A perturbative view on the subsurface water pressure response at hillslope scale

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[1] This work deals with a rearrangement of Richards Equation to obtain a better understanding of the components of the subsurface water and pressure flows and their interactions in a hillslope. The basic idea proposed in this paper is the normalization of the spatial and temporal coordinates according to the physical knowledge of the processes and the geometry of a hillslope. The pressure head and all the other quantities contained in Richards Equation are rescaled to obtain a dimensionless equation. The pressure head is also split into an equilibrium component and a transient component and expanded in a perturbation series. Subsequently, Richards Equation itself is conveniently split into two equations connected by a source/sink term. Results of these manipulations show that, for the first order approximation, the water and pressure flow is slope-normal, while slope-parallel effects only occur at successive orders. Integrating the perturbed equation obtained for the long-term pressure head component results in the well-known Boussinesq Equation for the water table level. Further integrations then lead to the hillslope-storage Boussinesq model and finally to a generalization of the O'Loughlin formula for the slope-parallel long-term subsurface flow. The introduced source/sink term is estimated at the first order of approximation to be proportional to the temporal variation of water pressure head at the bottom of the integration domain. The treatment of the equations clarifies some of the mechanisms acting in water pressure redistribution at hillslope scale, even without solving the equation.

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1. Introduction

[2] The recognized physical model for the subsurface flow of water is the three dimensional (3D) Richards Equation [*Richards*, 1931]. *Tracy* [2006, 2007] and *Basha* [1999] have provided analytical solutions to the 3D Richards equation. However, these papers present analytical solutions only to the linearized version of 3D Richards equation, i.e., for a constant diffusivity and a hydraulic conductivity that is a linear function of the moisture content and currently no analytical solutions exist to the 3D nonlinear Richards equation. Rather complex 3D models are also used [e.g., *Paniconi and Putti*, 1994; *Abbott et al.*, 1986; *Rigon et al.*, 2006, and references therein]. However, they are usually computationally expensive and cannot be used to model large catchments. Thus since the early studies of *Philip* [1955], several simplified approaches have been proposed in which the subsurface water flow has been decoupled in a vertical, usually unsaturated, one-dimensional (1D) flow, and a two dimensional (2D) lateral, usually but not necessarily saturated, flow. This 1D (unsaturated) plus 2D (saturated) decoupling was accepted as a "de-facto" physical situation [e.g., *Hilberts et al.*, 2007, and reference therein], and used to model situations where either infiltration processes or water table dynamics (with negligible recharge) were important. Accordingly, the recent literature contains plenty of analytical or numerical solutions for the vertical flow equation [*Broadbridge and White*, 1988; *Barry and Sander*, 1991; *Simunek et al.*, 1998] or for the horizontal flow equation (Boussinesq equation) [*Brutsaert*, 1994; *Troch et al.*, 2003, 2004; *Daly and Porporato*, 2004]. However, the treatment of some crucial aspects of the catchment hydrology, as for instance the evolution of variable source areas [e.g., *Hewlett and Hibbert*, 1967] or the redistribution of water table levels and pore pressures after a storm [*Iverson*, 2000], requires the coupling of the vertical and lateral dynamics. A convenient representation of the coupling of the 1D vertical and 2D lateral flows was indeed pursued by some authors [e.g., *Pikul et al.*, 1974; *Croton and Barry*, 2001; *Vogel and Disek*, 2006; *Hilberts et al.*, 2007], but on the basis of heuristic arguments. The present paper tries a top-down derivation of the above coupled equations from the 3D Richards Equation parent. We also argue that the 1D plus 2D representation clarifies some aspects of the hillslope fluxes which can be overlooked when dealing with the complete 3D Richards Equation.

[3] Our approach turns out to be very similar to the one introduced by *Wallach et al.* [1989, 1991] but actually evolves from the paper by *Iverson* [2000] which, facing some issues related to hillslope stability, proposed his two-timescale decomposition of 3D Richards Equation. He assumed that water evolves slope-normally only within the soil thickness H, which is usually between $\sim$0–10 m,

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Figure 1. Graphical description of a hillslope profile in a hillslope point. Equations written in the paper are valid in any point, but the actual phenomena are thought to be observed at the hillslope closure.

and laterally within the whole hillslope area A which can be $\sim$100–1000 m^2 (see Figures 1 and 2). The difference between the spatial extent of hillslope side and soil depth and an assumed similarity in vertical and lateral hydraulic properties cause different time spans in which water flows vertically and laterally and this is the seed of the ideas developed in this paper. *Iverson* [2000], linearizing Richards Equation, shows that the vertical flow provides transient water pressure head profiles during a rainfall event super-imposed on the lateral flow which affects the long term (mainly interstorm) redistribution of water table in a hillslope. *D'Odorico et al.* [2005] provide a generalization of *Iverson*'s scheme applied to variable hyetographs. In the present paper the works of *Iverson* [2000] and of *Wallach et al.* [1989, 1991] are completed with further derivations (especially of the integrated forms of the 2D equation which have been presented in the recent literature), and expanded to include non-linear, nonsteady state cases with any parameterization of soil water retention and conductivity curves [e.g., *VanGenuchten*, 1980]. At the same time, the work of *Hilberts et al.* [2007] is complemented with a rigorous derivation of the 1D–2D coupled equations.

2. Rewriting the Hillslope Subsurface Flow in New Coordinates

[4] This section reproduces, with slightly different heuristic arguments, the derivation of *Iverson* [2000]. With no further discussion Richards Equation [*Richards*, 1931; *Brutsaert*, 2005] is here assumed to describe the unsaturated subsurface hillslope flow. It can be written as:

$$\frac{\partial}{\partial z}\left[k_z\frac{\partial\psi}{\partial z}-k_z\cos\alpha\right]+\frac{\partial}{\partial x}\left[k_x\frac{\partial\psi}{\partial x}-k_x\sin\alpha\right]+\frac{\partial}{\partial y}\left[k_y\frac{\partial\psi}{\partial y}\right]=C\frac{\partial\psi}{\partial t}+S \tag{1}$$

where the x[L] and z[L] axes are respectively downward parallel and normal to the slope as shown in Figures 1 and 2, while y [L] is perpendicular to them; t[T] is time; $\psi(x, y, x, t)$ [L] is the soil water pressure; $k_x(\psi)$, $k_z(\psi)$ and $k_y(\psi)$ [L T^{-1}] are the hydraulic conductivities referred to the respective axes; $C(\psi)$[L^{-1}] is the soil water storage capacity and $S(\psi)$[T^{-1}] is a sink term. The hydraulic conductivity is constant in saturated conditions ($\psi > 0$) whereas it decreases with the absolute value of ψ in unsaturated conditions ($\psi < 0$). The capacity C represents the slope of soil water retention curve [*VanGenuchten*, 1980] in unsaturated conditions while it depends on soil compressibility in saturated conditions [*Bear*, 1972]. The sink term S represents water root uptake losses principally due to evapotranspiration and will be assumed to be negligible, for the sake of simplicity, in the following derivation. Besides the dependences upon space and time, all the above quantities depend also on a set of parameters characterizing the soil.

