CHAPTER – 1

COMPLEX NUMBERS

1. Classification of Numbers :

(i) Natural numbers : **D. U. H. T, 75.**

The set of all natural numbers (or counting numbers or the positive integers) is denoted by N.

$N = \{1, 2, 3, \ldots \ldots\}$.

(ii) Integers : The set of all integers are those numbers whose elements are the positive and negative whole numbers and the zero and it is denoted by I (or Z) :

$I = \{\ldots \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \ldots\}$.

(iii) Rational numbers : **D. U. H. T. 75.**

The set of all rational numbers are those numbers which can be expressed as the ratio of two integers a/b where $b \neq 0$ and it is denoted by Q

$Q = \{a/b : a, b \in I, b \neq 0\}$.

Example 1 : $1, -1, 2, -2, 0, 3/2, -7/9$ etc are rational numbers.

(iv) Irrational numbers : The set of all irrational numbers are those numbers whose decimal representations are non-terminating and non-repeating and it can not be repesented in the form p/q where $p, q \in I, q \neq 0$. It is denoted by Q'

Example 2 : $2+\sqrt{3}$, $3-\sqrt{5}$, $\sqrt{2}$, π, e etc are irrational numbers.

(v) Real numbers : **D. U. C. T. 75.**

The set of all rational and all irrational numbers is called the set of all real numbers and it is denoted by $\boldsymbol{R}$.

$\boldsymbol{R}=\{x\,;\ x\in\boldsymbol{Q} \text{ or } x\in\boldsymbol{Q}'\}=\boldsymbol{Q}\cup\boldsymbol{Q}'$.

Here it is clear that $\boldsymbol{Q}\cap\boldsymbol{Q}'=\phi$.

Example 3 : $0, 5, -2, 2/3, -8/7, \pi, e, 2+\sqrt{3}, -5+\sqrt{7}$ etc are real numbers.

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(vi) *Complex numbers :*

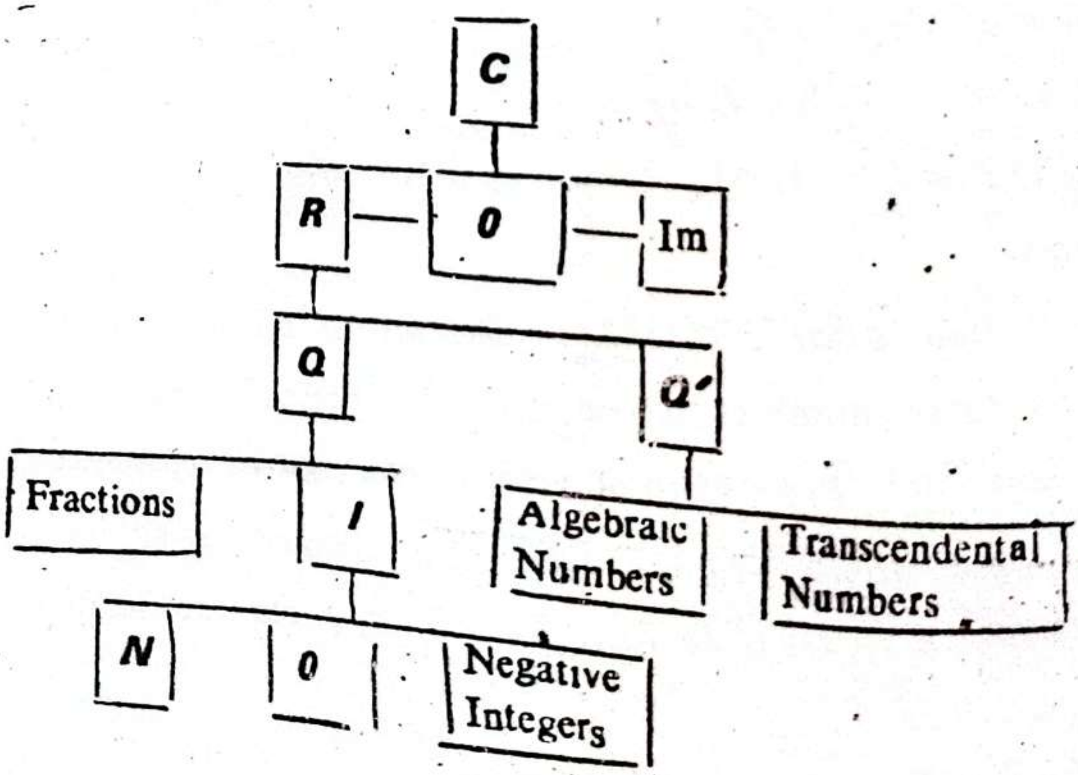
The set of all complex numbers are those numbers which can be represented in the form $a+ib$ where $a, b\in\boldsymbol{R}$ and $i=\sqrt{(-1)}$. It is denoted by $\boldsymbol{C}$:

$\boldsymbol{C}=\{a+ib:\ a, b\in\boldsymbol{R}, i=\sqrt{(-1)}\}=\boldsymbol{R}\cup \text{Im}$, where Im denotes the set of all imaginary numbers.

Here $\text{Im}=\{0\}\cup\{x+iy:\ x,\ y\in R, y\neq 0, i=\sqrt{(-1)}\}$.

Example 4 ; $0, 3, -5, 7/8, -27/19, 3+4i, -2-5i, 7-9i$ etc are complex numbers.

2. *A diagram* :



N.B. The definition of Algebraic and Transcendental numbers are given after some steps in th s chapter.

3. *Some definitions* :

(i) Set . Any collection of objects or any well defined list is called a set and its objects are called elements or members.

(ii) Null sot (or void set or empty set) :

A set that contains no elements is called the empty set or void set or null set and it is denoted by the symbol ϕ.

(iii) Sub set : A set A is said to be a subset of a set B if each element of A is also an element of B and is written $A \subset B$.

(iv) Union of sets : The union of two sets A and B is denoted by $A \cup B$ and it is defined by :

$A \cup B = \{x : x \in A \text{ or } x \in B\}$.

The union of A and B is also denoted by $A+B$.

(v) Intersection of sets : The intersection of two sets A and B is denoted by $A \cap B$ and it is demated by AB

$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

The intersection of A and B is also defined by : AB.

(vi) Complement : The complement of a set A is denoted by A^c and it is defined by : $A^c = \{x : x \notin A\}$.

(vii) Countability of a set : If there exists a one to one co rrespondence, to the members or elements of a set S with the set of integers $N = \{1, 2, 3, \ldots\ldots\}$, then the set S is called ***Countable*** or ***denumerable.*** If it is not countable, then it is call-

non-countable or ***non-denumerable***.

4. *Introduction*; We know in the real field R, the following types of equations have no solutions ;

$x^2+1=0$, $x^2+2=0$, $x^2+7=0$, $x^2+1/3=0$ etc. To solve these types of equations, the complex number system was introduced.

Historical Note : Swiss mathematician Leonard Euler (1707–83) introduced the ***imaginary unit*** i in 1748.

Definition : $i=\surd(-1)$ where $i^2=-1$.

Definition : If $a>0$, then $\surd(-a)=i\surd a$.

Definition : If $a, b>0$, then $\surd(-a)\,\surd(-b)=(i\surd a)\,(i\surd b)$
$=i^2\surd a\,\surd b=-\surd(ab)$.

N. B. If $a, b>0$, then $\surd(-a)\,\surd(-b)\neq\surd\{(-a)\,(-b)\}$.

Example : If $n\in I$, then $i^n\in\{1, -1, i, -i\}$.

5. *Complex Number* : The number having the form $a+ib$ is called a complex number where $a, b\in R$. We can also define a complex number $a+ib$ in an ordered pair of real numbers (a, b) i, e. $a+ib=(a, b)$.

Historical Note : The German mathematician Carl Friedrich Gauss (1777–1855), one of the greatest mathematician of the world, introduced the complex number system in 1832. The ordered pair system of complex number was introduced by the Irish mathematician William Rowan Hamilton (1805–65) in 1835.

6. *Real and Imaginary parts af a complex number*;

Let $z=(x, y)=x+iy$. Then x is called the ***real part*** of z

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and it is denoted by $\mathrm{Re}(z)$ *i. e.* $\mathrm{Re}(z)=x$. Again the ***imaginary part*** of z is y and it is denoted by $\mathrm{Im}(z)$ *i. e.* $\mathrm{Im}(z)=y$.

7. The fundamental operations of complex numbers : The following four definitions are given as like as the fundamental laws of algebra of real numbers using $i^2=-1$:

(i) Definition of sum : Let $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$ then $z_1+z_2=(x_1, y_1)+(x_2, y_2)=(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2)=(x_1+x_2, y_1+y_2)$.

(ii) Definition of difference : Let $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$, then $z_1-z_2=(x_1, y_1)-(x_2, y_2)=(x_1+iy_1)-(x_2+iy_2)=(x_1-x_2)+i(y_1-y_2)=(x_1-x_2, y_1-y_2)$.

(iii) Definition of $-z$; Let $z=(x, y)$, then $-z=(-x,-y)$. It is clear that $0-z=(0,0)-(x, y)=(0-x, 0-y)=(-x,-y)=-z$.

(iv) Definition of product ;

Let $z_1=(x_1, y_1)$, and $z_2=(x_2, y_2)$, then $z_1 z_2=(x_1, y_1)(x_2, y_2)$
$=(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$
$=(x_1x_2-y_1y_2, x_1y_2+y_1x_2)$.

8. Addition properties of complex numbers : The following properties hold in ***C***.

(i) Closure law for addition : $\forall z_1, z_2 \in \mathbf{C} \Rightarrow z_1+z_2 \in \mathbf{C}$.

(ii) Commutative law for addition ; $\forall z_1, z_2 \in C \Rightarrow z_1+z_2=z_2+z_1$.

(iii) Associative law for addition : $\forall z_1, z_2, z_3 \in C \Rightarrow z_1+(z_2+z_3)=(z_1+z_2)+z_3$.

(*iv*) ***Identity for addition*** : There exists exactly one complex number, $0=0+i0 \in \mathbf{C} \Rightarrow z+0=0+z=z, \forall z \in \mathbf{C}$. Here o is called the additive identity of $\mathbf{C}$:

(*v*) ***Inverse for addition :*** For each $z \in \mathbf{C}$, there exists exactly one $-z \in \mathbf{C}$ snch that $z+(-z)=(-z)+z=0$. Here $-z$ is called the inverse of z.

9. ***Multiplication properties of complex numbers*** ;

The following properties hold in $\mathbf{C}$:

(*i*) ***Closure law for multiplication*** : $\forall z_1, z_2 \in \mathbf{C} \Rightarrow z_1 z_2 \in \mathbf{C}$.

(*ii*) ***Commutative law for multiplication*** : $\forall z_1, z_2 \in \mathbf{C} \Rightarrow z_1 z_2 = z_2 z_1$.

(*iii*) ***Associative law for multiplication*** : $\forall z_1, z_2, z_3 \in \mathbf{C} \Rightarrow z_1 (z_2 z_3) = (z_1 z_2) z_3$.

(*iv*) ***Identity for multiplication*** : There exists exactly one complex number, $1=1+i0 \in \mathbf{C} \Rightarrow z.1=1.z=z, \forall z \in \mathbf{C}$. Here 1 is called the multiplicative identity of $\mathbf{C}$.

(*v*) ***Inverse for multiplication*** : Far each $z \in \mathbf{C}$, where $z \neq 0$, there exists exactly one $\frac{1}{z} \in \mathbf{C}$ such that $z.\frac{1}{z}=1$. Here $\frac{1}{z}$ is called the inverse of z.

10. ***The definition of division :***

Let $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$, then $\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)}$

$$= \frac{(x_1, y_1)}{(x_2, y_2)} \cdot \frac{(x_2, -y_2)}{(x_2, -y_2)} = \frac{x_1 + i\, y_1}{x_2 + iy_2} \cdot \frac{x_2 - i\, y_2}{x_2 - i\, y_2}$$

$$= \frac{x_1 x_2 + y_1\, y_2 + i(y_1 x_2 - x_1\, y_2)}{x_2^2 + y_2^2} \quad \text{where } x_2^2 + y_2^2 \neq 0.$$

Example 5 : Express $\dfrac{(1+2i)^2}{(2+i)^2}$ in the form $A+iB$. Also find its modulus and argument. ***D. U. 83.***

Solution : $\dfrac{(1+2i)^2}{(2+i)^2} = \dfrac{1+4i+4i^2}{4+4i+i^2} = \dfrac{1+4i-4}{4+4i-1} = \dfrac{-3+4i}{3+4i}$

$$= \frac{(-3+4i)\,(3-4i)}{(3+4i)\,(3-4i)} = \frac{-9+24i+16}{3^2+4^2} = \frac{7+24i}{25} = 7/25 + 24/25i.$$

Here $\left|\dfrac{(1+2i)^2}{(2+i)^2}\right| = \left|7/25 + 24/25i\right| = \dfrac{\sqrt{(7^2+24^2)}}{25} = \dfrac{25}{25} = 1,$

its principal argument $= \tan^{-1} \dfrac{24/25}{7/25} = \tan^{-1} 24/7$ and general argument $= \tan^{-1} \dfrac{24}{7} + 2n\,\pi,$

where $n = 0, \pm 1, \pm 2, \pm 3, \quad \ldots \quad \ldots$

11 *Equality of two complex numbers* :

(i) $a + ib = 0 \Rightarrow a = 0,\ b = 0,$

(ii) $a + ib = c + ib \Rightarrow a = c,\ b = d.$

Two complex numbers are said to be equal if and only if their real and imaginary parts are equal.

12. *The distributive law* : The distributive law holds in ***C*** : $\forall z_1, z_2, z_3 \in C,\ z_1\,(z_2 + z_3) = z_1\, z_2 + z_1\, z_3$. Here it is clear that $z_1\,(z_2+z_3) = (z_2+z_3)\, z_1$.

13. *Absolute value or modulus* :

J. U. H. 86 ; R. U. H. 85.

Let $z = x + iy$ be a complex number, then the absolute value or modulus of z is denoted by $|\,z\,|$ and is given by

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$|z| = |x+iy| = \sqrt{(x^2+y^2)}$. Here it is clear that $|z|$ is the distance from the origin and it is a non negative real number.

Example 6 : $|3+4i| = \sqrt{(3^2+4^2)} = 5$, $|3-4i| = \sqrt{(3^2+(-4)^2)} = 5$, $|-3+4i| = 5$, $|-3-4i| = 5$.

14. *Order relation* : Order relation exists only in the real number system and it does not exist in the complex number system. Therefore, ***inequality*** can be applied in the ordered real number system. Also it can be applied in the set of moduluses of complex numbers since the set of moduluses of complex numbers are non-negative real numbers.

Thus in complex variable "greater than" and "less than" have no meaning.

Example 7 : Which is the greater $3+4i$ or $6-8i$.

Solution : Inequality can not be applied there since they are complex numbers. But $|6-8i| = 10 > |3+4i| = 5$.

15. *Complex plane or Argand diagram or Argand plane or Gaussian plane or diagram* :

We know a complex number $x+iy$ can be considered as an ordered pair (x,y) where $x,y \in R$ and it canbe represented by points in the xy-plane which is called the complex plane or Argand diagram or Gaussian

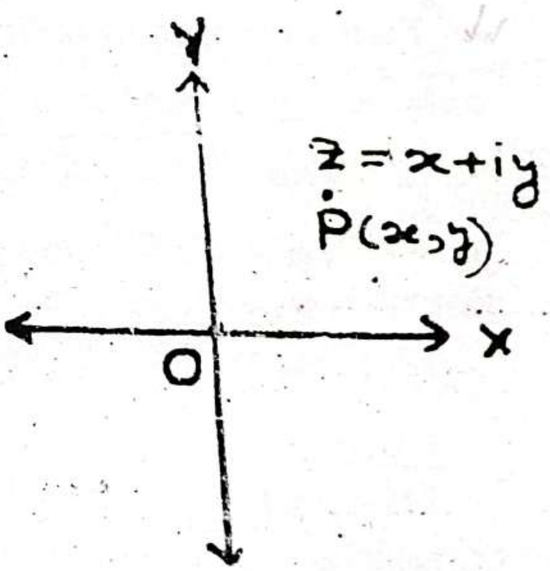


diagram. Thus the plane on which the complex numbers are represented is called the complex plane. In complex plane, to each complex number there exists one and conversely to each point in the complex plane there exists one and only one complex number.

Real axis : The points on the $x-$axis$=\{(x, 0): \ x \in \boldsymbol{R}\}$ which are real numbers since $(x, 0)=x+i0=x \in \boldsymbol{R}$. For this reason in the complex plane $x-$axis is called the real axis.

Imaginary axis : The points on the $y-$axis$=\{(0, y): \ y \in \boldsymbol{R}\}$ which are pure imaginary numbers since $(0, y)=0+iy=iy \in$ Im. For this reason in the complex plane $y-$axis is called the imaginary axis.

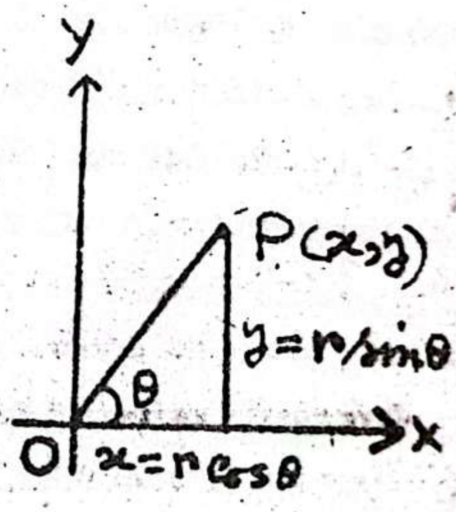
Zero : We have $(0, 0)=0+i0=0$ which is a complex number. It is clear that zero is the only number which is at a time real and imaginary since it lie on the real and imaginary axis at the origin.

Historical Note : French mathematician Jean Robert Argand (1768 – 1822) first explained the complex plane in 1806.

6. *Polar (or trigonometric) form of complex numbers :*

Let $z=x+iy$ be a complex number which is represected by the vector $\boldsymbol{OP}$.

Let $OP=|\boldsymbol{OP}|=r$ and any angle θ (positive which the vector makes with the positive $x-$axis, then we have $x=r\cos\theta$, $y=r\sin\theta$,



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Here $OP = |OP| = \sqrt{(x^2+y^2)} = r$ is called the ***modulus*** or ***absolute value*** of $z = x+iy$ and it is denoted by $|z|$ or mod z. Again $\theta = \tan^{-1} y/x + 2n\pi$, $n \in I$ is called the ***amplitude*** or ***argument*** or ***Phase*** of $z = x+iy$ and it is denoted by amp z or arg z. The form $z = x+iy = r\ (\cos\theta + i \sin\theta)$ is called the ***Polar form*** of the complex number, where r and θ is called the ***Polar coordinates***.

17. ***Modulus*** : ***J. U. H. 87 ; R. U. H. 85.***

Let (r, θ) be the polar coordinates corresponding to the complex number $z = x+iy = (x, y)$, then $r = \sqrt{(x^2+y^2)}$ is called the ***modulus*** or ***absolute value*** of z where $r = |z| = \sqrt{(x^2+y^2)}$.

18. ***Argument or amplitude or phase*** :

J. U. H. 85 ; R. U. H. 85.

Let (r, θ) be the polar coordinates corresponding to the complex number $z = (x, y) = x + iy$, then $\theta = \tan^{-1} y/x + 2n\pi \cdots (1)$ is called the ***argument*** or ***amplitude*** or ***phase*** of z where $n \in I$ and $-\pi < \theta \leq \pi$. If $n = 0$, then $(1) \Rightarrow \theta = \tan^{-1} y/x$ is called the ***principal argument*** of z and it is denoted by Arg z i. e. $\theta = \text{Arg } z = \tan^{-1} y/x$. It is clear that $\arg z = \text{Arg } z + 2\pi n$, $n \in I$.

N. B. In the following chapters, by arg z we will mean the principal argument, *i. e.* $\arg z = \theta = \tan^{-1} y/x$, where $-\pi < \theta \leq \pi$, unless otherwise stated.

$\theta = \tan^{-1} y/x + 2n\pi$, $n \in I$ is

sometimes called the ***general argument***.

19. ***Principal value of the argument*** :

Let $\theta = \arg z$, where $z \neq 0$. Then the particular value of θ such that $\theta \in]-\pi, \pi]$ or $-\pi<\theta\leq\pi$ is called the principal value of the argument of the complex number z and it is denoted by Arg z.

Example 8 : Let $z=1+\sqrt{3}i$, then $\arg z=\tan^{-1}\sqrt{3}/1+2n\pi$ $=\pi/3+2n\pi$, where $n=0, \pm 1, \pm 2, \pm 3, \cdots \cdots$ and $\text{Arg } z=\theta=\pi/3$.

Example 9 : Let $z=1-i$, then $\arg z=\tan^{-1}\left(\frac{-1}{1}\right)+2n\pi$ $=-\pi/4+2n\pi$, where $n=0, \pm 1, \pm 2, \pm 3, \cdots \cdots$ and $\text{Arg } z=\theta$ $=-\pi/4$.

Example 10 : Let $z=-2-2i$, then $\arg z=\tan^{-1}\frac{-2}{-2}$ $+2n\pi=-3\pi/4+2n\pi$, where $n=0, \pm 1, \pm 2, \pm 3, \cdots \cdots$ and $\text{Arg } z=\theta=-3\pi/4$.

Example 11 : $\arg i=\pi/2+2n\pi$, $n \in I$; $\text{Arg } i=\pi/_2$; $\arg(-1)$ $=\pi+2n\pi$, $n \in I$; $\text{Arg }(-1)=\pi$. n ; $\arg(-1+i)=3\pi/_4+2n\pi$, $n \in I$ and $\text{Arg }(-1+i)=\pi/_4$

N. B. (i) arg z is not unique but Arg z is unique.

(ii) if $z=0$, then arg z and Arg z do not exist since $z=0 \Rightarrow x=0, y=0$, then $\tan^{-1} y/x$ does not exists.

20. The distance between two points :

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Then $z_1-z_2=(x_1-x_2)+i(y_1-y_2)$ and $|z_1-z_2|$ $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. Here $|z_1-z_2|$ denotes the distance between the points z_1 and z_2.

21. Complex conjugate :

The complex conjugate of the number $z=x+iy$ is denoted by $\bar{z}$ and is defined by $\bar{z}=\overline{x+iy}=x-iy$.

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Again the complex conjugate of the number $z = x - iy$ is $\bar{z} = x + iy$. The complex conjugate of z is called the ***reflection*** or ***image*** of z with respect to the real axis.

Historical Note : The French mathematician Augustin Cauchy (1789–1857) introduced the name "Conjugate" in 1821 in his Cours d' Analyse algébrique.

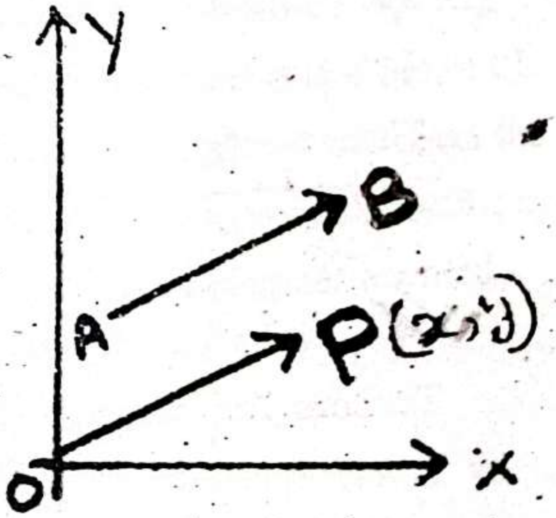
22. Complex conjugate coordinates : We have if $z = x + iy$, then

$$\bar{z} = x - iy, \quad x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z}).$$

Now if the coordinates $(z, \bar{z})$ denote a point, then these coordinates are called ***complex conjugate coordinates*** or simply ***conjugate coordinates*** of this point.

23. Complex numbers in vector form :

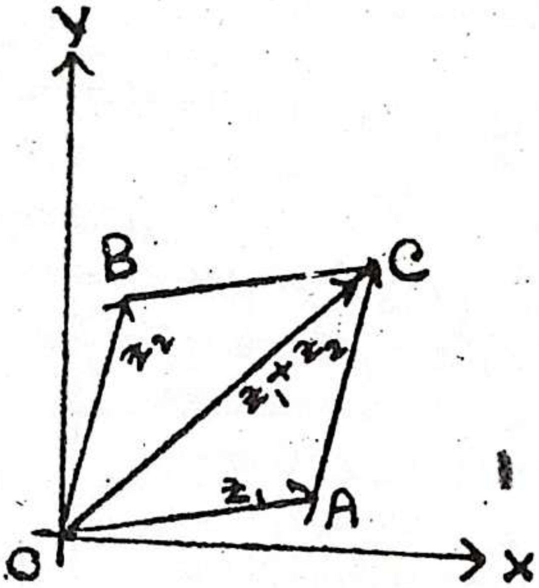
The complex number $z = x + iy$ can be represented by the directed line segment or vector ***OP*** whose initial Point O is called the origin and whose terminal Point P is the point (x, y). If the vectors ***OP*** and ***AB*** have the same length (or magnitude) and direction,



then they are considered equal *i. e.* ***OP*** = ***AB*** = x + *i*y. (see in the Fig.)

24. Geometrical representation of the sum, difference product and quotient of two complex numbers :

(1) Addition or sum : **D. U. H. T. 80.**



Let A and B be two points $z_1=(x_1, y_1)=x_1+iy_1$ and $z_2=(x_2, y_2)=x_2+iy_2$ in the Argand plane. Now we complete the parallelogram OACB.

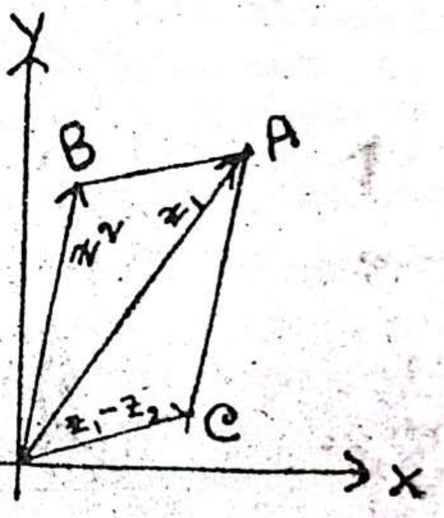
Then $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$ represents the point C, see in the Fig. In vector form :

$$\vec{OC}=\vec{OA}+\vec{AC}=\vec{OA}+\vec{OB}=\vec{OB}+\vec{BC}.$$

2. Subtraction or difference :

Let A and B be the points $z_1=(x_1, y_1)=x_1+iy_1$ and $z_2=(x_2, y_2)=x_2+iy_2$ in the Argand plane.

Now we complete the parallelogram OCAB.

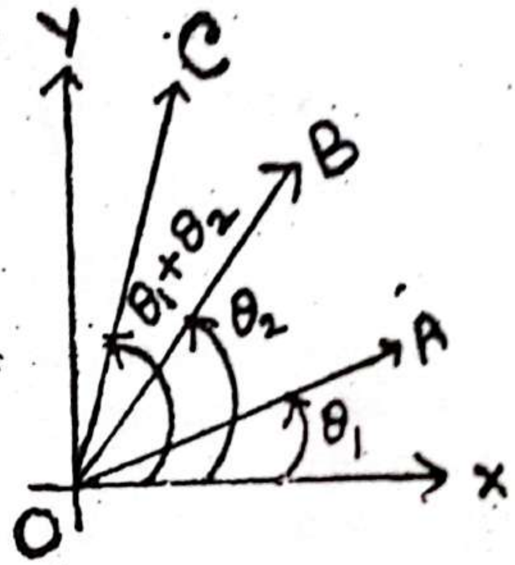


Then $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$ represents the point C, see in the Fig. In vector form $\vec{OC} = \vec{OA} - \vec{CA} = \vec{OA} - \vec{OB} = \vec{BA}$.

3. Product or multiplication :

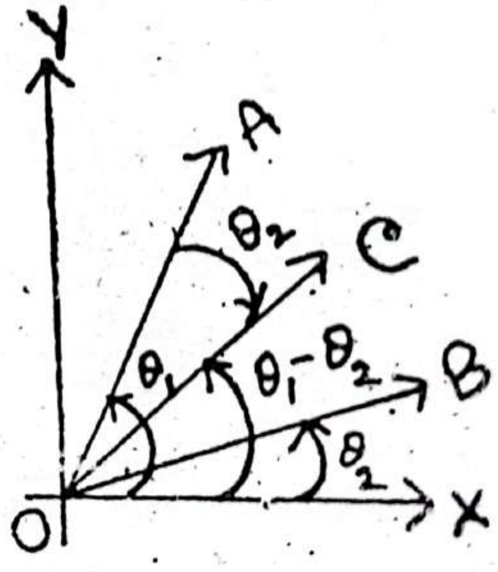
D. U. H. T. 90 ; C. U. 68.

Let A and B be the points $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ where $OA = r_1$, $OB = r_2$ and $\angle XOA = \theta_1$, $\angle XOB = \theta_2$, see in the Fig. Then $z_1 z_2 = r_2 r_1\{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}$ represents the point C where $OC = r_1 r_2$ and $\angle XOC = \theta_1 + \theta_2$, see in the Fig



Hence, the modulus and amplitude of the product of two complex nnmbers are equal respectively to the product of their moduli and the sum of the amplitudes of their factors.

4. Quotient or division :

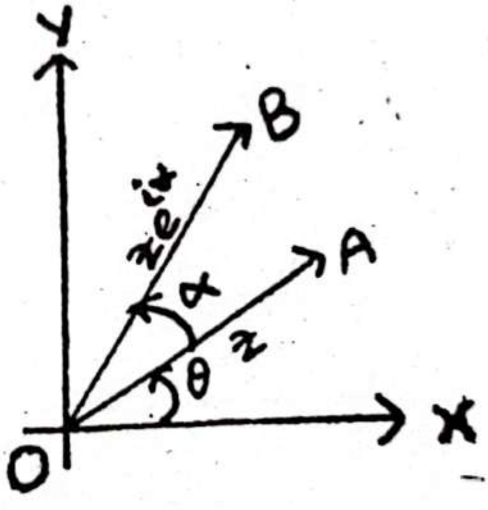


Let A and B be the points $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ where $OA=r_1$, $OB=r_2$ and $\angle XOA=\theta_1$, $\angle XOB=\theta_2$, see in the Fig.

Then $z_1/z_2=r_1/r_2\{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\}$ represents the point C where $OC=r_1 r_2$ and $\angle XOC=\theta_1-\theta_2$, see in the Fig.

Hence, the modulus and amplitude of the quotient of two complex numbers are equal respectively to the quotient of their moduli and the difference of the amplitudes of the numerator and denominator.

25. *Interpretation of* $ze^{i\alpha}$:

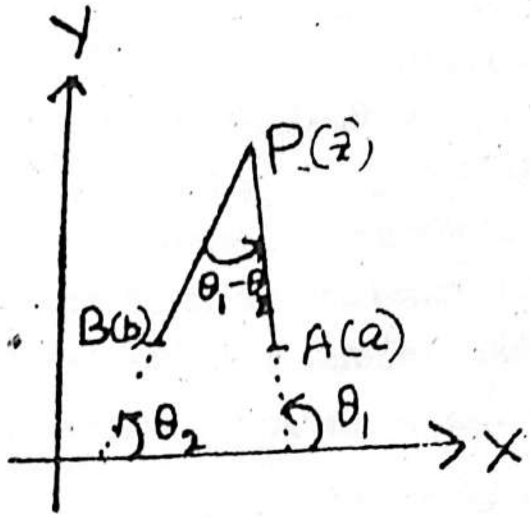


Let $z=re^{i\theta}$ be represented by the vector $\vec{OA}$, see in the Fig.

Then $ze^{i\alpha}=re^{i\theta}\cdot e^{i\alpha}=re^{i(\theta+\alpha)}$ is represented by the vector $\vec{OB}$ where θ and α are real.

Hence $ze^{i\alpha}$ i. e. multiplication of a vector z by $e^{i\alpha}$ amounts the rotation of z through an angle α in the positive direction Here $z\,i \Rightarrow$ the rotation of z through an angle $\pi/2$ since $zi=ze^{\pi i/2}$.

26. *Interpretation of* arg $\frac{z-a}{z-b}$:



Let z, a, b represent the points P, A, B respectively on the Argand plane. Then $\overrightarrow{AP}=z-a$, $\overrightarrow{BP}=z-b$, arg $\overrightarrow{AP}=\theta_1$, arg $\overrightarrow{BP}=\theta_2$ and $\angle BPA=\theta_1-\theta_2=\arg(z-a)-\arg(z-b)=\arg\frac{z-a}{z-b}$ see in the Fig. where we have considered only the principal argument. Thus arg $\frac{z-a}{z-b}$ gives the angle between the lines AP and BP in the positive sense. Similarly, arg $\frac{z-b}{z-a}$ gives the angle in the negative sense.

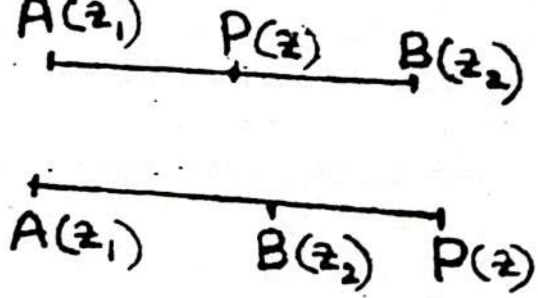
(*i*) *Condition for perpendicularity* :

If the lines AP and BP are perpendicular, then arg $\frac{z-a}{z-b}$ $=\pi/2$ & arg $\frac{z-b}{z-a}=-\pi/2 \Rightarrow \frac{z-a}{z-b}$ and $\frac{z-b}{z-a}$ are purely imaginaries.

(ii) Condition for collinearity ;

If the points A, B, P are collinear, then $\arg \frac{z-a}{z-b}=0$ or $\pi \Rightarrow \frac{z-a}{z-b}$ is purely real.

27. Equation of a straight line joining the points z_1 and z_2 in the Argand plane :



Let P (z) be any point on the line AB where z_1 and z_2 are points of A and B respectively, see in the Fig.

If P (z) lies inside, then $\arg \frac{z-z_1}{z-z_2}=\pi \Rightarrow \frac{z-z_1}{z-z_2}$ is purely real. Again if P (z) lies outside, then $\arg \frac{z-z_1}{z-z_2}=0 \Rightarrow \frac{z-z_1}{z-z_2}$ is purely real. Thus in both cases $\frac{z-z_1}{z-z_2}$ is purely real $\Rightarrow$

$$\Rightarrow \frac{z-z_1}{z-z_2}=\overline{\left(\frac{z-z_1}{z-z_2}\right)}=\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2} \Rightarrow$$

$$(z-z_1)\left(\bar{z}-\bar{z}_2\right)-(z-z_2)\left(\bar{z}-\bar{z}_1\right)=0.$$

$$\Rightarrow \left(\bar{z}_1-\bar{z}_2\right)z-(z_1-z_2)\bar{z}+\left(z_1\bar{z}_2-z_2\bar{z}_1\right)=0.$$

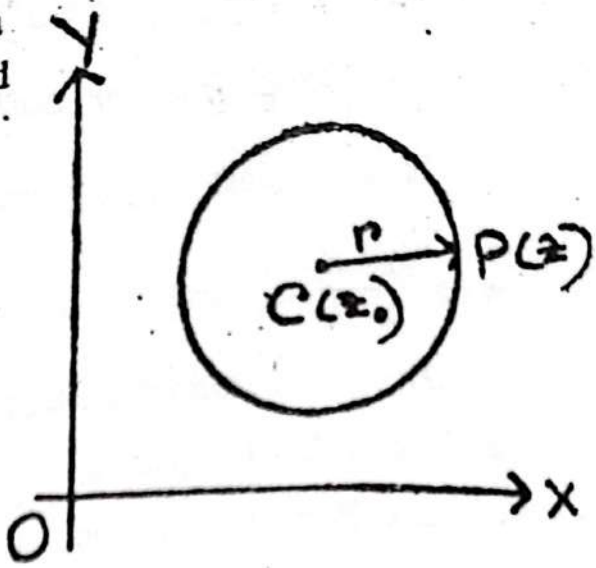
$$\Rightarrow \bar{a}z-a\bar{z}+c=0$$

which represents the equation of a straight line where

$\bar{a}=\bar{z}_1-\bar{z}_2 \therefore a=z_1-z_2$ and $c=z_1\bar{z}_2-z_2\bar{z}_1$

28. *Equation of a circle with centre at z_0 and radius r :*

Let P (z) be any point on the circle with centre C (z_0) and r be its radius.



$|z-z_0| = r \Rightarrow |z-z_0|^2 = r^2$

$\Rightarrow (z-z_0)\overline{(z-z_0)} = r^2$

$\Rightarrow (z-z_0)(\bar{z}-\bar{z}_0) = r^2$

$\Rightarrow z\bar{z} - \bar{z}_0 z - z_0\bar{z} + (z_0\bar{z}_0 - r^2) = 0$

$\Rightarrow z\bar{z} + \bar{a}z + a\bar{z} + \lambda = 0,$

which represents the equation of a circle where $a = -z_0$, $\bar{a} = -\bar{z}_0$ anb $z_0\bar{z}_0 - r^2 = \lambda$ (real).

29. *Dot product of two complex numbers :*

The dot product or scalar product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is defined by $z_1 o z_2 = |z_1|\,|z_2|\cos\theta$ $= x_1x_2 + y_1y_2 = \text{Re}\{\bar{z}_1 z_2\} = \frac{1}{2}\{\bar{z}_1 z_2 + z_1\bar{z}_2\}$ where θ is the angle between z_1 and z_2 and $0 \leq \theta \leq \pi$.

(*i*) If $z_1 \neq 0$, $z_2 \neq 0$, then $z_1 o z_2 = 0 \Rightarrow z_1$ and z_2 are perpendicular.

(*ii*) The projection of z_1 on z_2 is $\dfrac{|z_1 o z_2|}{|z_2|}$ and the projecton of z_2 on z_1 is $\dfrac{|z_1 o z_2|}{|z_1|}$.

Example 12 : If $z_1 = 4-3i$ and $z_2 = -3+4i$, then $z_1 o z_2 = \text{Re}(\bar{z}_1 z_2) = \text{Re}\{(4+3i)(-3+4i)\} = \text{Re}\{-24+7i\} = -24$ and

$$\cos\theta = \frac{z_1 o z_2}{|z_1|\cdot|z_2|} = \frac{-24}{|4-3i|\;|-3+4i|} = \frac{-24}{(5)(5)} = \frac{-24}{25}.$$

Example 13 : Show that $z_1 o z_2 = z_2 o z_1$.

Solution : Try yourself.

Example 14 : If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 \circ z_2$

$= \text{Re}\{\bar{z}_1 z_2\} = \text{Re}\{(r_1 e^{-i\theta_1})(r_2 e^{i\theta_2})\} = \text{Re}\{r_1 r_2 e^{i(\theta_2-\theta_1)}\}$

$= r_1 r_2 \cos(\theta_2 - \theta_1)$.

