1a) $\langle\psi|\psi\rangle = \langle u+\lambda^* v | u+\lambda v\rangle$

$= \langle u|u\rangle + \lambda^*\langle v|u\rangle + \lambda\langle u|v\rangle + \lambda\lambda^*\langle v|v\rangle$

$= \langle u|u\rangle + \lambda^*\langle v|u\rangle + \lambda\langle u|v\rangle + |\lambda|^2\langle v|v\rangle$

$\langle\psi|\psi\rangle = \langle u|u\rangle + |\lambda|^2\langle v|v\rangle$

$\Rightarrow \lambda^*\langle v|u\rangle = \lambda\langle u|v\rangle = 0$

$\Rightarrow$ $|u\rangle$ and $|v\rangle$ are orthogonal

or $\langle\psi|\psi\rangle = \langle u|u\rangle + |\lambda|^2\langle v|v\rangle$

$\Rightarrow \lambda^*\langle v|u\rangle = -\lambda\langle u|v\rangle$

$\lambda^*\langle v|u\rangle = -\lambda\langle v|u\rangle^*$

let $\lambda = r_1 e^{i\theta_1}$, $\langle v|u\rangle = r_2 e^{i\theta_2}$

$$r_1 e^{-i\theta_1} r_2 e^{i\theta_2} = -r_1 e^{i\theta_1} r_2 e^{-i\theta_2}$$

$$e^{-i(\theta_1-\theta_2)} = -e^{i(\theta_1-\theta_2)}$$

$$e^{-2(\theta_1-\theta_2)} = -1$$

n is an integer

$$\Rightarrow -2(\theta_1-\theta_2) = (1+2n)\pi = \pm\pi$$

$$\Rightarrow |\theta_1-\theta_2| = \pi/2$$

$\Rightarrow \lambda$ is perpendicular to $\langle v|u\rangle$ in the complex plane

b) $\langle \psi|\psi\rangle = \langle u|u\rangle + \lambda^* \langle v|u\rangle + \lambda \langle u|v\rangle + |\lambda|^2 \langle v|v\rangle$

$$= \langle u|u\rangle + \lambda^* \langle v|u\rangle + \lambda \langle v|u\rangle^* + |\lambda|^2 \langle v|v\rangle$$

$\langle \psi|\psi\rangle = \langle u|u\rangle + 2|\lambda||\langle u|v\rangle| + |\lambda|^2 \langle v|v\rangle$

$$\Rightarrow \lambda^* \langle v|u\rangle + \lambda \langle v|u\rangle^* = 2|\lambda||\langle u|v\rangle|$$

let $\lambda = r_1 e^{i\theta_1}$, $\langle v|u\rangle = r_2 e^{i\theta_2}$

$$r_1 e^{-i\theta_1} r_2 e^{i\theta_2} + r_1 e^{i\theta_1} r_2 e^{-i\theta_2} = 2 r_1 r_2$$

$$r_1 r_2 e^{-i(\theta_1-\theta_2)} + r_1 r_2 e^{i(\theta_1-\theta_2)} = 2 r_1 r_2$$

$$\Rightarrow 2 = e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)}$$

$$\Rightarrow e^{i(\theta_1-\theta_2)} = e^{-i(\theta_1-\theta_2)} = 1$$

$$\Rightarrow \theta_1 - \theta_2 = 0$$

$$\Rightarrow \theta_1 = \theta_2$$

$\Rightarrow \lambda$ is a positive scalar multiple of $\langle v|u\rangle$

thus λ and $\langle v|u\rangle$ are colinear in the complex plane.

2a)

$$|e_1'\rangle = \frac{|v_1\rangle}{\sqrt{\langle v_1|v_1\rangle}}$$

$$= \frac{|v_1\rangle}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}\left(|e_1\rangle + 2|e_2\rangle + |e_3\rangle\right)$$

$$|v_2'\rangle = |v_2\rangle - |e_1'\rangle\langle e_1'|v_2\rangle$$

$$= |v_2\rangle - |e_1'\rangle\left(\frac{1}{\sqrt{6}}(3+4-1)\right)$$

$$= \left(3-\frac{6}{6}\right)|e_1\rangle + (2-2)|e_2\rangle + (-1-1)|e_3\rangle$$

$$= 2|e_1\rangle - 2|e_3\rangle$$

$$|e_2'\rangle = \frac{|v_2'\rangle}{\sqrt{\langle v_2'|v_2'\rangle}}$$

$$= \frac{\sqrt{2}}{4}\cdot\left(2|e_1\rangle - 2|e_3\rangle\right)$$

$$= \frac{\sqrt{2}}{2}|e_1\rangle - \frac{\sqrt{2}}{2}|e_3\rangle$$

$$|v_3'\rangle = |v_3\rangle - |e_1'\rangle\langle e_1'|v_3\rangle - |e_2'\rangle\langle e_2'|v_3\rangle$$

$$= |v_3\rangle - \frac{3}{\sqrt{6}}|e_1'\rangle - \frac{5\sqrt{2}}{2}|e_2'\rangle$$

$$= \left(4 - \frac{3}{6} - \frac{10}{4}\right)|e_1\rangle + \left(0 - \frac{6}{6} - 0\right)|e_2\rangle +$$

$$\left(-1 - \frac{3}{6} + \frac{10}{4}\right)|e_3\rangle$$

$$= |e_1\rangle - |e_2\rangle + |e_3\rangle$$

$$|e_3'\rangle = \frac{|v_3'\rangle}{\sqrt{\langle v_3|v_3\rangle}}$$

$$= \frac{1}{\sqrt{3}}|e_1\rangle - \frac{1}{\sqrt{3}}|e_2\rangle + \frac{1}{\sqrt{3}}|e_3\rangle$$