[5] The analysis developed by *Iverson* [2000] starts with a nondimensionalization of (1) by rescaling the coordinates with respect to the soil depth H, measured normal to the slope and shown in Figure 1, and a characteristic hillslope length L. This length can be chosen as the mean distance of any point in the hillslope to the channel where it drains, the

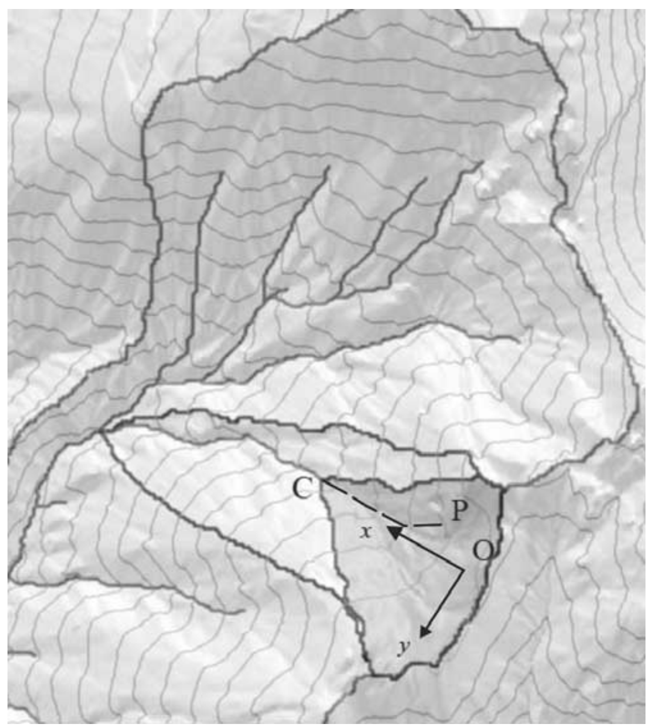

Figure 2. Figure represents a map of a small catchment, river network and a hillslope (hollow type, in gray). The distance of any point (P in the figure) in the hillslope to the channel head (C in the figure) is evaluated along the path drawn following the steepest descent (the dashed line). The characteristic length of the hillslope L (the length of x axis in Figure) is the mean of hillslope to channel distance for any point in the hillslope. The x axis used in the paper is downward parallel to mean topographic gradient, the axis y is normal to x (parallel to contour lines in a planar hillslope) and the z axis orthogonal to the x and y axes downward.

$$z^* = \frac{z}{H} \quad x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad t^* = \frac{t}{T_o} \tag{2}$$

The rationale of this choice is to highlight the effects at the hillslope scale where the subsurface flow occurs, with a proper choice of the axes shown in Figure 2, mainly in the x direction within the whole catchment area A, while slope-normal flow occurs within soil thickness H. The width of the hillslope has the same magnitude order of the characteristic length and then y is also rescaled in the same way of x. As *Wallach et al.* [1989, 1991] suggest: "Typical characteristic dimensions in the horizontal directions may be: physical dimensions of the domain, characteristic wavelength of the soil topography, characteristic wavelength of changes in horizontal soil properties (...). Typical lengths in the vertical direction may be one of the following: depth to a physical boundary, characteristic thickness of a profile (...). Any other choice complicates the algebra and eliminates itself in the final solution". Possible small variations of topography introduce smaller spatial scales which can have relevant local effects but are usually negligible when the whole hillslope is considered, as suggested by *Philip* [1991a, 1991b, 1991c] for a completely unsaturated flow, and by numerical simulations from the authors (not reported in the present paper).

[6] The phenomena are observed (averaged) at the timescale T_o. Time units are, at the moment, arbitrarily chosen by the observer and the physical challenge is to choose a timescale which captures the relevant variations in flow dynamics at the hillslope scale. To complete the dimensionless rewriting, it is necessary to rescale also the pressure fields and the hydraulic parameters according to

$$\psi^* = \frac{\psi}{H} \quad k_\eta{}^* = \frac{k_\eta}{k_0} \quad \eta = x, y, z \quad C^* = \frac{C}{C_0} \tag{3}$$

In fact, during a rainfall event, ψ has an order of magnitude comparable with H and although the hydraulic conductivity is generally anisotropic, the components k_x, k_y, k_z are rescaled with the same reference value k_0. *Iverson* [2000] proposes the saturated hydraulic conductivity k_S as k_0, but k_0 can be any typical value of hydraulic conductivity. The soil water capacity C is rescaled with a reference capacity value C_0 which can be related to an averaged slope of soil water retention curve.

[7] Appropriate observation timescale values are introduced for T_o by *Iverson* [2000]. In particular, he heuristically distinguishes a scale T_S [T] for slope-normal flow:

$$T_S = \frac{H^2}{D_0} \tag{4}$$

and a scale T_L [T] for slope-parallel lateral flow

$$T_L = \frac{L^2}{D_0} \quad D_0 \equiv \frac{k_0}{C_0} \tag{5}$$

where D_0 [L^2 T^{-1}] is a reference soil water diffusivity and corresponds to the ratio of the reference hydraulic conductivity k_0 to the reference soil water capacity C_0 ($D_0 = k_0/C_0$). Another possible choice for the normalization of the vertical flow would be $T_M = H/(dk/d\theta)$ which accounts for the rate of vertical infiltration processes at intermediate and long timescale. However, in view of some developments made in the next section, $T_o = T_S$ will be chosen throughout the paper.

[8] Thus after (1) is multiplied by H/k_0, Richards Equation is written in the following dimensionless form:

$$\frac{\partial}{\partial z^*}\left[k^*_z \frac{\partial \psi^*}{\partial z^*} - k^*_z \cos\alpha\right] + \epsilon^2 \frac{\partial}{\partial x^*}\left[k^*_x \frac{\partial \psi^*}{\partial x^*} - \epsilon^{-1} k^*_x \sin\alpha\right] + \epsilon^2 \frac{\partial}{\partial y^*}\left[k^*_y \frac{\partial \psi}{\partial y^*}\right] = C^* \frac{\partial \psi^*}{\partial t^*} \tag{6}$$

where $\epsilon = H/L$ and it is considered $\epsilon \ll 1$ [*Iverson*, 2000]. The factor ϵ^2 can also be seen as the ratio between the two reference timescales defined by (5) and (4).

3. Setting a New Framework for Solving Richards Equation at Hillslope Scales

[9] This section transforms Richards Equation into a coupled system of equations, one for the water table and one for the vertical infiltration. This is made possible by decomposing the solution into an equilibrium plus a transient part, and a subsequent perturbative expansion of the same equations. An interesting assumption, in line with *Iverson*'s and *D'Odorico et al.*'s treatment of (6), is that the unknown ψ^* can be written as the sum of two components:

$$\psi^* = (z^* - d^*)\cos\alpha + \psi^*_S \tag{7}$$

where $d^*(x, y, t) = d(x, y, t)/H$ and $d(x, y, t)$ [L] is the water table depth (measured from soil surface downward), (z^*-d^*) cos α can be recognized to be the dimensionless hydrostatic pressure distribution, and ψ^*_S is a dimensionless component of water pressure head which vanishes, $\psi^*_S \to 0$, for large times, ($t^* \to \infty$), and which we call in the following "short term" dimensionless pressure. The variable $d^*(x^*, y^*, t)$ does not depend on z^*. The definition in (7) suggests that (6) can be rewritten as a system of two coupled equations:

