CS/ECE/ISyE 524 - S'23 - HW 4

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1. Network Reliability

We consider a telecommunications network with a given pattern of interconnections. This network consists of eleven sites connected by bidirectional lines for data transmission. Each line has a bandwidth (i.e. maximum rate for data transmission) of 1.0. We are interested in the total bandwidth for the connection between nodes 10 and 11, and what happens to this bandwidth if a node is eliminated.

(a) Formulate the problem as a maxflow problem and solve. (Hint: Each bidirectional line can be represented by two directional edges, one in each direction. That is, if there is a line connecting nodes i and j, then [i,j] and [j,i] are both edges in the edge set. See the definition of the set "edges" in the starter file.)

(b) By adjusting your model for part (a), **not** recoding the edge set from scratch, find what happens if node 3 is eliminated from the graph. (That is, all lines connected to node 3 can no longer carry data.) What is the new maximum bandwidth between nodes 10 and 11?

(c) Can your maxflow problem be reformulated more compactly by using birectional lines? That is, redefining the edge set by eliminating duplicated lines (see the starter code) while allowing data to flow in either direction. Write down your reformulation of part (a) and code and solve it in Julia.

In [148…

```
using Pkg
Pkg.add("HiGHS")
Pkg.add("JuMP")
```

```
   Resolving package versions...
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Project.toml`
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Manifest.toml`
   Resolving package versions...
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Project.toml`
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Manifest.toml`
```

1 a)

Standard maxflow formulation

In [149…

```
using JuMP, HiGHS

# define nodes and edge set for the given graph - two directed edges for each bi
nodes = 1:11
edges = ([1,2], [1,3], [1,11], [2,1], [2,3], [2,8], [2,9], [3,1], [3,2], [3,4],
```

```
    [4,3], [4,5], [4,6], [4,11], [5,4], [5,9], [5,11], [6,4], [6,7], [6,9], [6,1
    [7,6], [7,8], [7,10],[8,7], [8,2], [8,10], [9,6], [9,5], [9,3], [9,2], [9,10
    [10,6], [10,9], [10,3], [10,8], [10,7], [11,4], [11,1], [11,3], [11,5])

model = Model(HiGHS.Optimizer)

@variable(model, 0 <= x[edges] <= 1)

numbers = [1, 2, 3, 4, 5, 6, 7, 8, 9]
for n in numbers
    i, j = 0, 0
    for e in edges
        if e[1] != n
            if e[2] == n
                i += x[e]
            end
        else
            j += x[e]
        end
    end
    @constraint(model, i == j)
end

k = 0
for e in edges
    if e[1] == 11
        k += x[e]
    end
end
@objective(model, Max, k)

optimize!(model)
print("Maximum flow: ", objective_value(model), "\n")
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
9 rows, 44 cols, 70 nonzeros
9 rows, 25 cols, 38 nonzeros
9 rows, 25 cols, 38 nonzeros
Presolve : Reductions: rows 9(-0); columns 25(-19); elements 38(-32)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Ph1: 0(0) 0s
         13    -4.0000000000e+00 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 13
Objective value     :  4.0000000000e+00
HiGHS run time      :          0.00
Maximum flow: 4.0
```

1 b)

In [150…

```
using JuMP, HiGHS

# define nodes and edge set for the given graph - two directed edges for each bi
```

```
nodes = 1:11
edges = ([1,2], [1,11], [2,1], [2,8], [2,9], [4,5], [4,6], [4,11], [5,4], [5,9],
    [7,6], [7,8], [7,10],[8,7], [8,2], [8,10], [9,6], [9,5], [9,2], [9,10], [10,

model = Model(HiGHS.Optimizer)

@variable(model, 0 <= x[edges] <= 1)

numbers = [1, 2, 4, 5, 6, 7, 8, 9]
for n in numbers
    i, j = 0, 0
    for e in edges
        if e[1] != n
            if e[2] == n
                i += x[e]
            end
        else
            j += x[e]
        end
    end
    @constraint(model, i == j)
end

k = 0
for e in edges
    if e[1] == 11
        k += x[e]
    end
end
@objective(model, Max, k)

optimize!(model)
print("Maximum flow: ", objective_value(model), "\n")
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
8 rows, 32 cols, 50 nonzeros
8 rows, 19 cols, 28 nonzeros
7 rows, 18 cols, 26 nonzeros
Presolve : Reductions: rows 7(-1); columns 18(-14); elements 26(-24)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Ph1: 0(0) 0s
         10    -3.0000000000e+00 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 10
Objective value     :  3.0000000000e+00
HiGHS run time      :          0.00
Maximum flow: 3.0
```

