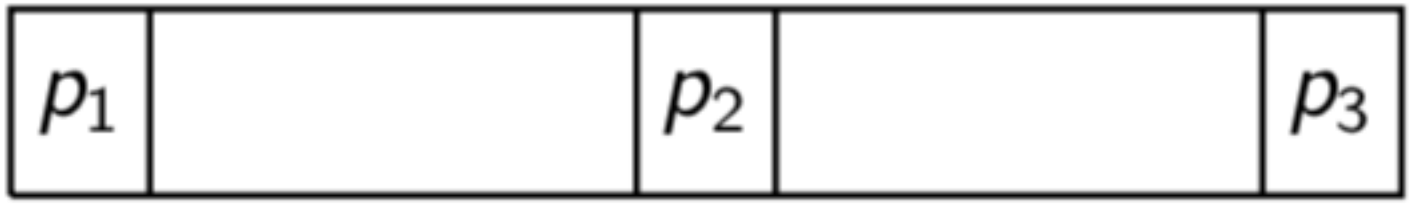
不能使用master theorem来计算

- That is, $C_{worst}(n) = n + (n - 1) + \cdots + 3 + 2 \in \Theta(n^2)$.

Quicksort Improvements: Median-of-Three

- It would be better if the pivot was chosen randomly.

- A cheap and useful approximation to this is to take the median of three candidates, A[lo], A[hi], and $A[\lfloor (lo + hi)/2 \rfloor]$.



