COMP90038
Algorithms and Complexity

Lecture 17: Hashing

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Recap

- We talked about using some memory space (in the form of extra tables, arrays, etc.) to speed up our computation.
    - Memory is cheap, time is not.

- Sorting by counting

- Horspool’s Algorithm

Sorting by counting

- Lets go through this example carefully:
  - The keys are: [1 2 3 4 5]
  - The data is: [5 5 1 5 4 1 2 3 5 5 1 5 5 3 5 1 3 5 4 5]
- Lets count the appearances of each key:

Key	1	2	3	4	5
Occurrences					

- Lets add up the occurrences

Occurrences					
Cumulation					

Sorting by counting

- Lets sort the data:

Key	1	2	3	4	5
Cumulation	4	5	8	10	20
P[20]					19
P[10]				9	
P[19]					18
P[8]			7		

Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Unsorted	5	5	1	5	4	1	2	3	5	5	1	5	5	3	5	1	3	5	4	5
Sorted								3		4									5	5

Horspool's algorithm

- Lets go through this example carefully:
  - The pattern is TAACG (A=1, T=2, G=3, C=4 → P[.] = [2 1 1 4 3], m =5)
  - The string is GACCGCGTGAGATAACGTCA
- This algorithm creates the table of shifts:

function $\textsc{FindShifts}(P[\cdot], m)$
  for $i \leftarrow 0$ to $alphasize - 1$ **do**
    $Shift[i] \leftarrow m$
  for $j \leftarrow 0$ to $m - 2$ **do**
    $Shift[P[j]] \leftarrow m - (j + 1)$

	A	T	G	C
After first loop	5	5	5	5
j=0	5	4	5	5
j=1	3	4	5	5
j=2	2	4	5	5
j=3	2	4	5	1

Horspool’s algorithm

- We append a **sentinel** at the end of the data to guarantee completion
  - The string is now GACCGCGTGAGATAACGTCATAACG

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
STRING	G	A	C	C	G	C	G	T	G	A	G	A	T	A	A	C	G	T	C	A	T	A	A	C	G
T[.]	3	1	4	4	3	4	3	2	3	1	3	1	2	1	1	4	3	2	4	1	2	1	1	4	3
FAILED (C!=A)	T	A	A	C	G																				
IS 'CG' SOMEWHERE ELSE?	T	A	A	C	G																				
(NO, SHIFT BY G)																									
FAILED (A!=G, SHIFT BY A)						T	A	A	C	G															
FAILED (A!=G, SHIFT BY A)								T	A	A	C	G													
FAILED (A!=G, SHIFT BY A)										T	A	A	C	G											
FAILED (C!=G, SHIFT BY C)												T	A	A	C	G									
FOUND AT 16													T	A	A	C	G								

Horspool's algorithm

- For this algorithm, at the end of **while True do** iteration, [*i k*] are:

```
function HORSPOOL(P[·], m, T[·], n)
    FINDSHIFTS(P, m)
    i ← m − 1
    while True do
        k ← 0
        while k < m and P[m − 1 − k] = T[i − k] do
            k ← k + 1
        if k = m then
            if i ≥ n then
                return −1
            else
                return i − m + 1
        i ← i + Shift[T[i]]
```

i	k
9	2
11	0
13	0
15	0
16	0

Hashing

- **Hashing** is a standard way of implementing the abstract data type “dictionary”, a collection of <attribute name, value> pairs. For example an student record:
  - Attributes: Student ID, Name, data of birth, address, major, etc…

- Implemented well, it makes data retrieval very fast.

- A **key** identifies each record. It can be anything: integers, alphabetical characters, even strings
  - It should map efficiently to a positive integer.
  - The set *K* of keys need not be bounded.

