1a) $\det(\underline{V} - \lambda \underline{I}) = 0$

$$\det\begin{pmatrix} -\lambda & v_0 \\ -v_0 & -\lambda \end{pmatrix} = 0$$

$$\Rightarrow \lambda^2 + v_0^2 = 0$$

$$\Rightarrow \lambda = \pm\sqrt{-v_0} = \pm i v_0$$

for $\lambda = i v_0$

$$v_0 \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow -ia + b = 0 \Rightarrow b = ia$$

$$\Rightarrow |v_1\rangle = \alpha\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ normalizing } |v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

for $\lambda = -i v_0$

$$v_0 \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow ia + b = 0 \Rightarrow b = -ia$$

$$\Rightarrow |v_2\rangle = \alpha\begin{pmatrix} i \\ 1 \end{pmatrix} \text{ normalizing } |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

The eigen Values are purely imaginary as they are +/-i

b) $|e_i'\rangle = T|e_i\rangle$

$|e_1'\rangle = -1/\sqrt{2}\begin{pmatrix} i \\ 1 \end{pmatrix}$ $\quad |e_2'\rangle = 1/\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\Rightarrow T = 1/\sqrt{2}\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}$

if T is unitary $TT^{+} = I$

$TT^{+} = 1/\sqrt{2}\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} 1/\sqrt{2}\begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1+1 & i-i \\ -i+i & 1+1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \therefore T$ is unitary

c) $$w_{ij} = \langle e_i | e^V | e_j \rangle$$

$$= \sum_{k,l} \langle e_i | e_k' \rangle \langle e_k' | e^V | e_l' \rangle \langle e_l' | e_j \rangle$$

$$e^V | e_l' \rangle = 1 + V + \frac{V^2}{2!} + \dots | e_l' \rangle$$

$$= 1 + \lambda_l + \frac{\lambda_l^2}{2!} \dots | e_l' \rangle$$

$$= e^{\lambda_l} | e_l' \rangle$$

$$= \sum_{k,l} e^{\lambda_l} \langle e_i | e_k' \rangle \langle e_k' | e_l' \rangle \langle e_l' | e_j \rangle$$

$$= \sum_{k} e^{\lambda_k} \langle e_i | e_k' \rangle \langle e_k' | e_j \rangle$$

$$\Rightarrow w = T \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} T^{-1}$$

$$w = \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix} \frac{1}{2}$$

$$= \frac{1}{2} \begin{pmatrix} -i e^{\lambda_1} & e^{\lambda_2} \\ -e^{\lambda_1} & i e^{\lambda_2} \end{pmatrix} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^{\lambda_1} + e^{\lambda_2} & i(e^{\lambda_1} - e^{\lambda_2}) \\ -i(e^{\lambda_1} - e^{\lambda_2}) & e^{\lambda_1} + e^{\lambda_2} \end{pmatrix}$$

$$w = \frac{1}{2} \begin{pmatrix} e^{-iv_0} + e^{iv_0} & i(e^{-iv_0} - e^{iv_0}) \\ -i(e^{-iv_0} - e^{iv_0}) & e^{-iv_0} + e^{iv_0} \end{pmatrix}$$

by inspection, $w^\dagger = w \Rightarrow w$ is hermitian

$$= \begin{pmatrix} \cos(v_0) & \sin(v_0) \\ -\sin(v_0) & \cos(v_0) \end{pmatrix}$$

2a)

$$f(x) = c\left(\frac{1 - ae^{-ix} + 1 - ae^{ix}}{1 - a(e^{ix} - e^{-ix}) + a^2} \right)$$

$$= C\left(\frac{2 - 2a\cos x}{1 - 2a\cos x + a^2} \right)$$

$\Rightarrow C = 1/2$

b)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-ae^{ix}} = \sum_{n=0}^{\infty} a^n e^{nix}$$

