1a) $$c_n = \frac{1}{2}\int_0^{2\pi} e^{-2\pi i n x/2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} e^{-i\beta x}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-ix(n+\beta)}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{e^{-ix(n+\beta)}}{-i(n+\beta)}\right)\Bigg|_0^{2\pi}$$

$$= \frac{i}{2\pi(n+\beta)}\left(e^{-2\pi i(n+\beta)} - 1\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{2\pi(n+\beta)}\left(e^{-i2\pi(n+\beta)} - 1\right) e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{i}{2\pi(n+\beta)} e^{-i2\pi n - i2\pi\beta + inx} - e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{i}{2\pi(n+\beta)} e^{in(x-2\pi)} e^{-i2\pi\beta} - e^{-inx}$$

b) $$f(\pi) = \sum_{n=-\infty}^{\infty} \frac{i}{2\pi(n+\beta)}\left(e^{-i2\pi(n+\beta)} - 1\right) e^{in\pi}$$

$$e^{in\pi} = -1 \text{ if } n \text{ is odd}$$
$$1 \text{ if } n \text{ is even}$$

$$f(\pi) = \sum_{n=-\infty}^{\infty} \frac{i}{2\pi(n+\beta)}\left(e^{-i2\pi(n+\beta)} - 1\right)\underbrace{(-1)^n}_{=(-1)^n}$$

let $$\frac{(-1)^n}{\beta+n} = g$$

$$f(\pi) = \sum_{n=-\infty}^{\infty} \frac{i}{2\pi} \left(e^{-i2\pi n} e^{-i2\pi\beta} - 1 \right) g$$

$$\overset{||}{1^n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{i}{2\pi} \left(e^{-i2\pi\beta} - 1 \right) g$$

$$= \frac{i}{2\pi} \left(e^{-i2\pi\beta} - 1 \right) \sum_{n=-\infty}^{\infty} g$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+\beta)} = \frac{f(\pi)}{\frac{i}{2\pi}\left(e^{-i2\pi\beta} - 1\right)}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+\beta} = a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+\beta} + \frac{(-1)^n}{(\beta-n)}$$

$$a_0 = \frac{1}{\beta} \qquad = a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n \left[\beta - \cancel{n} + \cancel{n} + \beta \right]}{(\beta-n)(n+\beta)}$$

$$= \frac{1}{\beta} + \sum_{n=1}^{\infty} \frac{2\beta (-1)^n}{\beta^2 - n^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\beta^2 - n^2} = \frac{1}{2\beta} \left(\frac{f(\pi)}{\frac{i}{2\pi}\left(e^{-i2\pi\beta} - 1\right)} - \frac{1}{\beta} \right) \qquad f(\pi) = e^{-i\pi/2}$$

$$\text{at } \beta = \tfrac{1}{2} \quad = \frac{2\pi f(\pi)}{i\left(e^{-i\pi} - 1\right)} - 2$$

$$= \frac{2\pi e^{-i\pi/2}}{i(e^{-i\pi} - 1)} - 2$$

$$= \frac{2\pi(-i)}{i(-2)} - 2$$

$$= \pi - 2$$

$$\doteq 1.14159$$

from wolfram alpha, the sum converges to 1.14159

$\therefore$ the sum formula is correct

(c) from a $f(x)=e^{-i\beta x}=\sum_{n=-\infty}^{\infty}\frac{i}{2\pi(n+\beta)}(\underbrace{e^{-i2\pi n}}_{=1}e^{-i2\pi\beta}-1)e^{inx}$

$$=\frac{(e^{-i2\pi\beta}-1)}{2\pi}i\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n+\beta}$$

$$=\left(\frac{(e^{-i2\pi\beta}-1)}{2\pi}i\right)\left(\frac{1}{\beta}+\sum_{n=1}^{\infty}\underbrace{\frac{e^{inx}}{n+\beta}+\frac{e^{-inx}}{\beta-n}}_{\text{re writing this}}\right)$$