30. Cross product of two complex numbers :

The cross product or vector product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is defined by $z_1 \times z_2 =$

$|z_1|\,|z_2| \sin\theta = x_1 y_2 - y_1 x_2 = \text{Im}\{\bar{z}_1 z_2\}$

$= \frac{1}{2i}\{\bar{z}_1 z_2 - z_1 \bar{z}_2\}$. where θ is the angle between z_1 and z_2 and $0 \leqslant \theta \leqslant \pi$.

(i) If $z_1 \neq 0$, $z_2 \neq 0$, then $z_1 \times z_2 = 0 \Rightarrow z_1$ and z_2 are parallel.

(ii) The area of a parallelogram is $|z_1 \times z_2|$ where z_1 and z_2 are its sides.

Example 15 : If $z_1 = 4 - 3i$ and $z_2 = -3 + 4i$, then $z_1 \times z_2 =$

$\text{Im}\{\bar{z}_1 z_2\} = \text{Im}\{(4+3i)(-3+4i)\} = \text{Im}\{-24+7i\} = 7$.

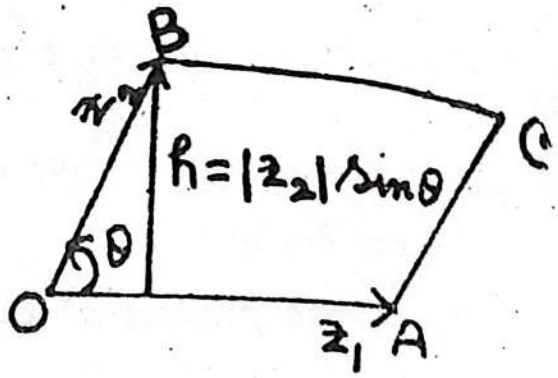
Example 16 : Show that $z_1 \times z_2 = -z_2 \times z_1$.

Solution : Try yourself.

Example 17 : If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 \times z_2$

$= \text{Im}\{\bar{z}_1 z_2\} = \text{Im}\{r_1 r_2 e^{i(\theta_2-\theta_1)}\} = r_1 r_2 \sin(\theta_2 - \theta_1)$.

31. Area of a parallelogram :



The area of a parallelogram having sides z_1 and z_2 =(base) (height) (see in the Fig) = (OA) (h) = $(|z_1|)(|z_2|)\sin\theta = |z_1||z_2|\sin\theta = |z_1 \times z_2|$.

32. Area of a triangle :

The area of a triangle OAB is $\frac{1}{2}|z_1 \times z_2|$ where z_1 and z_2 are the sides of a parallelogram OACB, see in the above Fig. Here $\triangle OAB = \frac{1}{2}|(x_1 y_2 - y_1 x_2)|$.

The area of a trianle having vertices $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

33 (i) Theorem 1 : If $z_p = r_p(\cos\theta_p + i\sin\theta_p)$ where $p = 1, 2, \cdots n$, then $z_1 z_2 \cdots z_n =$ $r_1 r_2 \cdots r_n\{\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)\}$.

Proof : Try yourself.

33. (i) Theorem 2 : If $z = r(\cos\theta + i\sin\theta)$, then show that $z^n = r^n(\cos n\theta + i\sin n\theta)$ where $n \in \{2, 3, \cdots, n\}$.

Proof : Putting $z_1 = z_2 = \cdots = z_n = z$, $r_1 = r_2 = \cdots = r_n = r$ and $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$. then the above theorem $\Rightarrow z^n = r^n(\cos n\theta + i\sin n\theta)$.

N. B. The above theorem is true for every integral values of n.

34. De moivres theorem for positive integral values :

Theorem 3 : $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $\forall\ n \in \mathbb{N}$.

Proof : We have if $z=r\ (\cos\theta+i\sin\theta)$, then $z^n=r^n(\cos n\theta+i\sin n\theta)$ where $n\in \boldsymbol{N}$.

That is, $r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta)$

$\Rightarrow (\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$

35. Deo moivre's theorem for negative Integers.

Theorem 4 : For all negative integral values of n,

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$

Proof ; Try yourself.

36. Deo moivre's theorem for rational numbers :

Theorem 5 : $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.\ \forall\, n\in \boldsymbol{Q}$

Proof : Try yourself.

37. General Deo moivre's theorem (Theorem 5) :

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,\ \forall\, n\in \boldsymbol{R}.$

Proof : Try yourself.

38. Euler Formula ; Show that : $e^{i\theta}=\cos\theta+i\sin\theta$, which is called Euler's formula. We have

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^2}{3!}+\ \dots\ \dots\ \dots\ \dots \qquad (1)$$

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ \dots\ \dots\ \dots\ \dots \qquad (2)$$

$$\text{and } \sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\ \dots\ \dots\ \dots\ \dots \qquad (3)$$

Now putting $x=i\theta$ in (1) we get

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)}{4!}+\ \dots\ \dots\ \dots$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\cdots\right).$$

$=\cos\theta+i\sin\theta$, by (2) and (3). Similarly, $e^{-i\theta}=\cos\theta-i\sin\theta$.

It is clear that $e^z=e^{x+iy}=e^x(\cos y+i\sin y)$.

39. Deo moivre's theorem in Euler's Form :

$$(\cos\theta+i\sin\theta)^n=\left(e^{i\theta}\right)^n=e^{in\theta}$$

N.B. cis θ : It is sometimes written cis θ for $\cos\theta+i\sin\theta$.

Example 18 : Show that : (i) $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$,

(ii) $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$, (iii) $\tan\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{i\,(e^{-i\theta}+e^{-i\theta})}$.

(iv) $\sec\theta=\dfrac{2}{e^{i\theta}+e^{-i\theta}}$, (v) $\operatorname{cosec}\theta=\dfrac{2i}{e^{i\theta}-e^{-i\theta}}$

(vi) $\cot\theta=\dfrac{i(e^{i\theta}+e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}$

Solution : We have $e^{i\theta}=\cos\theta+i\sin\theta$ and $e^{-i\theta}=\cos\theta-i\sin\theta$, then : Try yourself.

40. Complex polynomials and complex polynomial equations : An expression of the form

$$a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots\cdots+a_{n-1}z+a_n \quad \cdots \quad \cdots \quad (1)$$

is called a ***Complex polynomial*** of degree n, where $a_0 \neq 0, a_1, a_2, \cdots, a_n \in \boldsymbol{C}$ and $n \in \boldsymbol{N}$.

If $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0 \quad \cdots \quad (2)$,

then it is called the ***Complex polynomial equation*** of degree n, where $a_0 \neq 0, a_1, a_2, \cdots, a_n \in \boldsymbol{C}$ and $n \in \boldsymbol{N}$.

If $z_1, z_2, \cdots, z_n$ are the n roots of the equation (2),

then (2) $\Rightarrow a_0 (z - z_1)(z - z_2) \cdots (z - z_n) = 0 \quad \cdots \quad \cdots \quad (3)$.

The equation (3) is called the ***factor form*** of the polynomial equation.

41. Algebraic number : A number is called an ***algebraic number*** if it is a solution of any polynomial equation of the form

$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$, where $a_0, a_1, a_2, \cdots a_n \in I$.

Example 19 : $\sqrt{2}$ in an algebraic number since it satisfies the polynomial equation $z^2 - 2 = 0$ of integarl coefficients. similarly, $\sqrt{2} + \sqrt{3}$ is an algebraic number since it satisfies the polynomial equation $z^4 - 10z^2 + 1 = 0$ of integral coefficients.

42. Transcendental* number : A number is called a ***transcendetal number*** if it is not a solution of any polynomial equation with integral coefficients *i. e.* if it is not algebraic.

Example 20 : e and π are transcendental numbers. But it is not known numbers such as $e\pi$ and $e + \pi$ are transcendental or not.

43. The roots of complex numbers :

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Let z be a complex number. Then a number w is called an nth root of z if $w^n = z \Rightarrow w = z^{1/n}$ ... (1). Let $z = r\cos\theta + i\sin\theta$, then (1) $\Rightarrow w = r^{1/n}\left(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right)$... (2).

where $k = 0, 1, 2, \cdots, n-1 \Rightarrow$ there are n different values of $z^{1/n}$ where $z \neq 0$. (2) $\Rightarrow$

$$w = r^{1/n} e^{\left(\frac{2k\pi+\theta}{n}\right)i},$$

where $k = 0, 1, 2, \cdots, n-1$.

Example 21 : Show that : $e^{i\theta} = e^{2n\pi i + i\theta}$, where $n \in I$.

Solution : Try yourself.

44. *The nth roots of unity* : Let $z^n = 1$.

Then $z = (1)^{1/n} = \left\{\frac{(\cos 2k\pi + i\sin 2k\pi)}{n}\right\}^{1/n}$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{\frac{2k\pi}{n}}.$$

where $k = 0, 1, 2, \cdots, n-1$. Let $w = e^{\frac{2k\pi i}{n}}$, then the n roots are : $1, w, w^2, \cdots w^{n-1}$. The ***principal nth root of unity is 1.***

Geometrically these n roots represent the n vertices of a regular polygon of n sides inscribed in a circle of radius 1 (one) where centre is at the point (0, 0).

Here $|z| = \left|\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right|$

$= \sqrt{\cos^2\frac{2k\pi}{n} + \sin^2\frac{2k\pi}{n}} = 1$, which is the equation of a unit circle.

Example 22 : Show that : $e^{2n\pi i}=1$ where $n \in I$.

Solution : We have $e^{2n\pi i}=\cos 2n\pi+i\sin 2n\pi=1+i0=1$, where $n \in I$.

Example 23 ; If $w, w^2, w^3, \cdots, w^{n-1}, w^n=1$ are the n roots of $(1)^{1/n}$, where n is a positive integer then show that $1+w+w^2+\cdots+w^{n-1}=0$.

Solution : Try yourself.

45. The nth roots of (−1) : Let $z^n=-1$.

Then $z=(-1)^{1/n}=\{\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\}^{1/n}$

$$=\cos\frac{(2k+1)\pi}{n}+i\sin\frac{(2k+1)\pi}{n}=e^{\frac{(2k+1)\pi i}{n}},$$

where $k=0, 1, 2, 3, \cdots, n-1$. The ***principal nth root of*** (-1) is $e^{\frac{\pi i}{n}}$ if $k=0$. If n is odd, then the principal root is (-1) if $k=\frac{1}{2}(n-1)$.

46. Factors of z^n-1 : The n factors of z^n-1 are :

$z-e^{\frac{2k\pi i}{n}}$ where $k=0, 1, 2, 3, \cdots, n-1$.

(*i*) If n is even, then

$$z^n-1=(z-1)(z+1)\prod_{k=1}^{(n-2)/2}\left(z^2-2z\cos\frac{2k\pi}{n}+1\right).$$

(*ii*) If n is odd, then

$$z^n-1=(z-1)\prod_{k=1}^{(n-1)/2}\left(z^2-2z\cos\frac{2k\pi}{n}+1\right).$$

47. *Factors of z^n+1* : The n factors of z^n+1 are :

$$z-e^{\frac{(2k+1)\pi i}{n}} \quad \text{where } k=0, 1, 2, 3, \cdots, n-1.$$

(*i*) If n is even, then

$$z^n+1=\prod_{k=0}^{k=(n-2)/n} \{z^2-2z\cos\frac{(2k+1)\pi}{n}+1\}.$$

(*ii*) If n is odd, then

$$z^n+1=(z+1)\prod_{k=0}^{(n-3)/2} \{z^2-2z\cos\frac{(2k+1)\pi}{n}+1\}.$$

48. *Point sets in the Argand plane* :

(*i*) ***Neighbourhood*** : A neighbour hood of a point z_0 in the Argand plane is the set of all points z such that $|z-z_0|<\delta$ where $\delta>0$. It is also called a ***delta*** or ***δ-neighbourhood*** of z_0. A ***deleted*** or ***δ-neighbourhood*** of a point z_0 in the Argand plane is the set of all points z such that $0<|z-z_0|<\delta$, where $\delta>0$. In this case z_0 is omitted.

(*ii*) ***Limit point or cluster point or accumulation point*** : A point z_0 is called a limit point for a set S in the Argand plane if every deleted δ-neighbourhood of z_0 contains points of S other than z_0.

(*iii*) ***Closed set*** : A set S in the Argand plane is called closed if every limit point of the set S belongs to the set S.

(*iv*) ***Bounded set*** : A set S in the Argand plane is called bounded if there exists a positive constant M such that $|z|<M$ for every point z of S.

(v) ***Unbounded set*** : A set S in the Argand plane is called unbounded if it is not bounded.

(vi) ***Compact set*** : A set S in the Argand plane is called compact if it is both bounded and closed.

(vii) ***Interior point*** : A point z_0 is said to be an interior point of a set S in the Argand plane if there exists a δ-neighbourhood of z_0 all of whose points belonging to S.

(viii) ***Boundary (or frontier) point :*** A point z_0 is called a boundary point of a set S in the Argand plane if every δ-neighbourhood of z_0 contains points belonging to S and also points not belonging to S.

(ix) ***Exterior point*** : A point is called an exterior point of a set S in the Argand plane if it is not an interior point and also not a boundary point of the set S.

(x) ***Open set*** : A set S in the Argand plane is called an open set if it is a set which consists entirely of interior points.

(xi) ***Connected set*** : An open set S in the Argand plane is said to be connected if each pair of its point can be joined by some continuous chain of finite number of line segments all points of which lie in the set S.

(xii) ***Domain or open region :*** An open connected set S in the Argand plane is called a domain or open region.

(xiii) ***Interior of a set.*** The set of all interior points of a set S in the Argand plane is called the interior of the set and it is deonted by S_i.

(xiv) Frontier (or boundary) of a set :

The set of all frontier points of a set in the Argand plane is called the frontier of the set and it is denoted by S_f.

(xv) Exterior of a set ; The set of all exterior points of a set in the Argand plane is called the exterior of the set and it is denotod by S_e.

(xvi) Derived sets : The set of all the limit points of a set S in the Argand plane is called the derived set and it is denoted by $\bar{S}$:

(xvii) Closure of a set : The union $S_i \cup S_f$ is called the closure of the set S and it is denoted by $\bar{S}$ where $\bar{S} = S_i \cap S_f$, S_i = Interior of the set S and S_f = frontier of the set S.

(xviii) Closed region The closure of an open region or domain in the Argand plane is called a closed region.

(xix) Region : Let S be an open region or domain. If we take some or all or none of its limit, then we obtain a set R (say) which is called a region.

If all the limit points are added, then R is called a closed region. If none limit points are added, then R is called an open region

N. B. **In this book, by a region we will mean the open region unless otherwise any other region is stated.**

(xx) *Two important theorems* :

(1) *Bolzano-weierstrass theorem* **(*Theorem– 6*) :**

Every infinite bounded set on the Argand plane has at least one limit point.

(2) *Heine-Borel theorem* **(*Theorem –7*) :**

Let S be a compact set in the Argand plane.

If to each point of S is contained in one or more of the open sets $A_1, A_2, A_3, \cdots$, then there exists a finite number of sets $A_1, A_2, A_3, \cdots$, which will cover S.

49. *Theorem 8* . If $z_1, z_2, \cdots, z_n \in \boldsymbol{C}$, then show that :

(*i*) $\text{Re}\,(z_1 \pm z_2) = \text{Re}(z_1) \pm \text{Re}(z_2)$;

(*ii*) $\text{Re}\,(z_1 \pm z_2 \pm \cdots \pm z_n) = \text{Re}\,(z_1) \pm \text{Re}\,(z_2) \pm \cdots \pm \text{Re}\,(z_n)$;

(*iii*) $\text{Im}\,(z_1 \pm z_2) = \text{Im}\,(z_1) \pm \text{Im}\,(z_2)$;

(*iv*) $\text{Im}\,(z_1 \pm z_2 \pm \cdots \pm z_n) = \text{Im}\,(z_1) \pm \text{Im}\,(z_2) \pm \cdots \pm \text{Im}\,(z_n)$.

Proof : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2) \Rightarrow \text{Re}\,(z_1 \pm z_2) = x_1 \pm x_2 = \text{Re}\,(z_1) \pm \text{Re}\,(z_2)$

(*ii*), (*iii*) and (*iv*) : Try yourself.

50. *Theorem 9* ; Show that :

(*i*) $\text{Re}\,(z_1 z_2) = \text{Re}\,(z_1)\,\text{Re}\,(z_2) - \text{Im}\,(z_1)\,\text{Im}\,(z_2)$,

(*ii*) $\text{Im}\,(z_1 z_2) = \text{Re}\,(z_1)\,\text{Im}\,(z_2) + \text{Im}\,(z_1)\,\text{Re}\,(z_2)$.

Proof ; Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$.

(*i*) $\text{Re}\,(z_1 z_2) = x_1 x_2 - y_1 y_2 = \text{Re}\,(z_1)\,\text{Re}\,(z_2) - \text{Im}\,(z_1)\,\text{Im}\,(z_2)$

and (*ii*) $\text{Im}\,(z_1 z_2) = x_1 y_2 + y_1 x_2 = \text{Re}\,(z_1)\,\text{Im}(z_2) + \text{Im}\,(z_1)\,\text{Re}\,z_2)$.

51. *Theorem 10* : Let z be a complex number, then show that :

(i) z is real if $z=\bar{z}$ and

(ii) z is purely imaginary if $z=-\bar{z}$.

Proof ; Let $z=x+iy$, then $\bar{z}=x-iy$.

(i) Now if $z=\bar{z} \Rightarrow x+iy=x-iy \Rightarrow 2iy=0 \Rightarrow y=0 \Rightarrow$ $z=x \Rightarrow z$ is real.

(ii) Now if $z=-\bar{z} \Rightarrow x+iy=$ -

$\Rightarrow z=iy \Rightarrow z$ is purely imaginary

52. *Theorem 11* : Show that .

(i) $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$;

(ii) $\bar{\bar{z}}=z$;

(iii) $\overline{z+\bar{z}}=z+\bar{z}$ and

(iv) $\overline{z-\bar{z}}=-(z-\bar{z})$.

Proof : (i) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Then $\overline{z_1+z_2}=\overline{(x_1+iy_1)+(x_2+iy_2)}=\overline{(x_1+x_2)+i(y_1+y_2)}$

$=(x_1+x_2)-i(y_1+y_2)=(x_1-iy_1)+(x_2-iy_2)=\bar{z}_1-\bar{z}_2$.

(ii) Let $z=x+iy$, then $\bar{z}=x-iy \Rightarrow \bar{\bar{z}}=x+iy=z$.

(iii) and (iv) : Try yourself.

63. *Theorem 12* : Show that :

(i) $\overline{z_1 z_2}=\bar{z}_1\,\bar{z}_2$;

(ii) $\overline{z^2}=\left(\overline{z}\right)^2$

(iii) $\overline{z_1 z_2 \cdots z_n}=\overline{z_1}\ \overline{z_2} \cdots \overline{z_n}$;

(iv) $\overline{z^n}=\left(\overline{z}\right)^n$.

Proof : *(i)* Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Then $z_1 z_2=(x_1 x_2-y_1 y_2)+i\,(x_1 y_2+y_1 x_2) \Rightarrow$

$\overline{z_1 z_2}=(x_1 x_2-y_1 y_2)-i\,(x_1 y_2-y_1 x_2)=(x_1-iy_1)\,(x_2-iy_2)$

$=\overline{z_1}\ \overline{z_2}$.

(ii) If $z_1=z_2=z$, then *(i)* $\Rightarrow \overline{z^2}=\left(\overline{z}\right)^2$

(iii) using *(i)* again and again we have

$\overline{z_1 z_2 \cdots z_n}=\overline{z_1\,(z_2 z_3 \cdots z_n)}=\overline{z_1}\ \overline{z_2 z_3 \cdots\cdots z_n}$

$=\overline{z_1}\ \overline{z_2\,(z_3 z_4 \cdots z_n)}=\overline{z_1}\ \overline{z_2}\ \ \overline{z_3 z_4 \cdots z_n}= \cdots \cdots$

$=\overline{z_1}\ \overline{z_2} \cdots \overline{z_n}$.

(iv) If $z_1=z_2=\cdots=z_n=z$, *(iii)* $\Rightarrow \overline{z^n}=\left(\overline{z}\right)^n$.

54. *Theorem 13 :* Show that :

(i) $|z|^2=\left|\overline{z}\right|^2=z\overline{z}$; *(ii)* $|z_1 z_2 \cdots z_n|^2=$
$\overline{|z_1 z_2 \cdots z_n|}^2=(z_1 z_2 \cdots z_n)\,\overline{(z_1 z_2 \cdots z_n)}$.

Proof: *(i)* Let $z=x+iy$, then $z\overline{z}=(x+iy)\,(x-iy)=$
$x^2+y^2=|z|^2$. Again

$\left|\overline{z}\right|^2=|x-iy|^2=x^2+y^2 \Rightarrow |z|^2=\left|\overline{z}\right|^2=z\overline{z}$.

(ii) Try yourself.

55. *Theorem 14* : Show that :

D. U. M. SC. P.89.

(i) $|z_1 z_2| = |z_1|\,|z_2|$;

(ii) $|z_1 z_2 \dots z_n| = |z_1|\,|z_2| \quad |z_n|$. ***D.U.H.S.T.85.***

Salution : (i) We have $|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)}$

$= (z_1 z_2)(\bar{z}_1 \bar{z}_2) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 \Rightarrow$

$|z_1 z_2| = |z_1|\,|z_2|$.

(ii) We have $|z_1 z_2 \cdot z_n|^2 = (z_1 z_2 \cdots z_n)\overline{(z_1 z_2 \cdots z_n)}$.

$= (z_1 z_2 \cdots z_n)(\bar{z}_1 \bar{z}_2 \cdots \bar{z}_n) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) \cdots (z_n \bar{z}_n) =$

$|z_1|^2 |z_2|^2 \cdots |z_n|^2 \Rightarrow |z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$

56 *Theorem 15 :* Show that for any two complex numbers z_1 and z_2 :

(i) $|z_1 + z_2| \leqslant |z_1| + |z_2|$;

D.U.H.T. 87 ; D. U. H. 86 ; D. U, M. SC. P. 88, 89 ; R. U. 67 ; C. U. 68 , J. U. H. 87.

(ii) $|z_1 + z_2 + z_3| \leqslant |z_1| + |z_2| + |z_3|$;

(ii) $|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|$.

D. U. H. S.T. 85.

Proof : *(i)* We have $|z|^2 = z\bar{z}$, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2}$ $= \bar{z}_1 \bar{z}_2$, $z + \bar{z} = 2\text{Re}(z) \leqslant 2|z|$, $|z_1 \bar{z}_2| = |z_1|\,|z_2|$ and $|\bar{z}|$ $= |z|$

Now using these we have $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$=(\overline{z_1+z_2})(z_1+z_2)=\bar{z}_1 z_1+\bar{z}_2 z_2+\bar{z}_1 z_2+z_1\bar{z}_2=|z_1|^2+|z_2|^2+$
$\bar{z}_1 z_2+(\overline{\bar{z}_1 z_2})=|z_1|^2+|z_2|^2+2\text{Re}(z_1\bar{z}_2)$
$\leqslant |z_1|^2+|z_2|^2+2\left|z_1\bar{z}_2\right|=|z_1|^2+|z_2|^2+2|z_1|\left|\bar{z}_2\right|$
$=|z_1|^2+|z_2|^2+2|z_1||z_2|=(|z_1|+|z_2|)^2$
$\Rightarrow |z_1+z_2|\leqslant|z_1|+|z_2|$

(*ii*) Using (*i*) we have $|z_1+z_2+z_3|=|(z_1+z_2)+z_3|$
$\leqslant|z_1+z_2|+|z_3|\leqslant|z_1|+|z_2|+|z_3|$

(*iii*) Using (*i*) we have $|z_1+z_2+\cdots+z_n|$
$=|(z_1+z_2+\cdots+z_{n-1})+z_n|\leqslant|z_1+z_2+\cdots+z_{n-1}|+|z_n|$
$\leqslant|z_1+z_2+\cdots+z_{n-2}|+|z_{n-1}|+|z_n|\leqslant \cdots \cdots \cdots$
$\leqslant|z_1|+|z_2|+\cdots+|z_n|\Rightarrow|z_1+z_2+\cdots+z_n|$
$\leqslant|z_1|+|z_2|+\cdots+|z_n|$

Thus the theorem is proved.

56. ***Theorem 16 :*** Show that for any two complex numbers z_1 and z_2 :

$$|z_1-z_2|\leqslant|z_1|+|z_2|$$ ***D. U. H. 89.***

Proof (First method) : We have $|z_1+z_2|\leqslant|z_1|+|z_2|$

Now putting $-z_2$ for z_2, we get $|z_1-z_2|\leqslant|z_1|+|-z_2|$
$=|z_1|+|z_2|$ since $|-z_2|=|z_2|$.

(Second method) : We have $|z_1-z_2|^2=(z_1-z_2)\overline{(z_1-z_2)}$

$=(z_1-z_2)(\bar{z}_1-\bar{z}_2)=z_1\bar{z}_1+z_2\bar{z}_2-(z_1\bar{z}_2+\bar{z}_1 z_2)=|z_1|^2$
$+|z_2|^2-(z_1\bar{z}_2+\overline{z_1\bar{z}_2})=|z_1|^2+|z_2|^2-2\text{Re}(z_1\bar{z}_2)$

$$\leqslant |z_1|^2+|z_2|^2+2\left|z_1\bar{z}_2\right| \quad [\because \; -\text{Re}(z)\leqslant |z|]$$

$$=|z_1|^2+|z_2|^2+2|z_1||z_2|=(|z_1|+|z_2|)^2$$

$$\Rightarrow |z_1-z_2|\leqslant |z_1|+|z_2|.$$

57. Theorem 17: Show that for any two complex numbers z_1 and z_2 ;

(*i*) $|z_1-z_2|\geqslant ||z_1|-|z_2||\geqslant |z_1|-|z_2|.$

D. U. H. T. 87 ; R. U. H. 90 ; D. U. H. 86. 89 ; C. U. H. 81 ; D. U. M. SC. P. T. 90 ; D. U. M. SC. P. 88, 89 ; J. U. H.87.

(*ii*) $|z_1+z_2|\geqslant ||z_1|-|z_2||\geqslant |z_1|-|z_2|.$

Proof : (*i*) We have $|z_1-z_2|^2=(z_1-z_2)\overline{(z_1-z_2)}$

$$=(z_1-z_2)(\bar{z}_1-\bar{z}_2)=z_1\bar{z}_1+z_2\bar{z}_2-(z_1\bar{z}_2+\bar{z}_1z_2)$$

$$=|z_1|^2+|z_2|^2-2\text{Re}\,(z_1\bar{z}_2)\geqslant |z_1|^2+|z_2|^2-2\left|z_1\bar{z}_2\right|$$

$$=|z_1|^2+|z_2|^2-|z_1|\left|\bar{z}_2\right|=|z_1|^2+|z_2|^2-2|z_1||z_2|$$

$$=(|z_1|-|z_2|)^2=(||z_1|-|z_2||)^2$$

$$\Rightarrow |z_1-z_2|\geqslant ||z_1|-|z_2||$$

Again we have $||z_1|-|z_2||\geqslant |z_1|-|z_1|$. Thus

$$|z_1-z_2|\geqslant ||z_1|-|z_2||\geqslant |z_1|-|z_2|.$$

(ii) Now replacing z_2 by $-z_2$ in (*i*), we have

$$|z_1+z_2|\geqslant ||z_1|-|-z_2||\geqslant |z_1|-|-z_2| \Rightarrow$$

$$|z_1+z_2|\geqslant ||z_1|-|z_2||\geqslant |z_1|-|z_2| \text{ since}$$

$$|-z_2|=|z_2|.$$

58. ***Theorem 18 :*** Show that :

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

D. U. H. 85 ; D. U. H. T. 87, 90 ; D. U. M. SC. P. 84 ;

D. U. 63 ; R. U. 73.

Proof : We have $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\operatorname{Re}\left(z_1\bar{z_2}\right)$

and $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\operatorname{Re}\left(z_1\bar{z_2}\right)$. Then

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

Example 24: Show that :

$$|\alpha+\sqrt{(\alpha^2-\beta^2)}|^2+|\alpha-\sqrt{(\alpha^2-\beta^2)}|^2=|\alpha+\beta|\alpha-\beta|.$$

D. U. H. T. 87 ; D. U. H. 85.

Solution : We have $(|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|)^2$

$=|\alpha+\sqrt{(\alpha^2-\beta^2)}|^2+|\alpha-\sqrt{(\alpha^2-\beta^2)}|^2$

$+2|\alpha+\sqrt{(\alpha^2-\beta^2)}|\ |\alpha-\sqrt{(\alpha^2-\beta^2)}|$

$=2|\alpha|^2+2|\sqrt{(\alpha^2-\beta^2)}|^2+2|(\alpha+\sqrt{(\alpha^2-\beta^2)})(\alpha-\sqrt{(\alpha^2-\beta^2)})|$

$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\alpha^2-(\alpha^2-\beta^2)|$

$=2|\alpha|^2+2|(\alpha+\beta)(\alpha-\beta)|+2|\beta|^2$

$=2|\alpha|^2+2|\beta|^2+2|\alpha+\beta|\ |\alpha-\beta|$

$=|\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha+\beta||\alpha-\beta|$

$=(|\alpha+\beta|+|\alpha-\beta|)^2$

$\Rightarrow|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|=|\alpha+\beta|+|\alpha-\beta|.$

59. ***Theorem 19. :*** Show that :

$\left|\dfrac{z_1}{z_2+z_3}\right|\leq\dfrac{|z_1|}{||z_2|-|z_3||}$, where z_1, z_2, z_3 are complex numbers with $|z_2|\neq|z_3|$.

D. U. H. 87 ; D. U. H. T. 88.

Proof : We have $|z_2+z_3| \geq ||z_2|-|z_3||$, then

$$\frac{1}{|z_2+z_3|} \leq \frac{1}{||z_2|-|z_3||} \quad \ldots (1) \text{ if } |z_2| \neq |z_3|.$$

But $|z_1| \geq 0 \quad \ldots (2)$. Then by (1) and (2) we have

$$\frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{||z_2|-|z_3||} \text{ where } |z_2| \neq |z_3|.$$

60. ***Theorem 20*** : Show that :

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||} \text{ where } |z_3| \neq |z_4|$$

D. U. H. 88, 90.

Proof : We have $|z_1+z_2| \leq |z_1|+|z_2| \quad \ldots\ldots (1)$.

Again we have $|z_3+z_4| \geq ||z_3|-|z_4||$, then

$$\frac{1}{|z_3+z_4|} \leq \frac{1}{||z_3|-|z_4||} \ldots (2) \text{ where } |z_3| \neq |z_4|.$$

Then by (1) and (2) we have

$$|z_1+z_2| \cdot \frac{1}{|z_3+z_4|} \leq (|z_1|+|z_2|) \cdot \frac{1}{||z_3|-|z_4||}$$

$$\Rightarrow \frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

$$\Rightarrow \left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

61. ***Theorem 21*** : Show that ,

$|z|\sqrt{2} \geq |\text{Re}(z)|+|\text{Im}(z)|$, where z is any complex number.

D.U.H.88 ; D.U.H.T.88, 90.

Proof ; Let $z=x+iy$, then $\text{Re}(z)=x$ and $\text{Im}(z)=y$.

But we know

$$\frac{|x|^2+|y|^2}{2} \geq \left(\frac{|x|+|y|}{2}\right)^2 \Rightarrow \frac{x^2+y^2}{2} \geq \left(\frac{|x|+|y|}{2}\right)$$

$$\Rightarrow \frac{\sqrt{(x^2+y^2)}}{\sqrt{2}} \geqslant \frac{|x|+|y|}{2} \Rightarrow \sqrt{2(x^2+y^2)} \geqslant |x|+|y|.$$

$$\Rightarrow \sqrt{2}\,|z| \geqslant |\mathrm{Re}(z)| + |\mathrm{Im}(z)|.$$

62. ***Theorem 22*** : If $|z_1| = |z_2|$ and $\mathrm{amp}\, z_1 + \mathrm{amp}\, z_2 = 0$ then $z_2 = \bar{z}_1$. ***R.U.H. 85, 40 ; J.U.H 86.***

Proof : Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then $|z_1| = |z_2|$

$\Rightarrow |r_1 e^{i\theta_1}| = |r_2 e^{i\theta_2}| \Rightarrow r_1 = r_2 \cdots (1)$ since $|e^{i\theta_1}| = |e^{i\theta_2}| = 1.$

Again $\mathrm{amp}\, z_1 + \mathrm{amp}\, z_2 = 0 \Rightarrow \mathrm{amp}\, z_1 z_2 = 0 \Rightarrow \theta_1 + \theta_2 = 2n\pi$

$\Rightarrow \theta_2 = 2n\pi - \theta_1 \cdots (2)$ where $n = 0, \pm 1, \pm 2, \pm 3, \cdots \cdots$

Now by (1) and (2) we have $z_2 = r_2 e^{i\theta_2} = r_1 e^{i(2n\pi - \theta_1)}$

$= r_1 e^{2n\pi i} e^{-i\theta_1} = r_1 e^{-i\theta_1} = \bar{z}_1$ $\quad [\because e^{2n\pi i} = 1].$

63. ***Theorem 23*** : Show that the modulus of the quotient of two conjugate complex numbers is 1.

R. U. H. 85 : J. U. H. 86.

Proof ; Let $z = x + iy$ be a complex number, then its conjugate is $\bar{z} = x - iy$. Now

$$\left|\frac{z}{\bar{z}}\right| = \frac{|z|}{|\bar{z}|} = \frac{|x+iy|}{|x-iy|} = \frac{\sqrt{(x^2+y^2)}}{\sqrt{\{x^2+(-y)\}^2}} = \frac{\sqrt{(x^2+y^2)}}{\sqrt{(x^2+y^2)}} = 1.$$

Hence the theorem is proved.

Second proof : Let $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$

$$\text{Now } \left|\frac{z}{\bar{z}}\right| = \left|\frac{re^{i\theta}}{re^{-i\theta}}\right| = |e^{2i\theta}| = |\cos 2\theta + i \sin 2\theta|$$

$$= \sqrt{(\cos^2 2\theta + \sin^2 2\theta)} = 1.$$

64. Theorem 24 : Show that :

(*i*) $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z_1}}{\overline{z_2}}$ and (*ii*) $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$ where $z_2\neq 0$.

Proof : Let $z_1=r_1e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$. Then

$$\frac{z_1}{z_2}=\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}=\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

(*i*) $$\overline{\left(\frac{z_1}{z_2}\right)}=\frac{r_1}{r_2}e^{-i(\theta_1-\theta_2)}=\frac{r_1e^{-i\theta_1}}{r_2e^{-i\theta_2}}=\frac{\overline{z_1}}{\overline{z_2}}.$$

(*ii*) $$\left|\frac{z_1}{z_2}\right|=\left|\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}\right|=\frac{r_1}{r_2}=\frac{|z_1|}{|z_2|}\quad\text{since}\ \left|e^{i(\theta_1-\theta_2)}\right|=1.$$

Second Proof : $$\left|\frac{z_1}{z_2}\right|^2=\left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)}=\frac{z_1\overline{z_1}}{z_2\overline{z_2}}$$

$$=\frac{|z_1|^2}{|z_2|^2}\Rightarrow\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}.$$

65. Theorem 25 : Show that :

(*i*) $\arg(z_1z_2)=\arg z_1+\arg z_2$;

(*ii*) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg z_1-\arg z_2$;

(*iii*) $\arg(z_1z_2\cdots z_n)=\arg z_1+\arg z_2+\cdots+\arg z_n$;

(*iv*) $\arg z^n=n\arg z$;

(*v*) $\arg\overline{z}=-\arg z$.

Proof : (*i*) Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then we have $\arg z_1=\theta_1$ and $\arg z_2=\theta_2$.

Now we have $z_1z_2=r_1r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}$

$\Rightarrow\arg(z_1z_2)=\theta_1+\theta_2=\arg z_1+\arg z_2$.

(*ii*) Let $z_1 = r_1\ (\cos\theta_1 + i\sin\theta_1)$ and $z_2 =$ $r_2\ (\cos\theta_2 + i\sin\theta_2)$, then $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$.

Now $\dfrac{z_1}{z_2} = \dfrac{r_1\ (\cos\theta_1 + i\sin\theta_1)}{r_2\ (\cos\theta_2 + i\sin\theta_2)}$

$= \dfrac{r_1}{r_2}\ \{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$\Rightarrow \arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$

(*iii*), (*iv*) and (*v*) : Try yourself.

N. B. The above theorem is true for some of $\arg z_1$ and $\arg z_2$ and they are not true for all cases since $\arg z_1$ and $\arg z_2$ have many values and also they may not hold even if we use the principal values.

Example 25 : Show that : $\arg z + \arg \bar{z} = 2n\pi$, where $n \in I$.

Solution : We have $\arg z = \text{Arg } z + 2p\pi$ and $\arg \bar{z}$ $= \text{Arg } \bar{z} + 2q\pi = -\text{Arg } z + 2q\pi$ where $p, q \in I$.

Then $\arg z + \arg \bar{z} = \text{Arg } z + 2p\pi - \text{Arg } z + 2q\pi = 2(p+q)\pi$ $= 2n\pi$, where $n = p + q \in I$.

Example 26: Using Euler's formula show that :

$\arg(z_1\ z_2) = \arg z_1 + \arg z_2$ and $\arg\left(\dfrac{z_1}{z_1}\right) = \arg z_1 - \arg z_2$.

Solution ; Let $z_1 = r_1\, e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$.

Again $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \Rightarrow \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 +$ $\arg z_2$ and $z_1/z_2 = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \Rightarrow \arg(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$.

66. ***Theorem 26 :*** If P divides AB in the ratio m : n and O is any point, then show that : $(m+n)\,\boldsymbol{OP} = n\,\boldsymbol{OA} + m\,\boldsymbol{OB}$.

Proof : Try yourself.

67. ***Theorem 27*** : If P(z) divides the straight line joining $A(z_1)$ and $B(z_2)$ in the ratio $m_1 : m_2$, then show that :

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

proof ; Try yourself.

68. ***Theorem 28 :*** If z is the centriod of particles of masses $m_1, m_2, m_3, \cdots$ are located at points $z_1, z_2, z_3, \cdots$ respectively, then show that : $(m_1 + m_2 + m_3 + \cdots)z = m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots$

Proof : Try Tourself.

69. ***Theorem 29***: Show that any three complex numbers z_1, z_2, z_3 are connected by a relation of the form $a z_1 + b z_2 + c z_3 = 0$ where a, b, c are real numbers.

Proof : Try yourself.

Example 27 : If the vectors $\boldsymbol{OA}, \boldsymbol{OB}, \boldsymbol{OC}$ are connected by the relation $l\,\boldsymbol{OA} + m\,\boldsymbol{OB} + n\,\boldsymbol{OC} = \boldsymbol{0}$, where $l + m + n = 0$, then ***A, B, C*** are collinear.

Solution : Try yourself.

70 ***Theorem 30 :*** If the three points P, Q, R are collinear and O is any point, then show that :

OA. BC+OB. CA+OC. AB$=0$.

Proo. : Try yourself.

Example 28 : If the points z_1, z_2, z_3 are collinear, then show that :

$$z_1 |z_2-z_3| \pm z_2 |z_3-z_1| \pm z_3 |z_1-z_2| = 0.$$

Solution ; Try yourself.

71 ***Theorem 31*** : Show that the medians of a triangle with vertices at z_1, z_2, z_3 intersect in the point $\frac{1}{3}(z_1+z_2+z_3)$.