$$\frac{1}{1-ae^{-ix}} = \sum_{n=0}^{\infty} a^n e^{-nix}$$

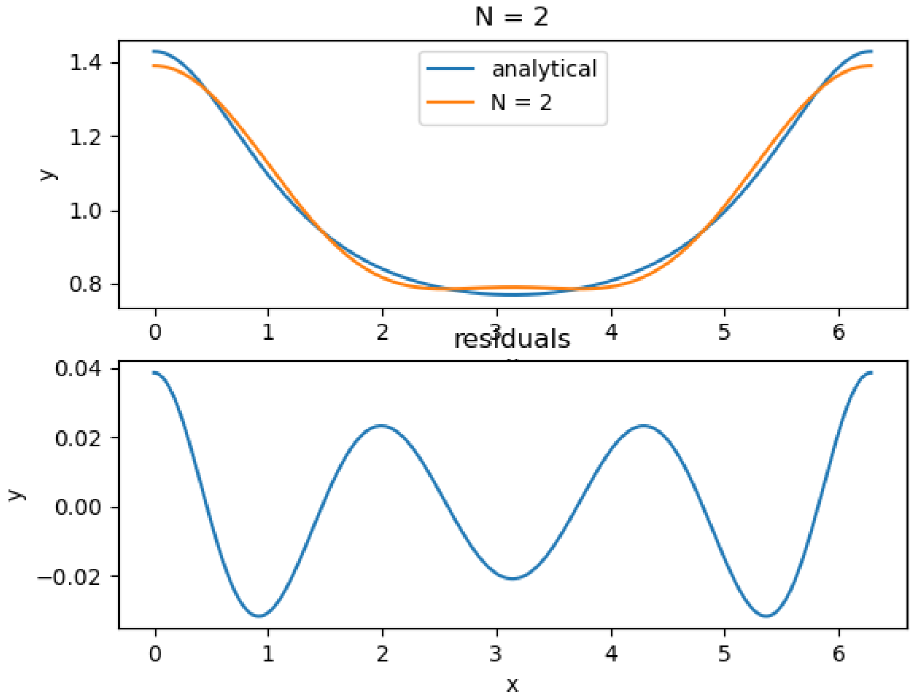
$$f(x) = \frac{1}{2}\left(\frac{1}{1-ae^{ix}} + \frac{1}{1-ae^{-ix}} \right)$$

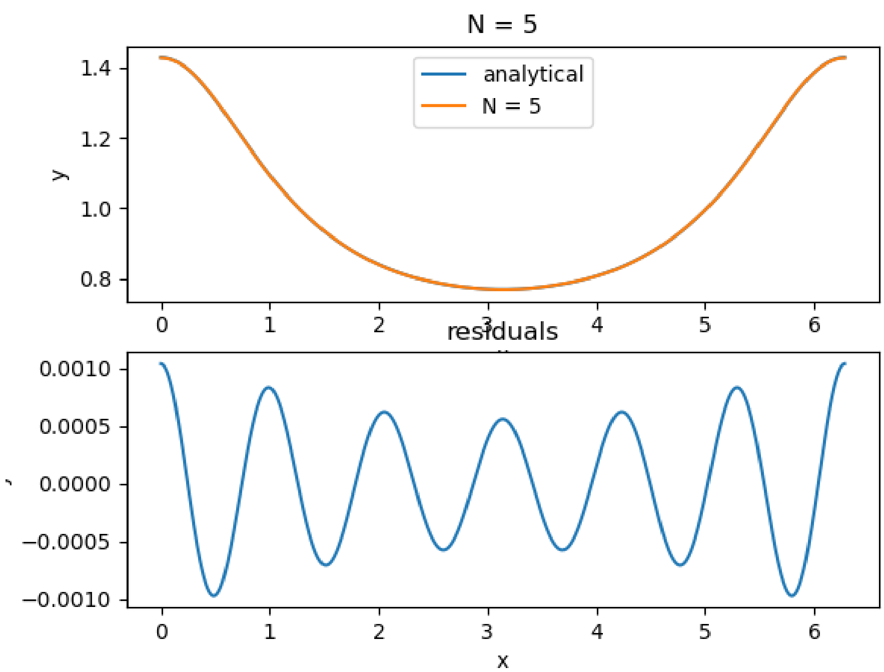
$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} a^n \left(e^{nix} + e^{-nix} \right) \right)$$

