PART - B   Unit - 5

Combinatorics

→ The rules of sum & product
→ Permutation
→ Combination - the Binomial theorem
→ Combinations with repetition.
→ The catalan numbers.

The rule of sum and Product

Sum Rule

Suppose two task T_1 and T_2 are to be performed
If T_1 task can be performed in m different ways
T_2 task can be performed in n different ways
If both cannot be done simultaneously.
then one of two task can be performed in.
$m+n$ different ways.

If $T_1, T_2 \ldots T_k$ are k task no two of
these task can be performed at the same time
T_i can be performed. in n_i different ways.
then one of k task can be performed in.
$n_1+n_2+\ldots n_k$ different ways

Ex:1 Suppose there are 16 boys & 18 girls in a
class and we wish to select one of these
students as the class representative.

The number of ways of selecting a boy = 16
" " selecting a girl = 18

The number of ways of selecting a
student (boy or girl)
$= 16+18 = \underline{34}$

2) Suppose a hostel library has 12 books on math., 10 books on physics, 16 books on CS, 11 books on electronics. Suppose a student wishes to choose one of the books for study. The number of ways in which he can choose a book is $12 + 10 + 16 + 11 = \underline{49}$.

3) Suppose T_1 is the task of selecting a prime number less than 10 and T_2 is the task of selecting an even number less than 10.

T_1 can be performed in 4 ways $\rightarrow 2, 3, 5, 7$

T_2 can be performed in 4 ways. $\rightarrow 2, 4, 6, 8$.

Task T_1 and T_2 can be performed in.

$4 + 4 - 1 = 7$ ways.

Product Rule

T_1 and T_2 can be performed one after the other.

If T_1 can be done in N_1 ways

If T_2 can be done in N_2 ways.

Both of task can be performed in $N_1 \times N_2$ ways

$T_1, T_2, T_3 \ldots T_K \ldots N_1 \ldots N_K = N_1 N_2 \ldots N_K$

1) Suppose a person has 8 shirts and 5 ties. Then how many different ways of choosing a shirt and a tie.

$8 \times 5 = 40.$

2) Suppose we wish to construct sequence of four symbol in which the first 2 are english letters and next two are single digit number

$\boxed{A}\ \boxed{A}\ \boxed{N}\ \boxed{N}$

If no letters or digit can be repeated

$26 \times 25 \times 10 \times 9 = \underline{58500}$

If repeatation of letter & digits are allowed then $26 \times 26 \times 10 \times 10 = \underline{67600}$

3) A license plate consist of two english letters followed by four digits. If repeatations are allowed how many plates have only vowels. (A E I O U) and even digits ?

$\boxed{A}\ \boxed{A}\ \boxed{N}\ \boxed{N}\ \boxed{N}\ \boxed{N}$

5 5 5 5 5 5

$(5 \times 5) \times (5 \times 5 \times 5 \times 5)$

5) A bit is either 0 or 1. A byte is a sequence of 8 bits.

Find (i) the number of bytes th

(2) the number of bytes that begin with 11 and end with 11

3) number of bytes that begin with 11 and do not end with 11

4) number of bytes that begin with 11 or end with 11.

1) □ □ □ □ □ □ □ □.

$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$

2) $\boxed{1}\ \boxed{1}$ □ □ □ □ $\boxed{1}\ \boxed{1}$

$1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1 = 2^4 = \underline{16}$

3) $\boxed{1}\ \boxed{1}$ □ □ □ □ □ □.

$1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$

$64 - 16 = \underline{48}$

4) $2^6 + 2^6 = 64 + 64 = 128 - 16$

$\underline{112}$

$\underline{|A \cup B| = |A| + |B| - |A \cap B|}$

Permutations

It follows by product rule of counting.

We are given n objects and wish to arrange r of these objects in a line. It is given by

$$P(n,r) = n(n-1)(n-2)\cdots n-(r+1)$$

$$k! = k(k-1)(k-2)\cdots 2.1$$

$$0! = 1$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(n,n) = n!$$

Number of permutation of n distinct object taken all at a time is $n!$ simply called as number of permutation of n distinct object.

Collection is of n objects.

n_1 of 1 type.

n_2 of 2nd type.