Proof ; Try yourself.

72 ***Theorem 32*** : If z_1, z_2, z_3, are the three vertices of an equilateral triangle, then show that

$$z_1^2+z_2^2+z_3^2=z_1 z_2+z_2 z_3+z_3 z_1.$$

Proof : Try yourself.

73 ***Theorem 33*** : If A, B, C are complex constants and $\bar{A}, \bar{B}, \bar{C}$ their conjugates, then show that the equation

$$(A+\bar{A})z\bar{z}+B\bar{z}+\bar{B}z+C+\bar{C}=0 \quad (1) \quad \text{will}$$

represent a circle if $B\bar{B} > (A+\bar{A})(C+\bar{C})$ ... ... (2).

J. U. H. 86 ; ***R. U.*** H. ***85.***

proof : Let $A=a_1+ia_2$, $B=b_1+ib_2$, $C=c_1+ic_2$ and $z=x+iy$ ... (3), Then $\bar{A}=a_1-ia_2$, $\bar{B}=b_1-ib_2$, $\bar{C}=c_1-ic_2$ and $\bar{z}=x-iy$ ... (4). Now by (1), (3) and (4) we have $2a_1(x^2+y^2)+2b_1x+2b_2y+2c_1=0$

$\Rightarrow a_1(x^2+y^2)+b_1x+b_2y+c_1=0$ which is the equation of a circle. The radius of this circle

$$=\left(\frac{b_1}{2a_1}\right)^2+\left(\frac{b_2}{2a_1}\right)^2-\frac{c_1}{a_1}$$ which is greater than zero $\Rightarrow$

$b_1^2+b_2^2-4a_1c_1>0 \Rightarrow b_1^2+b_2^2>4a_1c_1 \Rightarrow$

$B\bar{B}>(A+\bar{A})(C+\bar{C})$, by (3) and (4).

74. ***Theorem 34*:** Show that the equation of a circle or line in the Argand plane can be written as

$Az\bar{z}+Bz+\bar{B}\bar{z}+D=0$ where A and C are real constants and B may be a complex constant.

Proof : The general equation of a circle in the xy-plane can be written $a(x^2+y^2)+bx+cy+d=0$ ... ... (1)

But we have $x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$ ... ... (2)

Then by (1) and (2) we have

$$az\bar{z}+b\frac{z+\bar{z}}{2}+c\frac{z-\bar{z}}{2i}+d=0 \qquad [\because z\bar{z}=x^2+y^2]$$

$$\Rightarrow az\bar{z}+\left(\frac{b}{2}+\frac{c}{2i}\right)z+\left(\frac{b}{2}-\frac{c}{2i}\right)\bar{z}+d=0$$

$\Rightarrow Az\bar{z}+Bz+\bar{B}\bar{z}+D=0$ ... ... (3) where $A=a$,

$B=\frac{b}{2}+\frac{c}{2i}$, $\qquad \bar{B}=\frac{b}{2}-\frac{c}{2i}$.

and $D=d$, The equation (3) represents the general equation of a circle. If $a = A = 0$, then (3) represents the general equation of a straight line.

75 Theorem 35 : Show that

$$\left(\frac{z-z_1}{z-z_2}\right) \Big/ \left(\frac{z_3-z_1}{z_3-z_2}\right) = \left(\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\right) \Big/ \left(\frac{\bar{z}_3-\bar{z}_1}{\bar{z}_3-\bar{z}_2}\right)$$

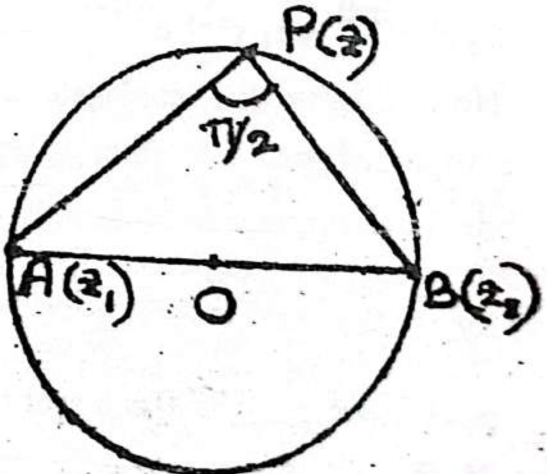
represents the equation of a circle through the points z_1, z_2, z_3.

Proof : Try yourself.

76 Theorem 36 : Show that :

$(z-z_1)\left(\bar{z}-\bar{z}_2\right)+(z-z_2)\left(\bar{z}-\bar{z}_1\right)=0$ represents the equation of a circle where A (z_1) and B (z_2) are the extremities of a diameter.

Proof : Let P (z) be a point on the circle. Here $\angle APB=\pi/2 \Rightarrow \arg \dfrac{z-z_1}{z-z_2} = \pi/2 \Rightarrow \dfrac{z-z_1}{z-z_2}$ is purely imaginary



ginary $\Rightarrow \text{Re}\left(\dfrac{z-z_1}{z-z_2}\right)=0 \Rightarrow 2\,\text{Re}\left(\dfrac{z-z_1}{z-z_2}\right)=0$

$$\Rightarrow \frac{z-z_1}{z-z_2}+\overline{\left(\frac{z-z_1}{z-z_2}\right)}=0 \Rightarrow \frac{z-z_1}{z-z_2}+\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}=0$$

$$\Rightarrow (z-z_1)\left(\bar{z}-\bar{z}_2\right)+(z-z_2)\left(\bar{z}-\bar{z}_1\right)=0.$$ Hence the theorem is proved.

77. Theorem 37; In an equation with real coefficients complex roots occur in conjugate pairs.

Proof : Let $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0 \cdots (1)$ where $a_0 \neq 0$, $a_1, \cdots, a_n \in \boldsymbol{R}$. Let $p+iq$ be a root of (1) where $p, q \in \boldsymbol{R}$, Now we will show that $p-iq$ is also a root of (1).

Let $p+iq = r\,e^{i\theta} \cdots$ (3) in polar form. Then by (1) and (2) we get

$$a_0 r^n e^{in\theta} + a_1 r^{n-1} e^{i(n-1)\theta} + \cdots + a_{n-1} re^{i\theta} + a_n = 0 \quad \cdots (3)$$

Now taking the conjugate of both sides of (3), we get

$$a_0 r^n e^{-in\theta} + a_1 r^{n-1} e^{-i(n-1)\theta} + \cdots + a_{n-1} re^{-i\theta} + a_n = 0 \quad \cdots \quad \cdots \quad (4)$$

By (1) and (4) we see that conjugate of (2), *i. e.* $p-iq = r\,e^{-i\theta}$ is also a root of (1). Thus the theorem is proved.

N. B. If $a_0, a_1, \cdots\cdots, a_n$ are not all real, then the above theorem is not correct.

Example 29 : Show that the conjugate pairs $\frac{1}{2}(1 \pm i\sqrt{3})$ and $\frac{1}{2}(-1 \pm i\sqrt{3})$ are the roots of $z^4 + z^2 + 1 = 0$.

Solution : Try yourself.

$= -\text{Im}\left(\frac{-}{z}\right)$, *(iv)* $\text{Im } z = y = \frac{2y}{2} = \frac{z-\bar{z}}{2}$

$= -\text{Im}\left(\frac{-}{z}\right)$, (v) $-\sqrt{(x^2+y^2} \leq x \leq \sqrt{(x^2+y^2)}$

$\Rightarrow -|z| \leq \text{Re}(z) \leq |z|$, (vi) $-\sqrt{(x^2+y^2)} \leq y \leq \sqrt{(x^2+y^2)}$

$\Rightarrow -|z| \leq \text{Im}(z) \leq |z|$; ***(vii)***, ***(viii)***, ***(ix)*** and ***(x)*** Try yourself.

Example 34 : ***Solve*** :

(i) $az^2+bx+c=0, a\neq 0$, *(ii)* $z^3=1$.

Solution : *(i)* $az^2+bz+c= \Rightarrow az^2+bz=-c$

$\Rightarrow 4a^2z^2+4abz=-4ac \Rightarrow 4a^2z^2+4abz+b^2=b^2-4ac$

$\Rightarrow (2az+b)^2=b^2-4ac \Rightarrow 2az+b=\pm\sqrt{(b^2-4ac)}$

$\Rightarrow 2az=-b\pm\sqrt{(b^2-4ac)} \Rightarrow z=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$

(ii) $z^3=1 \Rightarrow z^3-1=0 \Rightarrow (z-1)(z^2+z+1)= \Rightarrow$

$z=1, \ \frac{-1\pm\sqrt{(-3)}}{2}$.

Example 35 ; Show that if the amplitude of a complex number is $\pi/2$, then the complex number is purely imaginary but if the amplitude is o or π, then the complex number is purely real.

Solution : Let $z=r\ (\cos\theta+i\sin\theta)$ where $\text{amp} z=\theta$.

Now we have :

(i) if $\theta=\pi/2$, then $z=r\ (\cos\pi/2+i\sin\pi/2)=r\,i \Rightarrow$ z is purely imaginary.

(ii) if $\theta=0$, then $z=r\,(\cos 0+i\sin 0)=r \Rightarrow$ z is purely real.

(iii) if $\theta=\pi$, then $z=r(\cos\pi+i\sin\pi)=-r \Rightarrow$ z is purely real.

Example 36 : If $|z_1+z_2|=|z_1-z_2|$, then show that z_1/z_2 and z_2/z_1 are purely imaginary numbers.

Solution : We have $|z_1+z_2|=|z_1-z_2|$

$\Rightarrow \mathrm{Re}\left(z_1\bar{z}_2\right)=\mathrm{Re}\left(\bar{z}_1 z_2\right)=0 \Rightarrow z_1\bar{z}_2$ and $\bar{z}_1 z_2$ are imaginary numbers.

$$\text{Here } z_1/z_2=\frac{z_1\bar{z}_2}{z_2\bar{z}_2}=\frac{z_1\bar{z}_2}{|z_2|^2} \text{ and } \frac{z_2}{z_1}=\frac{\bar{z}_1 z_2}{z_1\bar{z}_1}=\frac{\bar{z}_1 z_2}{|z_1|^2}$$

$\Rightarrow$ they are imaginry numbers since $z_1\bar{z}_2$, $\bar{z}_1 z_2$ are imaginary and $|z_1|^2$, $|z_2|^2$ are real.

Example 37 : Show that the sum of the products of all the nth roots of unity taken 2, 3, 4, ... , (n−1) at a time is zero.

R. U. H. 86.

Solution : Let $z=(1)^{1/n} \Rightarrow z^n=1 \Rightarrow z^n-1=0$ ... ... (1)

Here $a_0=1, a_1=0, a_2=0, a_3=0, \ldots, a_{n-1}=0, a_n=-1$.

Let $z_1, z_2, \cdots, z_n$ be the roots of (1), then

$\Sigma z_1 z_2=\frac{a_2}{a_0}=0$, $\Sigma z_1 z_2 z_3=-\frac{a_3}{a_0}=0, \ldots$, $\Sigma z_1 z_2 \cdots z_{n-1}=(-1)^{n-1}\frac{a_{n-1}}{a_0}=0$.

Example 38 ; Show that $\left|\frac{z-1}{z+1}\right|=$constant and amp $\left(\frac{z-1}{z+1}\right)$ = constant are orthogonal circles;

R. U. H. 93.

Solution : We have $\left|\dfrac{z-1}{z+1}\right|=\text{constant}=k$ (say)

$$\Rightarrow \frac{(x-1)^2+y^2}{(x+1)^2+y^2}=k^2$$

$$\Rightarrow (1-k^2)x^2+(1-k^2)y^2-2(1+k^2)x+1-k^2=0 \quad \dots \quad \dots \quad (1)$$

Again we have $\operatorname{amp}\left(\dfrac{z-1}{z+1}\right)=\text{constant}=\lambda$ (say)

$$\Rightarrow \operatorname{amp}(z-1)-\operatorname{amp}(z+1)=\lambda \Rightarrow \tan^{-1}\frac{y}{x-1}-\tan^{-1}\frac{y}{x+1}=\lambda$$

$$\Rightarrow \tan^{-1}\left(\frac{\dfrac{y}{x-1}-\dfrac{y}{x+1}}{1+\dfrac{y^2}{x^2-1}}\right)=\lambda \Rightarrow \frac{2y}{x^2+y^2-1}=\tan\lambda \Rightarrow$$

$\mu(x^2+y^2-1)-2y=0 \quad \dots \quad (2) \qquad$ where $\mu=\tan\lambda$.

The eqnation (1) and (2) represent two circles. In (1) we have $g_1=-\left(\dfrac{1+k^2}{1-k^2}\right)$, $f_1=0$, $c_1=1$ and in (2) we have

$$g_2=0,\ f_2=-\frac{2}{\mu},\ c_2=-1$$

Here $2g_1 g_2+2f_1 f_2=0=c_1+c_2$. Hence the circles (1) and (2) cut orthogonally and they are orthogonal circles.

Example 39 : Show that the lines joining the points z_1, z_2 and z_3, z_4 are perpendicular provided that $\dfrac{z_1-z_2}{z_3-z_4}$ is pure imaginary. ***R. U. H. 86.***

Solution : Let θ be the angle between the lines joining the points z_1, z_2 and z_3, z_4. Then $\theta = \arg \dfrac{z_1 - z_2}{z_3 - z_4}$ If they are perpendicular *i. e.* $\theta = \pm \pi/2$, then $\arg \dfrac{z_1 - z_2}{z_3 - z_4} = \pm \pi/2$

$\Rightarrow \dfrac{z_1 - z_2}{z_3 - z_4}$ lies on the y-axis and therefore $\dfrac{z_1 - z_2}{z_3 - z_4}$ is pure imaginary.

Example 40 : Using polar coordinates measuring arguments in radians :

(*i*) show that $|z_1 + z_2| = |z_1| + |z_2|$ if $z_1 . z_2 \neq 0$ and $\arg z_2 = \arg z_1 \pm 2n\pi$, $n = 0, 1, 2, \ldots \ldots$;

(*ii*) show that $\operatorname{Re}\left(z_1 \overline{z_2}\right) = |z_1|\,|z_2|$ if $z_1 z_2 \neq 0$ and $\arg z_2 = \arg z_1 \pm 2n\pi$, $(n = 0, 1, 2, \ldots \ldots)$.

Solution : Try yourself.

Example 41 : Show that :

(*i*) $\arg\left(z_1 \overline{z_2}\right) = \arg z_1 - \arg z_2$; (*ii*) $\operatorname{Arg} \overline{z} = -\operatorname{Arg} z$, z is not real ; (*iii*) $\arg z = \operatorname{Arg} z + 2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$, $z \neq 0$; (*iv*) $\operatorname{Arg}(1 + i) = \pi/4$;

(*v*) $\operatorname{Arg}(\sqrt{3} - i) = -\pi/6$; (*vi*) $\operatorname{Arg}(3i) = \pi/2$.

Solution . Try yourself.

Example 42 ; Show that : if $n \in \mathbb{Z}$, then either

$$\arg \frac{z_2 - z_3}{z_1 - z_3} = \frac{1}{2} \operatorname{Arg} \frac{z_2}{z_1} + 2n\pi$$

or $\arg \dfrac{z_2 - z_3}{z_1 - z_3} = -\dfrac{1}{2} \operatorname{Arg} \dfrac{z_2}{z_1} + 2n\pi$, where z_1, z_2 and z_3 be three distinct points that lie on the circle of radius 1 about the origin.

Solution : Try yourself.

Example 43 : Show that ;

(*i*) $\operatorname{Re}\left(\alpha\bar{\beta}\right)=\operatorname{Re}\left(\bar{\alpha}\beta\right)$; (*ii*) $\operatorname{Im}\left(\bar{\alpha}\beta\right)=-\operatorname{Im}\left(\alpha\bar{\beta}\right)$
where $\alpha\ \beta\in \mathbf{C}$.

Soluton ; Let $\alpha=a+ib$ and $\beta=c+id$ then $\alpha\bar{\beta}=$

$(a, b)\ (c,-d)=(ac+bd, bc-ad)$ and $\bar{\alpha}\beta=(a,-b)\ (c, d)$
$=(ac+bd, ad-bc)$.

(*i*) $\operatorname{Re}\left(\alpha\bar{\beta}\right)=ac+bd=\operatorname{Re}\left(\bar{\alpha}\beta\right)$ and

(*ii*) $\operatorname{Im}\left(\alpha\bar{\beta}\right)=bc-ad=-(ad-bc)=-\operatorname{Im}\left(\bar{\alpha}\beta\right)$

Example 44 ; Show that ;

(*i*) $|\alpha+\beta|^2=|\alpha|^2+|\beta|^2+2\operatorname{Re}\left(\alpha\bar{\beta}\right)$;

(*ii*) $|\alpha-\beta|^2=|\alpha|^2+|\beta|^2-2\operatorname{Re}\left(\alpha\bar{\beta}\right)$ whre $\alpha, \beta\in \mathbf{C}$.

Solution : (*i*) $|\alpha+\beta|^2=(\alpha+\beta)\left(\overline{\alpha+\beta}\right)$

$=(\alpha+\beta)\left(\bar{\alpha}+\bar{\beta}\right)=\alpha\bar{\alpha}+\beta\bar{\beta}+\left(\alpha\bar{\beta}+\bar{\alpha}\beta\right)$

$=|\alpha|^2+|\beta|^2+2\operatorname{Re}\left(\alpha\bar{\beta}\right)$

(*ii*) Try yourself.

Example 45 : Show that : $\alpha\bar{\beta}+\bar{\alpha}\beta$

$=\operatorname{Re}\left(\alpha\bar{\beta}\right)=\operatorname{Re}\left(\bar{\alpha}\beta\right)$ where $\alpha, \beta\in \mathbf{C}$.

Solution : Try yourself.

Example 46 : Show that $|z|$ is real.

Solution : Try yourself.

Example 47 : Show that : $z\bar{z} > 0, z \in C$.

Solution : Let $z = (x, y)$ where $x, y \in R$.

Then $z\bar{z} = x^2 + y^2 \geqslant 0$.

Example 48 : Let P and Q be represent the complex numbers z and $\bar{z}$ respectively, then show that PQ is perpendicular to the x-axis.

Solution : Try yourself.

Example 49 : If $n = 2, 4, 6, 8, \ldots \ldots$, then show that

$$1 + \sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0 \text{ and } \sum_{k=1}^{n-1} \sin\frac{2k\pi}{n} = 0.$$

Solution : Try yourself.

Example 50 : Show that for $m = 2, 3, 4, \ldots \ldots$

$$\sin\frac{\pi}{m} \sin\frac{2\pi}{m} \sin\frac{3\pi}{m} \cdots \sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}.$$

Solution : Try yourself.

Example. 51 : Describe each of the following region geometrically :

(*i*) $|z + 3i| > 4$; ***D. U, H. T. 89.***

(*ii*) $|2z + 3| > 4$; ***D. U. H. T. 88.***

(*iii*) $1 < |z - 2i| \leqslant 2$; ***D. U. H. T. 88.***

(iv) $1 < |z+i| \leqslant 2$;

D. U.H. T. 89 ; D. U.M, SC, P. T. 89.

(v) $|z-4| > |z|$; D. U.H. T. 88

(vi) $1 < |z-2i| < 2$; D. U. H. T. 90.

(xii) $|z-i| = |z+i|$; D. U. H. 87.

(viii) $|z+2-3i| + |z-2+3i| < 10$; D. U. H. T. 89.

(ix) $|z-2| - |z+2| > 3$; D. U. H. T. 90.

(x) $\text{Re}\,(z^2) > 1$; D. U. H. T. 89.

(xi) $\text{Im}\,(z^2) > 0$; D. U. T. H. 88.

(xii) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$; D. U. H. T 88, 90 ; D. U. H. 86.

(xiii) $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$; D. U. H. 87.

(xiv) $|z| > 4$; D. U. H. 86.

(xv) $\text{Re}\left(\frac{-1}{z}\right) \leqslant \frac{1}{2}$; D. U. H. 88 ; D. U. M. SC. P. 88.

(xvi) $\text{Im}\,(z) > 1$; D. U. H. 88.

(xvii) $|z-2+i| \leqslant 1$; D. U. H. 88.

(xviii) $\pi/3 \leqslant \arg z \leqslant \pi/2$;

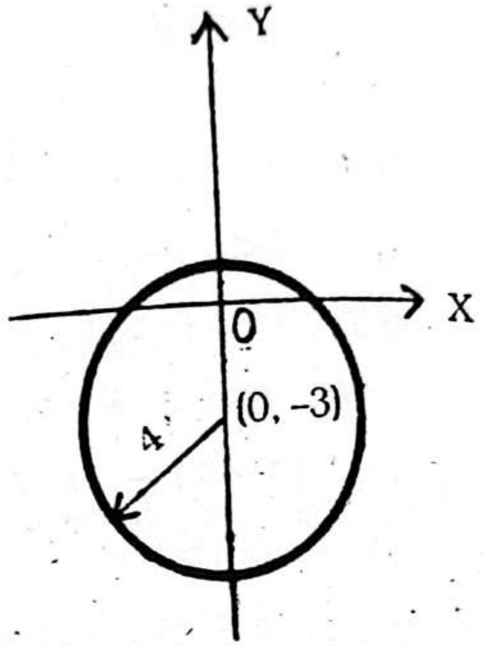
(xix) $0 < \text{Re}\,(iz) < 1$; D. U. H. T. 90.

(xx) $-\pi < \arg z < \pi,\ |z| > 2$; D. U. H. 88.

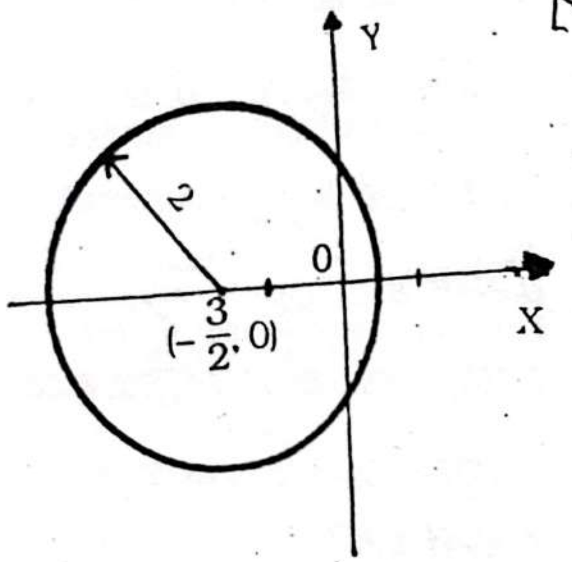
(xxi) $-\pi < \arg z < \pi$; D. U. H. 86 ; D. M. SC. P, 88

(xxii) $-\pi < \arg z < \pi,\ z \neq 0$; D. U. H, 88

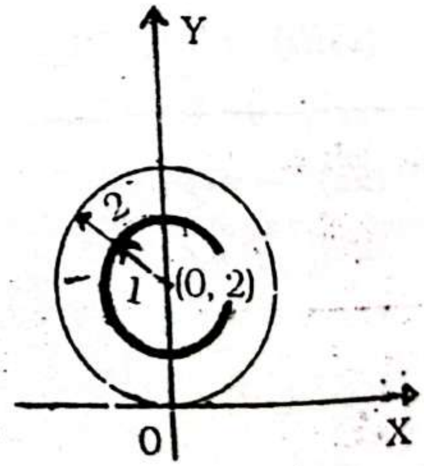
(xxiii) $0 < \arg z < 2\pi,\ |z| > 0$; D. U. M, SC, P, 89.



Solution : (*i*) $|z+3i|>4 \Rightarrow$ $x^2+(y+3)^2>16 \Rightarrow$ the set of all those points external to the circle $|z+3i|=4$ with centre $(0,-3)$ and radius $=4$,



(*ii*) $|2z+3|>4$ $\Rightarrow |z+3/2|>2 \Rightarrow$ $(x+3/2)^2+y^2>4 \Rightarrow$ the set of all those points external to the circle $|z+3/2|=2$ with centre$(-3/2, 0)$ and radius $=2$

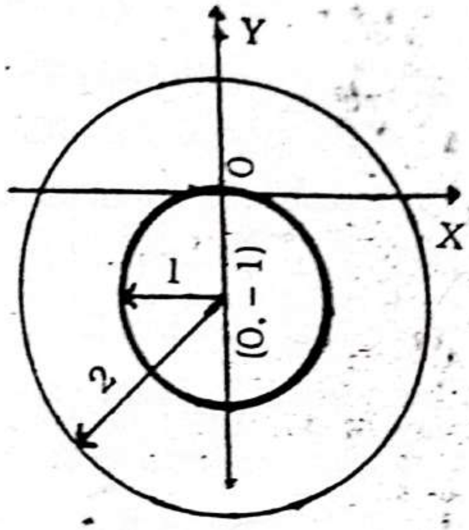


(*iii*) $1<|z-2i|\leqslant 2 \Rightarrow$ $1<x^2+(y-2)^2\leqslant 4$,

The given inequalities represent the set of all those points which are external to the circle $|z-2i|=1$ with centre $(0, 2)$ and internal to the and on the circle $|z-2i|=2$ with the same centre,

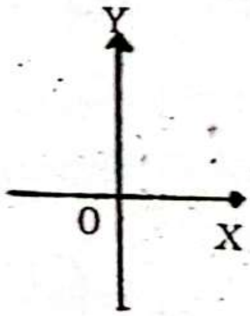
(iv) $1 < |z+i| \leq 2 \Rightarrow 1 < \sqrt{\{x^2+(y+1)^2\}} \leq 2 \Rightarrow$
$1 < x^2+(y+1)^2 \leq 4.$

The given inequalities represent the set of all those points which are external to the circle $|z+i|=1$ with center $(0,-1)$ and internal to the and on the circle $|z+i|=2$ with the same centre.



(v) $|z-4| > |z| \Rightarrow |z-4|^2 > |z|^2 \Rightarrow (z-4)(\bar{z}-4)$
$> z\bar{z} \Rightarrow z\bar{z}-4(z+\bar{z})+16 > z\bar{z} \Rightarrow -4(z+\bar{z})+16 > 0 \Rightarrow z+\bar{z} < 4$

$\Rightarrow 2x < 4 \Rightarrow x < 2 \Rightarrow$ the set of all points (x, y) such that $x<2$, That is, the set consists of all points (x, y) left hand side to the straight line $x=2$.



(vi) $1 < |z-2i| < 2 \Rightarrow 1 < x^2+(y-2)^2 < 4 \Rightarrow$ the set of all those points which are external to the circle $|z-2i|=1$ with centre $(0, 2)$ and internal to the circle $|z-2i|=2$ with the same centre.

(vii) $|z-i| = |z+i| \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2$

$\Rightarrow y^2-2y+1=y^2+2y+1 \Rightarrow y=0 \Rightarrow$ the x-axis or the real axis.

(viii) We have $|z+2-3i|+|z-2+3i|=10$ ... (1)

represents an ellipse whose foci are $(-2, 3)$ and $(2, -3)$ and the length of the major axis is 10. Hence $|z+2-3i|+|z-2+3i|$ <10 represents the set of all points interior to the ellipse (1).

(ix) Here $|z-2|-|z+2|=3$ ... (1) represents a hyperbola with foci are $(2, 0)$ and $(-2, 0)$ and the length of the transverse axis is 3, Hence $|z-2|-|z+2|>3$ represents the et of all poin ts external to the hyperbola (1).

(x) $\text{Re}\,(z^2)>1 \Rightarrow \text{Re}\,(x^2-y^2+2xy\,i)>1 \Rightarrow x^2-y^2>1$

$\Rightarrow$ the set of all points external to the rectangular hyperbola $x^2-y^2=1$:

(xi) $\text{Im}\,(z^2)>0 \Rightarrow \text{Im}\,(x^2-y^2+2xyi)>0 \Rightarrow xy>0$

$\Rightarrow$ the set of all points in the first and the third quadrants.

(xii) $\text{Re}\left(\frac{1}{z}\right)<\frac{1}{2} \Rightarrow \text{Re}\left(\frac{x-iy}{x^2+y^2}\right)<\frac{1}{2} \Rightarrow \frac{x}{x^2+y^2}<\frac{1}{2}$

$\Rightarrow x^2+y^2-2x>0 \Rightarrow (x-1)^2+y^2>1 \Rightarrow$ the set of all points external to the circle $(x-1)^2+y^2=1$ with centre $(1, 0)$ and radius is 1.

(xiii) $\text{Im}\left(\frac{1}{z}\right)<\frac{1}{2} \Rightarrow \text{Im}\left(\frac{x-iy}{x^2+y^2}\right)<\frac{1}{2} \Rightarrow$

$\frac{-y}{x^2+y^2}<\frac{1}{2} \Rightarrow x^2+y^2+2y>0 \Rightarrow x^2+(y+1)^2>1$

$\Rightarrow$ the set of all points external to the circle $x^2+(y+1)^2=1$ with centrre $(0, -1)$ and radius is 1;

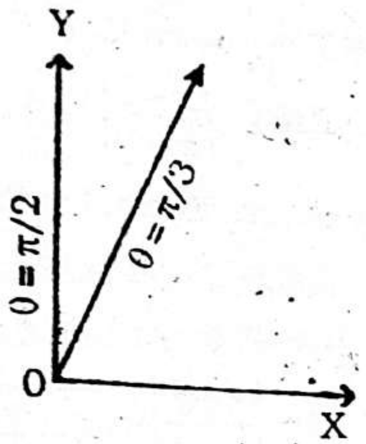
(xiv) $|z|>4 \Rightarrow x^2+y^2>16 \Rightarrow$ the set of all points external to the circle $x^2+y^2=16$ with centre $(0, 0)$ and radius $=4$.

(*xv*) $\mathrm{Re}\left(\frac{1}{z}\right)<\frac{1}{2} \Rightarrow$ the set of all points external to the and on the circle $(x-1)^2+y^2=1$.

(*xvi*) $\mathrm{Im}\, z>1 \Rightarrow y>1 \Rightarrow$ the set of all those points (x, y) such that $y>1$. That is, the set consists of all points (x, y) above the straight line $y=1$.

(*xvii*) $|z-2+i| \leqslant 1$ represents the set of all points interior to the and on the circle $|z-2+i|=1$.

(*xviii*) We have $\arg z=\theta$ where $z=r(\cos\theta+i\sin\theta)$ $=re^{i\theta}$. Hence $\pi/3 \leqslant \arg z \leqslant \pi/2 \Rightarrow$ the infinite region bounded by the lines $\theta=\arg z=\pi/3$ and $\theta=\arg z=\pi/2$ including these lines.



Others ; Try yourself.

Example 52 : Describe each of the following region geometrically :

(*i*) $\left|\frac{z-3}{z+3}\right|=2$; **D. U. M. SC. T. 80.**

(*ii*) $\left|\frac{z-3}{z+3}\right|>2$; (*iii*) $\left|\frac{z-3}{z+3}\right|<2$.

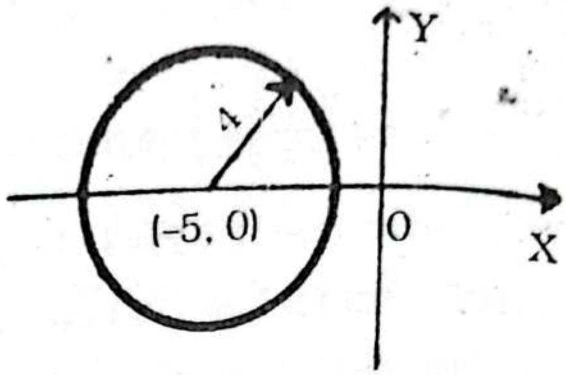
Solution : (*i*) $\left|\frac{z-3}{z+3}\right|=2 \Rightarrow |z-3|=2|z+3|$

$\Rightarrow |z-3|^2 = 4|z+3|^2$

$\Rightarrow (z-3)(\bar{z}-3) =$

$4(z+3)(\bar{z}+3)$

$\Rightarrow z\bar{z} + 5(\bar{z}+z) + 9 = 0$



$\Rightarrow (z+5)(\bar{z}+5) = 16$

$\Rightarrow |z+5| = 16 \Rightarrow$ the equation of a circle with centre $(-5, 0)$ and radius $= 4$.

(*ii*) $\left|\frac{z-3}{z+3}\right| > 2 \Rightarrow |z+5| < 4 \Rightarrow$ the set consisting of all the point internal to the circle $|z+5| = 4$.

(*iii*) $\left|\frac{z-3}{z+3}\right| < 2 \Rightarrow |z+5| > 4 \Rightarrow$ the set consisting of all the points external to the circle $|z+5| = 4$.

Example 53 : Describe each of the following region geometrically :

(*i*) $\left|\frac{z-3}{z+3}\right| = 3$; (*ii*) $\left|\frac{z-3}{z+3}\right| > 3$; (iii) $\left|\frac{z-3}{z+3}\right| < 3$.

D. U. M. SC. P. 89.

Solution : Try yourself.

Example 54 : Identify all the points in the complex plane determined by the following :

(*i*) $\frac{z-3}{z+3} < 3$; (*ii*) $|z-1| + |z+1| = 4$;

(*iii*) $0 < \arg z < 2\pi$, $|z| > 0$.

D. U. M. SC. P. 89.

Solution : (*i*) $\left|\frac{z-3}{z+3}\right|<3 \Rightarrow |z-3|<3|z+3| \Rightarrow$

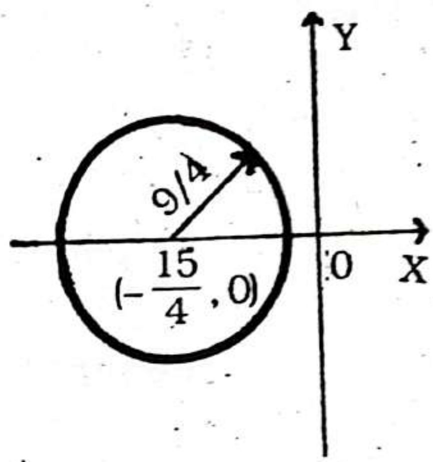
$|z-3|^2<9|z+3|^2 \Rightarrow (z-3)(\overline{z}-3)<9(z+3)(\overline{z}+3)$

$\Rightarrow z\overline{z}-3(z+\overline{z})+9<9\{z\overline{z}+3(z+\overline{z})+9\} \Rightarrow$

$8z\overline{z}+30(z+\overline{z})+72>0$

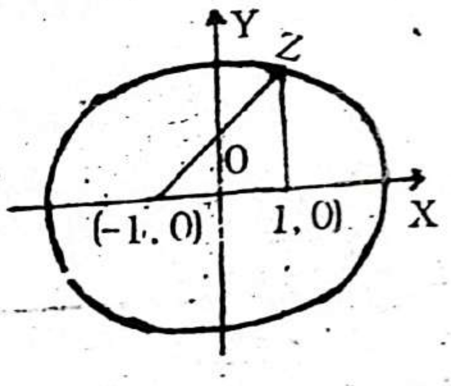
$\Rightarrow 8(x^2+y^2)+60x+72>0$

$\Rightarrow 2(x^2+y^2)+15x+18>0$



$\Rightarrow x^2+y^2+\frac{15}{2}x+9>0 \quad \dots \quad (1)$

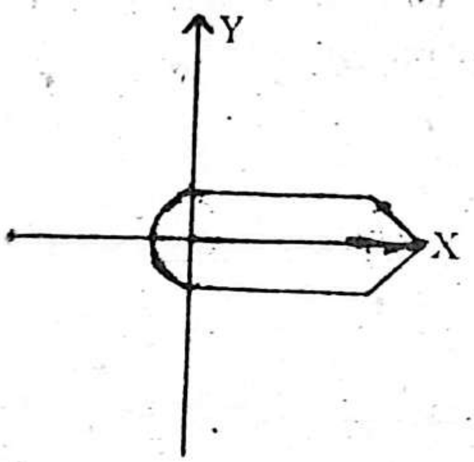
The equation represent a set which consists of all those points external to the circle $2(x^2+y^2)+15x+18=0$ with centre $(-15/4, 0)$ and radius $=\sqrt{\left(\frac{-15}{4}\right)^2-\frac{18}{2}}=\sqrt{\frac{225}{16}-9}=\sqrt{\frac{81}{16}}=\frac{9}{4}$.



(*ii*) The equation $|z-1|+|z+1|=4$ represents an ellipse whose foci are $(-1, 0)$ and $(1, 0)$ and the length of its major axis is 4 (*iii*) Here $0 < \arg z < 2\pi$, $|z| > 0$ represent the whole region excluding the origin and the positive part of the x-axis i. e. the points of the form (x, o) where $x > 0$.

Mathematically, the given set

$= \{(x, y) = x + iy : \ x \notin R_+ \cup \{0\}, y \in R\}$.



Example 55 : Show that the equation of a straight line through the points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is $\arg \dfrac{z - z_1}{z_1 - z_2} = 0$.

Solution : The equation of the straight line through the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2} \Rightarrow \arg(z - z_1) = \arg(z_1 - z_2) \Rightarrow \arg(z - z_1) - \arg(z_1 - z_2) = 0 \Rightarrow \arg \dfrac{z - z_1}{z_1 - z_2} = 0$.

Example 56 : Find the equation of a circle with centre $(-3, 4)$ and radius 2.

Solution : Let $z = (x, y)$ be any point on the circle, then the equation of the circle is $|(x, y) - (-3, 4)| = 2 \Rightarrow$

$|(x+iy)-(-3+4i)| \Rightarrow |z+3-4i| = 2$. In rectangular form this becomes $(x+3)^2+(y-4)^2=4$.

Example 57 : Find the equation of an ellipse with foci at $(0,-2)$ and $(0, 2)$ and its major axis is 10. ***D. U. 63.***

Solution : Let $z=(x, y)$ be any point on the ellipse. By definition, the sum of the distances from any point on the ellipse to the foci must be equal to the length of the major axis.

Then the required equation is

$$|(x, y)-(0,-2)| + |(x, y)-(0, 2)| = 10$$

$$\Rightarrow |z+2i| + |z-2i| = 10.$$

Exmple 58 : Find the equation of a hyperbola with foci at $(3, 0)$ and $(-3, 0)$ and its transverse axis is 4.

Solution : Let $z=(x, y)$ be any point on the hyperbola. By definition, the difference of the distances from any point on the hyperbola to the transverse axis. Then the required equation is

$$|(x, y)-(3, 0)| - |(x, y)-(-3, 0)| = 4$$

$$\Rightarrow |z-3| - |z+3| = 4.$$

Example 59 : Find the modulus and argument of the following complex nnmbers :

(*i*) $\dfrac{-2}{1+i\sqrt{3}}$; (*ii*) $\dfrac{1-i}{1+i}$. ***J. U. H. 87.***

Solution : (*i*) $\dfrac{-2}{1+i\sqrt{3}} = \dfrac{-2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{-2(1-i\sqrt{3})}{1+3}$

$=-1/2+i\sqrt{3}/2$. Then the requ,red modulus $=\frac{1}{2}\sqrt{\{(-1)^2+(\sqrt{3})^2\}}$

$=\frac{1}{2}\sqrt{4}=\frac{2}{2}=1$. The required argument (general)

$=2n\pi+\tan^{-1}\dfrac{\sqrt{3}/2}{-1/2}=2n\pi+\dfrac{2\pi}{3}$, $n=0, \pm 1, \pm 2, \ldots \ldots$

and the principal argument $=\dfrac{2\pi}{3}$.