$$\frac{\partial}{\partial z^*}\left[k^*_z \frac{\partial \psi^*_S}{\partial z^*}\right] + \epsilon^2 \frac{\partial}{\partial x^*}\left[k^*_x \frac{\partial \psi^*_S}{\partial x^*}\right] + \epsilon^2 \frac{\partial}{\partial y^*}\left[k^*_y \frac{\partial \psi^*_S}{\partial y^*}\right] = C^* \frac{\partial \psi^*_S}{\partial t^*} - C^* N^* \cos\alpha \tag{8}$$

$$-\epsilon^2 \frac{\partial}{\partial x^*}\left[k^*_x \frac{\partial d^*}{\partial x^*}\cos\alpha + \epsilon^{-1} k^*_x \sin\alpha\right] - \epsilon^2 \frac{\partial}{\partial y^*}\left[k^*_y \frac{\partial d^*}{\partial y^*}\cos\alpha\right] = -C^* \frac{\partial d^*}{\partial t^*}\cos\alpha + C^* N^* \cos\alpha \tag{9}$$

where C^*N^* cos α is a source-sink term and represents the water volume exchanges between the equations (8) and (9), i.e., between the water table motion and the vertical distribution of soil moisture.

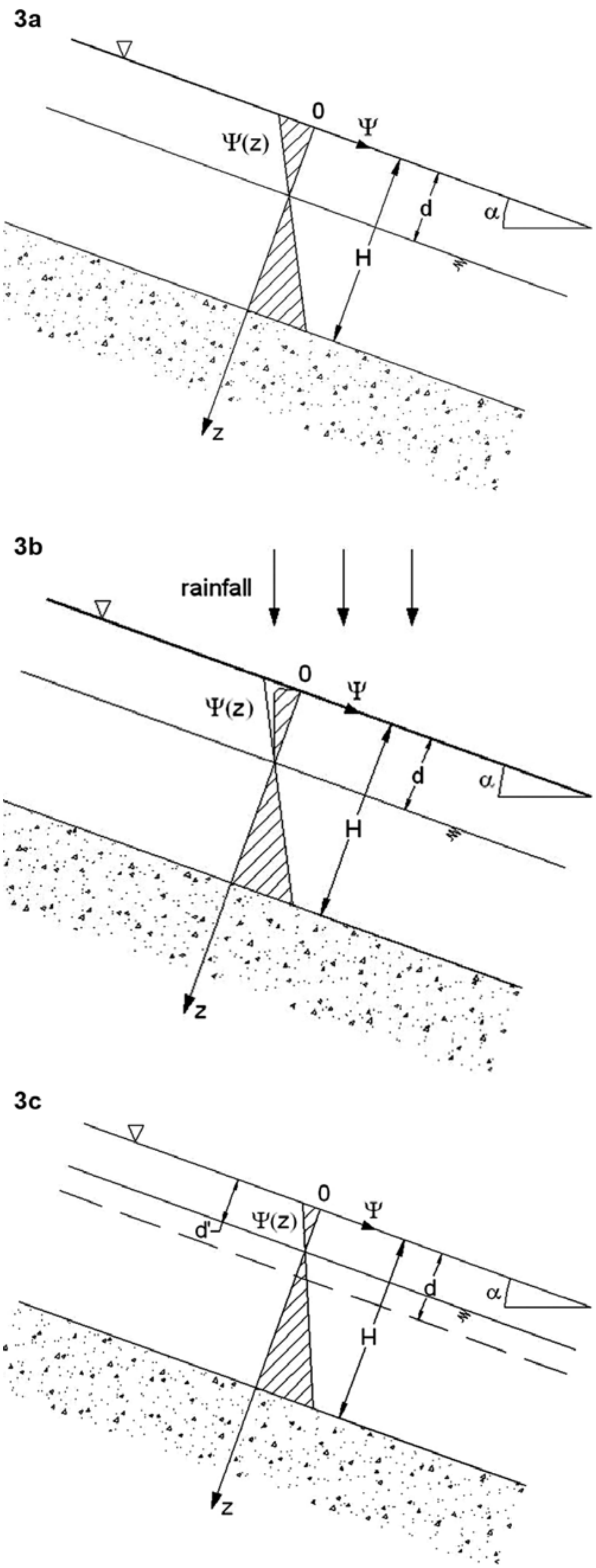

Figure 3. A qualitative example of short-term ψ slope-normal patterns due to a rainfall event: initial equilibrium profile with $\lambda = 0$ and water table depth d (in the text d_0 (x, y, 0)), soil thickness H (at the top)); transient regime and profile during rainfall event (in the middle); asymptotic equilibrium profile after rainfall event, d' (in the text d_0 (x, y, ∞)) is the new value of water table depth, the stored water volume is increased of the total rainfall depth since the begin, state (at the bottom). For an alternative view of the same concepts, see *Hilberts et al.* [2007] Figures 1 and 2.

3.1. Water Table Motion and Pressure Head Dynamics

[10] The water table uplift rate, N^*, is not known, however it can be quantified according to the following arguments, which are similar to those by *Hilberts et al.* [2007], and can be derived considering a slope portion as that illustrated in Figure 3. The soil water pressure head ψ^* has initially a local vertical equilibrium distribution:

$$\psi^*(x^*, y^*, z^*, 0) = [z^* - d^*(x^*, y^*, 0)] \cos\alpha \tag{10}$$

where $d^*(x^*, y^*, 0)$ is the initial water table depth (Figure 3a). During a rainfall event, the shallower layers near the surface get wet and ψ^* increases toward 0 (Figure 3b). After a relatively long time since the rainfall event, a new equilibrium profile is re-established and water table lifts up to a new value $d^*(x^*, y^*, \infty)$ (Figure 3). Then, at $t^* \to \infty$, it is:

$$\psi^*(x^*, y^*, z^*, \infty) - \psi^*(x^*, y^*, z^*, 0) = [d^*(x^*, y^*, 0) - d^*(x^*, y^*, \infty)] \cos\alpha \tag{11}$$

In fact, a relation similar to (11) can be assessed for any time interval $[t^*, t^* + \Delta t^*]$ at the soil bottom ($z^* = 1$):

$$\psi^*(x^*, y^*, 1, t^* + \Delta t^*) - \psi^*(x^*, y^*, 1, t^*) = [d^*(x^*, y^*, t^*) - d^*(x^*, y^*, t^* + \Delta t^*)] \cos\alpha \tag{12}$$

Eventually, dividing both sides of (12) by Δt^* and taking the limit $\Delta t^* \to 0$, the following equation is obtained:

$$\frac{\partial \psi^*}{\partial t^*}\Big|_{z^*=1} = -\frac{\partial d^*}{\partial t^*} \cos\alpha \tag{13}$$

This equation is still not a characterization of N^* but, as it is proved in the following sections, it is its main foundation.

[11] The derivative $\partial\psi^*/\partial t^*$ in (13) can be referred to any level in the saturated zone. The choice of putting $z^* = 1$ in (13) has been made for the sake of simplicity. This case, by inverting (7), the equation (13) implies to consider $\psi_S^* = 0$ at the bottom. However, both ψ^* and d^* vary inside the domain.

