Problem Set 1
Due Friday, September 23, 11:59 pm via OnQ

1. Express the following in the form $x + iy$.

$$\frac{\left(1 - i\sqrt{3}\right)^{6/5}}{(1-i)^{12/5}} \tag{1}$$

   (Note that though this problem can be done using Maple, Python, or Mathematica, you must work through by hand and show all your steps.)

2. Find the real and imaginary parts of the following in terms of x and y where $z = x + iy$:

$$\cos\left(e^{z}\right) \tag{2}$$

$$e^{\cos z} \tag{3}$$

3. Denote the n roots of unity by $\omega_0, \ldots, \omega_{n-1}$ with $\omega_0 = 1$.

   Assume that $n > 1$ and prove that

   (a)
$$\sum_{k=0}^{n-1} \omega_k = 0 \tag{4}$$

   and

   (b)
$$\prod_{k=0}^{n-1} \omega_k = (-1)^{n-1} \tag{5}$$

4. Consider the recurrence relation

$$z_{n+1} = z_n^2 + c \tag{6}$$

   where c is a complex numbe. The Mandelbrot set is the set of values of c such that the sequence generated by the recurrence relation with starting point $z = 0$ does not diverge.

   For this problem, we consider a variation of the problem in which

$$z_{n+1} = e^{z} + c \tag{7}$$

where once again, c is a complex number and we assume $z_1 = 0$. Write out this relation in terms of two real equations. That is, with $z_n = x_n + iy_n$ and $c = a + ib$, find expressions for x_{n+1} and y_{n+1} in terms of x_n, y_n, a and b.

Find the analog of the Mandelbrot set for this relation in the form of a scatter plot in the complex c plane. You can set this up using a computer language of your choice. Note that you can code up the recursion relation using complex numbers or as two coupled real sequences using the results from the first part of the problem.

Hints: You'll have to make some decisions about how to test whether a particular starting value leads to an infinite result (e.g., compute first N terms in the series and reject if the absolute value of the terms exceed some threshold value A where you'll have to decide what values to pick for N and A. Plot the starting values as a scatter plot in the complex plane or for a prettier picture, make a 2D histogram using colour for density (easy to do in python or matlab).

Begin with a large region of the complex plane, say, $|x| < 10$, $|y| < 10$. Discuss any periodicity that you notice in the figure.

The Mandelbrot set is an example of a fractal. The same seems to be true for our modified Mandelbrot example. Try zooming into a region of the complex plane near the boundary of the set and you'll see more and more detail.

Please include your code as an appendix and indicate any decisions you made (e.g., length of series to test for divergence; threshold).

5. The Julia set is the set of starting values z_0 such that the recurrence relation

$$z_{n+1} = z_n^2 + c \tag{8}$$

does not diverge when c is fixed. Generate the Julia set for $c = -0.83 + 0.18i$ and make a scatter plot in the complex plane. Different values of c will generate different patterns. You can find many beautiful examples online.