Hashing

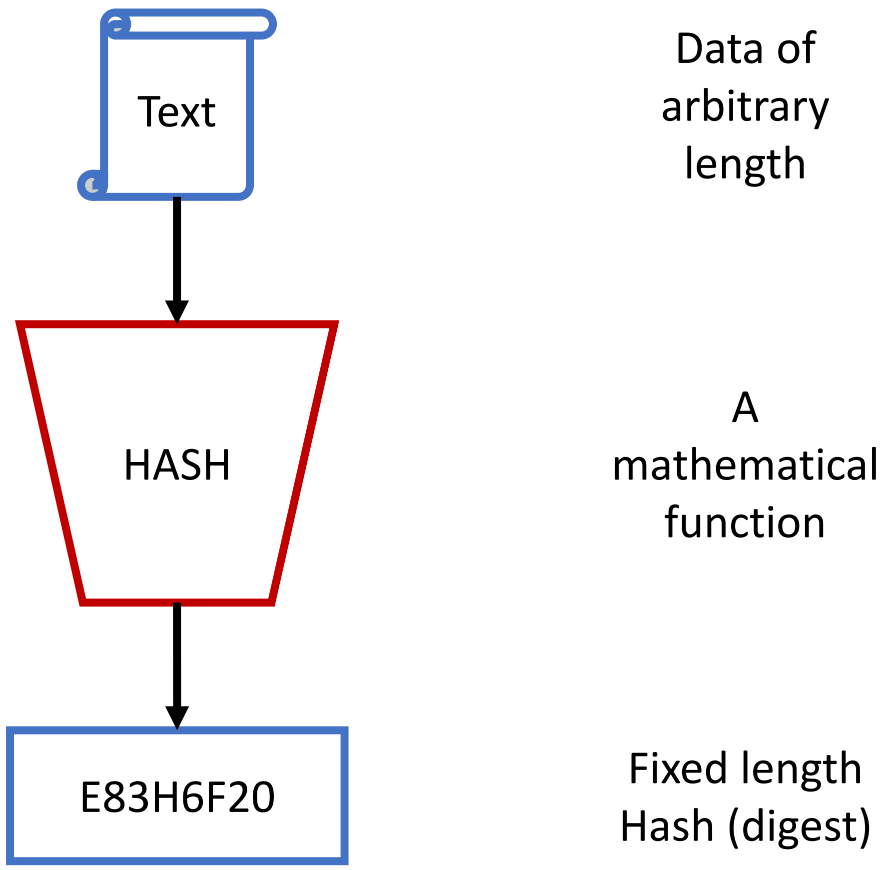
- We will store our records in a **hash table** of size m.
  - m should be large enough to allow efficient operation, without taking up excessive memory.

- The idea is **to have a function *h* that takes the key *k***, and determines an index in the hash table. This is the **hash function.**
  - A record with key k should be stored in location $h(k)$.

- The **hash address** is the value of $h(k)$.
  - Two different keys could have the same hash address (a collision).

Hashing

- Few example application are:
  - The MD5 algorithm used for data integrity verification.
  - The blockchain structure used in crypto currencies



The Hash Table

- We can think of the **hash table** as an abstract data structure supporting operations:
  - Find
  - Insert
  - Lookup (search and insert if not found)
  - Initialize
  - Delete
  - Rehash

- The challenges in implementing a table are:
  - Design a robust hash function
  - Handling of same addresses (collisions) for different key values

The Hash Function

- The hash function:
  - Must be easy (cheap) to compute.
  - Ideally distribute keys evenly across the hash table.

- Examples:
  - If the keys are integers, we could define $h(n) = n \bmod m$. If m=23:

n	19	392	179	359	262	321	97	468
h(n)	19	1	18	14	9	22	5	8

  - If the keys are strings, we could define a more complex function.

Hashing of strings

- Assume:
  - this table of 26 characters.
  - a hash table of size 13
  - the hash function:

$$h(s) = \left(\sum_{i=0}^{|s|-1} \mathrm{a}_i \right) \mod 13$$

  - and the following list of keys:
    [A, FOOL, AND, HIS, MONEY, ARE, SOON, PARTED]

char	a	char	a	char	a
A	0	J	9	S	18
B	1	K	10	T	19
C	2	L	11	U	20
D	3	M	12	V	21
E	4	N	13	W	22
F	5	O	14	X	23
G	6	P	15	Y	24
H	7	Q	16	Z	25
I	8	R	17		

Calculating the addresses

						SUM	*h*(*s*)
A							
0						0	0
F	O	O	L				
5	14	14	11			44	5
A	N	D					
0	13	3				16	3
H	I	S					
7	8	18				33	**7**
M	O	N	E	Y			
12	14	13	4	24		67	2
A	R	E					
0	17	4				21	8
S	O	O	N				
18	14	14	13			59	**7**
P	A	R	T	E	D		
15	0	17	19	4	3	58	6

The same address!!!

A more complex hash function

- Assume a binary representation of the 26 characters
  - We need **5 bits** per character (0 to 31)
- Instead of adding, we **concatenate** the binary strings
- Our hash table is of size 101 (***m* is prime**)
- Our key will be ‘MYKEY’

char	a	bin(a)
A	0	00000
B	1	00001
C	2	00010
D	3	00011
E	4	00100
F	5	00101
G	6	00110
H	7	00111
I	8	01000

char	a	bin(a)
J	9	01001
K	10	01010
L	11	01011
M	12	01100
N	13	01101
O	14	01110
P	15	01111
Q	16	10000
R	17	10001

char	a	bin(a)
S	18	10010
T	19	10011
U	20	10100
V	21	10101
W	22	10110
X	23	10111
Y	24	11000
Z	25	11001