3.2. The Perturbative Expansion of Equations of Infiltration and Water Table

[12] Because the ϵ^2 term tends to be small at any hillslope closure, we can consider that the dynamics of (8) and (9) is dominated by the terms which are not multiplied by this factor. The correct algebraic procedure to present this fact, as suggested in *Holmes* [1995], is the expansion of the fields

ψ^*, ψ^*_S and d^* (and, as a consequence, the N^*) as follows:

$$\psi^* = \psi^*_0 + \epsilon^2 \psi^*_1 + O(\epsilon^4) \tag{14}$$

$$\psi^*_S = \psi^*_{S0} + \epsilon^2 \psi^*_{S1} + O(\epsilon^4) \tag{15}$$

$$d^* = d^*_0 + \epsilon^2 d^*_1 + O(\epsilon^4) \tag{16}$$

$$N^*(\psi^*) = N^*_0(\psi^*_0) + \epsilon^2 N^*_1(\psi^*_0, \psi^*_1) + O(\epsilon^4) \tag{17}$$

where

$$\psi^*_0 = \psi^*_{S0} + (z^* - d^*_0)\cos\alpha \tag{18}$$

$$\psi^*_i = \psi^*_{Si} - d^*_i \cos\alpha \qquad i = 1, 2, 3, \ldots \tag{19}$$

Then substituting these expansion of the fields in the equations (8) and (9), and considering the terms multiplied by the same power of ϵ^2, we can properly separate the dominant terms. At the ϵ^0-order approximation, (8) and (9) become respectively:

$$\frac{\partial}{\partial z^*}\left[k_z^* \frac{\partial \psi^*_{S0}}{\partial z^*}\right] = C^* \frac{\partial \psi^*_{S0}}{\partial t^*} - C^* N^*_0 \cos\alpha \tag{20}$$

$$C^* \frac{\partial d^*_0}{\partial t^*} \cos\alpha = C^* N^*_0 \cos\alpha \tag{21}$$

and the ϵ^0-order expansion of (13) substituted in (21) gives the ϵ^0-order water table uplift:

$$N_0^* = -\frac{1}{\cos\alpha} \frac{\partial \psi^*_0}{\partial t^*}\Big|_{z^*=1} \tag{22}$$

This shows that, at this order approximation, the recharge term N^* is constant in z^* and that the water table starts to grow when the variation of the pressure head at the bottom of the integration domain is relevant.

[14] Once reintroduced in the original equations (8) and (9) the solution $\psi^*_{S0}(x^*, y^*, z^*, t^*)$ and $d^*_0(x^*, y^*, t^*)$, only the terms multiplied by a power of ε survive and we obtain the ε^2-order approximation of the original equations:

$$\frac{\partial}{\partial z^*}\left[k_z^* \frac{\partial \psi^*_{S1}}{\partial z^*}\right] + \frac{\partial}{\partial x^*}\left[k_x^* \frac{\partial \psi^*_{S0}}{\partial x^*}\right] + \frac{\partial}{\partial y^*}\left[k_y^* \frac{\partial \psi^*_{S0}}{\partial y^*}\right] = C^* \frac{\partial \psi^*_{S1}}{\partial t^*} - C^* N^*_1 \cos\alpha \tag{23}$$

$$-\frac{\partial}{\partial x^*}\left[k_x^* \frac{\partial d^*_0}{\partial x^*}\cos\alpha - \epsilon^{-1} k_x^* \sin\alpha\right] - \frac{\partial}{\partial y^*}\left[k_y^* \frac{\partial d_0^*}{\partial y^*}\cos\alpha\right] = -C^* \frac{\partial d^*_1}{\partial t^*}\cos\alpha + C^* N^*_1 \cos\alpha \tag{24}$$

The unknowns in (23) and (24) are those with subscript "1", while the other quantities derive from the solution of the first order equations (20) and (21). Also, for this expansion order, the use of (13) allows for the determination of the recharge rate N^*_1, by inversion of (24). At this order, the recharge depends upon the vertical infiltration and on the lateral fluxes generated by the gradients in the previous order field ψ^*_0 (which could be z-dependent).

[15] The term $\epsilon^{-1}\sin\alpha$ is an advection term and should have been treated as an ϵ-order term. The presence of such term, means that, in steep slopes, the ψ^*_{S1} evolution would be dominated by topography-driven advection, while in relatively gentle slopes, the quantity $\epsilon^{-1}\sin\alpha$ would be negligible in any case. In principle, this would call for a different expansion of the field ψ. However, this expansion would have made the algebra more cluttered without adding any further information. In fact, it can be proven by algebraic manipulations that the correction term ψ_{S1} accounts properly with no loss of physical information of all the above contributions.

4. Integrated Versions of the Subsurface Flow

[16] Hillslope flow dynamics are usually presented in integrated forms which respond to the needs to characterize "on average" the properties of a hillslope, since the complete local characterization of soil hydraulic properties can never be obtained and it is often not even required.

4.1. Integrated Slope-Normal Flow

[17] This subsection discusses the link between slope-normal infiltration and the water table recharge dynamics.

[18] At the ϵ^0 order, water flow is slope-normal. This is expressed by the coupled equations (20) (in ψ^*_{S0}) and (21) (in d^*_0) or, equivalently, by the one-dimensional Richards Equation in ψ^*_0:

$$\frac{\partial}{\partial z^*}\left[k_z^* \frac{\partial \psi^*_0}{\partial z^*} - k_z^* \cos\alpha\right] = C^* \frac{\partial \psi^*_0}{\partial t^*} \tag{25}$$

The commonly used boundary conditions associated with (25) can be written as:

$$-k_z^* \frac{\partial \psi^*_0}{\partial z^*} + k_z^* \cos\alpha = I^*(t^*)\cos\alpha \quad z^* = 0 \tag{26}$$

with

$$I^*(t^*) = i^*(t^*) \mathrm{H}\left(\psi^*_{\mathrm{sat}} - \psi^*_0|_{z^*=0}\right) + \frac{k_S}{k_0} \cdot \left[-\frac{1}{\cos\alpha}\frac{\partial \psi_0^*}{\partial z^*}\Big|_{z^*=0} + 1\right] \mathrm{H}\left(\psi^*_0\,|_{z^*=0} - \psi^*_{\mathrm{sat}}\right) \tag{27}$$

and

$$-k_z^* \frac{\partial \psi_0^*}{\partial z^*} + k_z^* \cos\alpha = \lambda(t^*) k_z^* \cos\alpha \quad z^* = 1 \tag{28}$$

where H(x) is the Heaviside step function ($H(x)$ = 1, $x \geq 0$; $H(x)$ = 0, $x < 0$), ψ^*_{sat} is a dimensionless reference pressure head value near saturation ($\psi^*_{sat} \sim 0$) and the parameter λ is a function of time, and has been introduced

λ is a ratio of

are impermeable bottom, for $\lambda = 0$, and only gravitational flux at the bottom, for $\lambda = 1$.