2 b) $T = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{23} & A_{33} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A_{11} = \frac{1}{\sqrt{6}} \quad A_{12} = \frac{2}{\sqrt{6}},\ A_{13} = \frac{1}{\sqrt{6}}$

using the same process for the next 2 collumns

$$T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

T is all real $\therefore T^\dagger = T^T$

$$T^\dagger = \begin{pmatrix} \frac{1}{\sqrt{6}} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

$$TT^\dagger = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{11} = 1/\sqrt{6} \cdot 1/\sqrt{6} + 1/\sqrt{2}\, 1/\sqrt{2} + 1/\sqrt{3}\, 1/\sqrt{3}$$
$$= 1/6 + 1/2 + 1/3 = 1$$

$$A_{12} = 1/\sqrt{6} \cdot 2/\sqrt{6} + 1/\sqrt{2} \cdot 0 + 1/\sqrt{3} \cdot -1/\sqrt{3}$$
$$= \frac{1}{3} - \frac{1}{3} = 0$$

$$A_{13} = 1/\sqrt{6} \cdot 1/\sqrt{6} + 1/\sqrt{2} \cdot -1/\sqrt{2} + 1/\sqrt{3} \cdot 1/\sqrt{3}$$
$$= 1/6 + 1/3 - 1/2$$
$$= 0$$

$$A_{21} = 2/\sqrt{6} \cdot 1/\sqrt{6} + 0 \cdot 1/\sqrt{2} + -1/\sqrt{3} \cdot 1/\sqrt{3}$$
$$= 1/3 - 1/3 = 0$$

$$A_{22} = \frac{2}{\sqrt{6}} \cdot \frac{2}{\sqrt{6}} + 0 \cdot 0 + \frac{-1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{3}}$$

$$= \frac{4}{6} + \frac{1}{3} = 1$$

$$A_{23} = \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} + 0 \cdot \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$$

$$= \frac{1}{3} - \frac{1}{3} = 0$$

$$A_{31} = \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$$

$$= \frac{1}{6} + \frac{1}{3} - \frac{1}{2} = 0$$

$$A_{32} = \frac{1}{\sqrt{6}} \cdot \frac{2}{\sqrt{6}} + \frac{-1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{3}}$$

$$= \frac{1}{3} - \frac{1}{3} = 0$$

$$A_{33} = \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} + \frac{-1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$$

$$= \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1$$

$$\therefore \; TT^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\therefore$ T is unitary

3a) Assume $K|v\rangle = \lambda|v\rangle$

$$\langle v|K|v\rangle^* = \langle v|K^\dagger|v\rangle$$

$$\langle v|\lambda|v\rangle^* = -\langle v|K|v\rangle$$

$$\lambda^* \langle v|v\rangle^* = -\langle v|\lambda|v\rangle$$

$$\lambda^* \langle v|v\rangle = -\lambda \langle v|v\rangle$$

$$\Rightarrow \lambda^* = -\lambda$$

$$\Rightarrow x - iy = -x - iy$$

$$\Rightarrow x = 0$$

$\therefore$ λ is purely imaginary

3b) $A^{+} = (1-K)^{+}\left((1+K)^{+}\right)^{-1}$

$= (1-K^{+})(1+K^{+})^{-1}$

$= (1+K)(1-K)^{-1}$

$AA^{+} = (1+K)^{-1}(1-K)(1+K)(1-K)^{-1}$

$(1-K)(1+K) = (1+K)(1-K)$

$AA^{+} = (1+K)^{-1}(1+K)(1-K)(1-K)^{-1}$

For any operator B $\quad BB^{-1} = I$

$= I\,I$

$AA^{+} = I$

3c) $K|v\rangle = \lambda|v\rangle$

$$A|v\rangle = (1+K)^{-1}(1-K)|v\rangle$$
$$= (1+K)^{-1}|v\rangle - \lambda|v\rangle$$
$$= (1-\lambda)(1+K)^{-1}|v\rangle$$

$(1+K)^{-1}$ is the inverse of $(1+K)$

$$(1+K)|v\rangle = |v\rangle + K|v\rangle$$
$$= (1+\lambda)|v\rangle$$

$$(1+K)^{-1}(1+K) = |v\rangle \Rightarrow (1+K)^{-1}|v\rangle = \frac{1}{(1+\lambda)}|v\rangle$$

$$A|v\rangle = (1-\lambda)\frac{1}{(1+\lambda)}|v\rangle$$

$\frac{1-\lambda}{1+\lambda}$ is a scalar $\therefore$ $A|v\rangle = \mu|v\rangle$

$\therefore$ $|v\rangle$ is also a eigenvector of A

5d) λ is purely imaginary. Let $\lambda = \alpha i$, $\alpha \in \mathbb{R}$

$$\mu = \frac{1-\lambda}{1+\lambda} = \frac{(1-i\alpha)^2}{1+\alpha^2} = \frac{(1-\alpha^2)-2\alpha i}{1+\alpha^2}$$

$$\mu\mu^* = \frac{(1-\alpha^2)-2\alpha i}{1+\alpha^2} \cdot \frac{(1-\alpha^2)+2\alpha i}{1+\alpha^2}$$

$$= \frac{(1-\alpha^2)^2+4\alpha^2}{(1+\alpha^2)^2}$$

$$= \frac{1-2\alpha^2+\alpha^4+4\alpha^2}{(1+\alpha^2)^2}$$

$$= \frac{1+2\alpha^2+\alpha^4}{(1+\alpha^2)^2}$$

$$= \frac{(1+\alpha^2)^2}{(1+\alpha^2)^2}$$

$$= 1$$

$\therefore$ μ has unit modulus