1 Hamming code (7,4)

Assume a codeword
T = $b_6b_5b_4b_3b_2b_1b_0$ ← we put 3 checkbits on positions that are powers of 2: $2^0 = 1, 2^1 = 2, 2^2 = 4$, all other bits are message bits m_i

T = $\boxed{b_6}\boxed{b_5}b_4\boxed{b_3}b_2b_1b_0 \quad \Rightarrow \quad$ T = $(c_1c_2m_3c_3m_2m_1m_0)$

To encode the message M = $(m_3m_2m_1m_0)$ we use a **generator matrix G** so that codeword is computed as $T = M \cdot G$, where T is a (1×7) vector, M is (1×4) vector and G is a (4×7) matrix:

this is how M is encoded

$$\begin{bmatrix} c_1 & c_2 & m_3 & c_3 & m_2 & m_1 & m_0 \end{bmatrix} = \begin{bmatrix} m_3 & m_2 & m_1 & m_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

So the checkbits are computed as (**note the modulo 2 addition - XOR !!!**):

use 1st column

use 2nd column

$$c_1 = \begin{bmatrix} m_3 & m_2 & m_1 & m_0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = m_3 \oplus m_2 \oplus m_0 \qquad c_2 = \begin{bmatrix} m_3 & m_2 & m_1 & m_0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = m_3 \oplus m_1 \oplus m_0$$

use 3rd column

$$c_3 = \begin{bmatrix} m_3 & m_2 & m_1 & m_0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = m_2 \oplus m_1 \oplus m_0$$

each check bit is computed using 3 out of 4 message bits !!

For $m = 4$ bit message we have $2^4 = 16$ legal codewords (we only show computation for the first four):

$$M_0 \cdot G = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = T_0$$

row · column 1 = $0\cdot 1 \oplus 0 \cdot 1 \oplus 0 \cdot 0 \oplus 0 \cdot 1 = 0 \oplus 0 \oplus 0 \oplus 0 = 0$

$$M_1 \cdot G = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = T_1$$

row · column 1 = $0\cdot 1 \oplus 0 \cdot 1 \oplus 0 \cdot 1 \oplus 1 \cdot 1 = 0 \oplus 0 \oplus 0 \oplus 1 = 1$

NOTE: basically just a copy of the last row of the matrix

$$M_2 \cdot G = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} = T_2$$

$$M_3 \cdot G = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = T_3$$

row · column 1 = $0\cdot 1 \oplus 0 \cdot 1 \oplus 1 \cdot 0 \oplus 1 \cdot 1 = 0 \oplus 0 \oplus 0 \oplus 1 = 1$

NOTE: just XOR of the last two rows

Exercise 1.1 *Continue the example above and compute the remaining 12 codewords. For full marks show your computations using the generator matrix G.*

Assume a codeword T is transmitted over a noisy channel, and another word $R = (b_6b_5b_4b_3b_2b_1b_0)$ of same length is received. The receiver uses a **check matrix H** to verify if the codeword was received correctly by recomputing $H \cdot R = S$, where R is a (1×7) vector, S is (3×1) vector and H is a (3×7) matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_6 \\ b_5 \\ b_4 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

corresponds to bit position:

$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \Rightarrow s_1 \cdot 2^0 + s_2 \cdot 2^1 + s_3 \cdot 2^2 =$ position of error

Decoding examples and correcting 1-bit error: transmitted $T = (0101010)$

- first example received $R = (0101010)$ (correct)
- second example received $R = (1101010)$ (incorrect)

row 1 · column =

$= \underbrace{1 \cdot 0}_{} \oplus \underbrace{0 \cdot 1}_{} \oplus \underbrace{1 \cdot 0}_{} \oplus \underbrace{0 \cdot 1}_{} \oplus \underbrace{1 \cdot 0}_{} \oplus \underbrace{0 \cdot 1}_{} \oplus \underbrace{1 \cdot 0}_{} =$

$= 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 =$

$= 0$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

no error

$\Rightarrow$ Decoding:

$R = (0\,1\,0\,1\,0\,1\,0) \Rightarrow M = (0010)$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \oplus 1 \\ 1 \oplus 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

error!

correct this bit

$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\Rightarrow 1 \cdot 2^0 + 0 + 0 = \underline{\underline{1}}$

row 2 · column =

$\underbrace{0 \cdot 1}_{} \oplus \underbrace{1 \cdot 1}_{} \oplus \underbrace{1 \cdot 0}_{} \oplus \underbrace{0 \cdot 1}_{} \oplus \underbrace{0 \cdot 0}_{} \oplus \underbrace{1 \cdot 1}_{} \oplus 1 \cdot 0 =$

$= 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus 0 = 1 + 1 = 0$

Decoding corrected:

$R' = (0101010) \Rightarrow M = (0010)$

Decoding examples and correcting 1-bit error: transmitted $T = (0100101)$

- received $R = (1000101)$ (2-bit error in positions 1 and 2 !)

$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 = \underline{\underline{3}}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \oplus 1 \oplus 1 \\ 1 \\ 1 \oplus 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

error

can not correct 2-bit errors!

Decoded corrected is not equal to the transmitted codeword !!!

Exercise 1.2 *Decode R (that is, find the transmitted word T) and for each R answer the following:*

a. was R received error free ? If yes: decode R.

b. if not: decode corrected R. Did a 2-bit error occur in reality?

For full marks show your computations using the check matrix H.

i. transmitted $T = (0100101)$, *received* $R = (0100101)$

ii. transmitted $T = (1100110)$, *received* $R = (1100100)$

iii. transmitted $T = (1010101)$, *received* $R = (0010101)$

iv. transmitted $T = (1110000)$, *received* $R = (1110000)$

v. transmitted $T = (0110011)$, *received* $R = (0110011)$

vi. transmitted $T = (1001100)$, *received* $R = (0000100)$