[19] The surface condition (26) is imposed on the water flux, and means that the rate $I^*(t^*)$ provided by (27) all infiltrates the soil surface. When the pressure head $\psi_0{}^*$ at the soil surface is less than the threshold value ψ^*_{sat}, it corresponds to the rainfall intensity $i^*(t^*)$. Otherwise, when soil surface gets saturated and ponds ($\psi^*{}_0|_{z^*=0} > \psi^*{}_{sat}$), $I^*(t^*)$ depends on the solution of (25) and the surface boundary conditions is a Dirichlet condition in which water pressure head $\psi^*{}_0$ at the surface is controlled by the surface runoff height. The Heaviside function used in (27) corresponds to the switching of boundary conditions in many numerical models [*Simunek et al.*, 1998; *Bixio et al.*, 2000; *Cordano et al.*, 2004a]. In unsaturated conditions (i.e., for $\psi^* < 0$), equation (25) contains strong nonlinearities, due to the soil water retention and conductivity curves which are usually described by parametric laws such as *VanGenuchten* [1980] and *Mualem* [1976]. Once solved the equation (25) in ψ_0^* with (26) and (28) as boundary condition, a temporal behavior of d_0^* is obtained by applying (13) at the ε^0 order. Then, by definition, ψ_{S0}^* is the difference $\psi^*{}_0 - (z^* - d_0^*)\cos\alpha$ (equation (18), and then $\psi_S^* = 0$ at $z^* = 1$ is verified).

[20] Since it is a divergence, the integral of the left-hand side of (25) is equal to the difference of the two boundary conditions (26) and (28):

$$\int_0^1 \frac{\partial}{\partial z^*}\left[k_z{}^*\frac{\partial \psi^*{}_0}{\partial z^*} - k^*{}_z\cos\alpha\right]dz^* = I^*(t^*)\cos\alpha - \lambda(t^*)k_z{}^*\left.\right|_{z^*=1}\cos\alpha \tag{29}$$

and corresponds to the difference between the water fluxes that infiltrate soil and percolate through the bottom. The integral of the right-hand side of (25) is:

$$\int_0^1 C^*\frac{\partial \psi^*{}_0}{\partial t^*}dz^* = \int_0^1 \frac{\partial \theta_0{}^*}{\partial t^*}\,dz^* = \frac{\partial V^*{}_0}{\partial t^*} \tag{30}$$

where $\theta^*{}_0(\psi^*{}_0)$ is the ε^0 order component of the rescaled water content θ^* ($\theta^* = \theta/C_0H$) and $V^*{}_0(x^*, y^*, t^*)$ is the ε^0 order estimation of the dimensionless water volume stored in the soil column $V^*(x^*, y^*, t^*)$, defined as follows:

$$V^* = \frac{V}{C_0H^2} \quad V = \int_0^H \theta dz$$
$$V^* = \int_0^1 \theta^* dz^* \quad V^*{}_0 = \int_0^1 \theta^*{}_0 dz^* \tag{31}$$

with $V(x, y, t)$ [L] as the total water volume which is stored in the soil. Thus after (29) and (30), the integration of (25) gives:

$$\frac{\partial V^*{}_0}{\partial t^*} = I^*(t^*)\cos\alpha - \lambda(t^*)k^*{}_z\left.\right|_{z^*=1}\cos\alpha \tag{32}$$

The above (32) is a trivial application of the divergence theorem which becomes interesting when it is split into two components, one related to the transient infiltration (i.e., to $\psi^*{}_{S0}$) and the other to the local equilibrium vertical profile of soil moisture (i.e., to $d^*{}_0$). *Hilberts et al.* [2005] establish a relation between the total unsaturated plus saturated stored water volume and the water table depth d^*, valid for the equilibrium profile, i.e., for $\psi_{S0}^* = 0$ and $\psi_0^* = (z^* - d_0^*)\cos\alpha$. In reality, during a rainfall event $V^* = V_e^* + V_{ne}^*$ where $V^*{}_e$ is the equilibrium dimensionless water volume:

$$V_e{}^* = \frac{V_e}{C_0H^2}V_e = \int_0^H \theta((z-d)\cos\alpha)dz \tag{33}$$

[21] After *Hilberts et al.* [2005] (equations (6) to (12)), it results:

$$\frac{\partial V_e{}^*}{\partial t^*} = -f^*\frac{\partial d^*}{\partial t^*}\frac{\partial V_{e0}{}^*}{\partial t^*} = -f^*\frac{\partial d_0{}^*}{\partial t^*} \tag{34}$$

where the new quantity f^*, named soil water storage-dependent porosity, can be obtained by generalizing the equation (12) of *Hilberts et al.* [2005]:

$$f^* = -\frac{\partial V^*{}_e}{\partial d^*} = -\frac{1}{C_0H}\frac{\partial}{\partial d}\int_0^H \theta((z-d)\cos\alpha)\quad dz = \frac{\theta((H-d)\cos\alpha) - \theta(-d\cos\alpha)}{C_0H} \tag{35}$$

and θ [dimensionless] is here the volumetric water content evaluated for the indicated values of ψ [*VanGenuchten*, 1980].

[22] The expression for the transient volume can be obtained by adding (21) and (32) which gives:

$$\frac{\partial V^*{}_0}{\partial t^*} = I^*(t^*)\cos\alpha - \lambda(t^*)k_z{}^*\left.\right|_{z^*=1}\cos\alpha - f^*\frac{\partial d_0{}^*}{\partial t^*}\cos\alpha + f^*N_0{}^* \tag{36}$$

and leads to:

$$\frac{\partial V_{ne0}{}^*}{\partial t^*} = i^*(t^*)\cos\alpha - \lambda(t^*)k_z{}^*\left.\right|_{z^*=1}\cos\alpha - P_0{}^* \tag{37}$$

where $P_0{}^*(x^*, y^*, t^*) \equiv -f^*N^*{}_0$ is the ε^0-order dimensionless water table recharge rate.

[23] The volume V_{ne0}^*, which is mostly contained in the unsaturated zones near surface, constitutes a buffer between rainfall and water table recharge. Its time derivative accounts for possible transient deviations from the hydrostatic pressure distribution.

4.2. Integrating the Slope-Parallel Flow (Water Table Dynamics)

[24] The results of the ϵ^2-order approximation are usually presented in literature in integrated forms. In particular, this section shows that the equations (21) and (24) are linked to a well known case of the Boussinesq Equation.

[25] In fact, integrating the equation (24) within the soil thickness, after some algebraic manipulations, shown in Appendix A, it is obtained:

$$-\frac{\partial}{\partial^x}\left[T_x{}^*\frac{\partial d^*}{\partial^*}\cos\alpha + \epsilon^{-1}T_x{}^*\sin\alpha\right] - \frac{\partial}{\partial y^*}\cdot\left[T_y{}^*\frac{\partial d^*}{\partial y^*}\cos\alpha\right] = -\epsilon^{-2}f^*\frac{\partial d^*}{\partial t^*} - \epsilon^{-2}P^* + \mathrm{O}(\epsilon^2) \tag{38}$$

$_x(x^*, y^*, t^*)$ and $T^*_y(x^*, y^*, t^*)$ are the dimension-
x and y directions defined in (A1) and (A2) respectively, f^* is the drainable porosity and P^* is the dimensionless water table recharge rate. The equation (38) is a general form of the Boussinesq equation in d^*, whose analytical and semi-analytical solutions were recently provided for a variety of hillslope geometries [*Brutsaert*, 1994; *Troch et al.*, 2003, 2004; *Hilberts et al.*, 2005, 2007]. A particular case of the Boussinesq Equation has also been solved by *Daly and Porporato* [2004]. *Basha and Maalouf* [2005] reviewed and accurately discussed the solutions of the subsurface flow in a sloping aquifer under various conditions. In most cases, the water table is recharged by rainfall infiltration and P^* can be approximated with P^*_0. Furthermore, if the slope-parallel flow occurs mostly in saturated zones, the transmissivities T^*_x and T^*_y can only be considered as functions of d^* and simplified as follows:

$$T_\eta{}^* = \frac{k_S(1-d^*)}{k_0} \eta = x, y \tag{39}$$

with the assumption that k_S is now the slope-normally averaged saturated conductivity.