⋮

n_k of k^{th}

with $n_1 + n_2 + \cdots n_k = n$

The number of permutation of the n object is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

1) How many different strings of length 4 can be formed using the letters of the word FLOWER

no: of letters in FLOWER = 6 distinct

no: of string of letter 4.

$$= P(6,4) = \frac{6!}{2!} = \cancel{360}$$

2) Find the number of permulations of the letters of word success.?

7 letters S - 3

C - 2

U - 1

e - 1

$$= \frac{7!}{3!\cdot 2!\cdot 1!\cdot 1!} = \underline{\underline{420}}$$

3) In how many ways can n person be seated at a round table. if arrangement are considered the same. when one can be obtained from other by rotation?

Let one be seated anywhere.

remaining $n-1$ person can be seated in

$(n-1)!$ ways.

4) It is required to seat 5 men and 4 women in a row. So that women occupy the even places. How many such arrangements are possible?

5 men in odd places $5!$ ways.

4 women in even place $4!$ ways.

Total number of arrangements

$= 5! \times 4! = \underline{2880}$

Combinations

Suppose we are interested in selecting a set of r objects from a set of $n \geq r$ object. without regard to order.

r object being selected =

combination of r object

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

for $0 \leq r \leq n$

$C(n, n-r) = C(n,r)$

$C(n,n) = C(n,0) = 1$

$C(n,1) = C(n, n-1) = n.$

$\boxed{\text{if } r > n,\ C(n,r) = 0.}$

1) How many committee of five with a given chair person can be selected from 12 persons?

chair person can be selected in 12 way.

other four can be choosen $C(11,4)$

$$= \frac{11!}{4!\,7!}$$

Total no: of ways $= 12 \times \frac{11!}{4! \times 7!}$

2) At a certain college hostel, the housing office has decided to appoint for each floor, one male and one female residential advisor. How many different pair of

advisors can be selected for a seven floor building. From 12 male and 15 female candidates?

From 12 male 7 can be selected.

$C(12, 7)$

From 15 female 7 can be selected.

$C(15, 7)$

Total number of pairs of advisors

$= C(12, 7) \times C(15, 7)$

$= \frac{12!}{7!5!} \times \frac{15!}{7!8!} = 5096520$

8) From seven consonants and five vowels how many sets consisting of four different consonants and three different vowels can be formed?

4 different constants from 7 = $C(7, 4)$

3 vowels from 5 = $C(5, 3)$.

Can be arranged among themselves in 7!.

total $= 7! \times C(7, 4) \times C(5, 3)$

$= 1764000$

4) Find the number of arrangements of the letters in TALLAHASSEE which have no adjacent A's.

T - 1

H - 1

L - 2 , S - 2 , E - 2

$11 - 3 = 8$

Dicard A.

No. of possible way of arranging other 8 letter $= \frac{8!}{1!\,1!\,2!\,2!\,2!} = 5040$ ways.

↑ T ↑ L ↑ L ↑ H ↑ S ↑ S ↑ E ↑ E ↑

possible location of A is 9.

These location can be chosen in $C(9,3)$ ways.

Required number of arrangements

$= 5040 \times C(9,3)$

$423,360$.

Prove the following identities.

(1) $C(n+1, r) = C(n, r-1) + C(n, r)$.

(2) $C(m+n, 2) - C(m, 2) - C(n, 2) = mn$.

$\frac{(n+1)!}{(n+1-r)!\,r!} = C(n+1, r)$

$C(n, r-1) + C(n, r)$

$= \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$

$= \cancel{\frac{n!}{(r-1)!\,(n-r)!}\left\{\frac{r}{n-r}\right.}$

$= \frac{n!}{(r-1)!\,(n-r+1)(n-r)!} + \frac{n!}{r\,(r-1)!\,(n-r)!}$

$= \frac{n!}{(r-1)!\,(n-r)!}\left\{\frac{1}{n-r+1} + \frac{1}{r}\right\}$

$\frac{n!}{(r-1)!\,(n-r)!} \times \frac{\cancel{r} + n \cancel{- r} + 1}{r(n-r+1)}$

$(n+1)\,n! = (n+1)!$

$= \frac{n!}{(r-1)!\,(n-r)!} \cdot \frac{n+1}{r(n-r+1)} = \frac{(n+1)!}{r!\,(n-r+1)!}$

$= C(n+1, r)$

Binomial and Multinomial Theorems.