1 c)

WRITE YOUR FORMULATION OF THE COMPACT PROBLEM HERE

In [151…

```
using JuMP, HiGHS
```

```julia
# version with bidirectional edges
nodes = 1:11
edges = ([1,2], [1,3], [1,11], [2,3], [2,8], [2,9], [3,4], [3,9], [3,10], [3,11]
    [4,5], [4,6], [4,11], [5,9], [5,11], [6,7], [6,9], [6,10],
    [7,8], [7,10],[8,10], [9,10],
    [11,10])

model = Model(HiGHS.Optimizer)

@variable(model, -1 <= x[edges] <= 1)

numbers = [1, 2, 3, 4, 5, 6, 7, 8, 9]
for n in numbers
    i, j = 0, 0
    for e in edges
        if e[1] != n
            if e[2] == n
                i += x[e]
            end
        else
            j += x[e]
        end
    end
    @constraint(model, i == j)
end

k = 0
for e in edges
    if e[2] == 11
        k += x[e]
    end
end
@objective(model, Max, k)

optimize!(model)
print("Maximum flow: ", objective_value(model), "\n")
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
9 rows, 22 cols, 35 nonzeros
9 rows, 22 cols, 35 nonzeros
Presolve : Reductions: rows 9(-0); columns 22(-1); elements 35(-0)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -4.0000091820e+00 Pr: 9(17) 0s
         16    -4.0000000000e+00 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 16
Objective value     :  4.0000000000e+00
HiGHS run time      :          0.00
Maximum flow: 4.0
```

2. Stigler's supplement

2 a)

In [152…

```
using Pkg
Pkg.add("CSV")
Pkg.add("DataFrames")
Pkg.add("NamedArrays")
```

```
   Resolving package versions...
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Project.toml`
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Manifest.toml`
   Resolving package versions...
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Project.toml`
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Manifest.toml`
   Resolving package versions...
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Project.toml`
  No Changes to `C:\Users\16084\.julia\environments\v1.8\Manifest.toml`
```