[26] Because of the coefficient ϵ^{-2} before the time derivative in (38), significant variations of d^* result to occur at the long timescale (i.e., T_L) instead of T_S. In fact, the derivative with respect to $\epsilon^2\ t^*$ is equivalent to considering the rate of time variation using T_L as T_0 for normalizing the time coordinate, t, as suggested by *Iverson* [2000].

4.3. The Hillslope-Boussinesq-Storage Model (hBs)

[27] *Troch et al.* [2003] and others averaged (38) along the slope width. In fact, if a hillslope can be thought as a unique flow tube with a width $w^* \equiv w/L$ (w [L]) and downhill coordinate x^*, the equation (38), after integration along the y^* direction (orthogonal to x^*), becomes:

$$\begin{aligned} &-\frac{1}{w^*}\int_{w^*}\frac{\partial}{\partial x^*}\left[T^*_x\frac{\partial d^*}{\partial x^*}\cos\alpha+\epsilon^{-1}T^*_x\sin\alpha\right]dy^* \\ &\quad-\frac{1}{w^*}\int_{w^*}\frac{\partial}{\partial y^*}\left[T^*_y\frac{\partial d^*}{\partial y^*}\cos\alpha\right]dy^* \\ &\quad=-\epsilon^{-2}\frac{1}{w^*}\int_{w^*}f^*\frac{\partial d^*}{\partial t^*}dy^*-\epsilon^{-2}\frac{1}{w^*}\int_{w^*}P^*dy^* \end{aligned} \tag{40}$$

The basis for obtaining the hBs model from (40) is to assume that variation around the mean of the quantities in the integrals above (in particular, the water table depth, d^*, and the recharge, P^*) are negligible, i.e., $d^* \sim d^*(x^*, t^*)$ and $P^* \sim P^*(x^*, t^*)$. Under these assumptions, it is:

$$\begin{aligned} &-\frac{1}{w^*}\frac{\partial}{\partial x^*}\left[w^*T_x{}^*\frac{\partial d^*}{\partial x^*}\cos\alpha+\epsilon^{-1}w^*T_x{}^*\sin\alpha\right] \\ &\quad+\left[T_x{}^*\frac{\partial d^*}{\partial x^*}\cos\alpha+\epsilon^{-1}T_x{}^*\sin\alpha\right]\frac{1}{w^*}\frac{\partial w^*}{\partial x^*} \\ &\quad=-\epsilon^{-2}f^*\frac{\partial d^*}{\partial t^*}-\epsilon^{-2}P^* \end{aligned} \tag{41}$$

Then, using (39):

$$\begin{aligned} &-\frac{r}{w^*}\frac{\partial}{\partial x^*}\left[w^*(1-d^*)\frac{\partial d^*}{\partial x^*}\cos\alpha\right. \\ &\quad\left.+\epsilon^{-1}w^*(1-d^*)\sin\alpha\right]+\left[(1-d^*)\frac{\partial d^*}{\partial x^*}\cos\alpha\right. \\ &\quad\left.+\epsilon^{-1}(1-d^*)\sin\alpha\right]\frac{r}{w^*}\frac{\partial w^*}{\partial x^*}=-\epsilon^{-2}f^*\frac{\partial d^*}{\partial t^*}-\epsilon^{-2}P^* \end{aligned} \tag{42}$$

where: $r = k_S/k_0$. The equation (42) constitutes the basis of the Hillslope-Boussinesq-Storage model [*Troch et al.*, 2003] and it is often solved with the following boundary conditions [*Troch et al.*, 2003, 2004; *Basha and Maalouf*, 2005; *Berne et al.*, 2005]:

$$(1-d^*)\frac{\partial d^*}{\partial x^*}\cos\alpha+\epsilon^{-1}(1-d^*)\sin\alpha=0 \quad x^*=0 \tag{43}$$

$$d^* = d^*\,|_{x^*=L^*}\,(t^*) \qquad x^* = L^* \tag{44}$$

These are a no-flux condition at the top ($x^* = 0$) and an inhomogeneous Dirichlet condition at the bottom ($x^* = L^*$ if L^* is the dimensionless length of the hillslope for example) which depends on surface or channel flow.

[28] Further simplifications of the Boussinesq equations are derived in Appendix B.

5. Discussions and Interpretations

[29] The coupled equations presented in the previous section are not solved in the present paper, however, solutions under a variety of boundary conditions exist in literature and are cited along the text. From the point of view of this paper, those solutions could be seen as approximations of the 3D fields.

[30] The perturbation expansion of the pressure field, in itself, is valid for any value of ϵ. However, the first orders are a good approximation of the exact solution of (1) only if ϵ is small and the gradients along the x and y are also small. The first is a condition which derives from the geometry of a hillslope, and assuming isotropic hydraulic properties: it can fail to be verified inside the hillslope, close to the hillslope apex. The latter is a dynamical condition, and can be verified only after integration.

[31] The integrated form of the 1D vertical infiltration equation introduces a new "source-sink" term P^*_0 whose characterization depends on the knowledge of the time derivative of ψ^*_0 at the bottom of the integration domain, which is actually not available if the original (25) is not solved, and thus, not all the required information is actually available at the integrated level. Thus the integral form of the vertical infiltration equation is more a theoretical tool to explore the connection with the Boussinesq equation than a practical one. However, many authors proposed directly several solutions of (25) at different degree of approximation. Among the others, *Broadbridge and White* [1988], *Segol* [1994] and *Barry et al.* [1993] proposed analytical solutions under simplified boundary conditions; *Ross and Parlange* [1994] compared exact and approximate solu-

[2000] and *Marinelli and*

of solution; the linear diffusion approximation of the equation (25) was utilized by *Iverson* [2000] and *D'Odorico et al.* [2005]. A complete and definitive integration of 1D Richards Equation is to be found in numerical models [e.g., *Simunek et al.*, 1998; *Warrick*, 2003; *Cordano et al.*, 2004a] in which possible soil heterogeneities and different kinds of boundary conditions can be treated.

[32] The water table dynamics (2D) equation is a diffusion-advection equation whose linear approximation is even present in textbooks [e.g., *Brutsaert*, 1994, 2005]. Recently, *Troch et al.* [2003, 2004] extended *Brutsaert*'s results to various convergent or divergent hillslopes, and confirmed the theoretical results with laboratory experiments and comparison with numerical solution of the whole 3D flow equation [*Paniconi et al.*, 2003; *Hilberts et al.*, 2005]. Furthermore, results compatible with ours have been recently introduced by *Hilberts et al.* [2007]: our approach reveals that their equations are the outcome of the application of a perturbation methodology to the 3D Richards Equation. Moreover, our approach is not limited to the capillary fringe, even if it is clear that the major contribution to flows derives from the wettest regions of the vadose zone.