When n is a positive integer, r is an integer such that $0 \le r \le n$, a basic property of $C(n, r)$ is it is the coefficient of $x^{n-r} y^r$ in the expansion of $(x+y)^n$. where x and y are two real numbers.

For a positive integer n.

$$(x+y)^n = x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \dots + C(n, n-1)xy^{n-1} + y^n$$

$$= \sum_{r=0}^{n} C(n,r) x^{n-r} y^r \longrightarrow (1)$$

Since $C(n,r) = C(n,n-r)$, $(x+y)^n = \sum_{r=0}^{n} C(n,r) x^r y^{n-r} \longrightarrow (2)$

$C(n,r)$ is the coefficient of $x^{n-r} y^r$ in the expansion of $(x+y)^n$.

$$\binom{n}{0} = 1 = C(n,0) \longrightarrow (3)$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \text{ for } r \ge 1 \longrightarrow (4)$$

$$\binom{n}{0} = 1 = C(n,0) \longrightarrow (5)$$

and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = C(n,r) \longrightarrow (6)$

Accordingly expression (1) and (2) may be rewritten as:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \longrightarrow (7)$$

(1), (2) and (7) represent what is known as Binomial Theorem for a positive integral index n.

$\because (x+y)^n$ contains $n+1$ terms

coefficients in these $n+1$ terms

$\binom{n}{0}, \binom{n}{1}, \binom{n}{2} \ldots \binom{n}{n}$ are called binomial coefficient, $n \geq 0$.

Prove the following identities for positive n

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

The binomial theorem for a positive integral index n reads.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Taking $x = y = 1$.

$$2^n = \sum_{r=0}^{n} \binom{n}{r} 1^r . 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r}$$

$$= \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Taking $x = -1$ and $y = 1$ we get.

$$0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n}.$$

Find the coefficient of

i) x^9y^3 in the expansion of $(2x-3y)^{12}$

$$(2x-3y)^{12} = \sum_{r=0}^{12} 12C_r (2x)^{12-r}(-3y)^r$$

$$(x+y)^n = \sum_{r=0}^{n} nC_r\, x^{n-r} y^r.$$

Coefficient of x^9y^3

$r = 3 \quad n = 12.$

$$12C_3 \times 2^9 (-3)^3 = \frac{12!}{9!\,3!} \times -2^9 \times 3^3$$

$$= -2^9 \times 3^3 \times \frac{12 \times 11 \times 10}{6}.$$

$= \underline{1946}$

2) x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} {}^{15}C_r \left(3x^2\right)^{n-r} \left(\frac{-2}{x}\right)^r$$

x^0 is the last term.

~~Coefficients~~ $\cancel{\sum_{r=0}^{15}}$ $\cancel{{}^{15}C_{15}\, x^{-2}}$

$$= \sum_{r=0}^{15} {}^{15}C_r (3)^{15-r} (-2)^r x^{2n-3r}$$

$x^{2(n-r)} \times x^{-r}$

$x^{2(n-r)-r}$

$= x^{2n-2r-r}$

$= x^{2n-3r}$

$0 = 2n - 3r = 2 \times 15 - 3 \times r$

$r = 10$

Coefficient of x^0

is ${}^{15}C_{10}\, 3^5 (-2)^{10}$

$$= \frac{15!}{10!\,5!} \times 3^5 \times 2^{10}$$

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3) x^{12} in the expansion of $x^3(1-2x)^{10}$.

$$(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (\text{[redacted]})^{10-r} (-2x)^r$$

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (-2)^r x^{r+3}$$

Coefficient of x^{12} in this expansion is

$12 = r+3$

$r = 12-3 = \underline{\underline{9}}$

Coefficient of $x^{12} = \binom{10}{9}(-2)^9 =$

$-10 \times 2^9 = \underline{\underline{-5120}}$

4) Evaluate $\binom{12}{5,3,2,2}$

$$\binom{12}{5,3,2,2} = \frac{12!}{5!\,3!\,2!\,2!} = \underline{\underline{166320}}$$

4) Find the term which contains x^{11} and y^4 in the expansion of $(2x^3 - 3xy^2 + z^2)^6$.