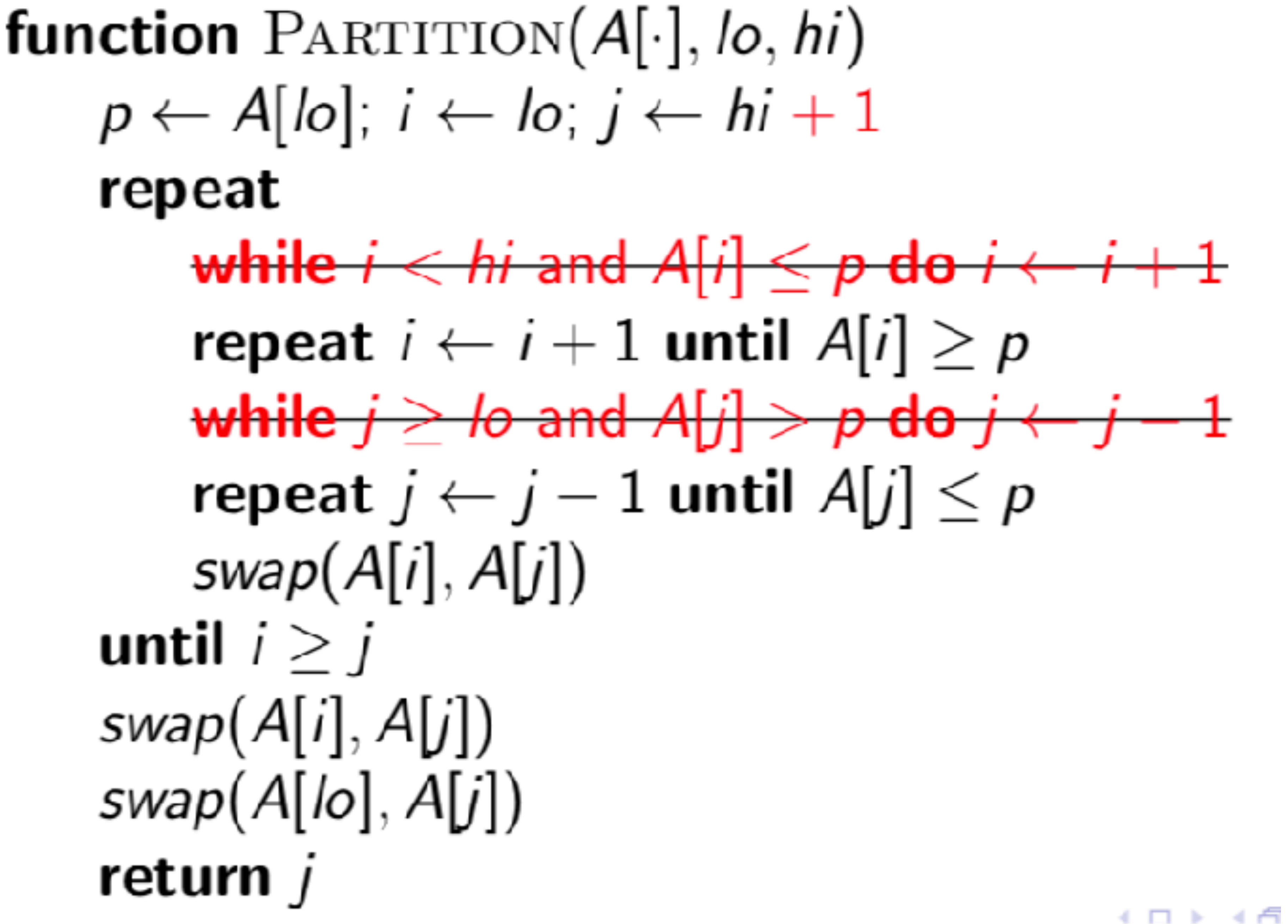
p_1 p_2 p_3

- Reorganise the three elements so that p_1 is the median, and p_3 is the largest of the three.

- Now run quicksort as before.

Quicksort Improvements: Median-of-Three

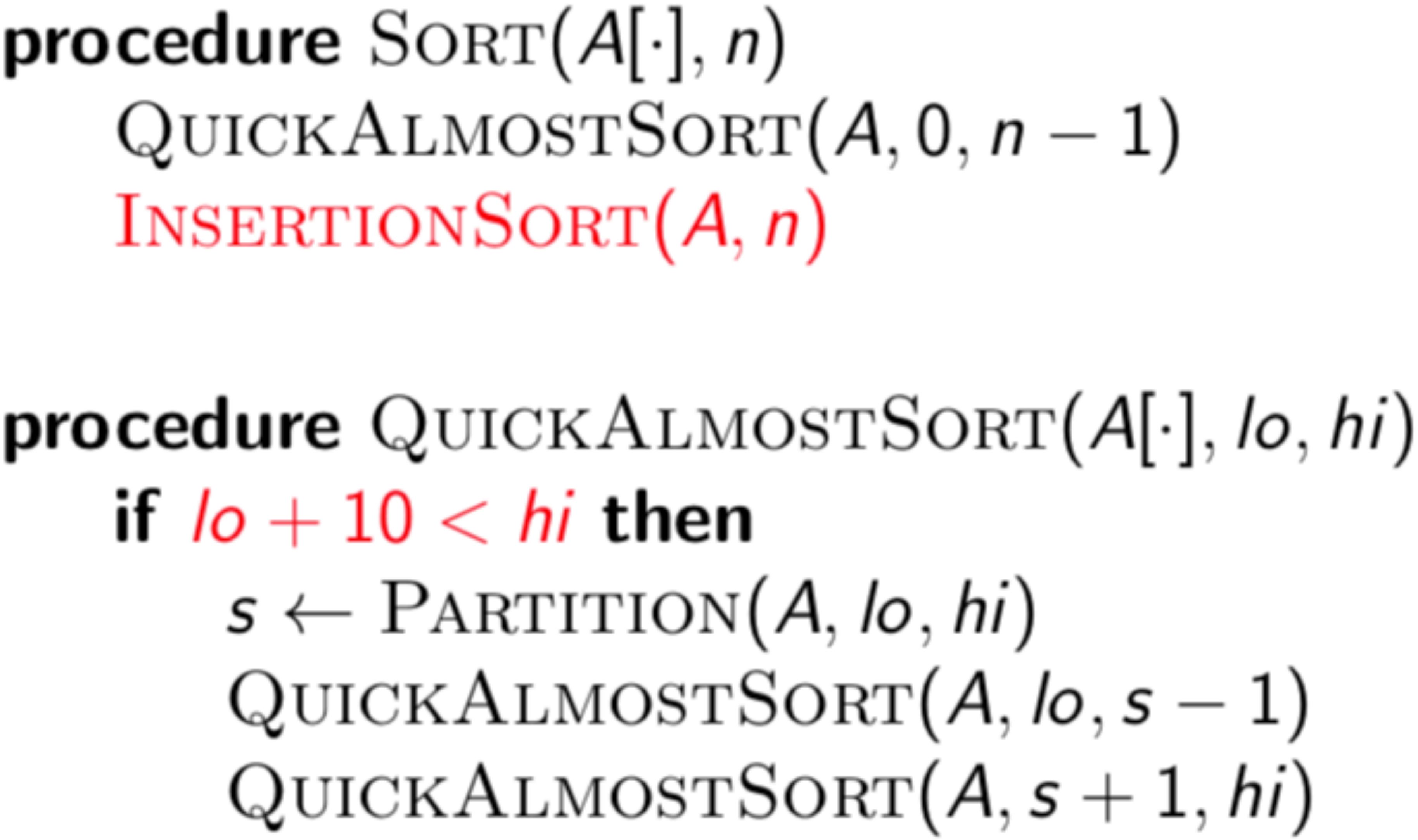
- In fact, with median-of-three, we can have a much faster version than before, simplifying tests in the innermost loops:



```
function PARTITION($A[\cdot], lo, hi$)
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi + 1$
    repeat
        ~~while $i < hi$ and $A[i] \leq p$ do $i \leftarrow i + 1$~~
        repeat $i \leftarrow i + 1$ until $A[i] \geq p$
        ~~while $j \geq lo$ and $A[j] > p$ do $j \leftarrow j - 1$~~
        repeat $j \leftarrow j - 1$ until $A[j] \leq p$
        $swap(A[i], A[j])$
    until $i \geq j$
    $swap(A[i], A[j])$
    $swap(A[lo], A[j])$
    return $j$
```