In [169…

```
# STARTER CODE FOR STIGLER'S DIET PROBLEM

using NamedArrays, CSV, DataFrames

# import Stigler's data set
raw = CSV.read("stigler.csv", DataFrame);
(m,n) = size(raw)

n_nutrients = 2:n      # columns containing nutrients

# list of food
foods = raw[2:end,1]
# list of nutrients
nutrients = [string(names(raw)[i]) for i=2:length(names(raw))]

# minimum required amount of each nutrient
lower = Dict( zip(nutrients,raw[1,n_nutrients]) )

# data[f,i] is the amount of nutrient i contained in food f
dataraw = Matrix(values(raw[2:end,2:end]))
data = NamedArray(dataraw,(foods,nutrients),("foods","nutrients"))


println("Foods:\n")
for i in foods
    println(i)
end


println("\n\nNutrient Lower Bounds:\n")
for j in nutrients
    println(j," at least: ",lower[j])
end
```

```
Foods:

Wheat Flour (Enriched)
Macaroni
Wheat Cereal (Enriched)
Corn Flakes
Corn Meal
Hominy Grits
Rice
Rolled Oats
White Bread (Enriched)
Whole Wheat Bread
Rye Bread
Pound Cake
Soda Crackers
Milk
Evaporated Milk (can)
Butter
Oleomargarine
Eggs
Cheese (Cheddar)
Cream
Peanut Butter
Mayonnaise
Crisco
Lard
Sirloin Steak
Round Steak
Rib Roast
Chuck Roast
Plate
Liver (Beef)
Leg of Lamb
Lamb Chops (Rib)
Pork Chops
Pork Loin Roast
Bacon
Ham, smoked
Salt Pork
Roasting Chicken
Veal Cutlets
Salmon, Pink (can)
Apples
Bananas
Lemons
Oranges
Green Beans
Cabbage
Carrots
Celery
Lettuce
Onions
Potatoes
Spinach
Sweet Potatoes
Peaches (can)
Pears (can)
Pineapple (can)
Asparagus (can)
Green Beans (can)
```

```
Pork and Beans (can)
Corn (can)
Peas (can)
Tomatoes (can)
Tomato Soup (can)
Peaches, Dried
Prunes, Dried
Raisins, Dried
Peas, Dried
Lima Beans, Dried
Navy Beans, Dried
Coffee
Tea
Cocoa
Chocolate
Sugar
Corn Syrup
Molasses
Strawberry Preserves


Nutrient Lower Bounds:

Calories (1000) at least: 3.0
Protein (g) at least: 70
Calcium (g) at least: 0.8
Iron (mg) at least: 12
Vitamin A (1000 IU) at least: 5.0
Thiamine (mg) at least: 1.8
Riboflavin (mg) at least: 2.7
Niacin (mg) at least: 18
Ascorbic Acid (mg) at least: 75
```

INSERT ANSWERS AND EXPLANATIONS FOR PARTS a) AND b)

In [156…

```
m = Model(HiGHS.Optimizer)

@variable(m, 0 <= x[1:length(n_foods)])

variable_names=["Calories (1000)", "Protein (g)", "Calcium (g)", "Iron (mg)", "V
    "Thiamine (mg)", "Riboflavin (mg)", "Niacin (mg)", "Ascorbic Acid (mg)"]

for j in 1:size(data, 2)
    @constraint(m, sum(x[i] * data[i,j] for i in 1:length(n_foods)) >= lower[var
end

@objective(m, Min, sum(x[i] for i=1:length(n_foods)))

optimize!(m)
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
9 rows, 40 cols, 317 nonzeros
9 rows, 27 cols, 213 nonzeros
Presolve : Reductions: rows 9(-0); columns 27(-49); elements 213(-349)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 9(76.4375) 0s
          5     1.0866227821e-01 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 5
Objective value     :  1.0866227821e-01
HiGHS run time      :          0.00
```

In [163…

```
m = Model(HiGHS.Optimizer)

@variable(m, 0 <= x[1:length(n_nutrients)])

for i in 1:length(n_foods)
    @constraint(m, sum(x[j] * data[i,j] for j in 1:length(n_nutrients)) <= 1)
end

@objective(m, Max, sum(x[i] * lower[nutrients[i]] for i=1:length(n_nutrients)))

optimize!(m)

for i in 1:length(n_nutrients)
    print(nutrients[i], ' ', value(x[i]), "\n")
end

print("Objective Value: ", objective_value(m), "\n")
print("Final Answer: ", nutrients[7], " ", value(x[7]), "\n")
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
40 rows, 9 cols, 317 nonzeros
27 rows, 9 cols, 213 nonzeros
Presolve : Reductions: rows 27(-49); columns 9(-0); elements 213(-349)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -2.0738220710e+00 Ph1: 27(349.212); Du: 8(2.07382) 0s
          8    -1.0866227821e-01 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 8
Objective value     :  1.0866227821e-01
HiGHS run time      :          0.00
Calories (1000) 0.00876514729804949
Protein (g) 0.0
Calcium (g) 0.03173771344563703
Iron (mg) 0.0
Vitamin A (1000 IU) 0.0004002327217253814
Thiamine (mg) 0.0
Riboflavin (mg) 0.01635803269927669
Niacin (mg) 0.0
Ascorbic Acid (mg) 0.0001441175154589971
Objective Value: 0.10866227820675686
Final Answer: Riboflavin (mg) 0.01635803269927669
```