A more complex hash function

	STRING					KEY	KEY mod 101
	M	Y	K	E	Y		
int	12	24	10	4	24		
bin(int)	01100	11000	01010	00100	11000		
Index	4	3	2	1	0		
32^(index)	1048576	32768	1024	32	1		
a*(32^index)	12582912	786432	10240	128	24	13379736	64

- By concatenating the strings, we are basically multiplying by 32
- Note that the hash function is a polynomial:

$$h(s) = a_{|s|-1}32^{|s|-1} + a_{|s|-2}32^{|s|-2} + \cdots + a_1 32 + a_0$$

Handling Long Strings as Keys

- What would happen if our key is the longer string ‘VERYLONGKEY’

$$h(VERYLONGKEY) = \left(21 \times 32^{10} + 4 \times 32^{9} + \cdots + 4 \times 32 + 24\right) \mod 101$$

- The stuff between parentheses quickly becomes a very large number quickly
  - DEC: 23804165628760600
  - BIN: 1010100100100011100001100110100011110100001001000011000

- Calculating this polynomial by brute force is very expensive

Horner’s rule

- Fortunately there is a trick, **Horner's rule**, that simplifies polynomial calculation.

$$p(x) = a_3 \times x^3 + a_2 \times x^2 + a_1 \times x + a_0$$

- By factorizing *x* we have that:

$$p(x) = (((a_3 \times x) + a_2) \times x + a_1) \times x + a_0$$

- If we apply the modulus we have:

$$p(x) = ((((a_3 \times x) + a_2) \times x + a_1) \times x + a_0) \mod m$$

Horner's rule

- We then can use the following properties of modular arithmetic:

$$\boxed{\begin{aligned} x \boxplus y &= (x + y) \mod m = ((x \mod m) + (y \mod m)) \mod m \\ x \boxtimes y &= (x \times y) \mod m = ((x \mod m) \times (y \mod m)) \mod m \end{aligned}}$$

- Given that modulus **distributes across all operations**, then we have:

$$p(x) = (((((a_3 \boxtimes x) \boxplus a_2) \boxtimes x) \boxplus a_1) \boxtimes x) \boxplus a_0$$

- The results of each **operation will not exceed *m*.**

Handling collisions

- The hash function should be as random as possible.

- However, in some cases different keys will be mapped to the same hash table address. For example $h(n) = n \bmod 23$

KEY	19	392	179	359	663	262	639	321	97	468	814
ADDRESS	19	1	18	14	19	9	18	22	5	8	9

- When this happens we have a **collision**.

- Different hashing methods resolve collisions differently.

Separate Chaining

(open hashing)

- Each element *k* of the hash table is a **linked list**, which makes collision handling very easy

ADDRESS	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
LIST		392				97			468	262					359				179	19			
										814									639	663			

- **Exercise**: add to this table [83 110 14]

m cells

- The **load factor** $\alpha = n/m$, where *n* is the number of items stored.
  - Number of probes in **successful** search ~(1 + α)/2.
  - Number of probes in **unsuccessful** search ~α.

$$S \approx 1 + \frac{\alpha}{2} \quad \text{and} \quad U = \alpha,$$

successful search

unsuccessful serach

Separate chaining: advantages and disadvantages

- Compared with **sequential search**, reduces the number of comparisons by the size of the table (a factor of *m*).
- Good in a dynamic environment, when (number of) keys are hard to predict.
- The chains can be ordered, or records may be “pulled up front” when accessed.
- Deletion is easy.
- However, separate chaining uses extra storage for links.

Open-Addressing Methods

- With **open-addressing** methods (also called **closed hashing**) all records are stored in the hash table itself (not in linked lists hanging off the table).

- There are many methods of this type. We focus on two:
  - **linear probing**
  - **double hashing**

- For these methods, the load factor $\alpha \leq 1$.

Linear probing

- In case of collision, try the next cell, then the next, and so on.

- Assume the following data (and its **keys**) arriving one at the time:

  [19(**19**) 392(**1**) 179(**18**) 663(**19**→**20**) 639(**18**→**21**) 321(**22**) …]

- Search proceeds in similar fashion
- If we get to the end of the table, we wrap around.
- For example, if key 20 arrives, it will be placed in cell 0.