$$=\frac{(\beta-n)e^{inx}+(\beta+n)e^{-inx}}{\beta^2-n^2}$$

$$=\frac{2\beta\cos(nx)-2ni\sin(nx)}{\beta^2-n^2}$$

$$f(x)=e^{-i\beta x}=\left(\frac{(e^{-i2\pi\beta}-1)}{2\pi}i\right)\left(\frac{1}{\beta}+\sum_{n=1}a_n\cos nx+b_n\sin nx\right)$$

where $a_n=\frac{2\beta}{\beta^2-n^2}$, $b_n=\frac{-in2}{\beta^2-n^2}$, $a_0=\frac{2}{\beta}$

using parsevals theorem

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(x)\Big/\left(\frac{(e^{-i2\pi\beta}-1)}{2\pi}i\right)\right|^2dx=\frac{|a_0|^2}{4}+\sum_n\frac{|a_n|^2+|b_n|^2}{2}$$

$$\frac{1}{2\pi}\int_0^{L}\left|\frac{e^{-iBx}\,2\pi}{i(e^{-i2\pi B}-1)}\right|^2 dx$$

$$e^{-i2\pi B}-1 = \cos 2\pi B - 1 - i\sin 2\pi B$$

$$|\cos 2\pi B - 1 - i\sin 2\pi B|$$

$$=\sqrt{\cos^2 2\pi B - 2\cos 2\pi B + 1 + \sin^2 2\pi B}$$

$$=\sqrt{2 - 2\cos 2\pi B}$$

$$=\sqrt{2 - 2 + 4\sin^2 \pi B}$$

$$= 2\sin \pi B$$

continuing the integral

$$2\pi\int_0^{2\pi}\frac{|e^{-iBx}|}{4\sin^2\pi B}\,dx$$

$$=\frac{\pi^2}{\sin^2 \pi B} = \pi^2\csc^2\pi B$$

$$\frac{|a_0|^2}{4} + \frac{1}{2}\sum_n |a_n|^2 + |b_n|^2 = \frac{(2/B)^2}{4} + 2\sum_{n=1}^{\infty}\frac{B^2+n^2}{(B^2-n^2)^2}$$

$$\Rightarrow \frac{1}{2}\left(\frac{\pi^2\csc^2\pi B^2 - 1}{B^2}\right) = \sum_{n=1}^{\infty}\frac{B^2+n^2}{(B^2-n^2)^2}$$

for $\beta = 1/2$ $\displaystyle\sum_{n=1}^{\infty} \frac{\beta^2 + n^2}{(\beta^2 - n^2)^2} = \frac{\pi^2 \csc^2 \pi/2 \, (0.5)^2 - 1}{2 \, 0.5^2}$

$$= 2\left(\pi^2/4 - 1\right) = \pi^2/2 - 2$$

$$\doteq 2.934802$$

from wolfram alpha, the series converges to 2.9348, $\therefore$ the solution is correct

2a) function is odd: sine components 0

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right)$$

$$a_n = \frac{2}{L}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{L}\left(\int_0^{\pi} x\cos(nx)\,dx - \int_{-\pi}^{0} x\cos(nx)\,dx\right)$$

integration by parts

$$\int x\cos(nx)$$

$$= \frac{x\sin(nx)}{n} + \int \frac{\sin(nx)}{n}\,dx$$

$$= \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}$$

$$= \frac{nx\sin(nx) + \cos(nx)}{n^2}$$

$$a_n = \frac{1}{\pi}\left(\left.\frac{nx\sin(nx)+\cos(nx)}{n^2}\right|_0^{\pi} - \left.\frac{nx\sin(nx)-\cos(nx)}{n^2}\right|_{-\pi}^{0}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right)$$

$$a_n = \frac{2((-1)^n - 1)}{\pi n^2}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x\,dx - \int_{-\pi}^{0} x\,dx$$

$$= \frac{1}{\pi} \left(\frac{x^2}{2}\Big|_0^{\pi} \right) - \left(\frac{x^2}{2}\Big|_{\pi}^{0} \right)$$

$$= \pi$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi n^2} \cos(nx)$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} ((-1)^n - 1)$$

only non 0 for n is odd

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\cos(nx)}{n^2} \qquad n = 2m+1$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos((2m+1)x)}{(2m+1)^2}$$

b) $$\frac{dy}{dx} = \frac{d}{dx}\left(\pi/2 - 4/\pi \sum_{m=0}^{\infty} \frac{\cos((2m+1)x)}{(2m+1)^2}\right.$$