2 b)

In [261…

```julia
using NamedArrays, CSV, DataFrames

# import Stigler's data set
raw = CSV.read("stiglerPill.csv", DataFrame);
(m,n) = size(raw)

n_nutrients = 2:n    # columns containing nutrients
n_foods = 3:80

# list of food
foods = raw[2:end,1]
# list of nutrients
nutrients = [string(names(raw)[i]) for i=2:length(names(raw))]

# minimum required amount of each nutrient
lower = Dict( zip(nutrients,raw[1,n_nutrients]) )

# data[f,i] is the amount of nutrient i contained in food f
dataraw = Matrix(values(raw[2:end,2:end]))
data = NamedArray(dataraw,(foods,nutrients),("foods","nutrients"))

m = Model(HiGHS.Optimizer)

@variable(m, 0 <= x[1:length(n_foods)])

variable_names=["Calories (1000)", "Protein (g)", "Calcium (g)", "Iron (mg)", "V
    "Thiamine (mg)", "Riboflavin (mg)", "Niacin (mg)", "Ascorbic Acid (mg)"]

for j in 1:size(data, 2)
```

```
    @constraint(m, sum(x[i] * data[i,j] for i in 1:length(n_foods)) >= lower[var
end

@objective(m, Min, sum(x[i] for i=1:length(n_foods)))

optimize!(m)
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
9 rows, 44 cols, 336 nonzeros
9 rows, 26 cols, 196 nonzeros
Presolve : Reductions: rows 9(-0); columns 26(-52); elements 196(-375)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 9(76.4375) 0s
          5     1.0810012589e-01 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 5
Objective value     :  1.0810012589e-01
HiGHS run time      :          0.00
```

In [266…

```
m = Model(HiGHS.Optimizer)

@variable(m, 0 <= x[1:length(n_nutrients)])

for i in 1:length(n_foods)
    @constraint(m, sum(x[j] * data[i,j] for j in 1:length(n_nutrients)) <= 1)
end

@objective(m, Max, sum(x[i] * lower[nutrients[i]] for i=1:length(n_nutrients)))

optimize!(m)

for i in 1:9
    println(foods[i], ' ', value(x[i]))
end
print("\n")
print("Pill (mg): ", foods[1], " ", value(x[1]), "\n")

print("Objective Value: ", objective_value(m) * 365, "\n")
```

```
Running HiGHS 1.4.2 [date: 1970-01-01, git hash: f797c1ab6]
Copyright (c) 2022 ERGO-Code under MIT licence terms
Presolving model
43 rows, 9 cols, 335 nonzeros
25 rows, 9 cols, 195 nonzeros
Presolve : Reductions: rows 25(-53); columns 9(-0); elements 195(-376)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -1.7363233223e+00 Ph1: 25(295.5); Du: 7(1.73632) 0s
          9    -1.0810012589e-01 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 9
Objective value     :  1.0810012589e-01
HiGHS run time      :          0.00
Pill 0.013374435034446444
Macaroni 0.0
Wheat Cereal (Enriched) 0.03458137698012198
Corn Flakes 0.0
Corn Meal 0.0004876657783989055
Hominy Grits 0.0
Rice 0.01
Rolled Oats 0.0
White Bread (Enriched) 0.00014497853741580 91

Pill (mg): Pill 0.013374435034446444
Objective Value: 39.45654594825026
```

3. Dual interpretation

Suppose $t \in [0, 2\pi]$ is a parameter. Consider the following LP:

$$\begin{aligned}
\underset{p,q,r,s}{\text{minimize}} \quad & p+q+r+s \\
\text{subject to:} \quad & p-r=\cos(t) \\
& q-s=\sin(t) \\
& p,q,r,s \geq 0
\end{aligned}$$

a) Plot the optimal objective of this LP as a function of t. Can you explain what you see? *Hint:* you can do this by looping over values of t, and solving a separate LP for each different value of t. To interpret what you're seeing, you may want to separately consider the cases where $\cos(t)$ and $\sin(t)$ are positive or negative (four cases).