(ii) $\dfrac{1-i}{1+i}=\dfrac{(1-i)^2}{1-i^2}=\dfrac{1-2i+i^2}{2}=-i$. The required modulus $=|-i|=1$. The required argument (general) $=2n\pi-\pi/2$, $n=0, \pm 1, \pm 2, \cdots$ and the principal argument $=-\pi/2$.

Example 60 **:** Find all the fifth roots of unity.

J. U. H. 87.

Solution **:** Let $z=(1)^{1/5}=(\cos 2n\pi+i\sin 2n\pi)^{1/5}$

$$= e^{2n\pi i/5}, \text{ where } n=0, 1, 2, 3, 4.$$

Then the required roots are $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$

or $1, w, w^2, w^3, w^4$ where $w=e^{2\pi i/5}$.

Example 61 **:** Find the argument and modulus of the complex number $\left(\dfrac{2+i}{3-i}\right)^2$. ***D. U. H. T. 91.***

Ans. modulus $=\frac{1}{2}$, principal argument $=\pi/2$ and general argument $=\pi/2+2n\pi$, $n\in I$.

Solution **;** Try yourself.

$\Rightarrow \arg (b-c)\left(\bar{a}-\bar{c}\right)=0 \Rightarrow (b-c)\left(\bar{a}-\bar{c}\right)$ is real and positive $\Rightarrow (b-c)\left(\bar{a}-\bar{c}\right)=|(b-c)\left(\bar{a}-\bar{c}\right)|$ ... ... (1)

Again we have $|b-c||a-c|=r^2 \Rightarrow$

$|b-c|\left|\bar{a}-\bar{c}\right|=r^2$ since $|z|=|\bar{z}|$

$\Rightarrow |(b-c)\left(\bar{a}-\bar{c}\right)|=r^2$ ... (2)

Now by (1) and (2), we have

$$(b-c)\left(\bar{a}-\bar{c}\right)=r^2 \Rightarrow b-c=\frac{r^2}{\bar{a}-\bar{c}} \Rightarrow b=c+\frac{r^2}{\bar{a}-\bar{c}}.$$

CHAPTER – 2

FUNCTIONS LIMITS AND CONTINUITY

81. Complex variable : *D. U. H. S. T. 85.*

Let S be a set of complex numbers.

If z denotes any one of the numbers of S, then z is called a complex variable.

If x and y are real variables, then $z = x + iy$ is called a complex variable.

82. Function : Let S_1 and S_2 be two sets of complex numbers. Now if for each complex variable z of S_1 there corresponds one or more values of a complex variable w of S_2, then w is called a function of z and it is denoted by $w = f(z)$ or $w = F(z)$ or $w = g(z)$ or $w = G(z)$ etc.

Here the set S_1 is called a ***domain*** of definition of the function $w = f(z)$ and S_2 is called the ***range*** of the function w.

N. B. In this book all functions will be considered complex functions unless otherwise any other functions stated.

83. Independent and dependent variable of a function

Let $w = f(z)$ be a function, then the variable z is called an independent variable and the variable w is called a dependent variable.

84. Value of a function :

Let $w = f(z)$ be a function, then the value of this function at $z = a$ is written $f(a)$.

85. Single-valued function :

A function $w = f(z)$ is called a single-valued in a domain S if only one value of w corresponds to each value of z in S.

86. Multiple-valued function : *R. U. H. 80.*

A function $w = f(z)$ is called a multiple-valued in a domain S if more than one value of w corresponds to each value of z in S.

Any multiple valued function can be considered as a collection of single-valued functions where each single-valued member is called a ***branch*** of the function.

Example 64 : If $w = f(z) = z^2 + 2$, then w is called a single valued function of z since to each value of z there is only one value of w.

Example 65 : If $w = f(z) = z^{1/2}$, then w is called a multiple-valued function of z since to each value of z there are two values of w.

Example 66 : If $w = f(z) = z^2$, then $f(1+i) = (1+i)^2$
$= 1 + 2i + i^2 = 1 + 2i - 1 = 2i.$

87. Even function : A function $f(z)$ is called an even function if $f(-z) = f(z)$.

Example 67 : The function $f(z) = z^2$ is an even function since $f(-z) = (-z)^2 = z^2 = f(z)$. Similarly, $\cos z$, $z^4 + z^2 + c$, etc are even functions.

88. Odd function : A function $f(z)$ is called an odd function if $f(-z) = -f(z)$.

Example 68 : The function $f(z)=z^3$ is an odd function since $f(-z)=(-z)^3=-z^3=-f(z)$. Similarly, sin z, tan z, z^3+z, etc are odd functions.

Example 69 : The functions $\cos z+\sin z$, z^4+z^3+5, etc are neither even nor odd.

N. B. Next we will consider all functions are single-valued function unless otherwise stated.

89. *Inverse function* : Let $w=f(z)$ be a function, then we can consider z as a function of w and it is denoted by $z=g(w)=f^{-1}(w)$. Here the function f^{-1} is called the inverse function of f. The functions $w=f(z)$ and $w=f^{-1}(z)$ are inverse functions of each other.

90. *Real and imaginary parts of* $w=f(z)$ ***corresponding to the complex variable*** $z=x+iy$.

Let $w=f(z)=u+iv$ be a single-valued function of $z=x+iy$.

Now replacing $x+iy$ for z, we have $u+iv=f(x+iy)$.

Then equating real and imaginary parts we have $u=u(x, y)$ and $v=v(x, y)$.

Example 70 : If $w=e^z$, then $u+iv=e^{x+iy}$ $=e^x(\cos y+i\sin y) \Rightarrow u=e^x\cos y=u(x, y)$ and $v=e^x\sin y=v(x, y)$

91. *The polynomial function* : A function of the form $P(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ is called a polynomial function of degree n where $a_0\neq 0, a_1, a_2, \cdots, a_n$ are complex constants and n is a positive integer.

92. *The rational algebraic function* : A function of the form $w=\dfrac{P(z)}{Q(z)}$ is called a rational algebraic function where P(z) and Q(z) are polynomials.

93. *The exponential function* : ***R. U. 76.***

A function of the form $w=e^z=e^{x+iy}=e^x e^{iy}$ $=e^x(\cos y+i\sin y)$ is called an exponential function where $e=2.71828\cdots$ is the natural base of logarithm.

It is clear that $e^{z_1}e^{z_2}=e^{z_1+z_2}$ and

$$\frac{e^{z_1}}{e^{z_2}}=e^{z_1-z_2}.$$

94. *Definition of* a^z : If a is real and positive, then a^z can be defined as follows :

$a^z=e^{z\ln a}=e^{z\log a}$ where ln a or log a is the ***natural logarithm*** of a.

N. B. In this book ln and log have the identical meaning but $\log_a$ is not a natural logarithm if $a\neq e$.

95. *Natural logarithm of* z :

If $z=e^w \Rightarrow w=\ln z$, which is called the ***natural logarithm*** of z. The natural logarithm function is the inverse of the exponential function and it can be defined by

$w=\ln z=\log r+i(2k\pi+\theta)$ where

$z=re^{i\theta}=re^{i(2k\pi+\theta)}$ and $k=0, \pm 1, \pm 2, \cdots\cdots$

It is clear that ln z is a multiple-valued function.

96. The principal value of log z :

The principal value or principal branch of ln z is defined as $\ln r + i\theta$ where $z = re^{i\theta}$ and $\theta \in [0, 2\pi[$ or $\theta \in]-\pi, \pi]$ etc where the interval must be a length of 2π.

97. Definition of a^w if a is real :

If a is real, then if $z = a^w \Rightarrow w = \log_a z$ where $a > 0$, $a \neq 0, 1$ and $a \neq e$. Also in this case we have $z = e^{w \ln a}$ and $w = \log_a z = \dfrac{\ln z}{\ln a}$.

98. Trigonometric or circular functions in terms of exponential functions :

The trigonometric or circular functions can be defined by the following :

(i) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ R. U. 76. (ii) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

(iii) $\sec z = \dfrac{1}{\cos z} = \dfrac{2}{e^{iz} + e^{-iz}}$, (iv) $\operatorname{cosec} z = \dfrac{1}{\sin z}$

$= \dfrac{2i}{e^{iz} - e^{-iz}}$, (v) $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

(vi) $\cot z = \dfrac{\cos z}{\sin z} = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$

Example 71 : Show that :

(i) $\sin^2 z + \cos^2 z = 1$, (ii) $\sec^2 z = 1 + \tan^2 z$, (iii) $\operatorname{cosec}^2 z = 1 + \cot^2 z$, (iv) $\sin(-z) = -\sin z$, (v) $\cos(-z) = \cos z$, (vi) $\tan(-z) = -\tan z$, (vii) $\cot(-z) = -\cot z$, (viii) $\sec(-z) = \sec z$, (ix) $\operatorname{cosec}(-z) = -\operatorname{cosec} z$,

(x) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$, (xi) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$, (xii) $\tan(z_1 \pm z_2) = \dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$, (xiii) $\cot(z_1 \pm z_2) = \dfrac{\cot z_2 \cot z_1 \mp 1}{\cot z_2 \pm \cot z_1}$

Solution : Try yourself.

99. Hyperbolic function : The hyperbolic functions are defined by the following :

(i) $\sinh z = \dfrac{e^z - e^{-z}}{2}$, R. U. 76 ; (ii) $\cosh z = \dfrac{e^z + e^{-z}}{2}$

(iii) $\operatorname{sech} z = \dfrac{1}{\cosh z} = \dfrac{2}{e^z + e^{-z}}$; (iv) $\operatorname{cosech} z = \dfrac{1}{\sinh z}$

$= \dfrac{2}{e^z - e^{-z}}$, (v) $\tanh z = \dfrac{\sinh z}{\cosh z} = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$,

(vi) $\coth z = \dfrac{e^z + e^{-z}}{e^z - e^{-z}}$

Example 72 : Show that :

(i) $\cosh^2 z - \sinh^2 z = 1$, (ii) $\operatorname{sech}^2 z = 1 - \tanh^2 z$,

(iii) $\operatorname{cosech}^2 z = \coth^2 z - 1$, (iv) $\sinh(-z) = -\sinh z$,

(v) $\cosh(-z) = \cosh z$, (vi) $\tanh(-z) = -\tanh z$,

(vii) $\operatorname{sech}(-z) = \operatorname{sech} z$, (viii) $\operatorname{cosech}(-z) = -\operatorname{cosech} z$,

(ix) $\coth(-z) = -\coth z$,

(x) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$.

(xii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

(xiii) $\coth(z_1 \pm z_2) = \dfrac{\cosh z_2 \coth z_1 \pm 1}{\coth z_2 \pm \coth z_1}$,

Solution : Try yourself.

100. Relation between the trigonometric or circular functions and the hyperbolic functions:

(*i*) $\sin iz = i \sinh z \Rightarrow \sinh z = -i \sin iz$,

(*ii*) $\cos iz = \cosh z$, (*iii*) $\tan iz = i \tanh z$

$\Rightarrow \tanh z = -i \tan iz$, (*iv*) $\operatorname{cosec} iz = -i \operatorname{cosech} z \Rightarrow$

$\operatorname{cosech} z = i \operatorname{cosec} iz$, (*vi*) $\sec iz = \operatorname{sech} z$,

(*vii*) $\cot iz = -i \coth z \Rightarrow \coth z = i \cot iz$,

(*ix*) $\sinh iz = i \sin z \Rightarrow \sin z = -i \sinh iz$

(x) $\cosh iz = \cos z$, (*xi*) $\tanh iz = i \tan z \Rightarrow \tan z = -i \tanh iz$.

Example 73: Show that: (*i*) $\overline{\sin z} = \sin \bar{z}$;

(*ii*) $\overline{\cos z} = \cos \bar{z}$; (*iii*) $\overline{\tan z} = \tan \bar{z}$;

(*iv*) $\overline{\operatorname{cosec} z} = \operatorname{cosec} \bar{z}$. (v) $\overline{\sec z} = \sec \bar{z}$.

(*vi*) $\overline{\cot z} = \cot \bar{z}$.

Solution: (*i*) We have $\sin z = \sin (x + iy)$

$= \sin x \cosh y + i \cos x \sinh y \Rightarrow \overline{\sin z} = \sin (x - iy) = \sin \bar{z}$.

Others: Try yourself.

Example 74: Find all the roots of $\sinh z = i$.

D. U. M. SC. P. 88.

Solution: We have $\sinh z = i \Rightarrow \dfrac{e^z - e^{-z}}{2} = i$

$\Rightarrow e^{2z} - 2ie^z - 1 = 0 \Rightarrow e^z = \dfrac{2i \pm \sqrt{4i^2 + 4}}{2} = i = e^{\pi i/2}$

$= e^{\pi i/2}\, e^{2n\pi i} = e^{(2n+1/2)\pi i} \Rightarrow z = (2n + \frac{1}{2})\pi i$ where

$n = 0, \pm 1, \pm 2, \pm 3, \cdots \cdots$

Example 75: Show that $\ln z = 2n\pi i + \frac{1}{2} \ln (x^2 + y^2) + i \tan^{-1} y/x$ where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$ and the principal value $= \frac{1}{2} \ln (x^2 + y^2) + i \tan^{-1} y/x$.

Solution 75: Try yourself.

101. Inverse trigonometric functions in terms of natural logarithms:

The inverse trigonometric functions are multiplevalued functions which can be expressed in terms of natural logarithms as follows;

(*i*) $\sin^{-1} z = \dfrac{1}{i} \ln (iz + \sqrt{1 - z^2}) + 2n\pi$;

(*ii*) $\cos^{-1} z = \dfrac{1}{i} \ln (z + \sqrt{(z^2 - 1)}) + 2n\pi$;

(*iii*) $\tan^{-1} z = \dfrac{1}{2i} \ln\left(\dfrac{1 + iz}{1 - iz}\right) + n\pi$;

(*iv*) $\operatorname{cosec}^{-1} z = \dfrac{1}{i} \ln\left(\dfrac{i + \sqrt{z^2 - 1}}{z}\right) + 2n\pi i$;

(*v*) $\sec^{-1} z = \dfrac{1}{i} \ln\left(\dfrac{1 + \sqrt{1 - z^2}}{z}\right) + 2n\pi$;

(*vi*) $\cot^{-1} z = \dfrac{1}{2i} \ln\left(\dfrac{z + i}{z - i}\right) + n\pi$,

where in each case $n = 0, \pm 1, \pm 2, \pm 3, \cdots \cdots$,

N. B. If $n = 0$, then the principal value can be obtained.

102. Inverse hyperbolic functions in terms of natural logarithms:

The inverse hyperbolic functions are multiple valued functions which can be expressed interms of natural logarithms as follows:

(i) $\sinh^{-1} z = \ln(z + \sqrt{z^2+1}) + 2n\pi i$;

(ii) $\cosh^{-1} z = \ln(z + \sqrt{z^2-1}) + 2n\pi i$;

(iii) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) + n\pi i$;

(iv) $\operatorname{cosech}^{-1} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right) + 2n\pi i$;

(v) $\operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) + 2n\pi i$;

(vi) $\coth^{-1} z = \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right) + n\pi i$,

where in each case $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

N B. If $n=0$, then the principal value can be obtained.

103. The functions of the forms z^a and $f(z)^{g(z)}$:

The function z^a is defined as $z^a = e^{a \ln z}$, where a may be complex. Again if $f(z)$ and $g(z)$ are two functions then the function $f(z)^{g(z)}$ is defined as $f(z)^{g(z)} = e^{g(z) \ln f(z)}$. Generally, the functions e^z and $f(z)^{g(z)}$ are multiple-valued functions.

104. Algebraic functions :

The function $w=f(z)$ is called an algebraic function of z if w is a solution of the polynomial equation $P_0(z) w^n + P_1(z) w^{n-1} + \cdots + P_{n-1}(z) w + P_n(z) = 0 \cdots$ (1) where $P_0(z) \neq 0$, $P_1(z), \cdots P_n(z)$ are polynomial in z and n is a positive integer.

Example 76 : The function $w = f(z) = z^{1/2}$ is an algebraic function since it is a solution of the polynomial equation $w^2 - z = 0$.

105. Transcendental functions :

The function which is not algebraic is called transcendental i.e. any function which can not be expressed as a solution of (1).

Example 77 : All trignometric, hyperbolic, logarithomic, inverse trigonometric, inverse hyperbolic etc functions are transcendental functions.

106. Limit at a finite point :

Let $f(z)$ be a single valued function which is defined in a neighbourhood of $z = z_0$ with the possible exception of $z = z_0$ itself. Then $f(z)$ is said to tend to the limit l as z tends to the value z_0 if corresponding to any positive number ϵ (however small) a positive number δ (which usually depends on ϵ) can be found such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and it is denoted by $\lim_{z \to z_0} f(z) = l$.

N. B. In above the limit l is independent of the path by which z tends to z_0.

Also, the limit l has not necessarily the same value as $f(z_0)$.

107. *Theorem 38* : If $\lim_{z \to z_0} f(z)$ exists, then it must be unique.

Proof : Try yourself.

108. Limit at infinity :

The single valued function $f(z)$ is said to tend to the limit l as z tends to infinity if corresponding to any positive number ϵ (however small) a positive number N can be found such that

$| f(z) - l | < \epsilon$ whenever $| z | > N$ and it is denoted by

$\lim_{n \to \infty} f(z) = l.$

109. *Infinite limit :*

The single valued function $f(z)$ is said to tend to the limit infinity as z tends to z_0 if corresponding to any positive number N (however large) a positive number δ can be found such that $| f(z) | > N$ whenever $| z - z_0 | < \delta$ and it is denoted by

$\lim_{z \to z_0} f(z) = \infty.$

110. *Four fundamental theorems on limit :*

Theorems 39, 40, 41 and 42 :

If $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ are exist, then

39. $\lim_{z \to z_0} \{f(z) + g(z)\} = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z),$

40. $\lim_{z \to z_0} \{f(z) - g(z)\} = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} g(z),$

41. $\lim_{z \to z_0} \{f(z)\, g(z)\} = \{\lim_{z \to z_0} f(z)\}\, \{\lim_{z \to z_0} g(z)\},$

42. $\lim_{z \to z_0} \{f(z)/g(z)\} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$ where $\lim_{z \to z_0} g(z) \neq 0.$

Proofs : Try yourself.

Example 78 : Show that $\lim_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist.

Solution : We have $z = x + iy$ and $\bar{z} = x - iy$. If $z \to 0$, then along the x-axis : $y = 0$ and $x \to 0$, so the required limit is

$\lim_{z \to 0} \dfrac{\bar{z}}{z} = \underset{x \to 0}{\text{Lt}} \dfrac{x}{x} = 1.$

Again if $z \to 0$, then along the y-axis : $x = 0$ and $y \to 0$,

so the required limit is $= \lim_{z \to 0} \dfrac{\bar{z}}{z} = \lim_{y \to 0} \dfrac{-iy}{iy} = -1.$

Thus the two approaches are not equal and the limit does not exist.

111. *Cotinuity :* **D. U. H. 93**

The single valued function $f(z)$ is said to be continuous at the point $z = z_0$ if $f(z_0)$ has a definite value and if $\lim_{z \to z_0} f(z) = f(z_0)$.

Second definition : The single-valued function $f(z)$ is said to be continuous at the point $z = z_0$ if for any $\epsilon > 0$ we can find $\delta > 0$ such that $| f(z) - f(z_0) | < \epsilon$ whenever $| z - z_0 | < \delta$.

112. *Discontinuity :* The function $f(z)$ is said to be discontinuous at the point $z = z_0$ if $f(z)$ is fails to be continuous at $z = z_0$.

113. *Removal discontinuity :* The function $f(z)$ is said to be removal discontinous at $z = z_0$ if $f(z)$ has a definite limit at $z = z_0$ but is not equal to $f(z_0)$.

114. *Continuity at infinity :* The continuity of $f(z)$ at $z = \infty$ can be examined by the continuity of $f(1/w)$ at $w = 0$ by replacing $z = 1/w$ in $f(z)$.

115. *Continuity in a region :*

A function $f(z)$ is said to be continuous in a region R if it is continuous at all points of the region R.

116. Four fundamental theorems on continuity :

Theorems 43, 44, 45 and 46 : If $f(z)$ and $g(z)$ are continuous at $z = z_0$, then the following functions are continuous at $z = z_0$.

(43) $f(z) + g(z)$; **(44)** $f(z) - g(z)$; **(45)** $f(z)\, g(z)$ and

(46) $f(z)/g(z)$ where $g(z_0) \neq 0$.

Proofs : Try yourself.

117. Theorem 47 : Every polynomial functions are continuous in a finite region.

Proof : Try yourself.

118. Theorem 48 : If $f(z)$ is continuous and has the value $f(z_1)$ at $z = z_1$. Again if $\phi(z)$ is continuous at $z = f(z_1)$, then $\phi(f(z))$ is continuous at $z = z_1$.

Proof : Try yourself.

119. Theorem 49 : If $w = f(z)$ is continuons at the point $z = z_0$ and $z = g(\xi)$ is continuous at $\xi = \xi_0$ and if $\xi_0 = f(z_0)$ then the composite function or the function of function $w = g(f(z))$ is continuous at $z = z_0$.

Proof : Try yourself.

120. Theorem 50 : If $f(z)$ is continuous in a closed region R and if it is bounded in R. i. e. if there exists a real constanst M such that $|f(z)| < M$ for all points z in the region R.

Proof : Try yourself.

121. Theorem 51 : The real and imaginary parts of a continuous function $f(z)$ are continuous.

Proof : Try yourself

Example 79 : The functions e^z, sin z and cos z are continuous in every finite region

Solution : Try yourself.

122. Uniform continuity : A function $f(z)$ is said to be uniformly continuous in a region R if corresponding to any $\epsilon > 0$ we can find $\delta > 0$ (which is a function of ϵ only) such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$ for every point z_0 in the region R.

N. B. In continuity, δ depend on both ϵ and the particular point z_0. But in uniform continuity, δ depends only on ϵ.

Second definition : A function $f(z)$ is said to be uniformly continuous in a region R if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$ for every points z_1 and z_2 in the region R.

123. Theorem 52 : If $f(z)$ is continuous in a closed region R, then it is uniformly continuous in R.

Proof : Try yourself.

Example 80 : If $f(z) = z^2$ then show that

(i) $\lim\limits_{z \to a} f(z) = a^2$; **D U. M. SC. P. 89.**

(ii) $f(z)$ is continuous at $z = a$;

(iii) $f(z)$ is uniformly continuous in the region $|z| < 1$.

D U M SC. p, 89

Solution : (i) : We have to show that given any $\epsilon > 0$, we can find $\delta > 0$ such that $|z^2 - a^2| < \epsilon$ whenever $0 < |z - a| < \delta$. Now if $\delta \leq 1$, then $0 < |z - a| < \delta \Rightarrow |z^2 - a^2| = |z - a|\,|z + a|$ $= |z - a|\,|(z - a) + 2a| < |z - a|\{|z - a| + |2a|\}$ $< \delta(1 + 2|a|)$. Now taking δ as 1 or $\epsilon/(1 + 2|a|)$ which ever is smaller. Thus $|z^2-a^2|<\epsilon$ whenever $0<|z-a|<\delta$ and we have $\lim_{z\to a} f(z)=a^2$.

(*ii*): By (*i*), $\lim_{z\to a} f(z)=a^2$. Again we have $f(a)=a^2$. Thus $\lim_{z\to a} f(z)=f(a) \Rightarrow f(z)$ is continuous at $z=a$.

(*iii*) We have to show that given any $\epsilon>0$, we can find $\delta>0$ such that $|z^2-a^2|<\epsilon$ when $|z-a|<\delta$ where δ in a function of ϵ only.

Suppose z and a are any two point in $|z|<1$, then

$|z^2-a^2|=|z-a||z+a|\leqslant|z-a|(|z|+|a|)$

$<2|z-a|$ ... (1). Now if $|z-a|<\delta$, then (1)

$\Rightarrow |z^2-a^2|<2\delta \Rightarrow |z^2-a^2|<\epsilon$ choosing $\delta=\epsilon/2$.

Thus $|z^2-a^2|<\epsilon$ when $|z-a|<\delta$.

Hence the given function is uniformly continuous in the region $|z|<1$.

Example 81 : Show that $f(z)=1/z$ is not uniformly continuous in the region $|z|<1$.

Solution : We consider $f(z)$ is uniformly continuous in the region $|z|<1$.

Then for any $\epsilon>0$ it is possible to find δ which lies between 0 and 1 such that $|f(z)-f(a)|<\epsilon$ when $|z-a|<\delta$ for all z and a in the region $|z|<1$.

Let $z=\delta$ and $a=\dfrac{\delta}{1+\epsilon}$ then $|z-a|=\left|\delta-\dfrac{\delta}{1+\epsilon}\right|$

$$=\left|\frac{\delta+\delta\epsilon-\delta}{1+\epsilon}\right|=\frac{\epsilon}{1+\epsilon}\delta<\delta.$$

But $\left|\dfrac{1}{z}-\dfrac{1}{a}\right|=\left|\dfrac{1}{\delta}-\dfrac{1+\epsilon}{\delta}\right|=\left|\dfrac{-\epsilon}{\delta}\right|=\dfrac{\epsilon}{\delta}>\epsilon$ since we have considered $0<\delta<1$.

Thus we have a contradiction and the given function is not uniformly continuous in $|z|<1$.

124 Complex sequence :

A complex sequence $<f(n)>$ or $<u_n>$ is a function whose domain is the set of natural numbers ***N*** and range is the subset of the set of complex numbers **C** In this book, by a sequence we will mean the complex sequence.

The nth term of the sequence $<f(n)>$ or $<u_n>$ is $f(n)$ or u_n.

Example 82 : $<i^n> = <i, i^2, i^3, i^4, \cdots>$ is a sequence.

125. Limit of a sequence :

A number l is said to be the limit of the sequence $<u_n>$ if for any positive number ϵ we can determine a positive number N (depending on ϵ) such that $|u_n-l|<\epsilon$ for all $n>N$ and it is denoted by $\lim_{n\to\infty} u_n=l$.

126. Convergent sequence :

If the limit of the sequence $<u_n>$ exists, then the sequence is called convergent.

127. Divergent sequence : If the limit of the sequence $<u_n>$ does not exist, then the sequence is called divergent.

128 Theorem 53 : If $\lim_{n\to\infty} u_n=l$, where l is finite then it must be unique

Proof : Try yourself.

129 Four fundamental theorems on limits of sequences ;

Theorems 54, 55, 56 and 57 : If the sequences $< a_n >$ and $<b_n>$ both are convergent, then

54. $\lim\limits_{n\to\infty} (a_n+b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

55. $\lim\limits_{n\to\infty} (a_n-b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$

56. $\lim\limits_{n\to\infty} (a_n\, b_n) = (\lim\limits_{n\to\infty} a_n)\, (\lim\limits_{n\to\infty} b_n)$

57 $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ where $\lim\limits_{n\to\infty} b_n \neq 0$.

Proofs : Try yourself.

130. *Infinite series :* Let $<u_n>$ be a sequence, then

$\sum\limits_{n=1}^{\infty} u_n = u_1+u_2+u_3+ \dots$ is called an infinite series.

Example 83 : $1+z+z^2+z^3+ \dots$ is an infinite series.

131. *nth partial sum*

Let $<u_n>$ be a sequence. Suppose $S_1 = u_1$, $S_2 = u_1+u_2$, $S_3 = u_1+u_2+u_3$, ..., $S_n = u_1+u_2+u_3+ \dots + u_n$, where S_n is called the nth partial sum of the first n terms of the sequence $<u_n>$.

If $\lim\limits_{n\to\infty} S_n = S$ exists, then the series $\sum\limits_{n=1}^{\infty} u_n$ is called ***convergent*** and S is called its sum. If it is not convergent, then it is called ***divergent***

132. *Theorem 58 :* If the series $u_1+u_2+u_3+ \dots \dots$ is convergent, then $\lim\limits_{n\to\infty} u_n = 0$.

Proof : Try yourself.

Example 84 : Show that $1+z+z^2+z^3+ \dots \dots = \dfrac{1}{1-z}$ if $|z|<1$.

Solution : Try yourself.

133. *Theorem 59 :* If $\lim\limits_{n\to\infty} z_n = l$, then show that $\lim\limits_{n\to\infty} \operatorname{Re}\{z_n\} = \operatorname{Re}\{l\}$ and $\lim\limits_{n\to\infty} \operatorname{Im}\{z_n\} = \operatorname{Im}\{l\}$.

Proof : Try yourself.

Example 85 : Show that if $|a|<1$, then

(i) $\sum\limits_{n=0}^{\infty} a^n \cos n\theta = \dfrac{1-a\cos\theta}{1-2a\cos\theta+a^2}$;

(ii) $\sum\limits_{n=0}^{\infty} a^n \sin n\theta = \dfrac{a\sin\theta}{1-2a\cos\theta+a^2}$.

Solution : Try yourself.

CHAPTER—3

ANALYTIC FUNCTIONS

134. Derivative or differential coefficient in a region or domain :

C. U. H. 83 ; D. U. H. T. 77. 85.

Let $f(z)$ be a single-valued function defined in a region or domain R of the Argand plane, then the derivative of $f(z)$ is defined as $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ .... (1)

provided that the limit exists and is independent of the manner in which $\Delta z \to 0$. If the limit (1) exists, then $f(z)$ is also called ***differentiable*** at z.

In the equation (1), sometimes we will use h instead of Δz.

135. Differentiable at a point :

D. U. H. T. 77, 85 ; D. U. H. 89, 90, 91 ; R. U. H. 86 ; J. U. H. 87 ; R. U. M. SC. P. 84 ; C. U. H. 83.

A single-valued function $f(z)$ is said to be differentiable at a point $z = z_0$ of the Argand plane

if the limit $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$ ... (2) exists.

136. $\epsilon - \delta$ definition of the derivative at a point ;

The single-valued function $f(z)$ has a derivative $f'(z_0)$ at $z = z_0$ of the Argand plane if given any $\epsilon > 0$, we can find $\delta > 0$ such that $\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - f'(z_0) \right| < \epsilon$

whenever $0 < |\Delta z| < \delta$.

137. *Another definition of the derivative at a point :* The single-valued function $f(z)$ has a derivative $f'(z_0)$ at the point $z = z_0$ of the Argand plane

if the limit $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ ... (3) exists.

N. B. The equation (3) $\Rightarrow$ the equation (2) if we use $z_0 + \triangle z$ instead of z in (3).

138. *Eight fundamental theorem on differention.*

Theorems 60, 61, 62, 63, 64, 65, 66 and 67 :

If $f(z)$, $g(z)$ and $h(z)$ are analytic functions of z in a region R, then

60. $\dfrac{d}{dz}\{f(z) + g(z)\} = \dfrac{d}{dz} f(z) + \dfrac{d}{dz} g(z) = f'(z) + g'(z)$;

61. $\dfrac{d}{dz}\{f(z) - g(z)\} = \dfrac{d}{dz} f(z) - \dfrac{d}{dz} g(z) = f'(z) - g'(z)$;

62. $\dfrac{d}{dz}\{cf(z)\} = c\dfrac{d}{dz} f(z) = c f'(z)$. where c is any constant ;

63. $\dfrac{d}{dz}\{f(z)\, g(z)\} = f(z) \dfrac{d}{dz} g(z) + g(z) \dfrac{d}{dz} f(z)$

$= f(z)\, g'(z) + g(z) f'(z)$;

64. $\dfrac{d}{dz}\left\{\dfrac{f(z)}{g(z)}\right\} = \dfrac{g(z)\dfrac{d}{dz} f(z) - f(z)\dfrac{d}{dz} g(z)}{\{g(z)\}^2}$

$= \dfrac{g(z)\, f'(z) - f(z)\, g'(z)}{\{g(z)\}^2}$ if $g(z) \neq 0$;

65. If $w = f(\xi)$ and $\xi = g(z)$, then $\dfrac{dw}{dz} = \dfrac{dw}{d\zeta} \cdot \dfrac{dz}{d\zeta}$

$= f'\{g(z)\}\, g'(z)$ which is called the chain rules for differentiation of composite functions ;

66. If $w=f(z)$, then $\dfrac{dw}{dz}=\dfrac{1}{dz/dw}$;

67. If $w=f(t)$ and $z=g(t)$, then $\dfrac{dw}{dz}=\dfrac{dw/dt}{dz/dt}=\dfrac{f'(t)}{g'(t)}$ where is a parameter.

Proof : The proofs are identical with those of elementary calculus and are not given here.

139. ***Differential coefficients of elementary functions*** ;

The following differential coefficients are identical with those of elementary calculus :

(*i*) $\dfrac{d}{dz}(\text{const})=0$; (*ii*) $\dfrac{d}{dz}z^n=nz^{n-1}$;

(*iii*) $\dfrac{d}{dz}e^z=e^z$; (*iv*) $\dfrac{d}{dz}a^z=a^z\log a$;

(*v*) $\dfrac{d}{dz}\log z=\dfrac{1}{z}$; (*vi*) $\dfrac{d}{dz}\log_a z=\dfrac{1}{z}\log_a e$;

(*vii*) $\dfrac{d}{dz}\sin z=\cos z$; (*viii*) $\dfrac{d}{dz}\cos z=-\sin z$;

(*ix*) $\dfrac{d}{dz}\tan z=\sec^2 z$; (x) $\dfrac{d}{dz}\sec z=\sec z\tan z$;

(*xi*) $\dfrac{d}{dz}\operatorname{cosec} z=-\operatorname{cosec} z\cot z$;

(*xii*) $\dfrac{d}{dz}\cot z=-\operatorname{cosec}^2 z$;

(*xiii*) $\dfrac{d}{dz}\sin^{-1}z=\dfrac{1}{\sqrt{1-z^2}}$; (*xiv*) $\dfrac{d}{dz}\cos^{-1}z=\dfrac{-1}{\sqrt{1-z^2}}$

(*xv*) $\dfrac{d}{dz}\tan^{-1}z=\dfrac{1}{1+z^2}$ (*xvi*) $\dfrac{d}{dz}\cot^{-1}z=\dfrac{-1}{1+z^2}$

(xvii) $\frac{d}{dz}\sec^{-1}z=\frac{1}{z\sqrt{z^2-1}}$;

(xviii) $\frac{d}{dz}\operatorname{cosec}^{-1}z=\frac{-1}{z\sqrt{z^2-1}}$; (xix) $\frac{d}{dz}\sinh z=\cosh z$;

(xx) $\frac{d}{dz}\cosh z=\sinh z$; (xxi) $\frac{d}{dz}\tanh z=\operatorname{sech}^2 z$;

(xxii) $\frac{d}{dz}\operatorname{sech} z=-\operatorname{sech} z\tanh z$;

(xxiii) $\frac{d}{dz}\operatorname{cosech} z=-\operatorname{cosech} z\coth z$;

(xxiv) $\frac{d}{dz}\coth z=-\operatorname{cosech}^2 z$;

(xxv) $\frac{d}{dz}\sinh^{-1}z=\frac{1}{\sqrt{1+z^2}}$;

(xxvi) $\frac{d}{dz}\cosh^{-1}z=\frac{-1}{\sqrt{z^2-1}}$;

(xxvii) $\frac{d}{dz}\tanh^{-1}z=\frac{1}{1-z^2}$;

(xxviii) $\frac{d}{dz}\coth^{-1}=\frac{1}{1-z^2}$;

(xxix) $\frac{d}{dz}\operatorname{sech}^{-1}z=\frac{-1}{z\sqrt{1-z^2}}$;

(xxx) $\frac{d}{dz}\operatorname{cosech}^{-1}z=\frac{-1}{z\sqrt{z^2+1}}$.

140. Higher order derivatives :

If $w=f(z)$ is an analytic function in a region R, then its derivative is denoted by $f'(z)$ or w' or $\frac{dw}{dz}$.

Again, if $f'(z)$ is an analytic function in a region R, then its derivative is denoted by $f''(z)$ or w'' or $\frac{d^2w}{dz^2}$. Similarly, if $f^{(n-1)}(z)$ is an analytic function in a region R, then its derivative is denoted by $f^{(n)}(z)$ or $w^{(n)}$ or $\frac{d^nw}{dz^n}$, where n is called the order of the derivative. Here $f'(z)$ is called the derivative of the first order, $f''(z)$, the derivative of the second order ... ..., $f^{(n)}(z)$, the derivative of nth the order.

141. *Theorem 68 :* If $f(z)$ is analytic in a region R, then $f'(z)$, $f''(z)$, $f'''(z)$, ..., $f^{(n)}(z)$ are also analytic in R.

Proof : The proof of this theorem is given in chapter 5.

142. *Analytic (or regular or holomorphic) function :*

C. U. H. T. 77, 85, 88, 91 ; C. U. H. 87, 88 ; R. U. H. 73, 75, 77, 82, 85 ; D.U. H. S. 85 ; R. U. M. SC. P. 84, 85, 86, 88; D. U. M. SC. P. T. 91 ; R. U. 80 ; D. U. M. SC. P. 78 ; C. U. H. 85, 89.

A single-valued function $f(z)$ is said to be analytic in a region R if the derivative $f'(z)$ exists at all points z of the region R.

143. *Analytic at a point :*

D. U. H. T. 85 ; R. U. M. SC. P. 85 ; D. U. H. 89, 91 ; J. U. H. 86, 87, 91.

A single-valued function $f(z)$ is said to be analytic at a point z_0 only if there is a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

144. *Analytic along a curve.* A single-valued function $f(z)$ is said to be analytic along a curve C only if $f(z)$ is analytic in a region R containing C.

145. *Entire or integral function :*

A function $f(z)$ is said to be entire function if $f(z)$ is analytic everywhere in the finite Argand plane i. e. $f(z)$ is analytic everywhere except at the point ∞.

Example 86 : The functions $f(z)=e^z$, $g(z)=\cos z$ and $h(z)=\sin z$ etc, are entire functions.

Example 87 : Show that the function $f(z)=|z|^2$ is continuous at every point but the derivative exists only at the point $z=0$ but nowhere else.

D. U. H. 90 ; D. U. H. T. 77, 85 ; R. U. H. 88 ; R. U.M. SC. P. 86.

Solution : We have $\lim\limits_{z \to z_0} f(z)=\lim\limits_{z \to z_0} |z|^2=|z_0|^2$

$=f(z_0) \Rightarrow f(z)$ is continuous at $z=z_0$. But here z_0 is any arbitrary point and therefore the given function is continuous at every point. Again we have $\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\dfrac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}$

$$=\frac{(z_0+\Delta z)\left(\bar{z}_0+\overline{\Delta z}\right)-z_0\bar{z}_0}{\Delta z}=\bar{z}_0+\overline{\Delta z}+z_0\frac{\overline{\Delta z}}{\Delta z} \quad \dots \quad (1)$$

Then $(1) \Rightarrow f'(z_0)=\lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

$$=\lim_{\Delta z \to 0}\left[\bar{z}_0+\overline{\Delta z}+z_0\frac{\overline{\Delta z}}{\Delta z}\right] \quad \dots \quad \dots \quad (2).$$ Now if

$z_0=0$, then (2) $\Rightarrow f'(0)=0 \Rightarrow$ the given function is differentiable at the point $z=0$. Now if $z_0 \neq 0$ and if we take the limit along the x-axis, then (2) $\Rightarrow f'(z_0)=\bar{z}_0+z_0=2x_0$ since in this direction $\Delta z=\Delta x \to 0$, $\overline{\Delta z}=\Delta x$ and $\Delta y=0$, and $z_0=x_0+iy_0$. Again if we take the limit along the y-axis, then (2) $\Rightarrow f'(z_0)=\bar{z}_0-z_0$ $=-2iy_0$ since in this direction $\Delta z=i\Delta z=\Delta y \to 0$, $\overline{\Delta z}=-i\Delta y$ and $\Delta x=0$. If $z_0 \neq 0$, then (2) does not tend to a unique limit and therefore the given function has derivative only at the point $z=0$ but nowhere else.