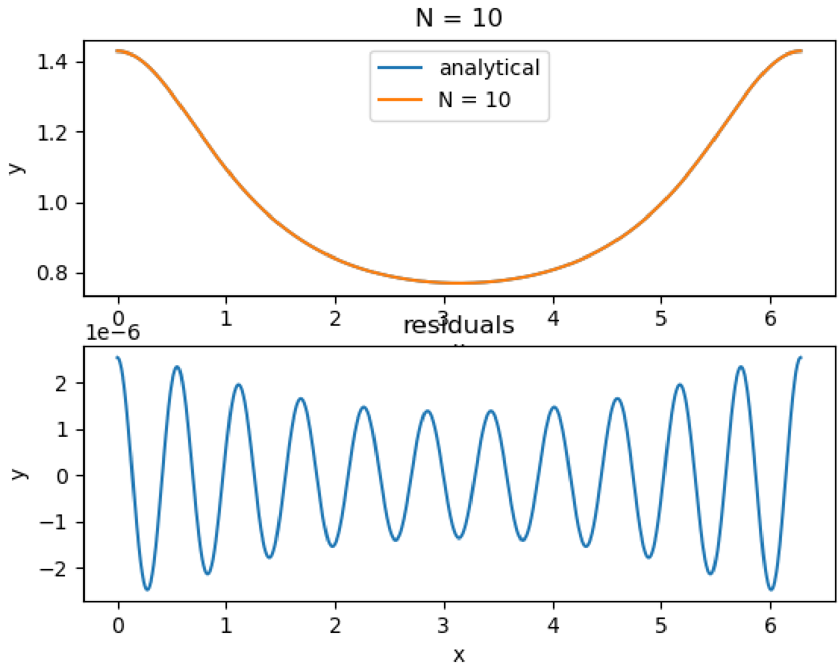
$$= \frac{1}{2}\sum_{n=0}^{\infty} a^n 2\cos(nx)$$