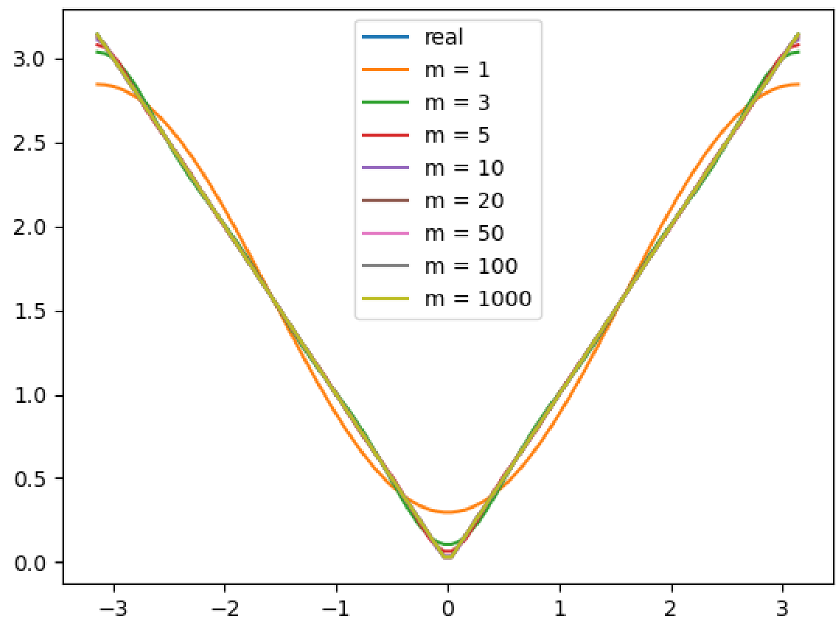
$$g(x) = \sum_{m=0}^{\infty} \frac{4}{\pi} \frac{\sin((2m+1)x)}{(2m+1)}$$

c) $$h(x) = \sum_{m=0}^{\infty} \frac{4}{\pi} \cos((2m+1)x)$$

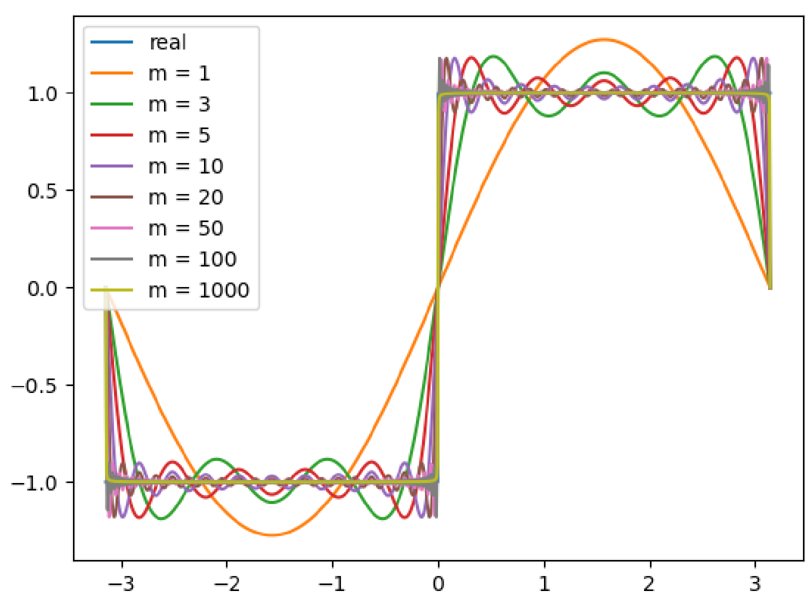
(simple differentiation of [illegible])

d)

All fourier series were plotted with different number of terms against the analytical solution



f(x)



g(x)

h(x), actual function not shown because it is not possible to show a real plot of a Dirac Delta. However, the function should be 2δ(X). At higher values for m, the peak was not visible as it got too narrow.