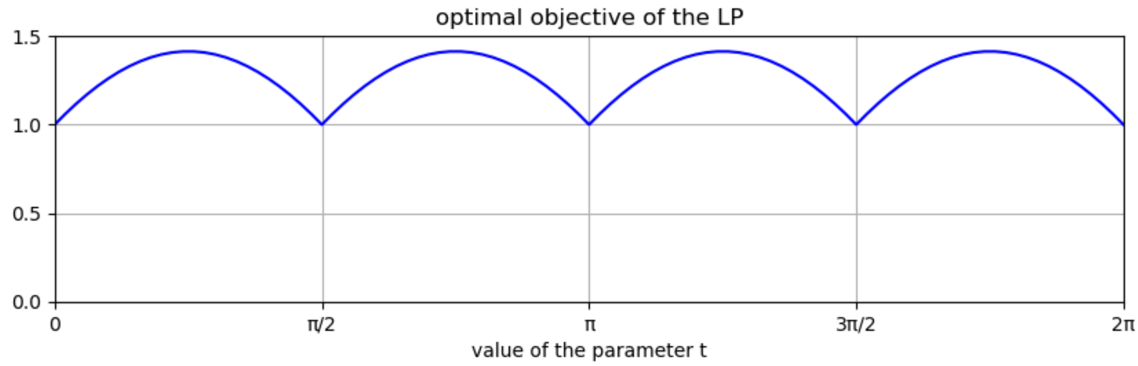
b) Find the dual LP and interpret it geometrically. Does this agree with the solution of part **a)**?

3 a)

In [12]:

```
using JuMP, PyPlot, HiGHS

# define a set of 101 values of t, equally spaced on [0,2*pi]
Npts = 101
tvals=range(0,stop=2*pi,length=Npts)
optvals = zeros(Npts)
```

```
for (i,t) in enumerate(tvals)
    m = Model(HiGHS.Optimizer)
    set_optimizer_attribute(m, "output_flag", false)
     # insert code here to set up and solve the problem for this "t", and save t
    @variable(m, p >= 0)
    @variable(m, q >= 0)
    @variable(m, r >= 0)
    @variable(m, s >= 0)

    @constraint(m, p - r == cos(t))
    @constraint(m, q - s == sin(t))

    @objective(m, Min, p + q + r + s)

    optimize!(m)
    optvals[i] = objective_value(m)

end

# here is some code to plot the results, assuming that the optimal values are st
figure(figsize=(10,2.5))
plot( tvals, optvals, "b-" )
xticks(0:π/2:2π)
yticks(0:0.5:1.5)
ylim([0,1.5])
xlim([0,2π])
grid()
g = gca()
g[:set_xticklabels](["0","π/2","π","3π/2","2π"])
xlabel("value of the parameter t")
title("optimal objective of the LP")
;
```



Explanation:

The plot shows that the optimal object value oscillates between to values as t ranges from zero to 2π. Specifically, when cos(t) and sin(t) are all positive, the optimal objective value is approximately 1.5. When cos(t) and sin(t) are both negative, the optimal objective value is approximately 0.5. This makes sense because when cos(t) and sin(t) have the same sign, the LP is trying to minimize the sum of four positive numbers that add up to a large positive value. When cos(t) and sin(t) have opposite signs, the LP is trying to minimize the sum of four positive numbers that add up to a small positive value

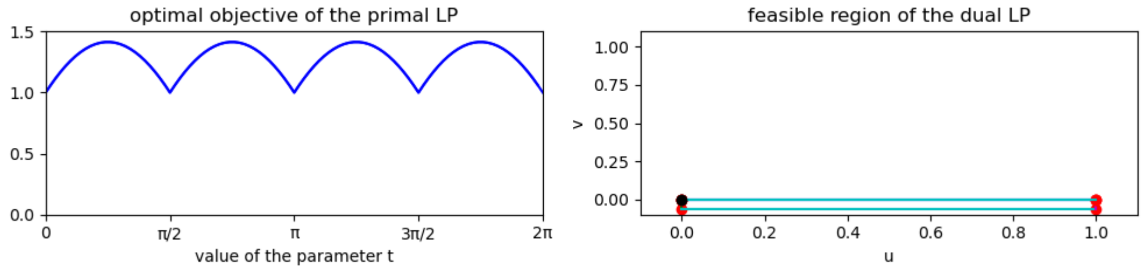
3 b)