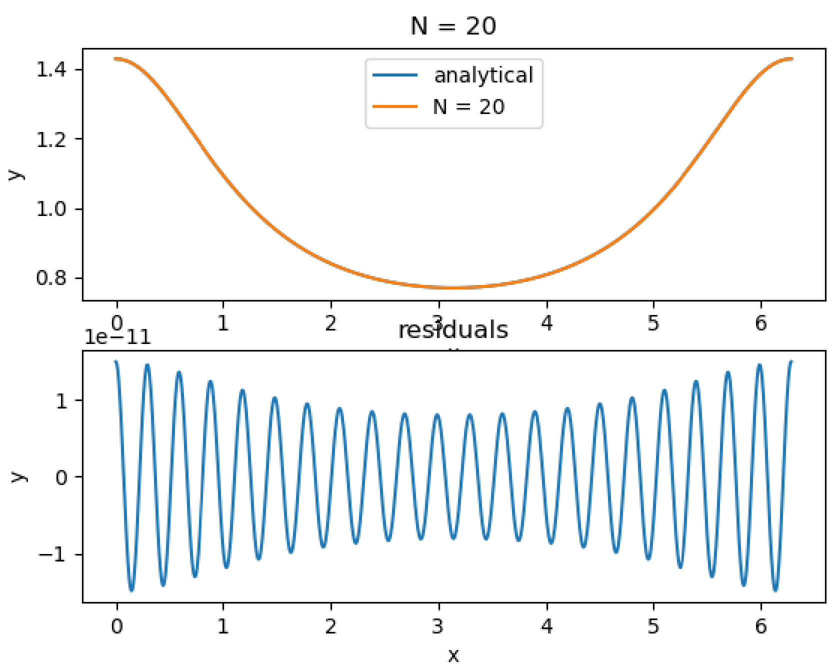
$$= \sum_{n=0}^{\infty} a^n \cos(nx)$$

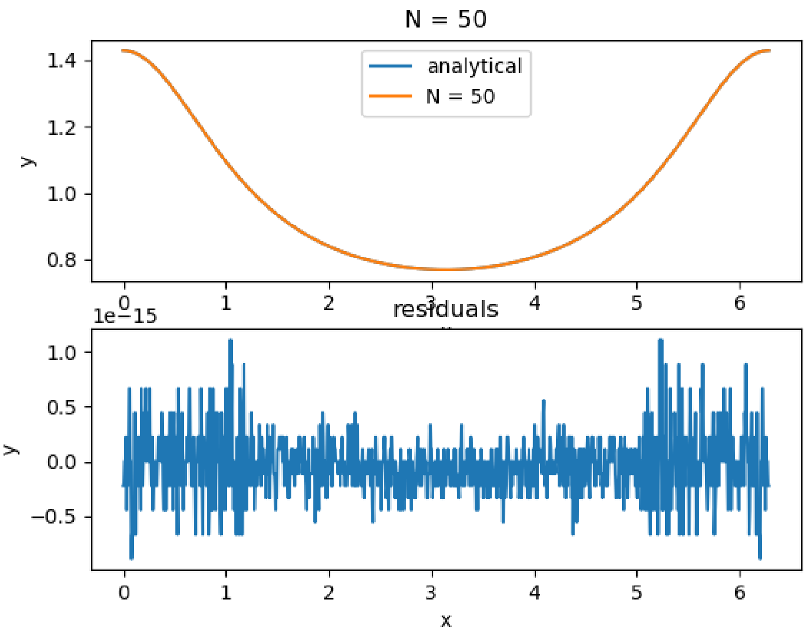
$$\Rightarrow c_n = a^n$$


N = 2
analytical
N = 2
y
residuals
y
x


N = 5
analytical
N = 5
y
residuals
x


N = 10
analytical
N = 10
y
1.4
1.2
1.0
0.8
0
1
2
3
4
5
6
residuals
1e−6
2
1
0
−1
−2
x


N = 20
analytical
N = 20
y
1.4
1.2
1.0
0.8
0
1
2
3
4
5
6
residuals
1e−11
1
0
−1
x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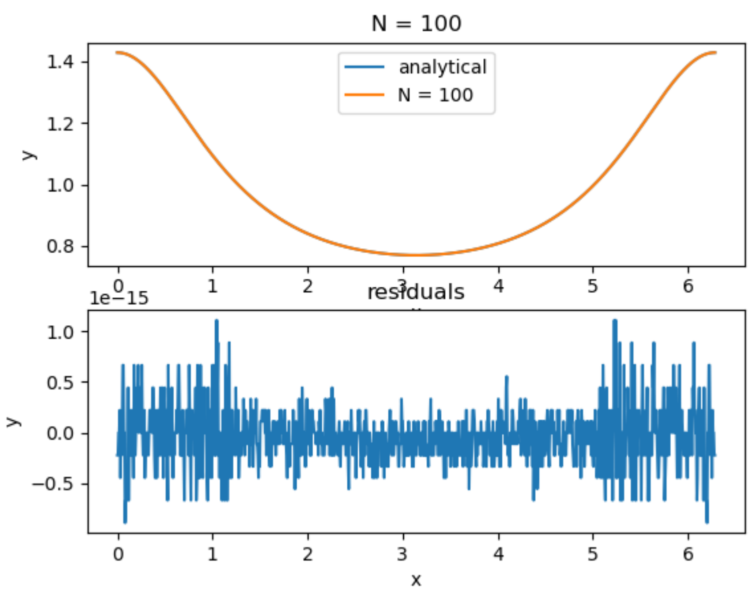
analytical
N = 50
y
x
1e−15
residuals
y
x


N = 100
analytical
N = 100
y
x
1e−15
residuals
y
x

d) $\int_0^{\pi} f(x)\,dx$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} a^n \cos(n\,x)\,dx$$

$$= \sum_{n=0}^{\infty} a^n \left. \frac{\sin(n\,x)}{n} \right|_0^{\pi}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{n} \left(0 - 0 \right)$$

$$= 0$$