146. Theorem 69 : If $f(z)$ is analytic at a point z_0 then it must be continuous at z_0. Give an example to show that the converse of this theorem is not necessarily true.

D. U. H. 88 ; R. U. H. 76, 80, 88 ; R. U. M. SC. P. 88.

Proof : Since $f(z)$ is analytic at z_0, then there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists. Now we have $f(z_0+h)-f(z_0)=\dfrac{f(z_0+h)-f(z_0)}{h}\cdot h$

Then $\lim\limits_{h\to 0}\{f(z_0+h)-f(z_0)\}=\lim\limits_{h\to 0}\dfrac{f(z_0+h)-f(z_0)}{h}\cdot\lim\limits_{h\to 0} h$
$=f'(z).0=0 \Rightarrow \lim\limits_{h\to 0} f(z_0+h)=f(z_0) \Rightarrow f(z)$ is continuous at z_0.

Second part : See the following example.

Example 88 : Show that $f(z)=\bar{z}$ is continuous at z_0 but not analytic at z_0.

Solution : Let $\in>0$. Then $|f(z)-f(z_0)|<\in \Rightarrow$

$\left|\bar{z}-\bar{z_0}\right|<\in \Rightarrow \left|\,|\bar{z}|-|\bar{z_0}|\,\right| \leq \left|\bar{z}-\bar{z_0}\right|<\in \Rightarrow |z-z_0|<\in \Rightarrow$

$|z-z_0|<\delta$ where $\delta=\in$. Thus $|f(z)-f(z_0)|<\in$ whenever $|z-z_0|<\delta$ and therefore $f(z)$ is continuous at $z=z_0$.

$$\text{Again } f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{\bar{z}-\bar{z_0}}{z-z_0}$$

$$=\lim_{\substack{x\to x_0\\ y\to y_0}}\frac{\overline{x+iy}-\overline{x_0+iy_0}}{(x+iy)-(x_0+iy_0)}=\lim_{\substack{x\to x_0\\ y\to y_0}}\frac{(x-x_0)-i(y-y_0)}{(x-x_0)+i(y-y_0)}$$

$$=\begin{cases}1, \text{ parallel to the } x-\text{axis where } y=y_0 \text{ and } x\to x_0\\ -1, \text{ parallel to the } y-\text{axis where } x=x_0 \text{ and } y\to 0\end{cases}$$

Thus the limit depends on the manner in which $z\to z_0$ and therefore the derivative does not exist, *i.e.* $f(z)$ is non-analytic at $z=z_0$.

Example 89 : Show that $\frac{d}{dz}\bar{z}$ does not exist any where.

R. U. 80.

Solution : Try yourself.

Example 90 : Show that $f(z)=\bar{z}$ is continuous at $z=0$ but not differentiable at $z=0$.

Solution : Let $\in>0$. Then $|f(z)-f(0)|<\in \Rightarrow$

$|\bar{z}-0|<\in \Rightarrow |\bar{z}|<\in \Rightarrow |z|<\in \Rightarrow |z-0|<\in \Rightarrow |z-0|$ $<\delta$ where $\delta=\in$. Thus $|f(z)-f(z_0)|<\in$ whenever $|z-0|<\delta$ and $f(z)$ is continuous at $z=0$.

Again, we have $f'(0)=\lim\limits_{z\to 0}\dfrac{f(z)-f(0)}{z-0}=\lim\limits_{z-0}\dfrac{\bar{z}}{z}$

$$=\begin{cases} 1 \text{ along the } x-axis \text{ where } y=0 \text{ and } x\to 0 \\ -1 \text{ along the } y-\text{axis where } x=0 \text{ and } y\to 0 \end{cases}$$

Here $f'(0)$ does not approaches the same answer and therefore $f'(0)$ does not exist.

147. Theorem 70 : Show that if a function $f(z)$ is analytic in a domain, then it is continuous in that domain. Give an example to show that the converse of this theorem is not necessarily true. ***R. U. H. 73, 85 ; R. U. M. SC. P. 84.***

Proof : See the above theorem 69.

148. Theorem 71 : Show that if a fnnction is differentiable at a point is continuous there. Show that the converse of this theorem is not necessarily true. ***R. U. H. 88.***

Proof : See the above theorem 69.

149. Cauchy-Riemann equations :

Necessary and sufficient conditions for the function $f(z)=u(x,y)+iv(x,y)$ to be analytic.

D. U. H. 83, 86, 89, 90 ; J. U. H. 85, 87 ; D. U. H. 86 ; D. U. 83 ; C. U. H. 82.

(i) Necessary condition for f(z) to be analytic (Thaorem 72) : D. U. H. T. 84, 87 ; R. U. M. SC, P. 84, 85 ; R. U. H. 73, 77, 82, 90 ; D. U. H 87, 91 ; R. U. 78 ; D. U. M. SC. P. 89 ; C. U. H. 86, 87, 88, 90.

If $f(z)=u(x,y)+iv(x,y)$ is analytic in a region R, then it is necessary that the four partial derivatives $u_x = \frac{\delta u}{\delta x}$, $v_x = \frac{\delta v}{\delta x}$, $u_y = \frac{\delta u}{\delta y}$, $v_y = \frac{\delta v}{\delta y}$ should exist and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$.

Proof : If $f(z) = u(x, y) + iv(x, y)$ is analytic, then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y} \quad \dots \dots \dots (1)$$

must exists and is unique. In this case it is independent of the manner in which $\Delta z \to 0$ or $\Delta x \to 0$ and $\Delta y \to 0$. At z :

(a) Let $\Delta y = 0$ and $\Delta x \to 0$ $\qquad$ $\Delta y = 0$ and $\Delta x \to 0$.

Then $(1) \Rightarrow f\ (z) = \lim_{\Delta x \to 0} \left\{ \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right\}$

which implies that u_x and v_x must exist and the limit becomes $f'(z) = u_x + iv_x \quad \dots \quad (2)$.

(b) Let $\Delta y = 0$ and $\Delta y \to 0$ $\qquad$ $\Delta x = 0$ and $\Delta y \to 0$

Then $(1) \Rightarrow f'(z) = \text{lime}_{\Delta y \to 0} \left\{ \frac{n(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right\}$

which imPlies that u_y and v_y must exist and the limit becomes $f'(z) = -iu_y + v_y \quad \dots \quad (3)$

Since f(z) is analytic, then (2) and (3) are identical and we have $u_x + iv_x = -iu_y + v_y \Rightarrow u_x = v_y$, $u_y = -v_x$.

Historical Note :

The equations $u_x = v_y$ and $u_y = -v_x$ are called the ***Cauchy-Riemann differential equations*** or simply the ***Cauchy-Riemann equations***, after the name of the two mathematicians, the French mathematician A. L. Cauchy (1789–1857) and the German mathematician G. F. B. Riemann (1826–1866).

(ii) *Sufficient condition for* f(z) *to be analytic* (*Theorem 73*) : *R. U. M. SC. P. 86, 88 ; R. U. H. 75 ; J U.H. 86, 88, 86 . R. U 80 ; C. U. H. 85, 89.*

The continuous single-valued function $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R if the four partial derivatives u_x, v_x, u_y, v_y exist, are continuous and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ in R.

Proof : Since u_x and u_y are continuous, then we have

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x, y)$$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x, y)\}$$

$$= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y \quad \left[\text{by the mean value theorem}\right]$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y \qquad (1) \quad \text{where } \epsilon_1 \to 0$$

and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Similarly, since v_x and v_y are continuons then we have

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial v}\Delta y + \epsilon_2 \Delta x + \eta_2 \Delta y \quad \cdots \quad (2)$$

where $\epsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Now adding (1) and (2) $\Rightarrow$ $\Delta w = \Delta u + i\Delta v$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \quad \cdots \quad (3)$$

where $\epsilon = \epsilon_1 + i\epsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ aud $\Delta y \to 0$

Now by the Cauchy-Riemann equations (3) $\Rightarrow$

$$\Delta w=\left(\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}\right)\Delta x+\left(-\frac{\delta v}{\delta x}+i\frac{\delta u}{\delta x}\right)\Delta y+\epsilon\Delta x+\eta\Delta y$$

$$=\left(\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}\right)\Delta x+i\left(\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}\right)\Delta y+\epsilon\Delta x+\eta\Delta y$$

$$=\left(\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}\right)(\Delta x+i\Delta y)+\epsilon\Delta x+\eta\Delta y$$

$$=\left(\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}\right)\Delta z+\epsilon\Delta x+\eta\Delta y \quad \cdots \quad (4)$$

Now dividing (4) by Δz and taking the limit as $\Delta z\to 0$, we have $\frac{dw}{dz}=f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}$ so that the derivative exists and is unique and therefore f(z) is analytic in R.

150. ***Theorem 74*** ; If f(z) is analytic in a region R, then $\frac{dw}{dz}=\frac{\delta w}{\delta x}=-i\frac{\delta w}{\delta y}$.

Proof : By the above theorem, we have $f'(z)=\frac{dw}{dz}$

$$=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}=\frac{\delta}{\delta x}(u+iv)=\frac{\delta w}{\delta x}.$$

Again. by the Cauchy-Riemann equations, we have $\frac{dw}{dz}$

$$=\frac{\delta v}{\delta y}-i\frac{\delta u}{\delta y}=-i\left(\frac{\delta u}{\delta y}+i\frac{\delta v}{\delta y}\right)=-i\frac{\delta}{\delta y}(u+iv)$$

$=-i\frac{\delta w}{\delta y}$. Thus we have $\frac{dw}{dz}=\frac{\delta w}{\delta x}=-i\frac{\delta w}{\delta y}$.

N, B, In some books, the equation $\frac{\delta w}{\delta x}=-i\frac{\delta w}{\delta y}$ is known as the Cauchy-Riemann equation in combine form.

Example 91 : Show that if a complex function $f(z)=u(x, y)+iv(x, y)$ $(z=x+iy)$, is differentiable at $z_0=x_0+iy_0$, then $f'(z_0)=u_x(x_0, y_0)+iv(x_0, y_0)$.

D U. H. 85.

Soultion : Since $f(z)$ is differentiable at $z_0=x_0+iy_0$ then $f'(z_0)$ exits and independent of the manner in which $z\to z_0$ and we have $f'(z)=\lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$

$$=\lim_{\substack{x\to x_0\\ y\to y_0}}\frac{u(x, y)+iv(x, y)-\{u(x_0, y_0)+iv(x_0, y_0)\}}{x+iy-(x_0+iy_0)} \quad \ldots \quad (1)$$

Now taking the limit through the point (x_0, y_0) and parallel to the x-axis, we have $y=y_0$ and $x\to x_0$. Then $(1)\Rightarrow$

$$f'(z_0)=\lim_{x\to x_0}\frac{u(x, y_0)+iv(x, y_0)-u(x_0, y_0)-iv(x_0, y_0)}{x-x_0}$$

$$=\lim_{x\to x_0}\frac{u(x, y_0)-u(x_0, y_0)+i\{v(x_0, y_0)-v(x_0, y_0)\}}{x_0-x_0}$$

$$=\lim_{x\to x_0}\frac{u(x, y_0)-u(x_0, y_0)}{x-x_0}+i\lim_{x\to x_0}\frac{v(x, y_0)-v(x_0, y_0)}{x-x_0}$$

$$=\frac{\delta u(x_0, y_0)}{\delta x}+i\frac{\delta v(x_0, y_0)}{\delta y}=u_x(x_0, y_0)+iv_x(x_0, y_0).$$

151. ***Theorem 75 :*** Show that an analytic function of z is necessarily independent of $\bar{z}$.

Proof : Let $w=f(z)=u(x, y)+i\,v(x, y)=F(z, \bar{z})$ where $x=\dfrac{z+\bar{z}}{2}$ and $y=\dfrac{z-\bar{z}}{2i}$.

Then $dF = \frac{\delta F}{\delta z} dz + \frac{\delta F}{\delta \bar{z}} d\bar{z} \Rightarrow \frac{\delta F}{\delta z} = \frac{\delta F}{\delta z} + \frac{\delta F}{\delta \bar{z}} \frac{d\bar{z}}{dz}$ ... (1).

But $\frac{d\bar{z}}{dz} = \begin{cases} 1 & \text{along the x-axis} \\ -1 & \text{along the y-axis} \end{cases}$ and is therefore

$\frac{d\bar{z}}{dz}$ does not exist. Therefore, in (1) $\frac{\delta F}{\delta z}$ exists only if

$\frac{\delta F}{\delta \bar{z}} = 0$, *i.e.* if it is independent of $\bar{z}$

152. Theorem 76 : If $f(z) = u(x, y) + iv(x, y)$ is analytic, then show that $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ using the condition

$\frac{\delta f}{\delta \bar{z}} = 0$.

Proof : Since $f(z)$ is independent of $\bar{z}$, then we have

$$\frac{\delta f}{\delta \bar{z}} = 0 \Rightarrow \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta \bar{z}} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta \bar{z}} = 0$$

$$\Rightarrow \frac{1}{2} \frac{\delta f}{\delta x} + \frac{i}{2} \frac{\delta f}{\delta y} = 0 \Rightarrow \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) + i \left(\frac{\delta u}{\delta y} + i \frac{\delta v}{\delta y} \right) = 0$$

$$\Rightarrow \left(\frac{\delta u}{\delta x} - \frac{\delta v}{\delta y} \right) + i \left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y} \right) = 0 \Rightarrow \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \text{ and}$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

153 Cauchy-Riemann equations in polar form :

Theorem 77 : Show that Cauchy-Riemann equations in polar form are :

$$\frac{\delta u}{\delta r}=\frac{1}{r}\frac{\delta v}{\delta \theta},\quad \frac{\delta v}{\delta r}=-\frac{1}{r}\frac{\delta u}{\delta \theta}$$

D. U. H. T. 77 ; J. U. H. 84 ; D. U. M. SC. P. 89 ; C. U. H. 83, 88.

Proof : We have $w=u+iv$ ... (1) and $z=x+iy$

$=r(\cos\theta+i\sin\theta)=re^{i\theta}$ ... ... (2).

Let $\left(\frac{dw}{dz}\right)_\theta$ denotes the value of $\frac{dw}{dz}$ when $\theta=$ Constant

and $\left(\frac{dw}{dz}\right)_r$, the value of $\frac{dw}{dz}$ when $r=$constant.

Now if w is analytic, then $\frac{dw}{dz}=\frac{\delta w}{\delta r}\left(\frac{dr}{dz}\right)_\theta$ ... (3)

and $\frac{dw}{dz}=\frac{\delta w}{\delta \theta}\left(\frac{d\theta}{dz}\right)_r$ ... (4)

Now (2) $\Rightarrow r=z\,e^{-i\theta}\Rightarrow\left(\frac{dr}{dz}\right)_\theta=e^{-i\theta}$ ... ... (5). Then

by (3) and (5) we have $\frac{dw}{dz}=\frac{\delta w}{\delta r}\,e^{-i\theta}$ ... ... (6).

Again (2) $\Rightarrow e^{i\theta}=\frac{z}{r}\Rightarrow ie^{i\theta}\left(\frac{d\theta}{dz}\right)_r=\frac{1}{r}\Rightarrow$

$\left(\frac{d\theta}{dz}\right)_r=-\frac{i}{r}\,e^{-i\theta}$ .. ... (7). Then by (4)

and (7) we have $\frac{dw}{dz}=-\frac{i}{r}\frac{\delta w}{\delta \theta}\,e^{-i\theta}$ ... ... (8)

Now by (6) and (8) we get $\frac{\delta w}{\delta r}\,e^{-i\theta}=-\frac{i}{r}\frac{\delta w}{\delta \theta}e^{-i\theta}$

$$\Rightarrow \frac{\delta w}{\delta r}=-\frac{i}{r}\frac{\delta w}{\delta \theta} \Rightarrow \frac{\delta u}{\delta r}+i\frac{\delta v}{\delta r}=-\frac{i}{r}\left(\frac{\delta u}{\delta \theta}+i\frac{\delta v}{\delta \theta}\right) \Rightarrow$$

$\frac{\delta u}{\delta r}=\frac{1}{r}\frac{\delta v}{\delta \theta}$, $\frac{\delta v}{\delta r}=-\frac{1}{r}\frac{\delta u}{\delta \theta}$ and the theorem is proved.

154. Theorem 78 : If $f'(z)=0$ in a region R, then the function $f(z)$ must be constant in R. ***D. U. H. T. 84.***

Proof : We have $f(z)=u+iv \Rightarrow f'(z)=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}$ $=u_x+iv_x=0$ ... (1) since it is given that $f'(z)=0$. Then (1) $\Rightarrow u_x=0$ and $v_x=0$ ... (2). But by Cauchy-Riemann equations (2) $\Rightarrow u_x=v_y=0$ and $u_y=-v_x=0$.

Thus $u_x=0$, $u_y=0 \Rightarrow u(x, y)=$constant ... (3). Again $v_x=0$, $v_y=0 \Rightarrow v(x, y)=$constant ... (4). Now by (3) and (4) $f(z)=u+iv=$constant. Hence the theorem is proved.

155. Theorem 79 : Show that an analytic function with constant modulus is constant.

R. U. M. SC. P. 85 ; D. U. H. 84 ; R. U. H. 76, 82.

Proof : Let $f(z)=u+iv$ ... (1) be an analytic function with constant modulus. Then $|f(z)|=\sqrt{u^2+v^2}=c$(say) $\Rightarrow$ $u^2+v^2=c^2$ ... (2). Now differentiating (2) with respect to x and y

$$\Rightarrow u\frac{\delta u}{\delta x}+v\frac{\delta v}{\delta x}=0 \ \ldots \ (3) \text{ and } u\frac{\delta u}{\delta y}+v\frac{\delta v}{\delta y}=0 \ \ldots \ (4)$$

Now by Cauchy-Riemann equations (4) $\Rightarrow -u\frac{\delta v}{\delta x}+v\frac{\delta u}{\delta x}$ $=0$ ... (5). Now squaring and adding (3) and (5) $\Rightarrow$

$$(u^2+v^2)\left\{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta v}{\delta x}\right)^2\right\}=0 \Rightarrow c^2|f'(z)|^2=0 \quad \ldots \ldots \quad (6)$$

since $f'(z)=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}$. Hence (6) $\Rightarrow f'(z)=0$ since $c\neq 0$

$\Rightarrow f(z)=$ constant. and the theorem is proved

156. *Theorem 80* : If $f(z)$ is analytic in a region R, then $f(z)$ is constant if Re $f(z)$ is constant.

Proof : Let $f(z)=u+iv$, then Re $f(z)=u$. But we have Re $f(z)=$ constant $=c$ (say) $\Rightarrow u=c$. Then by Cachy-Riemann equations, $\frac{\delta u}{\delta x}=\frac{\delta}{\delta x}(c)=0=\frac{\delta v}{\delta y}$ ... (1) and $\frac{\delta u}{\delta y}=\frac{\delta}{\delta y}(c)$ $=0=-\frac{\delta v}{\delta x}$ ... (2). Thus $f'(z)=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}=-i\left(\frac{\delta u}{\delta y}-i\frac{\delta v}{\delta y}\right)$ $=0$ by (1) and (2). Thus $f'(z)=0\Rightarrow f(z)$ is constant. Hence the theorem is proved.

157. *Theorem 81* : If $f(z)$ is analytic in a region R, then $f(z)$ is constant if Im $f(z)$ is constant.

Proof : Let $f(z)=u+iv$, then Im $f(z)=v$. But we have Im $f(z)=$ constant $=c$ (say) $\Rightarrow v=c$. Then by Cauchy-Riemann equations, $\frac{\delta v}{\delta x}=\frac{\delta}{\delta x}(c)=0=-\frac{\delta u}{\delta y}$ ... (1) and $\frac{\delta v}{\delta u}=\frac{\delta}{\delta y}(c)$ $=0=\frac{\delta u}{\delta x}$ ... (2). But $f'(z)=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}=-i\left(\frac{\delta u}{\delta y}-i\frac{\delta v}{\delta y}\right)$ $=0$ by (1) and (2). Thus $f'(z)=0\Rightarrow f(z)$ is constant.

Hence the theorem is proved.

158. *Theorem 82* : Show that $\left(\frac{\delta\phi}{\delta x}\right)^2+\left(\frac{\delta\phi}{\delta y}\right)^2=\left\{\left(\frac{\delta\phi}{\delta u}\right)^2+\left(\frac{\delta\phi}{\delta v}\right)^2\right\}|f'(z)|^2$ if $f(z)=u+iv$ is analytic, $f'(z)\neq 0$ and ϕ

is any function of x and y having differential coefficients of first and second order.

Proof : We have $\dfrac{\delta\phi}{\delta x}=\dfrac{\delta\phi}{\delta u}\dfrac{\delta u}{\delta x}+\dfrac{\delta\phi}{\delta v}\cdot\dfrac{\delta v}{\delta x}$ ... (1)

Again $\dfrac{\delta\phi}{\delta y}=\dfrac{\delta\phi}{\delta u}\cdot\dfrac{\delta u}{\delta y}+\dfrac{\delta\phi}{\delta v}\dfrac{\delta v}{\delta y}=-\dfrac{\delta\phi}{\delta u}\dfrac{\delta v}{\delta x}+\dfrac{\delta\phi}{\delta v}\dfrac{\delta u}{\delta x}$ .. (2)

by Cauchy-Riemann equations. Now squaring and adding (1) and (2) $\Rightarrow \left(\dfrac{\delta\phi}{\delta x}\right)^2+\left(\dfrac{\delta\phi}{\delta y}\right)^2=$

$$\left[\left(\frac{\delta\phi}{\delta u}\right)^2+\left(\frac{\delta\phi}{\delta v}\right)^2\right]\left[\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta v}{\delta x}\right)^2\right]=\left[\left(\frac{\delta\phi}{\delta u}\right)^2+\left(\frac{\delta\phi}{\delta v}\right)^2\right]|f'(z)|^2,$$

since $f'(z)=\dfrac{\delta u}{\delta x}+i\dfrac{\delta v}{\delta x}$ and $|f'(z)|^2=\left(\dfrac{\delta u}{\delta x}\right)^2+\left(\dfrac{\delta v}{\delta x}\right)^2$.

159. Orthogonal system (geometrical aspect) :

Theorem 83 : . Let $f(z)=u+iv$ be an analytic function where $u(x, y)=c_1=$ constant and $v(x, y)=c_2=$ constant represent two families of curves.

Then these system of families are orthogonal.

R. U. M. SC. P. 86 ; D. U. H. T. 75 ; R. U. H. 75.

Proof : Since $f(z)=u+iv$ is analytic, then u and v satisfy the Cauchy-Riemann equations *i. e.* $u_x=v_y$ and $u_y=-v_x$ ... (1)

Now if $u=$ constant $\Rightarrow du=0 \Rightarrow u_x\,dx+u_y\,dy=0 \Rightarrow m_1=-\dfrac{u_x}{u_y}$. Again if $v=$ constant $\Rightarrow dv=0 \Rightarrow v_x\,dx+v_y\,dy=0 \Rightarrow m_2=-\dfrac{v_x}{v_y}$. Now $m_1 m_2=\left(-\dfrac{u_x}{u_y}\right)\left(-\dfrac{v_x}{v_y}\right)=-\dfrac{u_x\,u_y}{u_y\,u_x}=-1$, by (1). Thus the families are orthogonal.

160. Orthogonal system : If each member of one family is perpendicular to each member of the other family at their point of intersection, then they form an orthogonal system.

Example 92 : Show that $f(z)=|z|^2$ is not analytic at the origin, although it is differentiable there.

R. U. H. 84 ; D. U. H. T. 77, 85.

Solution : We have $f'(0)=\lim\limits_{\triangle z\to 0}\dfrac{f(0+\triangle z)-f(0)}{\triangle z}$

$$=\lim_{\triangle z\to 0}\frac{|\triangle z|^2}{\triangle z}=\lim_{\triangle z\to 0}\frac{\triangle z\,\overline{\triangle z}}{\triangle z}=\lim_{\triangle z\to 0}\overline{\triangle z}=0\Rightarrow$$

$f(z)$ is differentiable at $z=0$.

We have $f(z)=u+iv=|z|^2=|x+iy|^2=x^2+y^2$

$\Rightarrow u=x^2+y^2$ and $\Rightarrow \dfrac{\delta u}{\delta x}=2x,\ \dfrac{\delta u}{\delta y}=2y$ and $\dfrac{\delta v}{\delta x}=0,\ \dfrac{\delta v}{\delta y}=0.$

Thus the Cauchy-Riemann equations are satisfied at the point $z=0$ but not in the neighbourhood $|z-0|<\delta$.

Therefore, $f(z)$ is differentiable at $z=0$, but not analytic at $z=0$.

N. B. We have $\triangle z=\triangle x+i\triangle y$ and $\overline{\triangle z}=\triangle x-i\triangle y$.

Now if $\triangle z\to 0$, then $\triangle x\to 0$, $\triangle y\to 0$ and in this case $\overline{\triangle z}\to 0$.

Example 93 : Show that the function $f(z)=z\,\mathrm{Re}\,z$ is not analytic at the point $z=0$ but it is differentiable there.

R. U. M. SC. 84 ; J. U. H. 87.

Solution : We have $f(z)=u+iv=z\,\mathrm{Re}\,z=(x+iy)x$

$=x^2+ixy \Rightarrow u=x^2$ and $v=xy \Rightarrow \frac{\delta u}{\delta x}=2x,\ \frac{\delta u}{\delta y}=0,\ \frac{\delta v}{\delta x}$

$=y$ and $\frac{\delta v}{\delta y}=x \Rightarrow \frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} \neq -\frac{\delta v}{\delta x} \Rightarrow u$ and v donot satisfy the Cauchy-Riemann equations in the Argand plane $\Rightarrow f(z)$ is not analytic at any point of the Argand plane

i.e. not analytic at the point $z=0$ or not differentiable in the neighbourhood of $z=0$. But at the origin we have

$\frac{\delta u}{\delta x}=0=\frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y}=0=-\frac{\delta v}{\delta x} \Rightarrow u$ and v satisfy the Cauchy-Riemann equations at the point $z=0$ (*i.e.* at $x=0, y=0$).

Thus $f(z)$ is differentiable at the point $z=0$.

Example 94 : Show that $\frac{d}{dz}\left(z^2\bar{z}\right)$ does not exist any where. ***R. U. H. 86.***

Solution : We have $\frac{d}{dz}\left(z^2\bar{z}\right)=\frac{d}{dz}(z|z|^2)$

$=|z|^2\frac{d}{dz}(z)+z\frac{d}{dz}|z|^2=|z|^2+z\frac{d}{dz}|z|^2$

But $\frac{d}{dz}|z|^2$ does not exist. Therefore $\frac{d}{dz}\left(z^2\bar{z}\right)$ does not exist any where,

(***Second Method*** **:** We have $\frac{d}{dz}\left(z^2\bar{z}\right)=z^2\frac{d}{dz}\left(\bar{z}\right)$

$+\bar{z}\frac{d}{dz}(z^2)=z^2\frac{d}{dz}\left(\bar{z}\right)+2\bar{z}z.$ But $\frac{d}{dz}\left(\bar{z}\right)$ does not exist, therefore $\frac{d}{dz}\left(z^2\bar{z}\right)$ does not exist any where.

Example 95 : Verify the Cauchy-Riemann equations are satisfied for the following functions :

(*i*) $f(z) = e^{z^2}$ ***D. U.H. 88.*** and

(*ii*) $f(z) = \cos 2z$. ***D. U. H. T. 83.***

Solution (i) : We have $f(z) = u + iv = e^{(x+iy)^2}$

$= e^{x^2-y^2+2xyi} = e^{x^2-y^2}(\cos 2xy + i \sin 2xy)$.

Then $u = e^{x^2-y^2} \cos 2xy$ and $v = e^{x^2-y^2} \sin 2xy$

$$\Rightarrow \frac{\delta u}{\delta x} = e^{x^2-y^2}(2x \cos 2xy - 2y \sin 2xy) = \frac{\delta v}{\delta y} \text{ and}$$

$$\frac{\delta u}{\delta y} = -e^{x^2-y^2}(2y \cos 2xy + 2x \sin 2x) = -\frac{\delta v}{\delta x}.$$

Thus u and v satisfy the Cauchy-Riemann equations.

(*ii*) We have $f(z) = u + iv = \cos(2x + 2yi) = \cos 2x \cosh 2y - i \sin 2x \sinh 2y \Rightarrow u = \cos 2x \cosh 2y$ and $v = -\sin 2x \sinh 2y$

$$\Rightarrow \frac{\delta u}{\delta x} = -2 \sin 2x \cos h\, 2y = \frac{\delta v}{\delta y} \text{ and } \frac{\delta u}{\delta y} = 2 \cos 2x \sinh 2y$$

$= -\frac{\delta v}{\delta x}$ Thus u and v satisfy the Cauchy-Riemann equations.

Example 96 : Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although Cauchy-Riemann equations are satisfied at the origin.

Solution (First Part) : Let $f(z) = \sqrt{|xy|} = u(x, y) + i\, v(x, y) \Rightarrow u(x, y) = \sqrt{|xy|}$ ... (1) and $v(x, y) = 0$ ... (2)

Now by (1) and (2), $u(x, 0)=0$, $u(0,y)=0$, $v(x, 0)=0$ and $v(0, y)=0$. Also $u(0, 0)=0$ and $v(0, 0)=0$. Thus at the origin,

$$\frac{\delta u}{\delta x}=\lim_{x\to 0}\frac{u(x, 0)-u(0, 0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0,\ \frac{\delta u}{\delta y}$$

$$=\lim_{y\to 0}\frac{u(0, y)-u(0, 0)}{y-0}=\lim_{y\to 0}\frac{0}{y}=0,\ \frac{\delta v}{\delta x}=\lim_{x\to 0}\frac{v(x, 0)-v(0, 0)}{x-0}$$

$$=\lim_{x\to 0}\frac{0}{x}=0 \text{ and } \frac{\delta v}{\delta y}=\lim_{y\to 0}\frac{v(0, y)-v(0, 0)}{y-0}=\lim_{y\to 0}\frac{0}{y}=0.$$

Thus at the origin $\frac{\delta u}{\delta x}=\frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y}=-\frac{\delta v}{\delta x}$ and the Cauchy-Riemann equations are satisfied at the origin.

***Second Part*:** Again we have $f(0)=0$, then $f'(0)$

$$=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{|xy|}}{x+iy}$$ Now along the line

$y=mx$, we have $f'(0)=\lim_{x\to 0}\frac{\sqrt{|mx^2|}}{(1+im)x}=\frac{\sqrt{|m|}}{1+im}$ which is not unique since it depends on the value of m. Thus $f'(0)$ does not exist at the origin i. e. $f(z)$ is not anlytic at the origin.

***Example 97*:** Show that the function $f(z)=u+iv$

$=\frac{(1+i)x^3-(1-i)y^3}{x^2+y^2}$ if $z\neq 0$ and $f(0)=0$ if $z=0$ is continuous and that the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. ***D. U. H. 87, 91 ; D. U. H. T. 75 ; J. U. H. 89, 90 ; C. U. H. 82, 87.***

***Solution (First part)*:** If $z\neq 0$, then we have $f(z)=u+iv$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2} \Rightarrow u(x, y) = \frac{x^3 - y^3}{x^2+y^2} \quad \ldots \ (1) \text{ and}$$

$$v(x, y) = \frac{x^3+y^3}{x^2+y^2} \quad \ldots \ (2).$$ Here it is clear that u and v are rational functions of x and y and they are finite for all values of $z \neq 0$. Therefore, they are continuous for all values of $z \neq 0$.

Now we will show that u and v are continuous at $z=0$ *i. e.* at $x=0$ and $y=0$. Now $\lim\limits_{z \to 0} f(z)$ exists if both $\lim\limits_{\substack{x \to 0 \\ y \to 0}} u(x, y)$ and $\lim\limits_{\substack{x \to 0 \\ y \to 0}} v(x, y)$ are exist.

Now along the line $y = mx$ we have $\lim\limits_{\substack{x \to 0 \\ y \to 0}} u(x, y)$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y^3}{x^2+y^2} = \lim_{x \to 0} \frac{x^3 - (mx)^3}{x^2+(mx)^2} = \lim_{x \to 0} x\left(\frac{1-m^3}{1+m^2}\right) = 0 \text{ and}$$

the limit exists since it is unique *i. e.* it does not depend on m.

Again along the line $y = mx$ we have $\lim\limits_{\substack{x \to 0 \\ y \to 0}} v(x, y)$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 + y^3}{x^2+y^2} = \lim_{x \to 0} \frac{x^3 + (mx)^3}{x^2+(mx)^2} = \lim_{x \to 0} \frac{x(1+m^3)}{1+m^2} = 0 \text{ and the}$$

limit exists since it is unique *i. e.* it does not depend on m.

Thus $\lim\limits_{z \to 0} f(z) = \lim\limits_{\substack{x \to 0 \\ y \to 0}} [u(x, y) + iv(x, y)] = \lim\limits_{\substack{x \to 0 \\ y \to 0}} u(x, y)$

$$+i \lim_{\substack{x \to 0 \\ y \to 0}} v(x, y) = 0 + i0 = 0 \Rightarrow \lim_{z \to 0} f(z) \text{ exists and also}$$

$$\lim_{z \to 0} f(z) = f(0) \Rightarrow f(z) \text{ is continuous at } z = 0.$$

Thus $f(z)$ is continuous for every value of z.

Second part : Now by (1) and (2), we have $u(x, 0) = x$, $v(x, 0) = x$, $u(0, y) = -y$ and $v(0, y) = y$.

Again at the origin we have $f(0) = 0 = u(0, 0) + i\, v(0, 0)$

$\Rightarrow u(0, 0) = 0$ and $v(0, 0) = 0$. Thus at the origin $\frac{\delta u}{\delta x}$

$$= \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x \to 0} \frac{x}{x} = 1, \frac{\delta u}{\delta y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y - 0}$$

$$= \lim_{y \to 0} \frac{-y}{y} = -1, \quad \frac{\delta v}{\delta x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x - 0} = \lim_{x \to 0} \frac{x}{x}$$

$$= 1 \text{ and } \frac{\delta v}{\delta y} = \lim_{y \to 0} \frac{v(0, y) = v(0, 0)}{y - 0} = \lim_{y \to 0} \frac{y}{y} = 1.$$

Thus at the origin $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ and the Cauchy-Reimann equations are satisfed at the origin.

Third Part : Again we have $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y^3 + i\,(x^3 + y^3)}{(x^2 + y^2)\,(x + i\,y)}$$ Now along the x-axis we have

$y = 0$ and $x \to 0$, then $f'(0) = \lim_{x \to 0} \frac{(1+i)x^3}{x^3} = 1 + i$, Again, along the line $y = x$, we have $f'(0) = \lim_{x \to 0} \frac{2i\,x^3}{(2x^2)(1+i)x} = \frac{i}{1+i} = \frac{1}{2}(1-i)$.

Thus $f'(0)$ does not exists since it depends on the manner in which $z \to 0$, *i. e.* the two approaches donot give the same answer.

Example 98 : Show that $f(z) = u + iv = x^2 + iy^3$ is not analytic anywhere but the Cauchy-Riemann equations are satisfied at the origin only.

Solution : We have $u + iv = x^2 + iy^3 \Rightarrow u = x^2$ and $v = y^3 \Rightarrow$

$$\frac{\delta u}{\delta x} = 2x,\ \frac{\partial u}{\delta} = 0, \frac{\delta v}{\delta x} = 0, \frac{\delta v}{\partial y} = 3\,y^2.$$

Here $\frac{\partial u}{\delta x} \neq \frac{\delta v}{\delta y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\delta v}{\delta x} \Rightarrow f(z) = u + iv$ is not analytic anywhere in any region. But if $x = 0$, $y = 0$, then $\frac{\delta u}{\delta x} = 0 = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\partial y} = 0 = -\frac{\delta v}{\delta x}$ and the Cauchy-Riemann equations are satisfied at the origin only.

Example 99 : The ·unction defined by the following is analytic for all finite values of z, except at $z = 0$;

$f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$. ***D. U. M. SC. P. 88.***

Solution : Now at the point $z = 0$, the Cauchy-Riemann equations are satisfied ; for at $z = 0$, we have $\frac{\delta u}{\delta x}$

$$= \lim_{x \to 0} \frac{e^{-x^{-4}}}{x} = 0,\ \frac{\delta v}{\delta x} = \lim_{x \to 0} \frac{0}{x} = 0,\ \frac{\delta u}{\partial y} = \lim_{y \to 0} \frac{e^{-y^{-4}}}{y} = 0,$$

$\frac{\delta v}{\partial y} = \lim_{y \to 0} \frac{0}{y} = 0$, Thus we have $\frac{\partial u}{\delta x} = \frac{\delta v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\delta v}{\delta x}$.

Now at $z = re^{\pi i/4}$, we have $f(z) = e^{-\left(re^{\pi i/4}\right)^{-4}} = e^{r^{-4}}$ which tends to infinity as $r \to 0$ and $f(z)$ is not analytic at $z=0$ since $f'(z)$ does not tends to a unique limit in the neighbour hood of $z=0$.

Example 100 : The function $f(z)$ is defined by the following is not analytic :

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4} \quad (z \neq 0) \text{ and } f(0)=0.$$

Solution : If is clear that $\lim\limits_{z \to 0} \dfrac{f(z)-f(0)}{z-0} = 0$ along any straight line. But on the curve $x=y^2$, $\dfrac{f(z)-f(0)}{z-0}$

$$= \frac{y^4}{y^4+y^4} = \frac{1}{2}.$$ Hence $f(z)$ is not analytic ar $z=0$.

Example 101: Show that $f(z)=z^3$ is analytic in the entire complex plane. ***R. U. M.SC. P. 86.***

Solution ; We have $f(z)=u+iv=(x+iy)^3$

$= x^3-3xy^2+i(3x^2y-y^3) \Rightarrow u = x^3-3xy^2$ and $v = 3x^2y-y^3$.

Thus $\dfrac{\delta u}{\delta x} = 3x^2-3y^2 = \dfrac{\delta v}{\delta y}$ and $\dfrac{\delta u}{\delta y} = -6xy = -\dfrac{\delta v}{\delta x}$

Thus the four first partial derivatives are continuous and also satisfy the Cauchy-Riemann equations at every point in the entire complex plane. Again $f'(z) = \dfrac{\delta u}{\delta x} + i\dfrac{\delta v}{\delta x} = 3x^2 - 3y^2$ $+ 6xyi = 3(x+iy)^2 = 3z^2$ exists in the entire complex plane.

Hence $f(z)$ is analytic in the entire complex plane

Example 102 : Is the function $2x-3y+i(3x+2y)$ is analytic ? justify by your answer. ***D. U. H. T. 88.***

Solution : Let $f(z)=u+iv=2x-3y+i(3x+2y)$.