[33] Our approach also shows that the traditional (and less traditional) 2D groundwater models are a good approximation to the complete 3D equation where large lateral hydraulic (or suction) gradients do not exist, i.e., the assumptions on which our derivations are based. These aspects must be considered in the application of the equation to case studies of which *Hilberts et al.* [2007] give an example.

[34] Finally, the subsurface flow equations in the form obtained in this paper highlight at least one further aspect of the flow phenomena at the hillslope scale which is worth discussing. This aspect is related to the actual value of the hydraulic capacity parameter C^*, which enters both the ϵ^0-order infiltration equations, (20) and (21), and the ϵ^2-order lateral flow equations, (23) and (24). This parameter, according to the experimental studies reported by *Brutsaert* [2005], and the common parameterizations given in literature [e.g., *VanGenuchten*, 1980], strongly decreases toward zero with the water content. Since it is known that the pressure waves generated by infiltration travel with celerity inversely proportional to the hydraulic capacity C and directly proportional to the hydraulic conductivity [*Rasmussen et al.*, 2000; *Cordano et al.*, 2004b], this would imply that pressure perturbations, generated at the soil surface by a rainstorm impulse, although attenuated in the vadose zone, according to (22), have a fast (compared to the saturated hydraulic conductivity) impact on the water table. Moreover, when $C^* \to 0$, i.e., in the capillary fringe and in the saturated zone, the equation governing the lateral flow is reduced to a form of the Laplace equation. This means that any upslope variation of the recharge on the top of the water table generated by rainstorms should almost instantaneously manifest at the closure of the hillslope. The singularity of the equations, determined by the vanishing control parameter C^*, implies that increasingly short time steps must be used to propagate the (nonlinear) parabolic waves generated by the transient infiltration.

6. Conclusion

[35] A mathematical perturbative formalism which treats the Richards Equation is presented. This formalism produces two coupled equations, one for the slope normal infiltration and one for the lateral hillslope-wide flows. In order to describe soil water subsurface flow at the hillslope scale, spatial nondimensionalization is taken essentially according to *Iverson* [2000] and *Hilberts et al.* [2005, 2007], where the lateral scales are larger than the vertical one, i.e., soil thickness. The proposed perturbation analysis is shown to lead to classical equations for groundwater flow with different level of approximation coupled to 1D infiltration equations. The paper also shows that, under quite general conditions, and like by *Iverson* [2000], the two equations work at different time-scales: a shorter one, T_S for infiltration, and a longer timescale, T_L, for lateral fluxes. In the paper, this is obtained by expanding the solutions of the equation in a fast and a slow components, which are subsequently and consequently expanded in perturbation series. Thus the vertical flow is well described by a 1D non linear diffusion equation (which could be, but not necessarily, linearized), and the lateral flow by a 2D equation, the main component of which can be expressed in term of water table movements. The latter, given also in a vertically integrated form, results to be the 2D Boussinesq equation which was studied in many recent papers. Once separated the two flows, the approach developed is useful to link them again and to characterize with the 1D motions the boundary conditions for the 2D one. This, in turn, allows to treat those hillslope phenomena in which the recharge rate or the transient pressure redistribution are important. The singularity in the coupling of the slope normal and lateral flow equations suggests that the phenomena require careful modeling. In the paper, generalizations of the Boussinesq, of the Hillslope-storage Boussinesq, and *O'Loughlin*'s equation (see Appendix B) and an accurate discussion of the hypothesis under which they are valid are also provided.

Appendix A: Deriving the Boussinesq Equation

[37] The equation (24) is referred to the water table dynamics and can be integrated within the soil thickness (i.e., for $0 \leq z^* \leq 1$) without any loss of information. Then, introducing the following integrals:

$$T_x^* = \int_0^1 k_x^* dz^* \tag{A1}$$

$$T_y^* = \int_0^1 k_y^* dz^* \tag{A2}$$

$$F^* = \int_0^1 C^* \cos\alpha dz^* \tag{A3}$$

$$M_1^* = -\int_0^1 C^* N_1^* \cos\alpha dz^* \tag{A4}$$

$$-\frac{\partial}{\partial x^*}\left[T_x^*\frac{\partial d_0^*}{\partial x^*}\cos\alpha+\epsilon^{-1}T_x^*\sin\alpha\right]-\frac{\partial}{\partial y^*}\left[T_y^*\frac{\partial d_0^*}{\partial y^*}\cos\alpha\right] = -F^*\frac{\partial d_1^*}{\partial t^*}-M_1^* \tag{A5}$$

where the dimensionless coordinate z^* varies, by definition, between 0 and 1, T_x^* and T_y^* are the dimensionless aquifer transmissivities in the x and y directions respectively, F^* is a (dimensionless integrated) soil water storage capacity and M_1^* is the integrated ϵ^2-order source-sink term which derives from the full solutions in ψ_0^* and ψ_1^*, i.e., at the ϵ^0 and ϵ^2 orders respectively.

[38] Then, a new quantity P_1^*, which is the ϵ^2-order component of the water table recharge, is introduced as:

$$P_1^* = M_1^* - \left(F^* - f^*\right)\frac{\partial d_1^*}{\partial t^*} \tag{A6}$$

where f^* is the drainable porosity. Combining (A6) and (A5), it becomes:

$$-\frac{\partial}{\partial x^*}\left[T_x^*\frac{\partial d_0^*}{\partial x^*}\cos\alpha+\epsilon^{-1}T_x^*\sin\alpha\right] -\frac{\partial}{\partial y^*}\left[T_y^*\frac{\partial d_0^*}{\partial y^*}\cos\alpha\right] = -f^*\frac{\partial d_1^*}{\partial t^*}-P_1^* \tag{A7}$$

In order to obtain the Boussinesq equation, the equation (21) at the ϵ^0 order is added to (A7) in the following way:

$$-\frac{\partial}{\partial x^*}\left[T_x^*\frac{\partial d_0^*}{\partial x^*}\cos\alpha+\epsilon^{-1}T_x^*\sin\alpha\right] -\frac{\partial}{\partial y^*}\left[T_y^*\frac{\partial d_0^*}{\partial y^*}\cos\alpha\right] = -f^*\frac{\partial d_1^*}{\partial t^*}-\epsilon^{-2}f^*\frac{\partial d_0^*}{\partial t^*}-\epsilon^{-2}P_0^* - P_1^* \tag{A8}$$

Then, gathering the perturbative expansion (16), (A8) can be rewritten in the unknown d^*:

$$-\frac{\partial}{\partial x^*}\left[T_x^*\frac{\partial d^*}{\partial x^*}\cos\alpha+\epsilon^{-1}T_x^*\sin\alpha\right] -\frac{\partial}{\partial y^*}\left[T_y^*\frac{\partial d^*}{\partial y^*}\cos\alpha\right] = -\epsilon^{-2}f^*\frac{\partial d^*}{\partial t^*}-\epsilon^{-2}P^*+\mathrm{O}(\epsilon^2) \tag{A9}$$

and P^* is the dimensionless water table recharge and is the sum of the first and the second order approximation of the integrated recharge:

$$P^* = P_0^* + \epsilon^2 P_1^* + \mathrm{O}(\epsilon^4) \tag{A10}$$

where P_0^* is the recharge component due to slope-normal rainfall infiltration and P_1^* is due to possible water transfer between saturated and unsaturated zones at the ϵ^2 order. The equation (A9) is the Boussinesq Equation with storage-dependent drainable porosity [*Hilberts et al.*, 2005, 2007].