Quicksort Improvements: Early Cut-Off

- A second useful improvement is to stop quicksort early and switch to insertion sort. This is easily implemented:

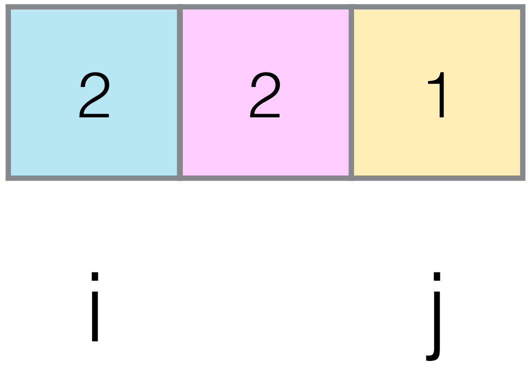


```
procedure SORT($A[\cdot], n$)
    QUICKALMOSTSORT($A, 0, n - 1$)
    INSERTIONSORT($A, n$)

procedure QUICKALMOSTSORT($A[\cdot], lo, hi$)
    if $lo + 10 < hi$ then
        $s \leftarrow$ PARTITION($A, lo, hi$)
        QUICKALMOSTSORT($A, lo, s - 1$)
        QUICKALMOSTSORT($A, s + 1, hi$)
```

Quicksort Properties

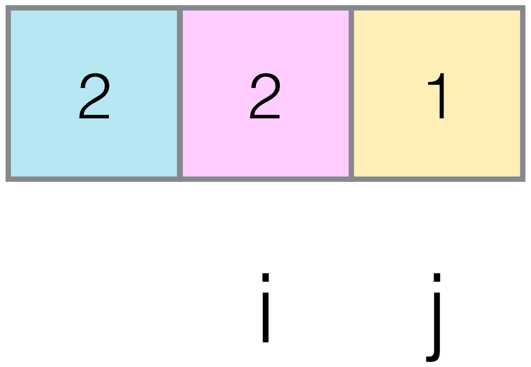
- With these (and other) improvements, quicksort is considered the best available sorting method for arrays of random data.
- A major reason for its speed is the very tight inner loop in PARTITION.
- Although mergesort has a better performance guarantee, quicksort is faster on average.
- In the best case, we get ⌈log2 n⌉ recursive levels. It can be shown that on random data, the expected number is $2 \log_e n \approx 1.38 \log_2 n$. So quicksort's average behaviour is very close to the best-case behaviour.
- Is quicksort stable? ?
- Is it in-place? yes