Then $u=2x-3y$ and $v=3x+2y \Rightarrow \frac{\partial u}{\partial x}=2, \frac{\partial u}{\partial y}=-3,$

and $\frac{\partial v}{\partial x}=3, \frac{\partial v}{\partial y}=2 \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \Rightarrow$

u and v satisfy the Cauchy-Riemann equations. Here the four partial derivatives are constants and therefore they are continuous. Hence the given function is analytic.

Example 103 : Show that the following two functions are not analytic.

(*i*) $w=f(z)=\text{Re } z$. ***D. U. H. T. 84*** : (*ii*) $W=f(z)=\text{Im } z$.

Solution : (*i*) We have $w=f(z)=u+iv=\text{Re } z=x$

$\Rightarrow u=x$ and $v=0 \Rightarrow \frac{\partial u}{\partial x}=1, \frac{\partial u}{\partial y}=0$ and $\frac{\partial v}{\partial x}=0, \frac{\partial v}{\partial y}=0$

$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \Rightarrow$ the given function is not analytic in any domain.

(*ii*) We have $w=f(z)=u+iv=\text{Im } z=y \Rightarrow u=0$ and $v=y$

$\Rightarrow \frac{\partial u}{\partial x}=0, \frac{\partial u}{\partial y}=0$ and $\frac{\partial v}{\partial x}=0, \frac{\partial v}{\partial y}=1 \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$ $=-\frac{\partial v}{\partial x} \Rightarrow$ the given function is not analytic in any domain.

Example 104 : If $f(z)$ is analytic in a region R, then show that $f(z)$ is constant if $\arg f(z)$ is constant.

Solution . Try yourself.

Example 105 : Show that in polar coordinates (r, θ), $\frac{dw}{dz} = e^{-i\theta}\frac{\delta w}{\delta r}$ if $w = f(z)$ is analytic.

Solution : Try yourself.

Example 106 : Test the nature of the function $f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}$ if $z \neq 0$ and $f(0) = 0$ in the region including the origin.

Solution : Try yourself.

161. Laplace's equation in two dimensions :

In two dimensions a second order partial differential equation $\frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} = 0$ or $\nabla^2\phi = 0$ is called Laplace's equation. The vector operator ∇^2 is called the ***Laplacian*** where $\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$

162. Harmonic function :

D. U. H. 86, 87, 89, 91 ; D. U. H. T. 83, 88, 90, 91 ; D. U. H. S. 85 ; J. U. H. 87, 91 ; D, U. M. SC. P. T. 91 ; C. U. M. SC. P. 78, 89 ; C. U. H. 90

A real-valued function $\phi(x, y)$ is called a harmonic function or simply harmonic in a region R if all its second-order partial derivatives are continuous in R and at each point of the region it satisfies the Laplace's equation $\frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} = 0$.

163. Conjugate harmonic function :

D. U. H. 86, 88 ; D. U, H. T. 90, 91 ; J. U. H. 88, 91 ; D. U. M. SC. P. 89.

If $f(z) = u + iv$ is an analytic function, then u and v are called conjugate harmonic functions or harmonic conjugate functions or simply conjugate functions.

N. B. Here v is called the conjugate harmonic function of u and u is called the conjugate harmonic function of $-v$.

164. *Theorem 84* : If $f(z)=u+iv$ is analytic in a region R and if u and v have continuous second order partial derivatives in R, then u and v are harmonic in R.

D. U. H. 86, 89, 91 ; R. U. H. 85 ; J. U. H. 86, 87 ;

D. U. M. SC. P. T. 91.

Proof : Since $f(z)=u+iv$ is analytic in R, then the Cauchy-Riemann eqnations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ ... (1) and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ... (2) are satisfied in R. If the second partial derivatives of u and v with respect to x and y are continuous, then $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$ ... (3)

and $\frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x}$ ... (4). Now by (1) and (2), (4)

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \Rightarrow$

$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow$ u is harmonic.

Again by (1) and (2), (3) $\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \Rightarrow -\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial y^2}$

$\Rightarrow \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

$\Rightarrow$ v is harmonic. Thus the theorem is proved.

165. *Laplace's equation in polar form:*

Theorem 85: Show that the Laplace's equation in polar form is: $\dfrac{\partial^2\psi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2}=0.$

D. U. M. SC. P. 89; J. **U. H. 91.**

proof: The Cauchy-Riemann equations in polar form are:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta} \text{ or } \frac{\partial v}{\partial\theta}=r\frac{\partial u}{\partial r} \quad \ldots \quad \ldots \quad (1) \text{ and}$$

$$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta} \text{ or } \frac{\partial u}{\partial\theta}=-r\frac{\partial v}{\partial r} \quad \ldots \quad \ldots \quad (2)$$

If the second partial derivatives of u and v with respect to r and θ are continuous, then

$$\frac{\partial^2 u}{\partial r\partial\theta}=\frac{\partial^2 u}{\partial\theta\partial r} \quad \ldots \quad \ldots \quad (3) \text{ and } \frac{\partial^2 v}{\partial r\partial\theta}=\frac{\partial^2 v}{\partial\theta\partial r} \quad \ldots \quad \ldots \quad (4)$$

Now, by (1) and (2), (4) $\Rightarrow \dfrac{\partial}{\partial r}\left(\dfrac{\partial v}{\partial\theta}\right)=\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial v}{\partial r}\right)$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial u}{\partial\theta}\right) \Rightarrow r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}=-\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 \quad \ldots \quad (5).$$

Again by (1) and (2), (3) $\Rightarrow \dfrac{\partial}{\partial r}\left(\dfrac{\partial u}{\partial\theta}\right)=\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial u}{\partial r}\right)$

$$\Rightarrow \frac{\partial}{\partial r}\left(-r\frac{\partial v}{\partial r}\right)=\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right) \Rightarrow -r\frac{\partial^2 v}{\partial r^2}-\frac{\partial v}{\partial r}=\frac{1}{r}\frac{\partial^2 v}{\partial\theta^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}=0 \quad \ldots \quad (6).$$ Thus (5) and (6)

$\Rightarrow \dfrac{\partial^2\psi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2}=0$, which is called Laplace's equation in polar form.

166. *Theorem 86*: If the function $u(x, y)$ is harmonic in a region R, then there exists another harmonic function $v(x, y)$ so that $f(z)=u+iv$ is analytic.

***Proof*:** Try yourself.

167. *Theorem 87*. If u is harmonic, then c u is harmonic where c is a constant.

***Proof*;** Since u is harmonic, then $\nabla^2 u=0$.

Now $\nabla^2(cu)=c\nabla^2 u=c.\ 0=0 \Rightarrow cu$ is harmonic.

168. *Theorem 88*: If u and v are harmonic, then $u+v$ and $u-v$ are harmonic.

***Proof*;** Since u and v are harmonic, then $\nabla^2 u=0$ and $\nabla^2 v=0$

Now $\nabla^2(u \pm v)=\nabla^2 u \pm \nabla^2 v=0 \Rightarrow u+v$ and $u-v$ are harmonic.

169. *Theorem 89*; If $u_1, u_2, \cdots, u_n$ are harmonic, then $c_1 u_1+c_2 u_2+\cdots+c_n u_n$ is harmonic where $c_1, c_2, \cdots, c_n$ are constants.

***Proof*:** Since $u_1, u_2, \cdots\cdots, u_n$ are harmonic, then $\nabla^2 u_1=0$, $\nabla^2 u_2=0, \cdots, \nabla^2 u_n=0$. Now we have $\nabla^2(c_1 u_1+c_2 u_2+\cdots+c_n u_n)=c_1 \nabla^2 u_1+c_2 \nabla^2 u_2+\cdots+c_n \nabla^2 u_n=c_1.\ 0+c_2.\ 0+\cdots+c_n.\ 0=0 \Rightarrow c_1 u_1+c_2 u_2+\cdots+c_n u_n$ is harmonic.

***Example 107*:** Show that $u=ax^3-bx^2y-cxy^2+dy^3$ is harmonic if $c=3a$ and $b=3d$.

***Solution*:** We have $u=ax^3-bx^2y-cxy^2+dy^3$.

Thus $\dfrac{\partial u}{\partial x}=3ax^2-2bxy-cy^2, \dfrac{\partial^2 u}{\partial x^2}=6ax-2by \quad \cdots \quad \cdots \quad (1)$

$\dfrac{\partial u}{\partial y}=-bx^2-2cxy+3dy^2, \dfrac{\partial^2 u}{\partial y^2}=-2cx+6yd \quad \cdots \quad \cdots \quad (2)$

Now adding (1) and (2) $\Rightarrow \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=2(3a-c)x+2(3d-b)y=0$ if $c=3a$ and $b=3d$. Thus u is harmonic, under the given condition.

Example 108 : Find the harmonic conjugate of the function $u = x^3 + 6x^2y - 3xy^2 - 2y^3$.

D. U. H. T. 91 ; J. U. H. 91 ; D. U. H. 87.

Solution : We have $\frac{\delta u}{\delta x} = 3x^2 + 12xy - 3y^2$ ... ... (1)

$\frac{\delta^2 u}{\delta x^2} = 6x + 12y$ ... (2), $\frac{\delta u}{\delta y} = 6x^2 - 6xy - 6y^2$ ... (3)

$\frac{\delta^2 u}{\delta y^2} = -6x - 12y$ ... (4). Now $(2)+4 \Rightarrow \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \Rightarrow u$ is harmonic. Since u is harmonic, therefore it will be possible to find the harmonic conjugate v of u. Now by Cauchy–Riemann equations, we have $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ ... (5) and $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$ ... (6).

Then by (1) and (5), we have $\frac{\delta v}{\delta y} = 3x^2 + 12xy - 3y^2$. Integrating this equation with respect to $y \Rightarrow v = 3x^2y + 6xy^2 - y^3 + F(x)$.

Differentiating this equation with respect to x, we get

$\frac{\delta v}{\delta x} = 6xy + 6y^2 + F'(x)$ (7). Now by (3), (6) and (7) $\Rightarrow 6xy + 6y^2 + F'(x) = -6x^2 + 6xy + 6y^2 \Rightarrow F'(x) = -6x^2 \Rightarrow$ $F(x) = -2x^3 + c \Rightarrow v = 3x^2y + 6xy^2 - y^3 - 2x^3 + c$.

Example 109 : Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic function. Find v such that $u + iv$ is analytic.

R. U. H. 86 ; C. U. H. 86.

Solution : We have $\frac{\delta u}{\delta x} = 3x^2 - 3y^2 + 6x$ ... ... (1)

$\frac{\delta^2 u}{\delta x^2} = 6x + 6$ ... (2), $\frac{\delta u}{\delta y} = -6xy - 6y$ ... ... (3) and

$\frac{\delta^2 u}{\delta y^2} = -6x - 6$ ... (4). Now $(2)+(4) \Rightarrow \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$

$\Rightarrow$ u is harmonic function. Again, by Cauchy–Riemenn equations we have $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = 3x^2 - 3y^2 + 6x$ ... (5)

and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} = 6xy + 6y$ ... (6).

Now integrating (5) with respect to y $\Rightarrow v = 3x^2y - y^3 + 6xy + F(x)$ ... (7). Now by (6) and (7) $\Rightarrow 6xy + 6y + F'(x) = 6xy + 6y \Rightarrow F'(x) = 0 \Rightarrow F(x) = c$ ... (8). Now by (7) and (8) $\Rightarrow v = 3x^2y - y^3 + 6xy + c$.

Example 110 : Show that $\psi(x, y) = \frac{1}{2}\log(x^2+y^2)$ is a harmonic in the region $C - \{(0, 0)\}$. Find the harmonic conjugate of this function such that $f(z) = \phi + i\psi$ is analytic and also find f (z) in terms of z. ***D. U. H. 86.***

Solution : We have $\psi = \frac{1}{2}\log(x^2+y^2)$.

Then $\frac{\delta\psi}{\delta x} = \frac{x}{x^2+y^2}$ ... (1), $\frac{\delta^2\psi}{\delta x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ ... (2),

$\frac{\delta\psi}{\delta y} = \frac{y}{x^2+y^2}$ ... (3) and $\frac{\delta^2\psi}{\delta y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$ ... (4)

Now (2)+(4) $\Rightarrow \frac{\delta^2\psi}{\delta x^2} + \frac{\delta^2\psi}{\delta y^2} = 0 \Rightarrow \psi$ is harmonic.

Now by Cauchy-Riemann equation(1)$\Rightarrow \frac{\delta\psi}{\delta x} = -\frac{\delta\phi}{\delta y} = \frac{x}{x^2+y^2}$

and integrating this with respect to y we get

$\phi = -x/x \tan^{-1} y/x + F(x) = -\tan^{-1} y/x + F(x)$ ... (5).

Now differentiating (5) with resect to x and using the Cauchy-Riemann equation $\frac{\delta\phi}{\delta x} = \frac{\delta\psi}{\delta y}$ ... ... (6).

Now we get $\frac{y}{x^2+y^2} + F'(x) = \frac{y}{x^2+y^2}$ [by (3), (5) and (6)]

$\Rightarrow F'(x) = 0 \Rightarrow F(x) = c$ (by integrating). Then (5) $\Rightarrow \phi = -\tan^{-1} y/x + c$. Again $f(z) = \psi + i\phi = -\tan^{-1} y/x + c + i/2 \log(x^2+y^2) = i\{\frac{1}{2}\log(x^2+y^2) + i\tan^{-1} y/x\} + c = i\log(x+iy) + c = i\log z + c$.

Example 111 : Show that $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic in some domain and determine the corresponding analytic function $u + iv$. ***D. U. H. 86 ; D.U.H.T.87 ; D. U. M. SC. P. 84, 88, 90 ; J. U. H. 88.***

Solution : We have $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$

Then $\frac{\delta u}{\delta x} = \cos x \cosh y - 2 \sin x \sinh y + 2x + 4y$

$= \phi_1(x, y)$ (say) ... ... (1)

$\frac{\delta^2 u}{\delta x^2} = -\sin x \cosh y - 2 \cos x \sinh y + 2$ ... ... (2)

$\frac{\delta u}{\delta y} = \sin x \sinh y + 2 \cos x \cosh y - 2y + 4x = \phi_2(x, y)$ (say) ... (3)

$\frac{\delta^2 u}{\delta y^2} = \sin x \cosh y + 2 \cos x \sinh y - 2$ ... ... (4)

Now (2)+(4) $\Rightarrow \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \Rightarrow$ u satisfies the Laplace's equation $\Rightarrow$ u is harmonic.

Second part : Now putting $x = z$ and $y = 0$ in (1) and (3), then we get $\phi_1(z, 0) = \cos z + 2z$ ... (5) and $\phi_2(z, 0) = 2 \cos z + 4z$ ... (6). But we have $f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$ ... (7)

Then by (5), (6) and (7) $\Rightarrow f'(z) = (\cos z + 2z) - i(2 \cos z + 4z)$
$= (1 - 2i) \cos z + 2(1 - 2i)z = (1 - 2i)(\cos z + 2z)$. Now integrating we have $f(z) = u + iv = (1 - 2i)(\sin z + z^2) + c$.

Example 112 : Show that $\psi = \log\{(x-1)^2 + (y-2)^2\}$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is anlytic and express $\phi + i\psi$ as a function of z. ***D. U. H. T. 82, 86, 90 ; J. U. H. 87.***

Solution (Method 1) : we have $\psi = \log\{(x-1)^2 + (y-2)^2\}$. Then

$\frac{\delta\psi}{\delta x} = \frac{2(x-1)}{(x-1)^2 + (y-2)^2}$ .. (1), $\frac{\delta^2\psi}{\delta x^2} = \frac{2(y-2)^2 - 2(x-1)^2}{\{(x-1)^2 + (y-2)^2\}^2}$ ... (2)

$\frac{\delta\psi}{\delta y} = \frac{2(y-2)}{(x-1)^2 + (y-2)^2}$ ... (3)

and $\frac{\delta^2\phi}{\delta y^2} = \frac{2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2}$ — (4).

Now (2)+(4) $\Rightarrow \frac{\delta^2\psi}{\delta x^2} + \frac{\delta^2\psi}{\delta y^2} = 0 \Rightarrow$ u satisfies the Laplaces's equation $\Rightarrow$ u is harmonic, Now by Cauchy-Riemann equation (1) $\Rightarrow \frac{\delta\psi}{\delta x}=-\frac{\delta\phi}{\delta y}$

$=\frac{2(x-1)}{(x-1)^2+(y-2)^2}$ and integrating this with respect to y

we get $\phi=\frac{-2(x-1)}{(x-1)}\tan^{-1}\frac{y-2}{x-1}+F(x)$

$=-2\tan^{-1}\frac{y-2}{x-1}+F(x)$ ... ... (5). Now differentiating

(5) with respect to x we get $\frac{\delta\phi}{\delta x}=\frac{2(y-2)}{(x-1)^2+(y-2)^2}+F'(x)$ ... (6)

But by Cauchy-Riemann equation we have $\frac{\delta\phi}{\delta x}=\frac{\delta\psi}{\delta y}$.

Then by (3) and (6) we get $F'(x)=0 \Rightarrow F(x)=c$ (constant)

and (5) $\Rightarrow \phi=-2\tan^{-1}\frac{y-2}{x-1}+c$. Thus $f(z)=\phi+i\psi$

$=-2\tan^{-1}\frac{y-2}{x-1}+i\log\{(x-1)^2+(y-2)^2\}+c$

$=2i\left[\frac{1}{2}\log\{(x-1)^2+(y-2)^2\}+i\tan^{-1}\frac{y-2}{x-1}\right]+c$

$=2i\log\{(x-1)+i(y-2)\}+c=2i\log(z-1-2i)+c.$

(Method 2) : Let $x-1=r\cos\theta$ and $y-2=r\sin\theta$, then

$\theta=\tan^{-1}\left(\frac{y-2}{x-1}\right)$ and $\psi=\log r^2=2\log r$. Now $\frac{\delta\psi}{\delta r}=\frac{2}{r}$ ... (1),

$\frac{\delta^2\psi}{\delta r^2}=\frac{-2}{r^2}$ ... (2), $\frac{\delta\psi}{\delta\theta}=0$ ... (3) and $\frac{\delta^2\psi}{\delta\theta^2}=0$ ... (4)

Thus $\frac{\delta^2\psi}{\delta r^2}+\frac{1}{r}\frac{\delta\psi}{\delta r}+\frac{1}{r^2}\frac{\delta^2\psi}{\delta\theta^2}=-\frac{2}{r^2}+\frac{2}{r^2}+0 \Rightarrow \psi$ is harmonic. Again the Cauchy-Riemann equations in polar form are

$\frac{\delta\phi}{\delta r}=\frac{1}{r}\frac{\delta\psi}{\delta\theta}$ ... (5) and $\frac{\delta\psi}{\delta r}=-\frac{1}{r}\frac{\delta\phi}{\delta\theta}$ ... ... (6).

Now by (1) and (6) $\Rightarrow \frac{2}{r}=\frac{-1}{r}\frac{\delta\phi}{\delta\theta}$ ... (7) and by (3) and (5) $\frac{\delta\phi}{\delta r}=0$ ... (8). Then (7) $\Rightarrow \frac{\delta\phi}{\delta\theta}=-2$

$\Rightarrow \phi=-2\theta+f(r)$ ... ... (9).

Now by (8) and (9) $\Rightarrow f'(r)=0 \Rightarrow f(r)=c$ and

(9) $\Rightarrow \phi=-2\tan^{-1}\left(\frac{y-2}{x-1}\right)+c.$

Example 113 . Show that $u=\frac{1}{2}\log(x^2+y^2)$ satisfies the Laplace's equation and find v if $f(z)=u+iv$ is analytic.

R. U. H. 74.

Solution : Let $x=r\cos\theta$ and $y=r\sin\theta$, then $u=\frac{1}{2}\log(r^2\cos^2\theta+r^2\sin^2\theta)=\frac{1}{2}\log(r^2)=\log r.$

Now $\frac{\delta u}{\delta r}=\frac{1}{r}$ ... (1), $\frac{\delta^2 u}{\delta r^2}=-\frac{1}{r^2}$ ... ... (2),

$\frac{\delta u}{\delta\theta}=0$ ... ... (3) and $\frac{\delta^2 u}{\delta\theta^2}=0$ ... ... (4). Thus

$\frac{\delta^2 u}{\delta r^2}+\frac{1}{r}\frac{\delta u}{\delta r}+\frac{1}{r^2}\frac{\delta^2 u}{\delta\theta^2}=-\frac{1}{r^2}+\frac{1}{r^2}+0=0 \Rightarrow u$ is harmonic.

Again the Cauchy-Riemann equations in polar co-ordinates are $\frac{\delta u}{\delta r}=\frac{1}{r}\frac{\delta v}{\delta\theta}$ ... (5); and $\frac{\delta v}{\delta r}=-\frac{1}{r}\frac{\delta u}{\delta\theta}$ ... ... (6).

Now by (1) and (5) $\Rightarrow \frac{1}{r}=\frac{1}{r}\frac{\delta v}{\delta\theta} \Rightarrow \frac{\delta v}{\delta\theta}=1 \Rightarrow$

$v=\theta+f(r)$ ... (7). By (3), (6) and (7) $\Rightarrow \frac{\delta v}{\delta\theta}=f'(r)=0 \Rightarrow$

$f(r)=c$ and (7) $\Rightarrow v=\theta+c=\tan^{-1}y/x+c$ since $\tan\theta=y/x$.

Example 114 . If $u=x^2-y^2$ and $v=\frac{-y}{x^2+y^2}$, then show that both u and v satisfy the Laplace's equation but $u+iv$ is not an analytic function of z.

D. U. H. 87 ; D. U. H. T. 91 ; J. U. H. 91.

Solution : We have $u=x^2-y^2$ and $v=\frac{-y}{x^2+y^2}$.

Thus $\frac{\delta u}{\delta x}=2x$, $\frac{\delta^2 u}{\delta x^4}=2$, $\frac{\delta u}{\delta y}=-2y$, $\frac{\delta^2 u}{\delta y^2}=-2$ and $\frac{\delta v}{\delta x}$

$= \frac{2xy}{(x^2+y^2)^2}$, $\frac{\delta^2 v}{\delta x^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$, $\frac{\delta v}{\delta y} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\delta^2 v}{\delta y^2}$ $= \frac{-2y(y^2-3x^2)}{(x^2+y^2)^3}$. From these we have $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0$ and $\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}=0 \Rightarrow$ u and v satisfy the Laplace's equation.

Again from above we have $\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} \neq -\frac{\delta v}{\delta x} \Rightarrow$ u and v do not satisfy the Cauchy-Riemann equations.

$\Rightarrow u+iv$ is not an analytic function of z.

Example 115 : If $\phi(x, y)$ and $\psi(x, y)$ satisfy the Laplace's equation, then show that $s+it$ is analytic where

$$s = \frac{\delta \phi}{\delta y} - \frac{\delta \psi}{\delta x} \quad \ldots \quad \ldots \quad (1) \quad \text{and} \quad t = \frac{\delta \phi}{\delta x} + \frac{\delta \psi}{\delta y} \quad \ldots \quad \ldots \quad (2).$$

Solution : Since ϕ and ψ satisfy the Laplace's equation we have $0 = \frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \phi}{\delta y^2}$ ... (3) and $0 = \frac{\delta^2 \psi}{\delta x^2} + \frac{\delta^2 \psi}{\delta y^2}$ ... (4)

$$\text{Now (1)} \Rightarrow \frac{\delta s}{\delta x} = \frac{\delta^2 \phi}{\delta x \delta y} - \frac{\delta^2 \psi}{\delta x^2} \quad \ldots \quad (5)$$

$$\text{and } \frac{\delta s}{\delta y} = \frac{\delta^2 \phi}{\delta y^2} - \frac{\delta^2 \psi}{\delta x \delta y} \quad \ldots \quad \ldots \quad (6)$$

Again (2) $\Rightarrow \frac{\delta t}{\delta x} = \frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \psi}{\delta x \delta y}$ ... (7) and $\frac{\delta t}{\delta y} = \frac{\delta^2 \phi}{\delta y \delta x}$ $+ \frac{\delta^2 \psi}{\delta y^2}$ ... (8). Now (4)+(5) $\Rightarrow \frac{\delta s}{\delta x} = \frac{\delta^2 \phi}{\delta y \delta x} + \frac{\delta^2 \psi}{\delta y^2} = \frac{\delta t}{\delta y}$, by (8). Again (6)−(3) $\Rightarrow \frac{\delta s}{\delta y} = -\left(\frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \psi}{\delta y \delta x}\right) = -\frac{\delta t}{\delta x}$, by (7)

Thus s and t satisfy the Cauchy-Riemann equations and therefore $s+it$ is analytic.

Example 116 : If ϕ and ψ are functions of x and y satisfying the Laplace's equation, then show that $s+it$ is analytic where $s = \frac{\delta \phi}{\delta y} - \frac{\delta \psi}{\delta x}$ ... (1) and $t = \frac{\delta \phi}{\delta x} + \frac{\delta \psi}{\delta y}$ ... (2).

Solution : Since ϕ and ψ satisfy the Laplace's equation, then we have $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ ... (3) and $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$ ... (4).

Now by (1) we have $\frac{\partial s}{\partial x}=\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial x^2}$ ... (5) and by (2) we have $\frac{\partial t}{\partial y}=-\frac{\partial^2\phi}{\partial y\partial x}+\frac{\partial^2\psi}{\partial y^2}$ ... ... (6)

Now $(5)-(6) \Rightarrow \frac{\partial s}{\partial x}-\frac{\partial t}{\partial y}=-\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0$, by (4)

$\Rightarrow \frac{\partial s}{\partial x}=\frac{\partial t}{\partial y}$. Again by (1) we have $\frac{\partial s}{\partial y}=\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\psi}{\partial y\partial x}$ ... (7)

and by (2) we have $\frac{\partial t}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\psi}{\partial x\partial y}$ ... (8)

Nom $(7)+(8) \Rightarrow \frac{\partial s}{\partial y}+\frac{\partial t}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$, by (3) $\Rightarrow$

$\frac{\partial s}{\partial y}=-\frac{\partial t}{\partial x}$. Thus s and t satisfy the Cauchy-Riemann equations and therefore $s+i\,t$ is analytic.

170. Construction of f(z) :

Theorem 90 : If $f(z)=u(x, y)+i\,v(x, y)$ is analytic, then

(*i*) $f(z)=u(z, o)+i\,v(z, o)$;

(*ii*) $f(z)=u(o, -iz)+i\,v(o, -iz)$;

(*iii*) $f(z)=2u(z/2, -i\,z/2)+\text{constant}$;

(*iv*) $f(z)=2i\,v(z/2, -iz/2)+\text{constant}$:

proof (i) : We have $f(z)=f(x+iy)=u(x, y)+iv(x, y)$ .. (1)

Nom putting $y=0$ in (1) $\Rightarrow f(x)=u(x, 0)+iv\ (x, 0)$ ... (2).

In (2) replacing x by z $\Rightarrow f(z)=u(z, 0)+i\,v(z, 0)$.

(ii), (iii) and (iv) : Try yourself.

N. B. If $f(z)=u(x, y)+iv(x, y)$ ... (1), then putting

$x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$ in the ***R. H. S.*** of (1) we can find f(z) interms of z.

171. *Construction of an analytic function :*

Theorem 91 : (***Milne Thomson Method*** , ***or simply Milne Method***)

If $f(z)=u(x, y)+iv(x,y)$ is analytic and if $\frac{\delta u}{\delta x}=\phi_1(x, y)$,

$\frac{\delta u}{\delta y}=\phi_2(x, y)$ and $\frac{\delta v}{\delta x}=\psi_1(x, y)$, $\frac{\delta v}{\delta y}=\psi_2(x, y)$, then

(*i*) $f'(z)=\phi_1(z, 0)-i\,\phi_2(z, 0)$;

(*ii*) $f(z)=\int[\phi_1(z, 0)-i\,\phi_2(z, 0)]\,dz+c$;

(*iii*) $f'(z)=\psi_1(z, 0)+i\,\psi_2(z, 0)$;

(*iv*) $f(z)=\int[\psi_1(z, 0)+i\psi_2(z, 0)]\,dz+c$.

Proof : (*i*) We have $f'(z)=\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}=\frac{\delta u}{\delta x}-i\frac{\delta u}{\delta y}$,

by Cauchy-Riemann equations $\Rightarrow f'(z)=\phi_1(x, y)-i\phi_2(x,y)$ ... (1)

Now putting $y=0$ in (1) $\Rightarrow f'(x)=\phi_1(x, 0)-i\,\phi_2(x, 0)$ ... (2)

Again replacing x by z, (2) $\Rightarrow f'(z)=\phi_1(z, 0)-i\,\phi_2(z, 0)$ ... (3)

(*ii*) Now integrating (3) with respect to $z \Rightarrow$
$f(z)=\int[\phi_1(z, 0)-i\,\phi_2(z, 0)]+c$.

(*iii*) and (*iv*) : Try yourself.

172 *Theorem 92 :*

If $f(z)=u(x, y)+iv(x, y)$ is analytic and if $\frac{\delta u}{\delta x}=\phi_1(x, y)$, $\frac{\delta u}{\delta y}=\phi_2(x, y)$ and $\frac{\delta v}{\delta x}=\psi_1(x, y)$, $\frac{\delta v}{\delta y}=\psi_2(x, y)$, then

(*i*) $f'(z)=\phi_1(0,-iz)-i\,\phi_2(0,-iz)$;

(*ii*) $f(z)=\int[\phi_1(0,-iz)-i\,\phi_2(0,-iz)]\,dz+c$;

(*iii*) $f'(z)=\psi_1(0,-iz)+i\,\psi_2(0,-iz)$;

(*iv*) $f(z)=\int[\psi_1(0,-iz)+i\psi_2(0,-iz)]\,dz+c$.

Proof : Try yourself.

173. *Remember the following :*

If a function u (or v) is given, then a corresponding analytic function $f(z)=u+iv$ will not exist unless u (or v) is harmonic. If this condition is fulfilled, then the function $f(z)=u+iv$ can be determined using Cauchy-Riemann equations.

174. Theorem 93 : Let $f(z)=u(x, y)+iv(x, y)$ be an analytic function and if u satisfies the Laplace's equation, then show that $dv=\frac{\delta v}{\delta x}dx+\frac{\delta v}{\delta y}dy$ is an exact differential and v can therefore be determined.

Proof : Try yourself.

175. Therorem 94 : Let $f(z)=u(x, y)+iv(x, y)$ be an analytic function and if v satisfies the Laplace's equation, then show that $du=\frac{\delta u}{\delta x}dx+\frac{\delta v}{\delta y}dy$ is an exact differential and u can therefore be determined.

Proof : Try yourself.

176. Construction of a conjugate function : If $f(z)=u(x, y)+iv(x, y)$ is analytic, then $u(x, y)$ and $v(x, y)$ are called conjugate functions. Now we have $du=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy$ ... (1) and $dv=\frac{\delta v}{\delta x}dx+\frac{\delta v}{\delta y}dy$ ... (2). Now by Cauchy Riemann equations (1) and (2) $\Rightarrow du=\frac{\delta v}{\delta y}dx-\frac{\delta v}{\delta x}dy$ ... (3) and $dv=-\frac{\delta u}{\delta y}dx+\frac{\delta u}{\delta x}dy$ ... (4). Here the right hand side of (3) and (4) are of the form $M\,dx+N\,dy$ and they are exact since u and v are harmonic.

(*i*) Now if u is given, then integrating (4), v can be determined with the addition of a constant term.

(*ii*) : Now if v is given, then integrating (3), u can be determined with the addition of a constant term.

Example 117 : Show that $u=3x^2y+2x^2-y^3-2y^2$ is harmonic and hence find its harmonic conjugate v if $f(z)=u+iv$ is analytic.

C. U. H. 89 ; R. U. H. 81.

Solution : We have $u=3x^2y+2x^2-y^3-2y^2$.

Thus $\dfrac{\delta u}{\delta x}=6xy+4x=\phi_1(x, y)$ ... (1) ; $\dfrac{\delta^2 u}{\delta x^2}=6y+4$ ... (2)

$\dfrac{\delta u}{\delta y}=3x^2-3y^2-4y=\phi_2(x, y)$ ... (3) ; $\dfrac{\delta^2 u}{\delta y^2}=-6y-4$ ... ... (4).

Now $(2)+(4) \Rightarrow \dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0 \Rightarrow$ u is harmonic.

Now using Milne's method we have

$f'(z)=\phi_1(z, 0)-i\,\phi_2(z, 0)$ $\left[\begin{array}{l}\text{Putting } x=z,\ y=0 \text{ in} \\ (1) \text{ and } (3)\end{array}\right]$

$=4z-3iz^2$. Now integrating we have $f(z)=2z^2-iz^3+ic$

$\Rightarrow f(z)=u+iv=2(x+iy)^2-i(x+iy)^3+ic=2(x^2-y^2+2xyi)$

$-i(x^3+3x^2yi-3xy^2-iy^3)+ic \Rightarrow v=4xy-x^3+3xy^2+c.$

Example 118 : Show that the function $u=2x-x^3+3xy^2$ is harmonic and also find the harmonic conjugate v if $f(z)=u+iv$ is analytic.

D. U. H. 91 ; J. U. H. 87, 89 ; C. U. H. 85.

Solution : We have $\dfrac{\delta u}{\delta x}=2-3x^2+3y^2=\phi_1(x, y)$ (say),

$\dfrac{\delta^2 u}{\delta x^2}=-6x$, $\dfrac{\delta u}{\delta y}=6xy=\phi_2(x, y)$ (say) and $\dfrac{\delta^2 \phi}{\delta y^2}=6x$.

Thus $\dfrac{\delta^2 \phi}{\delta x^2}+\dfrac{\delta^2 \phi}{\delta y^2}=0 \Rightarrow$ u is harmonic. By Milne method, we have $f'(z)=\phi_1(z, 0)-i\,\phi_2(z, 0)=2-3z^2 \Rightarrow f(z)=u+iv$

$=2z-z^3+c=2(x+iy)-(x+iy)^3+ic=2(x+iy)-(x^3+3x^2yi$

$-3xy^2-iy^3)+ic \Rightarrow v=2y-3x^2y+y^3+c$ where c is a constant.

Example 119 : Show that $u=\dfrac{y}{x^2+y^2}$ is harmonic and find its harmonic conjugate v and $f(z)=u+iv$ if $f(z)$ is analytic.

D. U. H. T. 76.

Solution : We have $\frac{\delta u}{\delta x}=\frac{-2xy}{(x^2+y^2)^2}=\phi_1(x, y)$ (say),

$\frac{\delta^2 u}{\delta x^2}=\frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$, $\frac{\delta u}{\delta y}=\frac{x^2-y^2}{(x^2+y^2)^2}=\phi_2(x, y)$ (say), $\frac{\delta^2 u}{\delta y^2}$

$=\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$. Then $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0 \Rightarrow u$ is harmonic.

Now by Milne method, we have, $f'(z)=\phi_1(z, 0)-i\,\phi_2(z, 0)$

$=\frac{-i}{z^2} \Rightarrow f(z)=\frac{i}{z}+c=u+iv=\frac{i}{x+iy}+c_1+ic_2=\frac{i(x-iy)}{x^2+y^2}+c_1$

$+ic_2 \Rightarrow v=\frac{x}{x^2+y^2}+c_2$.

Example 120 : Find the harmonic conjugate of the function

$u=e^{x^2-y^2}\cos 2xy$ and the corresponding analytic function

$f(z)=u+i\,v$. ***J. U. H. 87 ; D. U. H. T. 78.***

Solution : We have $\frac{\delta u}{\delta x}=2e^{x^2-y^2}(x\cos 2xy-y\sin 2xy)$

$=\phi_1(x, y)$ (say), $\frac{\delta^2 u}{\delta x^2}=2e^{x^2-y^2}[2(x^2-y^2)\cos 2xy-4xy\sin 2xy$

$+\cos 2xy]$, $\frac{\delta u}{\delta y}=-2e^{x^2-y^2}(y\cos 2xy+x\sin 2xy)=\phi_2(x, y)$(say),

$\frac{\delta^2 u}{\delta x^2}=2e^{x^2-y^2}[2(x^2-y^2)\cos 2xy+\cos 2xy-4xy\sin 2xy]$

and $\frac{\delta^2 u}{\delta y^2}=-2e^{x^2-y^2}[2(x^2-y^2)\cos 2xy+\cos 2xy-4xy\sin 2xy]$.

Now $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0 \Rightarrow u$ is harmonic. Thus the harmonic conjugate v of u exists

Now by Milne method, we have $f'(z)=\phi_1(z,0)-i\phi_2(z,0)$ $=2ze^{z^2}=e^{z^2}d(z^2) \Rightarrow f(z)=u+iv=e^{z^2}+c=e^{x^2-y^2+2ixy}$ $+c=e^{x^2-y^2}[\cos 2xy+i\sin 2xy+c_1+ic_2 \Rightarrow v=e^{x^2-y^2}\sin 2xy$ $+c_2$ where $c=c_1+ic_2$.

Example 121 : Show that the function $u=x^2-y^2-2xy-2x+3y$ is harmonic and find the harmonic conjugate v also $f(z)=u+iv$ if $f(z)$ is analytic.

D. U. M. SC. P. T. 91.

Solution : We have $\frac{\delta u}{\delta x}=2x-2y-2=\phi_1(x,y)$ (say),

$$\frac{\delta^2 u}{\delta x^2}=-2,\ \frac{\delta u}{\delta y}=-2y-2x+3=\phi_2(x,y)\text{ (say)},\ \frac{\delta^2 u}{\delta y^2}=-2,$$

Thus $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0 \Rightarrow u$ is harmonic.

Now by Milne method, we have $f'(z)=\phi_1(z,0)-i\phi_2(z,0)$ $=(2z-2)-i(-2z+3)=2(1+i)z-(2+3i)=f(z)=u+iv.$ $=(1+i)z^2-(2+3i)z+c=(1+i)(x^2-y^2+2xyi)-(2+3i)(x+iy)$ $+c_1+ic_2 \Rightarrow v=x^2-y^2+2xy-3x-2y+c_2$ where $c=c_1+ic_2$.

Example 122 : Show that the function $u=e^x(x\cos y-y\sin y)$ is a harmonic function and find the corresponding analytic function $f(z)=u+iv$. From it find v.

C. U. H. 85, 88 ; D. U. H. T. 88 ; J. U. H. 91.

Soluiton : We have $u=e^x(x\cos y-y\sin y)$.

Then $\frac{\delta u}{\delta x}=e^x(x\cos y-y\sin y+\cos y)=\phi_1(x,y)$ ... (1)

$$\frac{\delta^2 u}{\delta x^2}=e^x(x\cos y-y\sin y+2\cos y) \quad \ldots \quad (2)$$

$$\frac{\delta u}{\delta y}=e^x(-x\sin y-\sin y-y\cdot\cos y)=\phi_2(x,y)\quad\cdots\quad(3)$$

$$\frac{\delta^2 u}{\delta y^2}=e^x(-x\cos y+y\sin y-2\cos y)\quad\cdots\quad\cdots\quad(4)$$

Now $(2)+(4)\Rightarrow\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0\Rightarrow u$ is a harmonic function.