Appendix B: Further Simplifications

[39] *O'Loughlin* [1986] and *Dietrich and Montgomery* [1994] proposed an even more simplified model of hillslope subsurface flow assumed valid for relatively steep terrain. A generalization of their model is obtainable more rigorously by solving (42) under steady conditions and neglecting the diffusive term:

$$r\frac{\partial}{\partial x^*}\left[w^*\epsilon^{-1}\left(1-d^*\right)\sin\alpha\right] - r\left[\epsilon^{-1}(1-d^*)\sin\alpha\right]\frac{\partial w^*}{\partial x^*} = \epsilon^{-2}w^*P^* \tag{B1}$$

Since (B1) is a first-order differential equation, only the top boundary condition (43) controls the evolution of the solution (and not the bottom boundary condition, which has no influence). Then, integrating (B1) from the divides (x^* = 0) to the hillslope closure ($x^* = L^*$), it is:

$$r\left[w^*\epsilon^{-1}\left(1-d^*\right)\sin\alpha\right]_{x^*=L^*} = \int_0^{L^*}\epsilon^{-2}w^*P^*dx^* + \int_0^{L^*} r\left[\epsilon^{-1}\left(1-d^*\right)\sin\alpha\right]\frac{\partial w^*}{\partial x^*}dx^* \tag{B2}$$

To simplify the reading of (B2), the following quantities can be introduced:

$$I_{net}^* = \int_0^{L^*} w^* P^* dx^* \tag{B3}$$

where I_{net}^* is the mean dimensionless water table recharge rate, and:

$$B^* = \int_0^{L^*} r\left[\epsilon^{-1}\left(1-d^*\right)\sin\alpha\right]\frac{\partial w^*}{\partial x^*}dx^* \tag{B4}$$

where B^* is a quantity related to the average water table depth of the hillslope and its convergence or divergence (which is null for planar slopes, negative for convergent slopes, and positive for divergent slopes). In the work of *Troch et al.* [2004], $\partial w^*/\partial x^*$ was assumed to vary with an exponential law, bringing to further simplification of the equation. To sum up, (B2) leads to:

$$1-d^* = \epsilon^{-1}\frac{I_{net}^*+B^*}{rw^*\sin\alpha} \tag{B5}$$

[40] The equation is a generalization of the TOPOG formula [*O'Loughlin*, 1986; *Dietrich and Montgomery*, 1994]. Eventually, the original TOPOG expression is obtainable neglecting the hillslope convergence/divergence (i.e., putting B^* = 0). The TOPOG formula is a steady balance of the subsurface water in a hillslope, thus it has been used at relatively long timescale, and I_{net}^* corresponds to a pluri-daily averaged rainfall rate.

Table C1. Symbols With an Asterisk (e.g., I*) are Rescaled Dimensionless Quantities

Symbol	Dimension	Definition
A	[L^2]	surface area of the catchment
B^*		dimensionless quantity related to the average water table depth and the topography of the catchment, defined in equation (B4)
C	[L^{-1}]	soil water storage capacity
C_0	[L^{-1}]	reference soil water storage capacity (constant)
C^*		dimensionless soil water storage capacity
d	[L]	water table depth
d^*		water table depth rescaled with H
d^*_i		ϵ^{2i}-order component of d^*
f^*		storage-dependent drainable porosity [*Hilberts et al.*, 2005] rescaled with C_0H
F^*		integral of C^* cos α within the dimensionless rescaled soil thickness ($0 < z^* < 1$)
H	[L]	reference soil thickness (slope-normal spatial scale) (constant)
k_0	[L T^{-1}]	reference hydraulic conductivity (constant)
k_S	[L T^{-1}]	saturated hydraulic conductivity
k_x, k_y, k_z	[L T^{-1}]	hydraulic conductivities
k^*_x, k^*_y, k^*_z		hydraulic conductivities rescaled with k_0
$i(t)$	[L T^{-1}]	rainfall intensity (function of time)
$i^*(t^*)$		rainfall intensity (function of time) rescaled with k_0
$I(t)$	[L T^{-1}]	infiltration rate at the surface (function of time)
$I^*(t^*)$		infiltration rate (function of time) rescaled with k_0
I^*_{net}		mean dimensionless water table recharge rate in the whole catchment, defined in equation (B3)
L	[L]	mean hillslope to channel length of the hillslope (constant) as shown in Figure 2
L^*		length of the catchment L rescaled with the hillslope characteristic length L
M_1^*		integral of C^* N_1^* cos α within the dimensionless rescaled soil thickness ($0 < z^* < 1$)
N	[L T^{-1}]	pressure head rate related to the water table uplift
N^*		dimensionless pressure head rates related to the water table uplift
N^*_i		ϵ^{2i}-order component of N^*_d, $N^*_{\psi_S}$ respectively.
P^*		dimensionless water table recharge rate rescaled with k_0
P^*_i		ϵ^{2i}-order component of dimensionless water table recharge P^*
r		ratio k_S/k_0
s		volumetric moisture content
S	[T^{-1}]	root uptake sink term
t	[T]	time
t^*		rescaled time
T_o	[T]	observation timescale (constant)
T_S	[T]	short timescale [*Iverson*, 2000] (constant)
T_L	[T]	long timescale [*Iverson*, 2000] (constant)
T_M	[T]	intermediate timescale for vertical subsurface flow
T^*_x, T^*_y		dimensionless transmissivities in x and y directions respectively rescaled with k_0 H
V	[L]	total water volume stored within the soil thickness
V_e	[L]	total water volume stored within the soil thickness in equilibrium with the water table level [*Hilberts et al.*, 2005]
V^*, V^*_e		water stored volumes V, V_e rescaled with C_0 H^2
V^*_{ne}		rescaled non-equilibrium water stored volume ($V^*-V_e^*$)
V^*_i, V^*_{ei}, V^*_{nei}		ϵ^{2i}-order components of V^*, V^*_e and V^*_{nei} respectively
w	[L]	width of the hillslope (function of x)
w^*		width of the catchment w rescaled with L
x, y, z	[L]	space coordinates
x^*, y^*, z^*		dimensionless space coordinates
α		slope angle
ε		ratio H/L (constant)
θ		volumetric water content
θ_R		residual volumetric water content
θ_S		saturated volumetric water content
θ^*		volumetric water content rescaled with C_0 H
θ^*_i		ϵ^{2i}-order component of θ^*
λ		dimensionless parameter decreasing with bottom permeability
ψ	[L]	soil water pressure head
ψ_S	[L]	short-term pressure head component
ψ^*		dimensionless soil water pressure head
ψ^*_i		ϵ^{2i}-order component of ψ^*
ψ^*_{sat}		threshold value for surface rescaled pressure head (it is $\sim$0)
ψ^*_{Si}		ϵ_{2i}-order component of ψ^*_{Si}

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