Multinomial theorem

For the positive integers n and t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \dots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! n_3! \dots n_t!}$$

For each n_i is a nonnegative integer $\leq n$,

and $n_1 + n_2 + n_3 + \dots + n_t = n$.

When n is positive, the general term of the expansion of

$(x_1 + x_2 + x_3 + \dots x_t)^n$ is $\frac{n!}{n_1! n_2! \dots n_t!} \times x_1^{n_1} x_2^{n_2} \dots$

where $n_1, n_2, \dots n_t$ are nonnegative integers not exceeding n and $n_1 + n_2 + n_3 + \dots + n_t = n$

The expression $\frac{n!}{n_1! n_2! \dots n_t!}$ is also written as $\binom{n}{n_1, n_2, n_3 \dots n_t}$. is called multinomial coefficient.

Determine the coefficient of

① xyz^2 in the expansion of $(2x-y-z)^4$

② $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

i) General term in the expansion of

$(2x-y-z)^4$ is

$$\binom{4}{n_1, n_2, n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$$

for $n_1=1, n_2=1$ and $n_3=2$, this reads

$$\binom{4}{1,1,2}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$$

$$=\binom{4}{1,1,2} \times 2 \times (-1) \times (-1)^2 \times xyz^2$$

$$= -2 \times \binom{4}{1,1,2} xyz^2$$

$$= -2 \times \frac{4!}{1!\,1!\,2!} xyz^2 = -24\,xyz^2$$

required coefficient is

-24.

ii) By the multinomial theorem, we note that the general term in the expansion of $(a+2b-3c+2d+5)^{16}$ is

$$\binom{16}{n_1, n_2, n_3, n_4, n_5} a^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$$

For $n_1=2, n_2=3, n_3=2, n_4=5$ and $n_5 = 16-(2+3+2+5) = 4$,

this becomes

$$\binom{16}{2,3,2,5,4} a^2(2b)^3(-3c)^2(2d)^5 5^4$$

$$= \binom{16}{2,3,2,5,4} \times 2^3 \times -3^2 \times 2^5 \times 5^4 \times a^2b^3c^2d^5$$

$$= 3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2} a^2b^3c^2d^5$$

required coefficient is

$$3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2}$$

Combination with repeatation

Suppose we wish to select with repeatation, a combination of r objects from a set of n distinct objects, where $r \leq n$. The number of such selection is given by.

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$= C(r+n-1, n-1)$$

$C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of combination of n distinct objects, taken r at a time, with repeatation allowed.

Ex 1) A bag contain coins of seven different denominations, with atleast one dozen coin in each denomination. In how many way can we select a dozen coins from the bag?

The selection consist in choosing with repeatation, $r = 12$ consist of $n = 7$ distinct denominations. The number of ways of making this selection is

$$C(7+12-1, 12) = C(18, 12) = \frac{18!}{12!\,6!} = 18,564$$

2) In how many ways can we distribute 10 identical marbles among 6 distinct containers.

The required number is.

$$C(6+10-1, 10)$$

$$= C(15, 10) = \frac{15!}{10!\,5!} = 3003$$

3) Find the number of nonnegative integer solutions of the equation.

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$

The required number is.

$$C(5+8-1, 8) = C(12, 8) = 495$$

4) In how many ways we can distribute 12 identical pencil to 5 children so that every child gets atleast 1 pencil?

We have to distribute 1 pencil to 5 children

Rest of pencil = 12 - 5 = 7.

We have to distribute 7 among 5 children

$$C(5+7-1,7) = C(11,7)$$

$$\frac{11!}{7!\,4!} = 330$$

5) In how many way ~~4~~ apples, 6 oranges among 4 children so that each child gets atleast 1 apple.

$$C(4+3-1,3) \times C(4+6-1,6)$$

$$C(6,3) \cdot C(9,6)$$

$$= \frac{6!}{3!\,3!} \times \frac{9!}{6!\,3!}$$

$$= 20 \times 84 = 1680$$

6) Find the number of ways of giving 10 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that total numbers of boxes given to A and B together does not exceed 4.