Now using ***Milne's*** method, we have $f'(z)=\phi_1(z,0)-i\phi_2(z,0)$ $\left[\begin{array}{l}\text{putting } x=z,\ y=0 \text{ in}\\ (1) \text{ and } (3)\end{array}\right]$

$=e^z(z+1)$. Now integrating we have $f(z)=e^z(z+1)$ $-e^z+c=ze^z+c$. Again we have, $f(z)=u+iv$ $=(x+iy)\,e^{x+iy}+c=(x+iy)e^x(\cos y+i\sin y)+c_1+ic_2\Rightarrow$ $v=e^x(x\sin y+y\cos y)+c_2$ where $c=c_1+ic_2$.

Example 123 : Show that the function $u=2x(1-y)$ is harmonic and find a function v such that $f(z)=u+iv$ is analytic. Also find $f(z)$ in terms of z. ***D. U, H. 83 ; D. U. S. 83.***

Solution : We have $u=2x(1-y)$. Then $\dfrac{\delta u}{\delta x}=2(1-y)$ $=\phi_1(x,y)$ (say) $\cdots$ (1), $\dfrac{\delta^2 u}{\delta x^2}=0$ $\cdots$ (2),

$\dfrac{\delta u}{\delta y}=-2x=\phi_2(x,y)$ (say) $\cdots$ (2), $\dfrac{\delta^2 u}{\delta y^2}=0$ $\cdots$ $\cdots$, (4)

Now $(2)+(4)\Rightarrow\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0\Rightarrow u$ is harmonic.

Now putting $x=z$ and $y=0$ in (1) and (2) $\Rightarrow\phi_1(z,0)=2$ and $\phi_2(z,0)=-2z$. Then $f'(z)=\phi_1(z,0)-i\phi_2(z,0)=2+2iz$. Now integrating we get $f(z)=2z+iz^2+c\Rightarrow f(z)=u+iv=2(x+iy)+$ $i(x^2-y^2+2xyi)+c\Rightarrow v=x^2-y^2+2y+c_2$ where $c=c_1+ic_2$ (say).

Example 124 : Show that $u=e^{-x}(x\sin y-y\cos y)$ is harmonic. Find v such that $f(z)=u+iv$ is analytic.
D. U. H. T. 89 ; D. U. 81, 82 ; C. U. 82 ; R. U. M. SC. P. 85 ; D. U, M. SC. P. T. 90.

Solution : We have $u = e^{-x}(x \sin y - y \cos y)$.

Then $\frac{\delta u}{\delta x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y = \phi_1(x, y)$ ... (1)

$\frac{\delta^2 u}{\delta x^2} = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y$ ... ... (2),

$\frac{\delta u}{\delta y} = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y = \phi_2(x, y)$ ... (3),

$\frac{\delta^2 u}{\delta y^2} = -xe^{-x} \sin y + 2e^{-x} \sin y + e^{-x} \cos y$ ... ... (4)

Now (2)+(4) $\Rightarrow \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \Rightarrow$ u is harmonic.

Now putting $x = z$ and $y = 0$ in (1) and (3), we get $\phi_1(z, 0) = 0$ and $\phi_2(z, 0) = (z-1)e^{-z}$.

Then $f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0) = -i(z-1)e^{-z}$.

Now integrating we get $f(z) = i(z-1)e^{-z} + i\,e^{-z} + c = ize^{-z} + c$.

Again we have $f(z) = u + iv = i(x+iy)e^{-(x+iy)} + c$

$= i(x+iy)e^{-x}(\cos y - i \sin y) + c \Rightarrow v = x \cos y\, e^{-x} + y \sin y\, e^{-x} + c_2$

$= e^{-x}(x \cos y + y \sin y) + c_2$ where $c = c_1 + i\,c_2$.

Example 125 : If $w = f(z) = u + iv$ and $u - v = e^x(\cos y - \sin y)$ then find f(z) interms of z.

Solution , We have $f(z) = u + iv$ ... (1) aud $i\,f(z) = iu - v$ ... (2). Then (1)+(2) $\Rightarrow (1+i)f(z) = (u-v) + (u+v)i = P + iQ$ where $P = u - v = e^x(\cos y - \sin y)$ and $Q = u + v$.

Now $\frac{\delta P}{\delta x} = e^x(\cos y - \sin y) = \phi_1(x, y)$ (say) and $\frac{\delta P}{\delta y} = -e^x(\sin y + \cos y) = \phi_2(x, y)$ (say). Now by Milne method, we have $F'(z) = \phi_1(z,0) - i\phi_2(z, 0) = (1+i)e^z \Rightarrow F(z) = (1+i)e^z + (1+i)c \Rightarrow (1+i)f(z) = (1+i)e^z + (1+i)\,c \Rightarrow f(z) = e^z + c$

where we have considered $F(z) = (1+i)f(z)$ and the constant term in the integration $= (1+i)c$.

Example 126 : If $f(z) = u + iv$ is an analytic function then find $f(z)$ where $u - v = (x - y)(x^2 + 4xy + y^2)$.

Solution : We have $f(z) = u + iv$ ... ... (1) and $i f(z) = iu - v$ ... (2). Then $(1 + i) f(z) = (u - v) + (u + v) i = P + Qi$ where $P = u - v = (x - y)(x^2 + 4xy + y^2)$ and $Q = u + v$.

Now $\frac{\partial P}{\partial x} = (x^2 + 4xy + y^2) + (x - y)(2x + 4y)$

$= \phi_1 (x, y)$ (say)

and $\frac{\partial P}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$

$= \phi_2 (x, y)$ (say).

Now by Milne method, we have $F'(z) = \phi_1 (z, 0) - i\phi_2 (z, 0)$

$= 3z^2 - 3z^2 i = 3(1 - i) z^2 \Rightarrow F(z) = (1 - i) z^3 + (1 + i) c$

$\Rightarrow (1 + i) f(z) = (1 - i) z^3 + (1 + i) c \Rightarrow f(z) = - iz^3 + c$

where we have considered $F(z) = (1 + i) f(z)$ and the constant term in the integration $= (1 + i) c$.

Example 127 : Find the analytic function $f(z) = u + iv$ where $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$.

Solution : We have $\frac{\partial u}{\partial x} = \frac{2 \cos 2x \cosh 2y + 2}{(\cosh 2y + \cos 2x)^2}$

$= \phi_1 (x, y)$ (say)

$\frac{\partial u}{\partial y} = \frac{-2 \sin 2x \sin h2y}{(\cosh 2y + \cos 2x)^2} = \phi_2 (x, y)$ (say).

Then by Milne method, $f'(z) = \phi_1 (z, 0) - i\, \phi_2 (z, 0)$

$$= \frac{2\,(1+\cos 2z)}{(1+\cos 2z)^2} = \frac{2}{1+\cos 2z} = \frac{2}{2\cos^2 z} = \sec^2 z \Rightarrow f(z) =$$

$\tan z + c$.

Example 128 : If $f(z) = u + iv$ is analytic, then find $f(z)$ where $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ and $f(\pi/2) = 0$.

Solution : We have $u - v = \dfrac{\cos x + \sin x - (\cosh y - \sinh y)}{2\cos x - 2\cosh y}$

$= \dfrac{1}{2} + \dfrac{\sin x + \sinh y}{2(\cos x - \cosh y)}$. Now $f(z) = u + i\,v$ ...... (1)

and $i\,f(z) = u\,i - v$ .... (2). Then (1) + (2) $\Rightarrow$ $(1 + i)\,f(z)$

$= (u - v) + i\,(u + v) \Rightarrow F(z) = P + i\,Q$

where $F(z) = (1 + i)\,f(z)$, $P = u - v$

$= \dfrac{1}{2} + \dfrac{\sin x - \sinh y}{2\,(\cos x - \cosh y)}$ and $Q = u + v$.

Now $\dfrac{\partial P}{\partial x} = \dfrac{1 + \sin x \sinh y - \cos x \cosh y}{2\,(\cos x - \cosh y)^2} = \phi_1(x, y)$ (say)

and $\dfrac{\partial P}{\partial y} = \dfrac{-1 + \sin x \sinh y + \cos x \cosh y}{2\,(\cos x - \cosh y)^2} = \phi_2(x, y)$ (say).

Then by Milne method, we have

$F'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0) = \dfrac{(1 - \cos z) - (-1 + \cos z)i}{2(\cos z - 1)^2}$

$= \dfrac{(1 + i)}{2\,(1 - \cos z)} = \dfrac{1}{4}\,(1 + i)\,\operatorname{cosec}^2 z/2$

$\Rightarrow F(z) = (1 + i)\,f(z) = -\dfrac{1}{2}(1 + i)\cot z/2 + (1 + i)\,c$

$\Rightarrow f(z) = -\dfrac{1}{2}\cot z/2 + c$ ..... (3). But we have

$f(\pi/2) = 0$ .... (4) . Now (3) and (4) $\Rightarrow -\dfrac{1}{2}\cot \pi/4 + c = 0$

$\Rightarrow c = \frac{1}{2}$. Thus $f(z) = -\frac{1}{2} \cot z/2 + 1/2$

$= \frac{1}{2}(1 - \cot z/2)$.

Example 129 : Show that the following functions u are harmonic in any finite region R of the Argand plane and also find the harmonic conjugate v of each if $f(z) = u + iv$ is analytic :

(i) $u = \sin x \cosh y$; **(ii)** $u = y^3 - 3x^2y$; **R. U. 64.** **(iii)** $u = \cosh x \sin y$; **D. U. 64.** **(iv)** $u = e^{-x} \sin y$; **(v)** $u = 2xy + y^3 - 3x^2y$; **(vi)** $u = x^2 - y^2 + 2y$; **(vii)** $u = x^3 - 3xy^2$; **C. U. H. 90.** **(viii)** $u = \frac{1}{2} \log (x^2 + y^2)$; **R. U. H. 74.** **(ix)** $u = x^2 - y^2$; **(x)** $u = -\sin x \sinh y$ **(xi)** $u = x^3 - 3xy^2 + y$.

Solution : Try yourself.

Example 130 : Find $f(z)$ from the function $u(x, y)$ $= e^x \cos y$ and find the harmonic conjugate of u if $f(z) = u + iv$ is analytic. **D. U. H. 88.**

Solution : We have $u(x, y) = e^x \cos y$, then $u(z/2, -z\,i/2)$

$= e^{z/2} \cos(-z\,i/2) = e^{z/2} \cosh z/2 = \frac{e^{z/2}}{2} (e^{z/2} + e^{-z/2})$

$= \frac{1}{2}(e^z + 1) = \frac{1}{2}\{ e^x (\cos y + i \sin y) + 1 \}$. But $f(z) = u + iv$

$= 2u(z/2, -zi/2) + \text{constant} = e^x(\cos y + i \sin y) + \text{constant}$

$\Rightarrow v = e^x \sin y + \text{constant}$.

Example 131 : If $f(z) = u + iv$ is analytic then show that $v = 4xy - x^3 + 3xy^2 + c$ where $u = 3x^2y + 2x^2 - y^3 - 2y^2$. Show that u is harmonic and $f(z) = 2z^2 - iz^3 + i\,c$. **R. U. 81.**

Solution : Try yourself.

Example 132 : If $u = 2xy + 3xy^2 - 2y^3$, then show that it is not harmonic.

Solution : Try yourself.

Example 133 : Show that $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. From this, show that $f(z) = -ie^{iz^2} + ic$ and

$v = -e^{-2xy} \cos (x^2 - y^2) + c$.

Solution : Try yourself.

Example 134 : Show that $f(z) = z^2e^{-z} + c$, if $f(z) = u + iv$ is analytic and $u = e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \}$.

Solution : Try yourself.

Example 135 : If $f(z) = u + iv$ is analytic and $u = (x-1)^3 - 3xy^2 + 3y^2$, then show that $v = 3x^2y - 6xy + 3y - y^3 + c$.

Solution : Try yourself.

Example 136 : Show that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies the Laplace's equation. Also show that $f(z) = u + iv = z^3 + 3z^2 + c$ if $f(z)$ is analytic.

Solution : Try yourself.

Example 137 : If $f(z) = u + iv$ is analytic and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, then find $f(z)$ under the condition $f(\pi/2) = \dfrac{3-i}{2}$.

Solution : Try yourself. Ans $f(z) = \cot z/2 + \frac{1}{2}(1 - i)$.

Example 138 : If $f(z) = u + iv$ is analytic and $u + v$

$= \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, then show that

$f(z) = \frac{1}{2}(1 + i)\cot z + c$.

Example 139 : If $u(x, y)$ is harmonic, then show that $u(x + a, y + b)$ is harmonic where a and b are constants.

Solution : Try yourself.

Example 140 : If a is a real number, then show that the real and imaginary parts of $f(z) = \log(z - a)$ are harmonic functions in any region R not containing $z = a$.

Solution : Try yourself.

Example 141 : If u and v are conjugate harmonic functions, then show that (i) $du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$;

(ii) $dv = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$.

Solution : Try yourself.

Example 142 : If u and v are harmonic in a region R, then show that $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic in R.

177. Indeterminate forms :

The following are called indeterminate forms :

$0/0,\ \infty/\infty\ 0\times\infty,\ \infty/\infty,\ \infty^0,\ 0^0,\ 1^\infty$

L' Hospital's rule (Theorem 95) : If $f(z)$ and $g(z)$ are analytic at z_0 and $f(z_0) = g(z_0) = 0$

but $g'(z_0) \neq 0$, then $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{f'(z_0)}{g'(z_0)}$

Proof : We have $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)}$

$$= \lim_{z\to z_0} \frac{\dfrac{f(z)-f(z_0)}{z-z_0}}{\dfrac{g(z)-g(z_0)}{z-z_0}} = \frac{\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}}{\lim\limits_{z\to z_0} \dfrac{g(z)-g(z_0)}{z-z_0}} = \frac{f'(z_0)}{g'(z_0)}$$

where $g'(z_0) \neq 0$.

Historical Note : Giulaume Francois Marquis De L' Hospital (1661 – 1704) was an French amateur mathematician. He is called a friend and patron of mathematics. He studied under the Switze Mathematician Johann Bernoulli (1667 - 1748), who were the brother of the mathematician Jacob Bernoulli (1654 – 1705) and the father of the Mathematician Daniel Bernoulli (1700 – 1784).

Example 143 : Evalute

(i) $\lim\limits_{z\to 0} \dfrac{\sin z.}{z}$; (ii) $\lim\limits_{z\to 0} \dfrac{1-\cos z}{z^2}$;

(iii) $\lim_{z \to 0} \frac{\tan z}{z}$; (iv) $\lim_{z \to 0} (\cos z)^{1/z}$,

(v) $\lim_{z \to ai} \frac{z^6 + a^6}{z^{10} + a^{10}}$

Solution :

(i) $\lim_{z \to 0} \frac{\sin z}{z}$ (% form) $= \lim_{z \to 0} \frac{\cos z}{1} = 1$;

(ii) $\lim_{z \to 0} \frac{1 - \cos z}{z^2}$ (% form)

$$= \lim_{z \to 0} \frac{\sin z}{2z} \text{ (\% form)} = \lim_{z \to 0} \frac{\cos z}{2} = \frac{1}{2}.$$

(iii) $\lim_{z \to 0} \frac{\tan z}{z}$ (% form) $= \lim_{z \to 0} \frac{\sec^2 z}{1} = 1$

(iv) Let $P = (\cos z)^{1/z}$.

Then $\lim_{z \to 0} \log P = \lim_{z \to 0} \frac{\log \cos z}{z}$ (% form)

$$= \lim_{z \to 0} \frac{\frac{-\sin z}{\cos z}}{1} = 0 \Rightarrow P = e^0 = 1$$

(v) $\lim_{z \to ai} \frac{z^6 + a^6}{z^{10} + a^{10}}$ (% form

$$= \lim_{z \to ai} = \frac{6z^5}{10z^9} = \lim_{z \to ai} \frac{3}{5z^4} = \frac{3}{5a^4}$$

Theorem 96 : If $f(z)$ is analytic in a given region $\mathfrak{R}$ including the point z_0, then $f(z) = f(z_0) + (z - z_0) f'(z_0) + \eta (z - z_0)$ where $\eta \to 0$ as $z \to z_0$.

Proof : Since $f(z)$ is analytic in a given region $\mathfrak{R}$ and we have z_0 is a point in the region $\mathfrak{R}$, then $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =$ $f'(z_0)$ ..... (1) exists in $\mathfrak{R}$. Now (1)

$$\Rightarrow \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = 0$$

$$\Rightarrow \lim_{z \to z_0} \left\{\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right\} = 0 \text{ .... (2)}$$

Let $\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta$ .... (3), then

$\lim\limits_{z \to z_0} \eta = 0$ by (2) . Therefore (3) $\Rightarrow$ $f(z) = f(z_0) +$ $(z - z_0) f'(z_0) + \eta (z - z_0)$ where $\eta \to 0$ as $z \to z_0$.

178. Zero or root of a function :

D. U. H. 90; D. U. H. T. 78

If $f(z) = (z - z_0)^n g(z)$, where n is a positive integer and g (z) is analytic and $g(z_0) \neq 0$, then $z = z_0$ is called a zero of order n of the function $f(z)$. If $n = 1$, then z_0 is called a simple zero. If $n = 2$, then z_0 is called a double zero or zero of order 2. If $n = 3, 4, 5,$ ----- , then z_0 are called zeros of order 3, 4, 5, ---- respectively.

Example 144. : Let $f(z) = \frac{(z - 3)(z + 5)^2}{(z - 1)(z + 2)^3}$.

Here $z = 3$ is a simple zero and $z = -5$ is a zero of order 2.

179. Singular point or critical point or singularity of a function : **D. U. H. T. 89; D. U. H. 90.**

A point at which a function $f(z)$ fails or ceases to be analytic is called a singular point.

Or equivalently, a point $z = z_0$ is called a singular point of a function $f(z)$, if $f(z)$ is not differentiable at z_0 but if every neighbourhood of z_0 contains any other point at which $f(z)$ must be differentiable.

Example 145 : Let $f(z) = \dfrac{z^2 - 4}{(z - 1)(z - 3)}$.

Here $z = 1$ and $z = 2$ are two singular points since at these two points the function are not analytic.

180. Various types of singularities :

(i) Isolated singular point or singularity :

A singular point $z = z_0$ is called an isolated singular point of a function $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no other singularity of the function other than z_0. If there exists no such δ, in this case z_0 is called a **non – isolated singularity** of $f(z)$.

Example 146 : Let $f(z) = \dfrac{1}{(z - 1)(z - 3)}$, where $z = 1$ and $z = 3$ are the singular points of $f(z)$. Here $z = 1$ is an isolated singular point of $f(z)$ since there exists at least one circle $|z - 1| = \delta < 2$ which does not include the singularity $z = 3$. Similarly, the circle $|z - 3| = \delta' < 2$ does not include the singularity $z = 1$ and therefore $z = 3$ is also an isolated singularity of $f(z)$.

Ordinary point : A point z_0 is called an ordinary point of a function $f(z)$ if z_0 is not a singular point and we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point.

(iii) Pole : D. U. H. 83; D. U. H. T. 89; R. U. H. 76, 77.

A singular point $z = z_0$ is called a pole of order n of the function $f(z)$ if $\lim\limits_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, where n is a positive integer. If $n = 1$, then $z = z_0$ = is called a **simple pole.** If $n = 2$, then $z = z_0$ is called a **double pole** or **pole of order 2.** Again, if $n = 3$, then $z = z_0$ is called a **triple pole** or **pole of order** 3 and similarly we can define the poles of order 4, 5, 6, etc. If a is a zero of the function $f(z)$ of order n, then it is a pole of the function $1/f(z)$ of order n.

Example 147 : Let $f(z) = \dfrac{z - 3}{(z - 7)^5}$. Here $z = 7$ is a pole of order 5 of the function $f(z)$ since $\lim\limits_{z \to 7} (z - 7)^5 f(z)$

$$= \lim_{z \to 7} (z - 3) = 4 \neq 0.$$

Example 148 : Let $f(z) = \dfrac{z^2 + 5}{(z - 1)(z - 3)^4 (z - 4)}$. Here $z = 1$ and $z = 4$ are the simple poles of the function $f(z)$ and $z = 3$ is a pole of order 4 of $f(z)$.

(iii) Branch point : Let $f(z)$ be a multiple valued function. Then a singular point $z = z_0$ is called a branch point of the function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0.

Example 149 : Let $f(z) = \log z$. Then $f(z)$ has a branch point at $z = 0$.

Example 150 : Let $f(z) = \log(z^2 - 5z + 6)$. Then $f(z)$ has a branch points where $z^2 - 5z + 6 = 0 \Rightarrow z = 2$ and $z = 3$ are branch points.

Example 151 : Let $f(z) = (z-5)^{1/2}$. Then $f(z)$ has a branch point at $z = 5$.

(iv) Removal singularity : The singular point $z = z_0$ is called the removal singularity of the function $f(z)$

if $\lim\limits_{z \to z_0} f(z)$ exists.

Example 152 : Let $f(z) = \dfrac{\sin z}{z}$. Then the singular point $z = 0$ is called the removal singularity of $f(z)$ since

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1 \text{ exists.}$$

(vi) Singularity at infinity : The function $f(z)$ will have a singularity at $z = \infty$ if $w = 0$ is a singularity of $f(1/w)$.

Example 153 : Let $f(z) = az^2 + bz + c$ has a pole of second order at $z = \infty$ since $f(1/w) = \dfrac{cw^2 + bw + a}{w^2}$ has a pole of order 2 at $w = 0$.

Example 154 : For the function $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$, locate and name all the singularities in the finite z plane and also determine where $f(z)$ is analytic.

D. U. H. T. 86; R. U. M. SC. P. 84.

Solution : We have $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 \ (3z+2)^2}$ ------ (1)

Then in the finite z plane, the singularities of the function $f(z)$ are located at the points $z = 1$ and $z = -2/3$. Here $z = 1$ is a pole of order 3 and $z = -2/3$ is a pole of order 2. Now we will find the singularities at $z = \infty$. Let $z = 1/\omega$, then (1) $\Rightarrow$ $f(1/\omega) = \dfrac{2\,\omega^8 + \omega^4 + 1}{\omega^3\,(1-\omega)^3\,(3+2\omega)^2}$. Here $\omega = 0$ is a pole of order 3 for the function $f(1/\omega) \Rightarrow z = \infty$

is a pole of order 3 for the original function $f(z)$. Thus the given function has three singularities : (1) $z = 1$ is a pole of order 3; (2) $z = -2/3$ is a pole of order 2 and (3) $z = \infty$ is a pole of order 3. It follows that the given function is analytic every where in the finite z plane except at the points $z = 1$ and $z = -2/3$.

181. Some definition on curves and involving ∇ :

(i) Continuous curve or arc : Let $x = \phi(t)$ and $y = \psi(t)$ be two real functions of the real variable t which are continuous in the interval $t_1 \le t \ge t_2$. Then the parametric equation $z = x + iy = \phi(t) + i\,\psi(t) = z(t)$ ----- (1) in the interval $t_1 \le t \le t_2$, is called a continuous curve or arc in the Agrand plane joining the points $a = z(t_1)$ and $b = z(t_2)$.

(ii) Closed curve : In (1), if $t_1 \neq t_2$ but $z(t_1) = z(t_2)$ i. e. if $a = b$, then the curve (1) is called closed.

(iii) Simple closed curve : A closed curve which does not intersect itself any where in the Aragand plane is called a simple closed curve.

(iv) Smooth curve or arc : In (1), if $\phi(t)$ and $\psi(t)$ i. e. if $z(t)$ have continuous derivative in $t_1 \le t \le t_2$, then the curve (1) is called a smooth curve or arc.

(v) Sectionally (or piecewise) smooth curve or contour : A curve which is composed of a finite number of smooth arcs is called a sectionally smooth curve.

(vi) Complex .differential operator ∇ (del) : The (complex) differential operator ∇ is defined by $\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ $= 2\frac{\partial}{\partial \bar{z}}$.

(vii) Complex differential operator $\overline{\nabla}$ (del bar) : The (complex) differential operator is defined by $\overline{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$

$$= 2\frac{\partial}{\partial z}.$$

(viii) Gradient of a real function : The gradient of real function F(x, y) (which is a scalar) is defined by grad $F = \nabla F$

$$= \frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y} = 2\,\frac{\partial G}{\partial \bar{Z}} \text{ where } F(x, y) = F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right)$$

$$= G(z, \bar{z}).$$

(ix) Gradient of a complex function :

The gradient of a complex function $A = P + iQ$ (which is a vector) is defined by grad $A = \nabla A = \left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$ $(P + iQ)$

$$= \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) = 2\,\frac{\partial A}{\partial \bar{z}} \text{ where } A(z, \bar{z})$$

$$= P(x, y) + iQ(x, y).$$

(x) Divergence : The divergence of a complex function A (which is a vector) is defined by div $A = \nabla o A = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$

where o denotes dot product and $A = P(x, y) + i\,Q(x, y)$.

(xi) Curl : The curl of a complex function A (which is a vector) is defined by

$$\text{Curl } A = \nabla \times A = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \text{ where } A = P(x, y) + i\,Q(x, y)$$

and $\times$ denotes the cross product.

(xii) Laplacian : The Laplaciar operator is defined by $\nabla o \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z \partial \bar{z}}$ where 0 denotes the dot product.

182. Some important identities involving gradient, divergence and curl : If A, A_1 and A_2 are differentiable functions, then the following identities hold good

(1) $\nabla(A_1 + A_2) = \nabla A_1 + \nabla A_2$; (2) $\nabla o (A_1 + A_2) = \nabla o A_1 + \nabla o A_2$; (3) $\nabla \times (A_1 + A_2) = \nabla \times A_1 + \nabla \times A_2$; (4) $\nabla (A_1 A_2) = A_1 \nabla A_2 + A_2 \nabla A_1$; (5) $\nabla \times (\nabla A) = 0$ if A is real or generally, if Im { A } is harmonic; (6) $\nabla o (\nabla A) = 0$ if A is imaginary or genarally, if Re { A } is harmonic.

Theorem 97 : Show that (i) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$;

(ii) $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$ where $z = x + iy$ and $\bar{z} = x - iy$.

Proof : (i) ϕ is any continuously differentiable function, then we have $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial z}\dfrac{\partial z}{\partial x} + \dfrac{\partial \phi}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial x}$

$$= \frac{\partial \phi}{\partial z}\frac{\partial}{\partial x}(x + iy) + \frac{\partial \phi}{\partial \bar{z}}\frac{\partial}{\partial x}(x - iy)$$

$$= \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}.$$

(ii) Again we have $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \Phi}{\partial z}\dfrac{\partial z}{\partial y} + \dfrac{\partial \phi}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial y}$

$$= i\left(\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial \bar{z}}\right) \Rightarrow \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

Theorem 98 : (i) Show that : $\nabla = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial \bar{z}}$;

(ii) $\bar{\nabla} = \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial z}$ where $z = x + iy$ and $\bar{z} = x - iy$.

Proof: (i) We have $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$ and $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$

Then $\nabla = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} + i.\, i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$

$$= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = 2 \frac{\partial}{\partial \bar{z}}.$$

(ii) $$\bar{\nabla} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - i.\, i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} = 2 \frac{\partial}{\partial z}.$$

Theorem 99 : Show that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} = 4 \frac{\partial^2}{\partial \bar{z} \partial z}.$$

Proof : We have $\nabla = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = 2 \frac{\partial}{\partial \bar{z}}$ and

$\bar{\nabla} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} = 2 \frac{\partial}{\partial z}$. Then $\nabla^2 = \nabla \bar{\nabla} = \bar{\nabla} \nabla$

$$= \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(2 \frac{\partial}{\partial z}\right)\left(2 \frac{\partial}{\partial \bar{z}}\right) = \left(2 \frac{\partial}{\partial \bar{z}}\right)\left(2 \frac{\partial}{\partial z}\right)$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} = 4 \frac{\partial^2}{\partial \bar{z} \partial z}$$

where we have considered $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$

Expample 155 : Show that if u is harmonic, then $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Solution : If u is harmonic, then $\nabla^2 u = 0$ ---------- (1)

But we have $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$ ------------ (2)

Now by (1) and (2), we have $4 \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0 \Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$

Theorem 100 : Show that $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$

$$= \left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right) |f'(z)|^2$$

if $\omega = f(z) = u + iv$ is analytic, $f'(z) \neq 0$ and ϕ is any function of x and y having differential coefficients of first and second order.

Proof : We have $\omega = f(z) = u + iv$ and $\overline{\omega} = f(\overline{z}) = u - iv$.

Then $u = \frac{1}{2}(\omega + \overline{\omega})$ and $v = \frac{1}{2i}(\omega - \overline{\omega})$

Now $\frac{\partial}{\partial \omega} = \frac{\partial}{\partial u}\frac{\partial u}{\partial \omega} + \frac{\partial}{\partial v}\frac{\partial v}{\partial \omega} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$ and

$$\frac{\partial}{\partial \overline{\omega}} = \frac{\partial}{\partial u}\frac{\partial u}{\partial \overline{\omega}} + \frac{\partial}{\partial v}\frac{\partial v}{\partial \overline{\omega}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Then $\frac{\partial^2}{\partial \omega \partial \overline{\omega}} = \frac{1}{4}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \Rightarrow 4\frac{\partial^2\phi}{\partial \omega \partial \overline{\omega}} = \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} \quad \text{----- (1)}$$

Again we have $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Then $\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$\therefore z = x + yi$

$\overline{z} = x - iy$

$x = \frac{1}{2}(z + \overline{z})$

$y = \frac{1}{2i}(z - \overline{z})$

$$\Rightarrow 4\frac{\partial^2\phi}{\partial z \partial \overline{z}} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \quad \text{-------- (2)}$$

But $4\frac{\partial^2}{\partial \omega \partial \overline{\omega}} = 4\left(\frac{\partial}{\partial z}\frac{dz}{d\omega}\right)\left(\frac{\partial}{\partial \overline{z}}\frac{d\overline{z}}{d\overline{\omega}}\right)$

$$= 4 \left\{\frac{1}{f'(z)} \cdot \frac{1}{f'(\bar{z})}\right\} \frac{\partial^2}{\partial z \partial \bar{z}}$$

$$\Rightarrow |f'(z)|^2 \, 4 \frac{\partial^2 \phi}{\partial \omega \partial \bar{\omega}} = 4 \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \quad \text{--- (3)}$$

$\therefore f'(z)\, f'(\bar{z})$

$= f'(z)\, \overline{f'(z)}$

$= |f'(z)|^2$

Now by (1), (2) and (3) we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right) |f'(z)|^2 .$$

Example 156 : Show that $\psi = \log |f(z)|$ is harmonic in a region $\mathfrak{R}$ if $f(z)$ is analytic in $\mathfrak{R}$ and $f(z) f'(z) \neq 0$ in $\mathfrak{R}$.

R. U. H. 75, 77.

Solution : We have $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \psi = 4 \frac{\partial^2 \psi}{\partial z \partial \bar{z}}$

$$= 4 \frac{\partial}{\partial z \partial \bar{z}} \log |f(z)|$$

$$= 4 \frac{\partial}{\partial z \partial \bar{z}} [\tfrac{1}{2} \log | f(z)|^2] = 2 \frac{\partial}{\partial z \partial \bar{z}} [\log \{ f(z) f(\bar{z}) \}]$$

$$= 2 \frac{\partial}{\partial z \partial \bar{z}} [\log f(z) + \log f(\bar{z})] = 2 \frac{\partial}{\partial z} \left[0 + \frac{f'(\bar{z})}{f(\bar{z})}\right]$$

$= 0 \Rightarrow \psi$ is harmonic.

Example 157 : If $f(z)$ is an analytic function of z, then show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 |f'(z)|^2$.

R. U. H. 73; R. U. M. Sc. P. 84; D. U. H. 86, 90; D. U. M. Sc. P. 86, 90.

Solution : We have $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right) |f(z)|^2$

$= 4 \dfrac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^2$

$= 4 \dfrac{\partial^2}{\partial z \partial \bar{z}} [f(z) f(\bar{z})] = 4 \dfrac{\partial}{\partial z} [f(z) f'(\bar{z})]$

$= 4 f'(z) f'(\bar{z}) = 4 |f'(z)|^2$. |

Example 158 : If $f(z)$ is analytic , then show that

$\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$.

R. U. H. '88 ; R. U. M. Sc. P. '88

Solution : We have $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^p$

$= 4 \dfrac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^p = 4 \dfrac{\partial^2}{\partial z \partial \bar{z}} \{|f(z)|^2\}^{p/2}$

$= 4 \dfrac{\partial^2}{\partial z \partial \bar{z}} \{f(z) f(\bar{z})\}^{p/2}$

$= 4 \dfrac{\partial^2}{\partial z \partial \bar{z}} [\{f(z)\}^{p/2} \{f(\bar{z})\}^{p/2}]$

$= 4 \dfrac{\partial}{\partial z} [p/2 \{f(z)\}^{p/2} \{f(\bar{z})\}^{p/2-1} f'(\bar{z})$

$= 4 (p/2)(p/2) \{f((z)\}^{p/2-1} f'(z) \{f(\bar{z})\}^{p/2-1} f'(\bar{z})$

$= p^2 \{f(z) f(\bar{z})\}^{p/2-1} f'(z) f'(\bar{z})$

$= p^2 \{|f(z)|^2\}^{p/2-1} |f'(z)|^2$

$= p^2 |f(z)|^{p-2} |f'(z)|^2$.

Example 159 : If $f(z)$ is analytic, then show that

$\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |\text{Re } f(z)|^2 = 2 |f'(z)|^2$. Also find

$\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |\text{Im } f)z)|^2$.

Solution : Let $f(z) = u + iv$ --- (1), then $\overline{f(z)}$ $= f(\bar{z}) = u - iv$ ---- (2). Now (1) + (2)

$\Rightarrow$ $u = \frac{1}{2}\{ f(z) + f(\bar{z}) \} = \text{Re} f(z).$

But we have $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Re}\, f(z)|^2$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |u|^2$

$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} [\frac{1}{2}\{ f(z) + f(\bar{z}) \}]^2$

$= 2 \frac{\partial}{\partial z}\{ f(z) + f(\bar{z})\} f'(\bar{z}) = 2 f'(\bar{z}) \} \; |f'(\bar{z})$

$= 2 |f'(z)|^2$

Second part : Try yourself.

Example 160 : If $f(z)$ is analytic, then show that

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |u|^\rho = \rho(\rho - 1) \; |u|^{\rho-2} \; |f'(z)|^2.$

Solution : We have $f(z) = u + iv$ and $f(\bar{z})$

$= u - iv \Rightarrow u = \frac{1}{2}\{ f(z) + f(\bar{z}) \} = \bar{u}$ [since u is real]

Again $u^2 = u\bar{u} = |u|^2$. Now $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |u|^\rho$

$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} (|u|^2)^{\rho/2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}} (u^2)^{\rho/2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}} u^\rho$

$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{\frac{f(z) + f(\bar{z})}{2}\right\}^\rho$

$= \frac{4\rho}{2^\rho} \frac{\partial}{\partial z} \{ f(z) + f(\bar{z})\}^{\rho-1} f'(\bar{z})$

$= \frac{\rho(\rho - 1)}{2^{\rho-2}} \{ f(z) + f(\bar{z}) \}^{\rho-2} f'(\bar{z}) f'(\bar{z})$

$$= \rho(\rho-1)\left\{\frac{f(z)+f(\bar{z})}{2}\right\}2^{\rho-2}\ |f'(z)|^2$$

$$= \rho(\rho-1)\ |u|^{\rho-2}\ |f'(z)|^2.$$

Theorem 101 : If $f(z)$ is analytic function of z, then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^{\rho} = \rho^2\ |f(z)|^{\rho-2}\ |f'(z)|^2 \ldots (1)$$

R. U. M. Sc. P. 88.

Also, from it show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$,

R. U. H. 73; D. U. H. 86; 90; D. U. M. Sc. P. 84, 88; R. U. M. Sc. P. 84; R. U. H. 73 .

and $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^3 = 9\ |f(z)|\ |f'(z)|^2$. **R. U. 88.**

Proof : We have $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^{\rho}$

$$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\{|f(z)|^2\}^{\rho/2}$$

$$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\{f(z)\,f(\bar{z})\}^{\rho/2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}\left[\{f(z)\}^{\rho/2}\ \{f(\bar{z})\}^{\rho/2}\right]$$

$$= 4\frac{\partial}{\partial z}\left[\rho/2\ \{f(z)\}^{\rho/2}\ \{f(\bar{z})\}^{\rho/2-1}\ f'(\bar{z})\right]$$

$$= 2\rho\left[\rho/2\ \{f(z)\}^{\rho/2-1}\ f'(\bar{z})\ \{f(\bar{z})\}^{\rho/2-1}\ f'(\bar{z})\right]$$

$$= \rho^2\{f(z)\,f(\bar{z})\}^{\rho/2-1}\ \{f'(z)\,f'(\bar{z})\}$$

$$= \rho^2\{|f(z)|^2\}^{\rho/2-1}\ |f'(z)|^2 = \rho^2\ |f(z)|^{\rho-2}\,|f'(z)|^2.$$

Second Part : Now putting $\rho = 2$, in (1),

we have $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\ |f'(z)|^2.$

Third Part : Now putting $n = 3$, in (1)

we get $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^3 = 9\,|f(z)|\,|f'(z)|^2$.

Example 161 : If $f(z)$ is an analytic function such that $f'(z) \neq 0$, then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f'(z)| = 0$ ..... (1).

If $|f'(z)|$ is the product of a function of x and a function of y, then show that $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$ where α is real and β and γ are complex constants. **J. U. H. 89**

Solution : We have $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f'(z)|$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}} \log (|f'(z)|$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}} \log |f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}} \log \{f'(z)\, f'(\bar{z})\}$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}} \{\log f'(z) + \log f'(\bar{z})\}$$

$$= 2\frac{\partial}{\partial z}\left\{\frac{f''(\bar{z})}{f'(\bar{z})}\right\} = 0.$$

Second part : Let $|f'(z)| = P(x)\, Q(y)$. Then (1)

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log\{P(x)\, Q(y)\} = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \{\log P(x) + \log Q(y)\} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} \log P(x) + \frac{\partial^2}{\partial y^2} \log Q(y) = 0$$

$$\Rightarrow \frac{d^2}{dx^2} \log P(x) + \frac{d^2}{dy^2} \log Q(y) = 0 \text{ ---- } (2)$$

since P (x) and Q (y) are functions of one variable x and y respectively.

Let $\frac{d^2}{dx^2} \log P(x) = c$ (say) ---- (3), then $\frac{d^2}{dy^2} \log Q(y) = -c$ ---(4)

Now by successive integration (3)

$$\Rightarrow \log P(x) = \frac{1}{2} cx^2 + dx + e$$

$$\Rightarrow P(x) = \exp\left(\frac{1}{2} cx^2 + dx + e\right) \text{ -- (5) and (4)}$$

$$\Rightarrow Q(x) = \exp\left(-\frac{1}{2} y^2 + d'y + e'\right) \text{ --- (6)}$$

where d, e, d′ and e′ are real constants.

Now by (5) and (6) we have

$|f'(z)| = P(x) \cdot Q(y)$

$= [\exp(\frac{1}{2} cx^2 + dx + e)]\ [\exp(-\frac{1}{2} cy^2 + d'y + c')]$

$= \exp\{\frac{c}{2} (x^2 - y^2) + dx + d'y + (e + e')\}$ ---- (7)

Now putting $\beta = a_1 + ib_1$, $\gamma = c_1 + id_1$ and $z = x + iy$,

we have $|\exp(\alpha z^2 + \beta z + \gamma)| = \exp\{\alpha(x^2 - y^2) + a_1 x - b_1 y + c_1\}$ --- (8) since $|\exp(A + iB)| = \exp A$.

