Supplementary Materials
Realism of Visual, Auditory, and Haptic Cues in Phenomenal Causality

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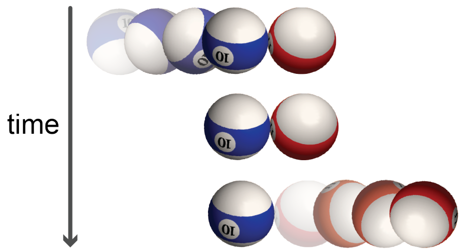

Fig. 1. An overview of the study in which the blue ball starts rolling towards the red ball, makes contact with the red ball, stops (temporal delay), and then the red ball rolls away

In the study, we considered two levels of realism for ball collisions across vision, audition, and vibrotactile feedback. Here we describe the dynamics for both visual conditions – one unrealistic and one realistic. In each trial, the blue ball moved towards the red ball, and then later the red ball moved away (Fig. 1). The amount of time between contact and the red ball starting to move away is an independent variable – temporal offset (delay). Here we will refer to the blue ball as 1 and the red ball as 2. Across all conditions, we treat both balls as rigid objects.

I. Unrealistic Visual Dynamics

In the unrealistic condition, the objects are rolling at a constant velocity and undergo an elastic collision. These balls are governed by the following piecewise functions (1 & 2) and rolling constraint (3) where x_{1_o} is the starting location of the center of the first ball, x_{2_o} is the starting location of the center of the second ball, $\dot{x}$ is the velocity, and r is the radius of each billiard ball. Also note, that in this study $\dot{x}_1$ is equal to $\dot{x}_2$ and is held constant, and thus is referred to as $\dot{x}$.

$$x_1(t) = \begin{cases} x_{1_o} + \dot{x}\ t & \text{if } 0 \leq t \leq t_c \\ x_{1_o} + \dot{x}\ t_c & \text{if } t_c < t < t_c + t_d \\ x_{1_o} + \dot{x}\ t_c & \text{if } t_c + t_d \leq t \end{cases} \tag{1}$$

$$x_2(t) = \begin{cases} x_{2_o} & \text{if } 0 \leq t \leq t_c \\ x_{2_o} & \text{if } t_c < t < t_c + t_d \\ x_{2_o} + \dot{x}\ (t - t_c - t_d) & \text{if } t_c + t_d \leq t \end{cases} \tag{2}$$

$$\theta(t) = \frac{x(t)}{r} \tag{3}$$

Here t_d is the temporal offset (delay) for that particular trial, and t_c is the time that it takes for the first ball to arrive at the point of contact with the second ball (collision). We can solve for t_c as shown below (4):

$$t_c = \frac{x_{2_o} - x_{1_o} - 2r}{\dot{x}} \tag{4}$$

II. Realistic Visual Dynamics

In the realistic condition, each ball was modeled as a sphere (body B) that rolls, slides, and undergoes an inelastic collision. There is drag from air that affects the balls' motion. While the balls are sliding, they receive forces from the ground due to friction. We considered gravity, normal force, friction, and the following force (5) and torque (6) from air:

$$F_{air} = -b\ A\ \dot{x}(t) \tag{5}$$

$$T_{air} = -b\ \dot{\theta}(t)\ I^{B/B_{cm}} \tag{6}$$

Here, b is the damping coefficient, A is the surface area of a sphere, and $I^{B/B_{cm}}$ is the inertia of ball B about its center of mass. The first ball begins rolling with a given set of initial conditions. We can represent the balls' rotation during rolling (7), where $\dot{\theta}_o$ is the initial angular velocity and m is the mass of each billiard ball, as follows:

$$\theta(t) = \tfrac{\dot{\theta}_o}{a}(e^{at} - 1)\ , \text{ where } a = -\tfrac{10b}{21}(1 + \tfrac{6\pi r^2}{m}) \tag{7}$$

In order to calculate the position in time, we use the same rolling constraint from before (3). After the balls collide (t_c), they remain static for the duration of the temporal offset (t_d). Then they begin to move according to an inelastic collision, in which the second ball begins to move and the first ball continues moving given conservation of momentum (8) and a coefficient of restitution e (9). Note that the velocity of ball 1 is taken from the moment of collision (t_c), as if the temporal delay had not occurred.

$$\dot{x}_1(t_c) = \dot{x}_2^{'}(t_c + t_d) + \dot{x}_1^{'}(t_c + t_d) \tag{8}$$

$$e = \frac{|\dot{x}_2^{'}(t_c + t_d) - \dot{x}_1^{'}(t_c + t_d)|}{|\dot{x}_1(t_c)|} \tag{9}$$

In our scenario, $\dot{x}_1(t_c)$, $\dot{x}_1^{'}(t_c + t_d)$, and $\dot{x}_2^{'}(t_c + t_d)$ are always positive; additionally, $\dot{x}_2^{'}(t_c + t_d) > \dot{x}_1^{'}(t_c + t_d)$. So, we can combine (8) and (9) to produce two equations that solve for velocity immediately after contact (10) for each of the balls:

$$\begin{aligned} \dot{x}_1^{'}(t_c + t_d) &= \tfrac{1}{2}\ \dot{x}_1(t_c)\ [1 - e] \\ \dot{x}_2^{'}(t_c + t_d) &= \tfrac{1}{2}\ \dot{x}_1(t_c)\ [1 + e] \end{aligned} \tag{10}$$

Next, both balls slip before returning to rolling. The balls slip from the moment of impact until t_s. This is calculated as follows (11), where u is the coefficient of kinetic friction and g is gravity:

$$t_s = \frac{2}{3} \frac{\dot{x}^{'}(t_c + t_d)}{ug} \tag{11}$$

During slip, the ball's motion is governed by the following equation of motion, where m is the mass of the ball:

$$\begin{aligned} x(t) = &-c_1[\dot{x}^{'}(t_c + t_d) + c_2]e^{-\frac{t-t_c-t_d}{c_1}} - \\ &c_2[t - t_c - t_d] + \\ &c_1\dot{x}^{'}(t_c + t_d) + x_o + c_1 c_2 \ , \end{aligned} \tag{12}$$

where $c_1 = \frac{m}{4\pi r^2 b}$ & $c_2 = \frac{mug}{b}$

Finally, after this time, both balls return to rolling (7 & 3), until the end of the trial time.