Linear probing

- **Exercise**: Add [83(**14**) 110(**18**) 497(**14**)] to the table

ADDRESS	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
LIST		392				97			468	262	814				359				179	19	663	639	321

- Again let *m* be the table size, and *n* be the number of records stored.
- As before, $\alpha = n/m$ is the load factor. Then, the average number of probes:
  - Successful search: $0.5 + 1/(2(1-\alpha))$
  - Unsuccessful: $0.5 + 1/(2(1-\alpha)^2)$

$$S \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \quad \text{and} \quad U \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

successful search

unseccessful search

Linear probing: advantages and disadvantages

- Space-efficient.
- Worst-case performance miserable; must be careful not to let the load factor grow beyond 0.9.
- Comparative behavior, m = 11113, n = 10000, α = 0.9:
  - Linear probing: 5.5 probes on average (success)
  - Binary search: 12.3 probes on average (success)
  - Linear search: 5000 probes on average (success)
- **Clustering** (large groups of contiguous keys) is a major problem:
  - The collision handling strategy leads to clusters of contiguous cells being occupied.
- Deletion is almost impossible.

Double Hashing

- To alleviate the clustering problem in linear probing, there are better ways of resolving collisions.

- One is **double hashing** which uses a second hash function *s* to determine an **offset** to be used in probing for a free cell.

- For example, we may choose $s(k) = 1 + k \bmod 97$.

- By this we mean, if $h(k)$ is occupied, next try $h(k) + s(k)$, then $h(k) + 2s(k)$, and so on.

  if get to the end of table, wrap around

- This is another reason why **it is good to have *m* being a prime number**. That way, using $h(k)$ as the offset, we will eventually find a free cell if there is one.

Rehashing

- The standard approach to avoiding performance deterioration in hashing is to keep track of the load factor and to **rehash** when it reaches, say, 0.9.
- Rehashing means allocating a larger hash table (typically about twice the current size), revisiting each item, calculating its hash address in the new table, and inserting it.
- This **“stop-the-world”** operation will introduce long delays at unpredictable times, but it will happen relatively infrequently.

An exam question type

- With the hash function $h(k) = k \bmod 7$. Draw the hash table that results after inserting in the given order, the following values

- When collisions are handled by:
  - separate chaining
  - linear probing
  - double hashing using $h'(k) = 5 - (k \bmod 5)$

Solution

Index	0	1	2	3	4	5	6
Separate Chaining							
Linear Probing							
Double Hashing							

Rabin-Karp String Search

not examable

- The Rabin-Karp string search algorithm is based on string hashing.

- To search for a string p (of length m) in a larger string s, we can calculate *hash*(p) and then check every substring $s_i \dots s_{i+m-1}$ to see if it has the same hash value. Of course, if it has, the strings may still be different; so we need to compare them in the usual way.

- If $p = s_i \dots s_{i+m-1}$ then the hash values are the same; otherwise the values are **almost certainly** going to be different.

- Since false positives will be so rare, the $O(m)$ time it takes to actually compare the strings can be ignored.

Rabin-Karp String Search

- Repeatedly hashing strings of length *m* seems like a bad idea. However, the hash values can be calculated **incrementally**. The hash value of the length-*m* substring of *s* that starts at position *j* is:

$$\mathrm{hash}(s, j) = \sum_{i=0}^{m-1} \mathrm{chr}(s_{j+i}) \times a^{m-i-1}$$

- where *a* is the alphabet size. From that we can get the next hash value, for the substring that starts at position *j*+1, **quite cheaply**:

$$\mathrm{hash}(s, j+1) = (\mathrm{hash}(s, j) - a^{m-1}\mathrm{chr}(s_j)) \times a + \mathrm{chr}(s_{j+m})$$

- modulo *m*. Effectively we just subtract the contribution of s_j and add the contribution of s_{j+m}, for the cost of two multiplications, one addition and one subtraction.

An example

- The data ‘31415926535’
- The hash function $h(k) = k \bmod 11$
- The pattern ’26’

STRING	3	1	4	1	5	9	2	6	5	3	5
31 MOD 11		9									
14 MOD 11			3								
41 MOD 11				8							
15 MOD 11					4						
59 MOD 11						4					
92 MOD 11							4				
26 MOD 11								4			

Why Not Always Use Hashing?

- Some drawbacks:
  - If an application calls for traversal of all items in sorted order, a hash table is no good.
  - Also, unless we use separate chaining, deletion is virtually impossible.
  - It may be hard to predict the volume of data, and rehashing is an expensive “stop-the-world” operation.

When to Use Hashing?

- All sorts of information retrieval applications involving thousands to millions of keys.
- Typical example: Symbol tables used by compilers. The compiler hashes all (variable, function, etc.) names and stores information related to each – no deletion in this case.
- When hashing is applicable, it is usually superior; a well-tuned hash table will outperform its competitors.
- **Unless** you let the load factor get too high, or you botch up the hash function. It is a good idea to print statistics to check that the function really does spread keys uniformly across the hash table.

Next lecture

- Dynamic programming and optimization.