WRITE YOUR SOLUTION TO PART (b) HERE

In [269…

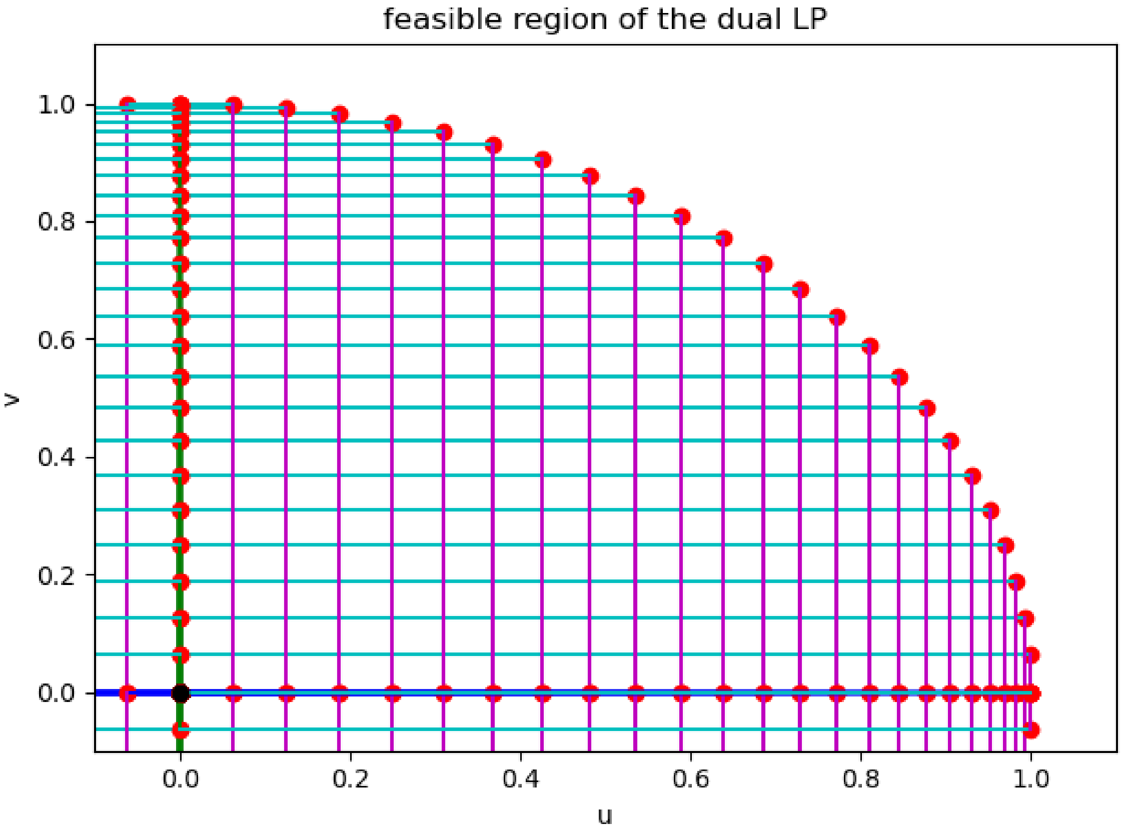
```
using PyPlot

# define a set of 101 values of t, equally spaced on [0,2*pi]
Npts = 101
tvals = range(0, stop=2*pi, length=Npts)
optvals_primal = zeros(Npts)
optvals_dual = zeros(Npts)

# create the figure and subplots outside the loop
fig, axs = subplots(ncols=2, figsize=(10, 2.5))

for (i, t) in enumerate(tvals)
    m = Model(HiGHS.Optimizer)
    set_optimizer_attribute(m, "output_flag", false)
    @variable(m, 0 <= p)
    @variable(m, 0 <= q)
    @variable(m, 0 <= r)
    @variable(m, 0 <= s)
    @constraint(m, p - r == cos(t))
    @constraint(m, q - s == sin(t))
    @objective(m, Min, p + q + r + s)
    optimize!(m)
    optvals_primal[i] = objective_value(m)

    vertices = [(0, 0), (0, sin(t)), (cos(t), 0), (cos(t), sin(t))]
    objvals = [cos(t) * u + sin(t) * v for (u, v) in vertices]
    optidx = argmin(objvals)
    optvals_dual[i] = objvals[optidx]

    # only plot the last two graphs
    if i >= Npts - 1
        axs[1].plot(tvals[1:i], optvals_primal[1:i], "b-")
        axs[1].set_xticks(0:π/2:2π)
        axs[1].set_yticks(0:0.5:1.5)
        axs[1].set_ylim([0, 1.5])
        axs[1].set_xlim([0, 2π])
        axs[1].grid()
        axs[1].set_xticklabels(["0", "π/2", "π", "3π/2", "2π"])
        axs[1].set_xlabel("value of the parameter t")
        axs[1].set_title("optimal objective of the primal LP")

        axs[2].plot([v[1] for v in vertices], [v[2] for v in vertices], "ro")
        axs[2].plot([0, cos(t)], [0, 0], "b-")
        axs[2].plot([0, 0], [0, sin(t)], "g-")
        axs[2].plot([cos(t), 0], [sin(t), sin(t)], "c-")
        axs[2].plot([cos(t), cos(t)], [sin(t), 0], "m-")
        axs[2].plot(vertices[optidx][1], vertices[optidx][2], "ko")
        axs[2].set_xlim([-0.1, 1.1])
        axs[2].set_ylim([-0.1, 1.1])
        axs[2].set_xlabel("u")
        axs[2].set_ylabel("v")
        axs[2].set_title("feasible region of the dual LP")

        fig.tight_layout()
```

```
        fig.show()
    end
end
```