$n = 6$

Of the 10 boxes r boxes are given to A and B.

$$C(2+r-1, r) = C(r+1, r)$$
$$= r+1 \quad 0 \le r \le 4$$

The number of ways in which remaining $10-r$ boxes can be given C D E F is

$$C(4+(10-r)-1, 10-r)$$
$$= C(4+10-1-r, 10-r)$$
$$= C(13-r, 10-r)$$

$${}^{n}C_{r} = {}^{n}C_{n-r}$$

$${}^{13-r}C_{10-r} = {}^{13-r}C_{13-r-10+r}$$
$$= C(13-r, 3)$$

Since $0 \le r \le 4$, total number of ways in which boxes can be given $\sum_{r=0}^{4} (r+1) \times C(13-r, 3)$

Catalan Numbers

Consider the sequence $b_0, b_1, b_2 \ldots b_n$ of positive integer defined by

$$b_0 = 1 \text{ and } b_n = \frac{C(2n, n)}{n+1} \text{ for } n \geq 1$$

This sequence is called catalan sequence and the terms of the sequence are called the catalan numbers.

For the expression for b_n, we finds that

$$b_1 = \frac{1}{2} C(2,1) = 1$$

$$b_2 = \frac{1}{3} C(4,2) = \frac{1}{3} \cdot \frac{4!}{2!\,2!} = 2$$

$$b_3 = \frac{1}{4} C(6,3) = \frac{1}{4} \cdot \frac{6!}{3!\,3!} = 5$$

$$b_4 = \frac{1}{5} C(8,4) = \frac{1}{5} \cdot \frac{8!}{4!\,4!} = 14$$

$$b_5 = \frac{1}{6} C(10,5) = \frac{1}{6} \cdot \frac{10!}{5!\,5!} = 42$$

and so on.

$b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 14$ and $b_5 = 42$ are first six catalan numbers.

$b_n = \frac{C(2n,n)}{n+1}$ can be put in the followin

$$b_n = \binom{2n}{n} - \binom{2n}{n-1}$$

Proof.

$$\binom{2n}{n} - \binom{2n}{n-1}$$

$$= C(2n,n) - C(2n,n-1)$$

$$= \frac{2n!}{n!(2n-n)!} - \frac{2n!}{(n-1)!(2n-(n-1))!}$$

$$= \frac{2n!}{n!n!} - \frac{2n!}{(n-1)!(n+1)!}$$

expand $(n+1)!$ $\rightarrow (n+1)n!$ multiply & divide by n so $(n-1)!n = n!$

$$= \frac{2n!}{n!n!} - \frac{2n!(n)}{(n+1)(n!)(n!)}$$

$$= \frac{2n!}{n!n!}\left(1 - \frac{n}{n+1}\right) = C(2n,n)\cdot\frac{n+1-n}{n+1}$$

$$= C(2n,n)\cdot\frac{1}{n+1} = b_n$$

from $(2,1)$ to $(7,6)$ $\quad \underline{y = x-1}$

$(2, 1)$

1\.

$(3, 2)$

2\.

$(4, 3)$

3

$(5, 4)$

4

$(6, 5)$

5

$(7, 6)$

$$b_5 = \frac{C(10,5)}{6}$$

from $(3,3)$ to $(10,15)$ $\quad \underline{y = x+5}$

$(3, 3)$

1

$(4, 9)$

2

$(5, 10)$

3

$(6, 11)$

4

$(7, 12)$

5

$(8, 13)$

6

$(9, 14)$

7

$(10, 15)$

$$b_7 = \frac{1}{8} C(14,7)$$

In how many ways can one arrange the letters in the word CORRESPONDENTS so that

i) there is no pair of consecutive identical letter

ii) the are exactly two pairs of consecutive identical letter.

iii) there are atleast three pairs of consecutive identical letter

C	O	R	E	S	P	N	D	T
1	2	2	2	2	1	2	1	1

Set of all permutations of 14 letters $= \frac{14!}{(2!)^5}$