CS 170 Spring 2016 — Efficient Algorithms and Intractable Problems — Alessandro Chiesa and Umesh Vazirani — HW 9

Due March 21, 11:59pm

1. (10 + 10 pts.) Max-Flow Variants

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a graph G and the additional variant constraints, show how to construct a graph G' such that

(1) If F is a flow in G satisfying the additional constraints, there is a flow F' in G' of the same size,

(2) If F' is a flow in G', then there is a flow F in G satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph G'.

(a) **Max-Flow with Vertex Capacities:** In addition to edge capacities, every vertex $v \in G$ has a capacity c_v, and the flow must satisfy $\forall v : \sum_{u:(u,v)\in E} f_{uv} \leq c_v$.

(b) **Max-Flow with Multiple Sources:** There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

(c) **Feasibility with Capacity Lower Bounds (extra credit) :** In addition to edge capacities, every edge (u,v) has a demand d_{uv}, and the flow along that edge must be at least d_{uv}. Instead of proving (1) and (2), design a graph G' and a number D such that if the maximum flow in G' is at least D, then there exists a flow in G satisfying $\forall(u,v) : d_{uv} \leq f_{uv} \leq c_{uv}$.

2. (20 pts.) Zero-Sum Primary

Bernie Sanders and Hillary Clinton are competing to win the votes of the "Fans of Florida" constituency in the Republican primary. This set of voters is guaranteed to vote for either Clinton or Sanders, and currently 10 million plan to vote for Sanders.

The table below indicates the number of voters Clinton would gain (and Sanders would lose), in millions, if the candidates choose the indicated strategy. For example, if Sanders takes the low road (makes personal attacks on Clinton, etc.) and Clinton takes the high road (focuses on the issues), Clinton will lose 2 million voters.

Find the optimal mixed strategy for the candidates and the expected outcome of that strategy. Show the linear equations you created. Feel free to use an online LP solver to solve the equations.

		Sanders: high road	low road	drop out
Clinton:	high road	4	-2	10
	low road	1	2	10

3. (20 pts.) Modeling - Linear Regression

One of the most important problems in the field of *statistics* is the *linear regression problem*. Roughly speaking, this problem involves fitting a straight line to statistical data represented by points – $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ – on a graph. Denoting the line by $y = a + bx$, the objective is to choose the constants a and b to provide the "best" fit according to some criterion. The criterion usually used is the *method of least squares*, but there are other interesting criteria where linear programming can be used to solve for the optimal values of a and b. For each of the following criteria, formulate the linear programming model for this problem:

(a) Minimize the sum of the absolute deviations of the data from the line; that is,

$$\text{Minimize } \sum_{i=1}^{n} |y_i - (a + bx_i)|$$

(*Hint:* define new variables $z_i = y_i - (a + bx_i)$, as well as non-negative variables z_i^+ and z_i^- such that $z_i = z_i^+ - z_i^-$. How to minimize $|z_i|$ by optimizing z_i^+ and z_i^-?)

(b) Minimize the maximum absolute deviation of the data from the line; that is,

$$\text{Minimize } \max_{i=1\ldots n} |y_i - (a + bx_i)|.$$

(*Hint:* use the above hint.)

4. (20 pts.) Boundedness and Feasibility of Linear Programs

Find necessary and sufficient conditions on the reals a and b under which the linear program

$$\begin{aligned} \max_{x,y} x + y \\ ax + by \leq 1 \\ x, y \geq 0 \end{aligned}$$

(a) Is infeasible

(b) Is unbounded

(c) Has a unique optimal solution

5. (20 pts.) Salt

Salt is an extremely valuable commodity. There are m producers and n consumers, each with their own supply $[a_1 \ldots a_m]$ and demand $[b_1 \ldots b_n]$ of salt.

Note: Solve parts $(b), (c)$ independently of each other.

(a) Each producer can supply to any consumer they choose. Find an efficient algorithm to determine whether it is feasible for all demand to be met.

(b) Each producer is willing to deliver to consumers at most c_i distance away. Each producer i has a distance $d_{i,j}$ from consumer j. Solve part (a) with this additional constraint.

(c) Each producer and consumer now belongs to one of p different countries. Each country has a maximum limits on the amount of salt that can be imported (e_k) or exported (f_k). Deliveries within the same country don't contribute towards this limit. Solve part (a) with this additional constraint.