Quicksort Stability



2	2	1
i		j

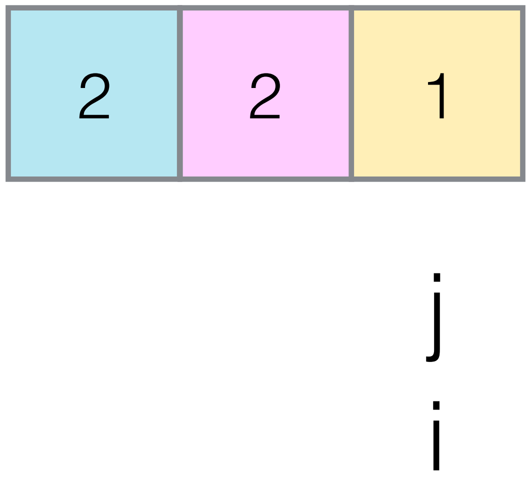
Quicksort Stability



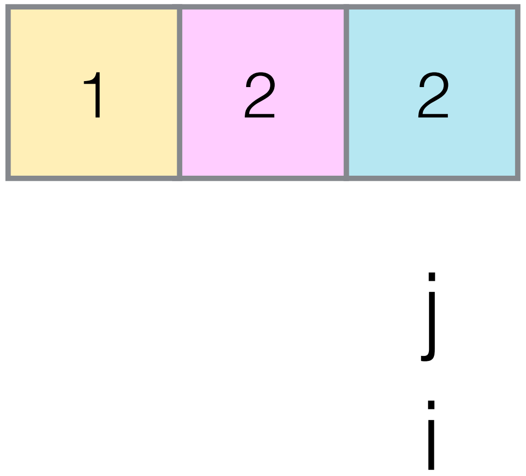
2 2 1

i j

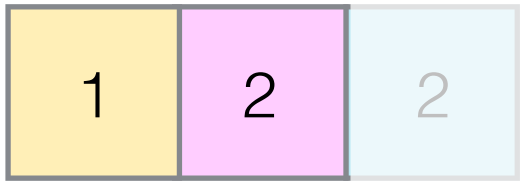
Quicksort Stability



Quicksort Stability

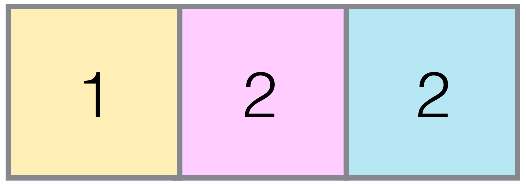


Quicksort Stability



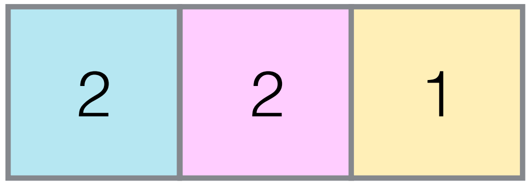
1	2	2

Quicksort Stability



1	2	2

This is where we finished



2	2	1

This is where we started

Not stable

Quicksort Properties

- With these (and other) improvements, quicksort is considered the best available sorting method for arrays of random data.
- A major reason for its speed is the very tight inner loop in PARTITION.
- Although mergesort has a better performance guarantee, quicksort is faster on average.
- In the best case, we get ⌈log2 n⌉ recursive levels. It can be shown that on random data, the expected number is $2 \log_e n \approx 1.38 \log_2 n$. So quicksort's average behaviour is very close to the best-case behaviour.
- Is quicksort stable? no
- Is it in-place? yes

Next up

- Tree traversal methods, plus we apply the divide-and-conquer technique to the closest-pair problem.