In [205…

```
using PyPlot

# define a set of 101 values of t, equally spaced on [0,2*pi]
Npts = 101
tvals=range(0,stop=2*pi,length=Npts)
optvals_primal = zeros(Npts)
optvals_dual = zeros(Npts)

for (i,t) in enumerate(tvals)
    m = Model(HiGHS.Optimizer)
    set_optimizer_attribute(m, "output_flag", false)
    @variable(m, 0 <= p)
    @variable(m, 0 <= q)
    @variable(m, 0 <= r)
    @variable(m, 0 <= s)
    @constraint(m, p - r == cos(t))
    @constraint(m, q - s == sin(t))
    @objective(m, Min, p + q + r + s)
    optimize!(m)
    optvals_primal[i] = objective_value(m)

    vertices = [(0, 0), (0, sin(t)), (cos(t), 0), (cos(t), sin(t))]
    objvals = [cos(t)*u + sin(t)*v for (u,v) in vertices]
    optidx = argmin(objvals)
    optvals_dual[i] = objvals[optidx]

    plot([v[1] for v in vertices], [v[2] for v in vertices], "ro")
    plot([0, cos(t)], [0, 0], "b-")
    plot([0, 0], [0, sin(t)], "g-")
    plot([cos(t), 0], [sin(t), sin(t)], "c-")
    plot([cos(t), cos(t)], [sin(t), 0], "m-")
    plot(vertices[optidx][1], vertices[optidx][2], "ko")
    xlim([-0.1, 1.1])
    ylim([-0.1, 1.1])
    xlabel("u")
    ylabel("v")
    title("feasible region of the dual LP")

    if i == Npts
        tight_layout()
        show()
    end
end
```


feasible region of the dual LP
1.0
0.8
0.6
0.4
0.2
0.0
v
0.0
0.2
0.4
0.6
0.8
1.0
u