Now by (7) and (8) we have $|f'(z)| = \exp(\alpha z^2 + \beta z + \gamma)$ where $\alpha = c/2$, $\beta = a_1 + ib_1$, $\gamma = c_1 + id_1$, $a_1 = d$, $-b_1 = d'$, $c_1 = e + e'$.

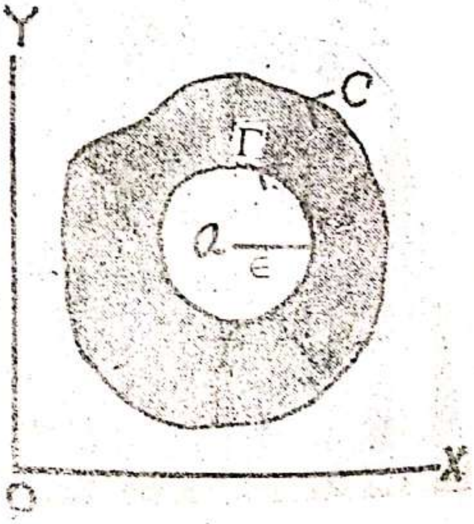
CHPTER-5

CAUCHY'S INTEGRAL FORMULAE AND RELATED THEOREMS

Cauchy's integral formula (Theorem–123) : Let $f(z)$ be analytic inside and on a simple closed curve C. If a is any point inside C, then $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$, where C is traversed in the positive sense, i. e. in the counter clockwise direction.

D. U. H. T. 75, 83, 85,88, 91; D. U. H. 86, 88, 89; J. U. H. 83, 85, 85, 87, 88; R. U. H. 73, 75, 77; C. U. H. 81, 86, 87; D. U. 72; R. U. M. Sc. P. 84, 86; R. U. 80, 82.

First proof : Here the function $\frac{f(z)}{z-a}$ is analytic inside and on the simple closed curve C except at the point z = a. Then by a corollary of Cauchy's fundamental integral theorem, we have $\oint_C \frac{f(z)}{z-a}\, dz = \oint_\Gamma \frac{f(z)}{z-a}\, dz \cdots (1)$, where Γ is



a circle of radius ε with centre at $z = a$ inside C. Now the equation of Γ is $|z-a| = \varepsilon \Rightarrow z - a = \varepsilon e^{i\theta} \Rightarrow z = a + \varepsilon e^{i\theta}$

where $0 \leq \theta \leq 2\pi$. Now putting $z = a + \varepsilon e^{i\theta}$, $dz = i\varepsilon e^{i\theta} d\theta$ and limit of θ from o to 2π in (1), we have $\oint_{\Gamma} \frac{f(z)}{z-a} dz$ $= \int_0^{2\pi} \frac{f(a + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = i \int_0^{2\pi} f(a + \varepsilon e^{i\theta}) d\theta$ ... (2). Now by (1) and (2), we have $\oint_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a + e^{i\theta}) d\theta$ ... (3). Now taking the limit $\varepsilon \to 0$ of both sides of (3) and using the continuity of $f(z)$, we get $\oint_C \frac{f(z)}{z-a} = \lim_{\varepsilon \to 0} i \int_0^{2\pi} f(a + \varepsilon e^{i\theta}) d\theta$

$$= i \int_0^{2\pi} \lim_{\varepsilon \to 0} f(a + \varepsilon e^{i\theta}) d\theta = i \int_0^{2\pi} f(a) d\theta = 2\pi i \, f(a)$$

$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$ and our required result is proved.

Second Proof : Since $f(z)$ is analytic inside C including the point a, then we have $f(z) = f(a) + (z - a) f'(a) + (z - a)\eta$ ... (1), where $\eta \to 0$ as $z \to a$.

Now $\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$

$$= \frac{1}{2\pi i} \oint_C \frac{f(a) + \{f(z) - f(a)\}}{z-a} dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(a)}{z-a} dz + \frac{1}{2\pi i} \oint_C \frac{f(z) - f(a)}{z-a} dz$$

$$= \frac{f(a)}{2\pi i} \oint_C \frac{dz}{z-a} + \frac{1}{2\pi i} \oint_C \frac{(z-a) f'(a) + (z-a)\eta}{z-a} dz \text{ [by (1)]}$$

$$= \frac{f(a)}{2\pi i}(2\pi i) + \frac{f'(a)}{2\pi i}\oint_C dz + \frac{1}{2\pi i}\oint_C \eta\, dz \quad \left[\because \oint_C \frac{dz}{z-a} = 2\pi i\right]$$

$$= f(a) + 0 + \frac{1}{2\pi i}\oint_C \eta dz \quad \left[\begin{array}{l}\text{The second part is zero by}\\ \text{the Cauchy's funda-mental}\\ \text{integral theorem}\end{array}\right]$$

$$= f(a) + \frac{1}{2\pi i}\oint_C \eta dz.$$

Now we take C so small that for all points on it $|\eta| < \varepsilon$, then

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz - f(a)\right| = \left|\frac{1}{2\pi i}\oint_C \eta dz\right| < \frac{\varepsilon}{2\pi}L \ldots (2),$$

where L is the length of C. In the equality of (2), the right-hand side tends to zero and the left-hand side is independent of ε. Therefore, it must be zero.

Thus $f(a) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-a}dz$.

Theorem -124 : If $f(z)$ is analytic inside and on the boundary C of a simply-connected closed region $\Re$, then

$$\frac{1}{2\pi i}\oint_C \left\{\frac{1}{z-a} \pm \frac{1}{z-b}\right\} f(z)\, dz$$

$$= \begin{cases} f(a) \pm f(b) & \text{if the points a and b are inside C} \\ f(a) & \text{if only the point a lies inside C} \\ \pm f(b) & \text{if only the point b lies inside C} \\ 0 & \text{if the points a and b are not inside C} \end{cases}$$

Proof : Try yourself.

Example 182 : Show that $\dfrac{1}{2\pi i}\oint_C \dfrac{e^z}{z-2}dz$

Final

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$$= \begin{cases} e^2 & \text{if C is the circle } |z| = 3 \\ 0 & \text{if C is the circle } |z| = 1. \end{cases}$$ D. U. H. '84.

Solution : First part : Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$. Again $z = a = 2$ lies inside the circle $|z| = 3$ and the Cauchy's integral formula can be applied there. Then $\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = f(2) = e^2$.

Second part : Here $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z| = 1$, then by the Cauchy's integral theorem, we have $\oint_C f(z)\, dz = 0 \Rightarrow \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = 0$.

Example 183 : Show that $\oint_C \frac{\sin 3z}{z+\pi/2} dz = 2\pi i$, where C is the circle $|z| = 5$. R. U. H. '81.

Solution : Here $f(z) = \sin 3z$ is analytic inside and on the circle $|z| = 5$. Again $z = -\pi/2$ lies inside the given circle and therefore by Cauchy's integral formula, we have $\oint_C \frac{\sin 3z}{z+\pi/2} dz = 2\pi i\, f(-\pi/2) = 2\pi i \sin 3(-\pi/2) = 2\pi i$.

Example 184 : Show that $\oint_C \frac{e^{3z}}{z-\pi i} dz$ R. U. H. '81, '86.

$$\begin{cases} -2\pi i & \text{if C is the circle } |z-1| = 4 \\ 0 & \text{if C is the ellipse } |z-2| + |z+2| = 6. \end{cases}$$

Solution : (First part) : Let $f(z) = e^{3z}$ and it is analytic inside and on the given circle $|z-1| = 4$. Again $z = \pi i$ is a point inside the given circle. Then by the Cauchy's integral formula, we have $\oint_C \frac{e^{3z}}{z-\pi i}\,dz = 2\pi i\, f(\pi i) = 2\pi i\, e^{3\pi i} = -\,2\pi i$

since $e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -\,1$.

Second part : Let $f(z) = \frac{e^{3z}}{z-\pi i}$. Here

$|z-2| + |z+2| = 6$ is an ellipse with foci at $(2, 0)$ and $(-2, 0)$ whose major axis has length 6. Therefore, the point $z = \pi i$ lies outside the ellipse. Thus $f(z)$ is analytic inside and on the ellipse C and by Cauchy's integral theorem $\oint_C f(z)\,dz = \oint_C \frac{e^{3z}}{z-\pi i}\,.\,dz = 0.$

Example 185 : show that

$\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz = \frac{\pi i}{32}$, where C is the circle $|z| = 1$.

Solution : Here $\sin^6 z$ is analytic inside and on the circle $|z| = 1$ and $z = \pi/6$ is a point inside the given circle. Then by the Cauchy's integral formula $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz =$ $2\pi i\,(\sin \pi/6)^6 = 2\pi i \left(\frac{1}{2}\right)^6 = \frac{\pi i}{32}$.

Example 186 : show that $\oint_C \frac{\cos z}{z-\pi}\,dz = -\,2\pi i$, where C is the circle $|z-1| = 3$.

Solution : Here $f(z) = \cos z$ and the point $z = a = \pi$ lies inside the given circle, then by the Cauchy's integral formula, we have $\frac{1}{2\pi i}\oint_C \frac{\cos z}{z-\pi}\,dz = \cos \pi = -\,1 \Rightarrow \oint_C \frac{\cos z}{z-\pi}\,dz$ $= -\,2\pi i.$

Example 187 : Show that $\oint_C \frac{dz}{z-2} = 2\pi i$, where C is the circles $|z-2| = 4$ or $|z-2| = 5$.

Solution : Here $f(z) = 1$ and the point $z = a = 2$ lies inside the both circles. Therefore the same result can be obtained and by the Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z-2} = 1 \Rightarrow \oint_C \frac{dz}{z-2} = 2\pi i.$$

Example 188 : Show that $\oint_C \frac{e^z}{z(z+1)}\,dz = 2\pi i(1 - e^{-1})$, where C is the circle $|z-1| = 3$.

Solution : We have $\oint_C \frac{e^z}{z(z+1)}\,dz = \oint_C \left(\frac{1}{z} - \frac{1}{z+1}\right) e^z dz$

$= \oint_C \frac{e^z}{z}\,dz - \oint_C \frac{e^z}{z+1}\,dz = 2\pi i e^0 - 2\pi i e^{-1} = 2\pi i(1 - e^{-1})$, by the Cauchy's integral formula.

Example 189 : Show that $\oint_C \frac{dz}{z-2} = 2\pi i$, where C is the square with vertices $2 \pm 2i, -2 \pm 2i$.

Solution : Try yourself.

Example 190 : Show that $\oint_C \frac{dz}{z+1} = 2\pi i$, where C is the circle $|z| = 2$.

Solution : Here $a = -1$, which lies inside C and $f(z) = 1$. Then by the Cauchy's integral formula, $\oint_C \frac{dz}{z+1} = 2\pi i\, f(-1)$ $= 2\pi i$.

Example 191 : Show that $\oint_C \frac{\cos^2 tz}{z^3} = -2\pi i t^2$, where C is the circle $|z| = 1$ and $t > 0$.

Solution : Let $f(z) = \cos^2 tz = \frac{1}{2}(1 + \cos 2tz)$ and we have $z = a = 0$. Then $f'(z) = -\frac{1}{2}(2t \sin 2tz)$ and $f''(z) = -\frac{1}{2}(4t^2 \cos 2tz) = -2t^2 \cos 2tz$. Again, we have $f''(0) = -2t^2$. Then $\oint_C \frac{\cos^2 tz}{z^3} dz = 2\pi i \frac{f''(0)}{2!} = -2\pi i t^2$.

Example 192 : Show that $\oint_C \frac{e^{tz}}{z^2+1} dz = 2\pi i \sin t$, where C is the circle $|z| = 3$ and $t > 0$.

J. U. H. 88; R. U. H. 75; D. U. H. 85; R. U. M. SC. P. 84; C. U. M. SC. P. 86.

Solution : Here $f(z) = e^{tz}$ is analytic inside and on the given circle $|z| = 3$. Again we have $\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$. Since $z = i$ and $z = -i$ are inside C, then by Cauchy's integral formula $\oint_C \frac{e^{tz}}{z^2+1} dz$

$$= \frac{1}{2i}\left\{\oint_C \frac{e^{tz}}{z-i} dz - \oint_C \frac{e^{tz}}{z+i} dz\right\}$$

$$= \frac{1}{2i}\{2\pi i\, f(i) - 2\pi i f(-i)\} = \pi(e^{it} - e^{-it}) = 2\pi i \sin t.$$

Example 193 : Show that $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1} dz$

$$= \begin{cases} 0 \text{ if C is the rectangle with vertices at } 2 \pm i, -2 \pm i \\ -\frac{1}{2} \text{ if C is the rectangle with vertices at } \pm i, 2 \pm i. \end{cases}$$

Solution : We have $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\, dz$

$$= \frac{1}{4\pi i}\oint_C \left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1}\right] dz$$

$$= \frac{1}{4\pi i}\left[\oint_C \frac{\cos \pi z}{z-1}\, dz - \oint_C \frac{\cos \pi z}{z+1}\, dz\right] \quad .... \quad (1)$$

(First part) : Here $\cos \pi z$ is analytic inside and on C and also both points $z = \pm 1$ lies inside the rectangle $2 \pm i, -2 \pm i$. Then by the Cauchy's integral formula $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}\, dz$

$= \frac{1}{4\pi i}[2\pi i \cos\pi - 2\pi i \cos(-\pi)] = \frac{1}{4\pi i}[-2\pi i + 2\pi i] = 0$, by (1).

Second part : Here only the point $z = 1$ lies inside the rectangle $\pm i, 2 \pm i$. Then using the Cauchy's integral formula and also the Cauchy's integral theorem, we have

$$\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\, dz = \frac{1}{4\pi i}[2\pi i \cos \pi - 0] = -\frac{1}{2}, \text{ by (1).}$$

Example 194 : show that $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 4\pi i$, where C is the circle $|z| = 3$.

Solution : Here $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$. Then by the Cauchy's integral formula $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

$$= \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz$$

$= 2\pi i(\sin 4\pi + \cos 4\pi) - 2\pi i\,(\sin \pi + \cos \pi) = 2\pi i(1) - 2\pi i(-1) = 4\pi i$, where $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic inside and on C and we have a = 2 and a = 1 both lies inside C.

Cauchy's integral formula for the first derivative of an analytic function (Theorem — 125) : Let $f(z)$ be analytic inside and on a simple closed curve C and a is a point inside C. Then $f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz.$

R. U. 80; R. U. H. 73, '80, '82, '85; R. U. M. SC. P. '84, '85; D. U. M. SC. P. '89; D. U. M. SC. P. T. '88; D. U. H. T. '90; D. U. H. '87, '90; J. U. H. '91.

Proof : Let a be any point inside C and a + h be a neighbouring point of a, which is also inside C. Now by Cauchy's integral formula, we have $\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)\,(z-a-h)}$

and also $\frac{f(a+h)-f(a)}{h} - \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^2}$

$$= \frac{h}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^2\,(a-a-h)} \quad \dots (1).$$

Now if $h \to 0$, then (1)

$$\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(a)}{h} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz = 0 \Rightarrow f'(a)$$

$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$ provided that $\left|\oint_C \frac{f(z)\,dz}{(z-a)^2\,(z-a-h)}\right|$

is bounded as $h \to 0$. Since $f(z)$ is analytic, then it is continuous on C and it is bounded. Suppose that $|f(z)| \leq M$ on C and that the shortest distance from a to C is δ, i. e. $|z - a| \geq \delta$. If L be the length of C and $|h| \leq \delta$, then $\left| \oint_C \frac{f(z)}{(z-a)^2 (z-a-h)} dz \right| < \frac{ML}{\delta^2 (d - |h|)}$ which is bounded as $|h| \to 0$. Thus $f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$. Hence the theorem is proved.

Cauchy's integral formula for the second derivative of an analytic function (Theorem — 126) : Let $f(z)$ be analytic inside and on a simple closed curve C and a is a point inside C. Then $f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz$.

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Proof : Let a be any point inside C and a + h be a neighbouring point of a, which is also inside C. Now by Cauchy's first derivative integral formula, we have

$$\frac{f'(a+h) - f'(a)}{h} = \frac{1}{2\pi i} \oint_C \frac{1}{h} \left\{ \frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2} \right\} f(z)\, dz = \frac{2!}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-a)^3} + \frac{h}{2\pi i} \oint_C \frac{3(z-a) - 2h}{(z-a-h)^2 (z-a)^3} f(z)\, dz \quad \ldots (1)$$

Now if $h \to 0$, then (1) $\Rightarrow \lim\limits_{h \to 0} \frac{f'(a+h) - f'(a)}{h} - \frac{2!}{2\pi i} \oint_C \frac{f(z) dz}{(z-a)^3} = 0 \Rightarrow f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z) dz}{(z-a)^3}$ provided that

$\left|\int_C \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3} f(z)\,dz\right|$ is bounded as $h \to 0$. Since $f(z)$ is analytic, then it is continuous on C and it is bounded. Suppose $|\{3(z-a)-2h\} f(z)| \le M$ and that the shortest distance from a to C is δ, i. e. $|z-a| \ge \delta$. If L be the length of C and $|h| < \delta$, then $\left|\oint_C \frac{\{3(z-a)-2h\} f(z)dz}{(z-a-h)^2(z-a)^3}\right| \le \frac{ML}{(\delta-|h|)^2\delta^3}$

which is bounded as $|h| \to 0$. Thus $f''(a)$

$= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz$. Hence the theorem is proved.

Theorem -127 : Show that $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}$ where $n = 0, 1, 2, 3, \ldots$.

D. U. H. '89, '90; J. U. H. '86, '88, '89, '90; R. U. H. '85; D. U. M. SC. P. '89.

Proof : BY the above three theorems show that the theorem is true for $n = 0, 1, 2$. In a similar way we can prove the theorem for $n = 3, 4, 5, \ldots$. The second proof is given below.

Cauchy's integral formula for the nth derivative of an analytic function (Theorem — 128) : Let $f(z)$ be analytic inside and on the boundary C of a simply-connected region $\Re$. Then $f(z)$ has, at every point $z = a$ of $\Re$, derivatives of all orders and their values are given by

$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$, where $n = 1, 2, 3, \ldots\ldots$

J. U. H. '86, '88, '89, '90; R. U. H. '85; D. U. H. '89, '90; D. U. M. SC. P. '89.

Proof : Using mathematical induction, we will prove the theorem. If n = 1, then $f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$ and the theorem is true for n = 1. Now we assume that the theorem is true for n = m. Then we have $f^{(m)}(a)$

$$= \frac{m!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{m+1}}\,dz. \text{ Now } \frac{f^{(m)}(a+h) - f^{(m)}(a)}{h}$$

$$= \frac{1}{h}\,\frac{m!}{2\pi i}\left(\oint_C \frac{f(z)}{(z-a-h)^{m+1}}\,dz - \oint_C \frac{f(z)}{(z-a)^{m+1}}\,dz\right)$$

$$= \frac{1}{h}\,\frac{m!}{2\pi i}\oint_C \left[\frac{1}{(z-a)^{m+1}}\right]\left\{\left(1 - \frac{h}{z-a}\right)^{-(m+1)} - 1\right\} f(z)\,dz$$

$$= \frac{1}{h}\,\frac{m!}{2\pi i}\oint_C \left[\frac{1}{(z-a)^{m+1}}\left\{(1 + (m+1)\frac{h}{z-a} + \right.\right.$$

$$\left.\left.\frac{(m+1)(m+2)}{2!}\left(\frac{h}{z-a}\right)^2 + \ldots - 1\right\} f(z)\,dz \ldots (1).$$

Now taking $h \to 0$ in both sides of (1), we have

$$\lim_{h\to 0} \frac{f^{(m)}(a+h) - f^{(m)}(a)}{h} = \frac{m!\,(m+1)}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{m+2}}\,dz \Rightarrow$$

$$f^{(m+1)}(a) = \frac{(m+1)!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{m+2}} \ldots (2).$$

The above result (2) shows that the theorem is true for n = m + 1. But it is shown that it is true for n = 1 and hence it must be true for n = 1 + 1 = 2. But it is true for n = 2, then it is true for n = 2 + 1 = 3 and so on. Therefore, it is true for all integral values of n and we have

$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$. From the above result it is clear that if $f'(a)$ exists, then $f''(a)$, $f'''(a)$, ... $f^{(n)}(a)$ are exist. Thus if a function of a complex variable has a first derivative (i. e. analytic) in a simply connected region $\Re$, then all its higher derivatives exist in $\Re$ (i. e. analytic in $\Re$). Hence the theorem is proved.

Theorem -129 : If $f(z)$ is analytic in a region $\Re$, then $f'(z)$, $f''(z)$, ... are analytic in $\Re$.

Proof : Use the above theorems.

N. B. The above theorem is not necessarily true for functions of real variables.

Theorem-130 : If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region $\Re$, then u and v are harmonic in $\Re$.

Proof : If $f(z) = u + iv$ is analytic in a region $\Re$, then $f'(z)$, $f''(z)$, ... are also analytic in $\Re$. Since $f''(z)$ is analytic $\Rightarrow f''(z)$ is continuous $\Rightarrow$ the second partial derivatives of u and v are continuous. Therefore, by Theorem—84, u and v are harmonic in $\Re$. Hence the theorem is proved.

Example 195 : Using Leibnitz's rule for differentiation under the integral sign we will find the Cauchy's integral formulae :

We have $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$. Then $f'(a) = \frac{d}{da} f(a)$

$$= \frac{d}{da}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{z-a}\,dz\right\}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz,\ f''(a) = \frac{d}{da}f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{(z-a)^2}dz\right\}$$

$$= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz \ \ldots,$$

$$f^{(n)}(a) = \frac{d}{dz}f^{(n-1)}(a) = \frac{(n-1)!}{2\pi i}\oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{(z-a)^{n+1}}dz\right\}$$

$$= \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz.$$

Example 196 : Show that $\frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^3}dz = (t-\frac{t^2}{2})\ e^{-t}$

where C is any simple closed curve enclosing $z=-1$ and $t > 0$. **R. U. H. '74; C. U. M. SC. P. '87.**

Solution : Here $f(z) = ze^{zt}$ is analytic inside and on C and $a = -1$ is any point inside C. Then using the Cauchy's integral formula $\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{1}{n!}\left[\frac{d^n}{dz^n}f(z)\right]$ at $z = a$

for $n = 2$, we have $\frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^3}dz = \frac{1}{2!}\left[\frac{d^2}{dz^2}(ze^{tz})\right]$ at $z =$

$-1 = \frac{1}{2}\left[\frac{d}{dz}(e^{tz}+tze^{tz})\right]$ at $z = -1$ $= \frac{1}{2}\left[2te^{tz}+t^2ze^{tz}\right]$ at z

$= -1 = \left[t-\frac{t^2}{2}\right]\ e^{-t}$.

Example 197 : Show that $\oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i e^{-2}}{3}$, where C is the circle $|z| = 3$. **D. U. H. '91.**

Solution : Here $f(z) = e^{2z}$ is analytic inside and and on the given circle and $z = a = -1$ is a point inside

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the given circle. Then Cauchy's integral formula $f^{(n)}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}\,dz$ ... (1) is applicable there for n = 3, we have $f'''(z) = 8e^{2z} \Rightarrow f'''(a) = f'''(-1) = 8e^{-2}$ ... (2). Now putting $n = 3$, $f(z) = e^{2z}$ and $a = -1$ in (1) and using (2)

$$\Rightarrow 8e^{-2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz \Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi i e^{-2}}{3}.$$

Example 198 : Show that $\oint_C \frac{e^{iz}}{z^3}\,dz = -\pi i$, where C is the circle $|z| = 2$.

Solution : Here $f(z) = e^{iz}$ is analytic inside and on the circle $|z| = 2$ and $z = a = 0$ is a point inside the given circle. Then using the Cauchy's integral formula $\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n} f(z)\right]$ at $z = a$ for $n = 2$, we have

$$\oint_C \frac{e^{iz}}{z^3}\,dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2} e^{iz}\right]_{\text{at } z=0} = \pi i\,[i^2 e^{iz}]_{\text{at } z=0} = -\pi i.$$

Example 199 : Show that $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{21\pi i}{16}$ where C is the circle $|z| = 1$.

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Solution : Here $f(z) = \sin^6 z$ is analytic inside and on the circle $|z| = 1$ and $z = a = \pi/6$ is a point inside the given circle. Then using the Cauchy's integral formula

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n} f(z)\right]_{\text{at } z=a} \text{ for } n = 2,$$

we have $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\sin^6 z\right]_{\text{at } z=\pi/6}$

$= \pi i\left[\frac{d}{dz}(6\sin^5 z\cos z)\right]$ at $z = \pi/6$

$= \pi i\,[-6\sin^6 z + 30\sin^4 z\cos^2 z]$ at $z = \pi/6$

$= \pi i\left[-6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2\right] = \frac{(-6+90)\pi i}{64} = \frac{21\pi i}{16}.$

Example 200 : If $z = 1$ is a point inside a simple closed curve C, then show that $\oint_C \frac{5z^2-3z+2}{(z-1)^3}\,dz = 10\pi i.$

Solution : Here $5z^2 - 3z + 2$ is analytic inside and on C. Then by the Cauchy's integral formula $\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$ $= \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n}f(z)\right]_{\text{at } z=a}$ for $n = 2$, we have $\oint_C \frac{5z^2-3z+2}{(z-1)^3}\,dz$

$= \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}(5z^2-3z+2)\right]$ at $z = 1$

$= \pi i\left[\frac{d}{dz}(10z-3)\right]_{\text{at } z=1} = \pi i\,[10] = 10\pi i.$

Ezample 201 : Show that $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{n!\,z^{n+1}}\,dz.$

C. U. H. '90; R. U. H. '73.

Solution : Let $f(z) = e^{az} \Rightarrow f^{(n)}(z) = a^n e^{az}$ ... (1).

Now putting $z = 0$ in (1) we get $a^n = f^{(n)}(0)$ ... (2)

But by Cauchy's integral formula,

$$f^{(n)}(b) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-b)^{n+1}}\,dz \Rightarrow f^{(n)}(0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz$$

... (3). Now by (2) and (3) we have $\frac{a^n}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz$ ... (4)

and multiplying both sides of (4) by $\frac{a^n}{n!}$ we get $\left(\frac{a^n}{n!}\right)^2$

$$= \frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{n!z^{n+1}}\,dz.$$

Hence our required result is proved.

Example 202 : Show that $\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta.$

C. U. H. '90.

Solution : We have $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\cdot\oint_C \frac{a^n e^{az}}{n!\,z^{n+1}}\,dz$ ... (1)

Now taking summation over n in (1) from o to ∞, then

we have $\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_C \frac{a^n e^{az}}{n!z^{n+1}}\,dz$

$$= \frac{1}{2\pi i}\oint_C e^{az}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n \frac{1}{n!}\,\frac{dz}{z} = \frac{1}{2\pi i}\oint_C e^{az}\,e^{a/z}\,\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\oint_C e^{a\left(z+\frac{1}{z}\right)}\,\frac{dz}{z} \;\ldots\; (2).$$

Now let C be a circle of unit radius having centre at the origin, then $|z| = 1 \Rightarrow z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $z + \frac{1}{z} = e^{i\theta} + e^{-i\theta}$

$= 2\cos\theta$ and $0 \le \theta \le 2\pi$. Using these (2) $\Rightarrow \sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2$

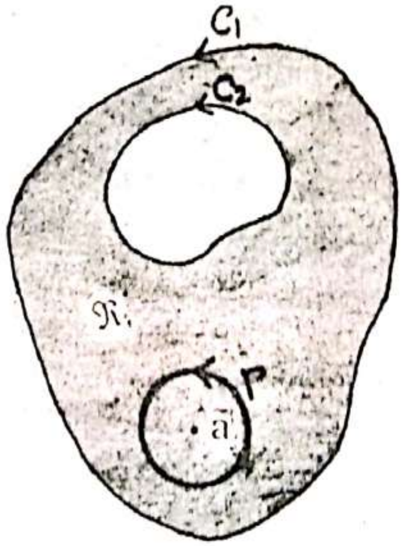
$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{2a\cos\theta}\,\frac{ie^{i\theta}\,d\theta}{e^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta.$$ Hence the required result is proved.

Cauchy's integral theorem for multiply-connected region (Theorem —131) : Let $f(z)$ be analytic inside a multiply-connected region $\Re$ and on the boundaries bounded by the simple closed curves C_1 and C_2. If a is a point in $\Re$, then $f(a)$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-a}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz,$$ where C represents the entire boundary of $\Re$ (i. e. the boundaries C_1 and C_2).

Proof : In $\Re$ we construct a circle Γ having centre at the point $z = a$ so that Γ lies entirely in $\Re$, see in the Fig. Suppose $\Re'$ consists of the set of points in $\Re$ which are exterior to Γ. Then by the Çaucly's integral theorem for multiply-connected regions, we have



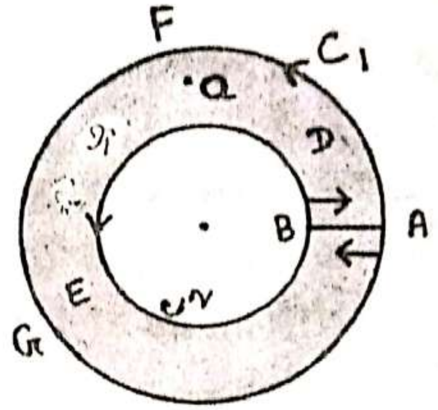
$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z-a}\,dz = 0$$

... (1). But by the Cauchy's integral formula for simply-connected regions, we have $\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z-a}\,dz = f(a)$ ... (2).

Now (1) + (2) $\Rightarrow f(a) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-a}dz$

$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$, where C represents the entire boundary of $\Re$. Hence the theorem is proved.

Cauchy's integral formula for a ring shaped region (Theorem —132) : If $f(z)$ is analytic in a region $\Re$ bounded by two concentric circles C_1 and C_2 and on the boundary, then $f(a) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-a}dz$, where a is any point in $\Re$ and C_2 lies inside C_1.



Proof : We construct a cross-cut AB where A and B are two points on the circles C_1 and C_2 resectively. Since $f(z)$ is analytic in the region $\Re$ bounded by AFGABEDBA, then by the Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\oint_{AFGABEDBA}\frac{f(z)}{z-a}dz, \text{ where a is in } \Re$$

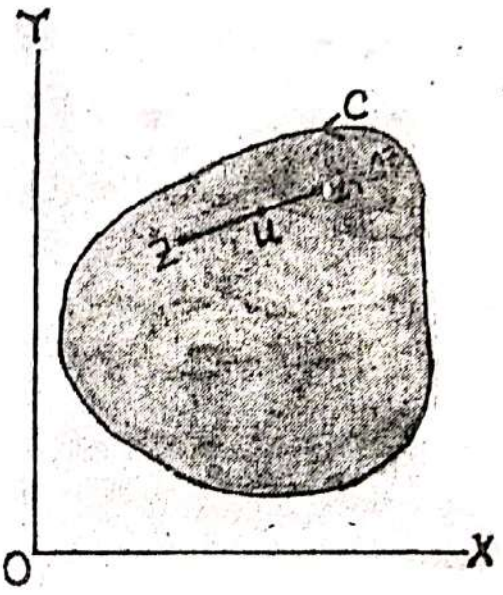
$$= \frac{1}{2\pi i}\oint_{AFGA} f(z)\,dz + \frac{1}{2\pi i}\int_{AB} f(z)\,dz + \frac{1}{2\pi i}\oint_{BEDB} f(z)dz + \frac{1}{2\pi i}\int_{BA} f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_{AFGA} f(z)\,dz - \frac{1}{2\pi i}\oint_{BDEB} f(z)\,dz,$$ [the integral along AB and BA cancel]

$= \frac{1}{2\pi i}\oint_{C_1} f(z)\, dz - \frac{1}{2\pi i}\oint_{C_2} f(z)\, dz$. Hence the theorem is proved.

Morrera's theorem or the converse of the Cauchy's theorem (Theorem — 133) : If $f(z)$ is a continuous function of z in a simply connected region $\mathfrak{R}$ and if $\oint_C f(z)\, dz = 0$ taken round every simple closed curve C in $\mathfrak{R}$, then $f(z)$ is analytic in $\mathfrak{R}$. **D. U. M. SC. P. '89; C. U. H. '89; D. U. H. T. '77.**

Proof : If the integral $\oint_C f(z)\, dz = 0$ is independent of C, then the function $F(z) = \int_a^z f(u)\, du$... (1) is independent of the path of integration joining the point a and z so long as this path is in $\mathfrak{R}$. Now we will show that (1) is analytic in $\mathfrak{R}$ and also $F'(z) = f(z)$.



Now we have $\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)$

$$= \frac{1}{\Delta z}\left\{\int_a^{z+\Delta z} f(u)\, du - \int_a^z f(u)du\right\} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(u) - f(z)]\, du \quad ... (2).$$

By Cauchy's integral theorem (2) is independent of the path joing the two points z and $z + \Delta z$ so long as this path is in $\mathfrak{R}$. Now we consider the path of integration from z to $z + \Delta z$ is a straight line provided that

$|\Delta z|$ may be considered so small as this path lies in $\mathfrak{R}$. Since $f(z)$ is continuous, then for all points u on this straight line path we have given $\varepsilon > 0$, we can find $\delta > 0$ such that $|f(u) - f(z)| < \varepsilon$ whenever $|u - z| < \delta$ and it will be certainly true if $|\Delta z| < \delta$. Then from (2) we have $\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| =$

$\frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(u) - f(z)]\, du \right| < \frac{1}{|\Delta z|}(\varepsilon\, |\Delta z|) = \varepsilon$ for $|\Delta z| < \delta$.

Then from (2), we have $\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) \Rightarrow F(z)$ is analytic and $F'(z) = f(z)$. But we have if $F(z)$ is analytic, then $F'(z)$ must be analytic and therefore $f(z)$ is analytic. Thus the theorem is proved.

Cauchy's inequality (Theorem —134) : If $f(z)$ is analytic inside and on a circle C with centre at $z = a$ and of radius r, then $\left| f^{(n)}(a) \right| \le \frac{M.n!}{r^n}$, where $n = 0, 1, 2, 3, \ldots$

R. U. H. '74, '81; D. U. M. SC. P. T. '91.

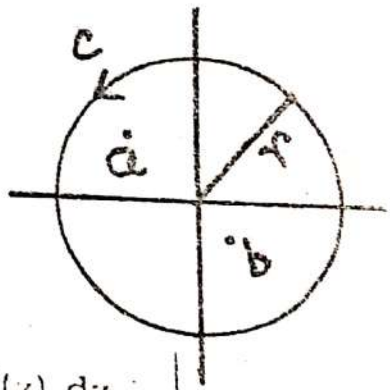
Proof : Since $f(z)$ is analytic inside and on the cirle C : $|z - a| = r$ ... (1). Then by the Cauchy's integral formula we have $f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}}\, dz$ ... (2), where $n = 0, 1, 2, 3.$ ... . Here the length of C is $2\pi r$ ... (3). Again $f(z)$ is analytic inside and on the given circle, then it is bounded i. e. $|f(z)| \le M$, ... (4), where M is an upper bound of $|f(z)|$ on C. Now by (1), (2), (3) and (4), we have $\left| f^{(n)}(a) \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - a)^{n+1}}\, dz \right|$

$\le \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{M.n!}{r^n}$, where $|i| = 1$ and $\oint_C dz = 2\pi r$.

Liouville's theorem (Theorem — 135) : If $f(z)$ is analytic for all z in the entire Argand (or complex) plane and is bounded, then $f(z)$ is a constant.

D. U. H '75, '88; R. U. H '75, '77, '80, '82 ; C. U. H '82, '88, '90; J. U. H. '87, '89, '90; R. U. M. SC. P. '84, '85.

First proof : Let M be the upper bound of $|f(z)|$, then $|f(z)| \le M$ for all z. Let a and b be any two points on the Argand plane. Now taking C to be a circle with centre at the origin and radius r greater than both $|a|$ and $|b|$. Then clearly a and b lie inside the circle C. Then by Cauchy's integral formula, we have $f(a) - f(b)$



$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-b}dz$$

$$= \frac{a-b}{2\pi i}\oint_C \frac{f(z)}{(z-a)(z-b)}dz$$

$$\Rightarrow |f(a) - f(b)| = \frac{|a-b|}{2\pi}\left|\oint_C \frac{f(z)\,dz}{(z-a)(z-b)}\right|$$

$$\le \frac{|a-b|\,M(2\pi r)}{2\pi(r-|a|)(r-|b|)} = \frac{|a-b|Mr}{(r-|a|)(r-|b|)} \quad \dots (1)$$

Now taking $r \to \infty$ in (1), we have $|f(a) - f(b)| = 0 \Rightarrow f(a) = f(b)$ for any two arbitrary points a and b in the Argand plane and this holds for all values of a and b. Therefore, $f(z)$ is a constant.

Second proof : Let M be an upper bound of $|f(z)|$, then $|f(z)| \le M$ for all z. Now if C is a circle of radius r with centre at z = a, then by Cauchy's inequality, we have $|f^{(n)}(a)| \le \frac{M.n!}{r^n}$ ... (1), where n = 0, 1, 2, 3, ... Now replacing a by z and putting n = 1 in (1) we have $|f'(z)| \leq \frac{M}{r}$ ... (2). But (2) is true for every r and making $r \to \infty$ in (2), we have $|f'(z)| = 0$ $\Rightarrow f'(z) = 0 \Rightarrow f(z) =$ constant and the theorem is proved.

Fundamental theorem of algebra (Theorem — 136) : Every polynomial equation $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$ with degree $n \geq 0$ and $a_n \neq 0$ has at least one root.

R. U. H. '82; J. U. H'87, '88, '90, '91.

Proof : If $f(z) = 0$ has no root, then $F(z) = \frac{1}{f(z)}$ is finite and analytic for all values of z. Therefore $|F(z)| = \frac{1}{|f(z)|}$ is bounded and by Liouville's theorem $F(z) = \frac{1}{|f(z)|}$ is constant $\Rightarrow f(z)$ is constant, which contradicts our hypothesis. Therefore, the polynomial equation $f(z) = 0$ must have atleast one root.

Theorem–137 : Show that every polynomial equation $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$ has exactly n roots where $a_n \neq 0$ and the degree $n \geq 1$.

J. U. H. '88.

Proof . By the fundamental theorem of algebra, the given polynomial equation $f(z) = 0$ has at least one root. Let z_1 be that root and we have $f(z_1) = 0 \Rightarrow f(z) - f(z_1) = a_n(z^n - z_1^n) + a_{n-1}(z^{n-1} - z_1^{n-1}) + \ldots + a_1(z - z_1) = (z - z_1) f_1(z)$ ... (1), where $f_1(z)$ is a polynomial of degree n–1. Again, by the fundamental theorem of algebra $f_1(z) = 0$ has at least one root and we denote this root by z_2. Then $f_1(z) = (z - z_2) f_2(z)$ ... (2), where $f_2(z)$ is a polynomial of degree n–2. Now by (1) and (2), we have $f(z) = (z - z_1)(z - z_2) f_2(z)$. Continuing in this way we can show that $f(z)$ has exactly n zeros.

Gauss's mean value theorem (Theorem — 138) : Let $f(z)$ be analytic inside and on a circle C having centre at the point $z = a$ and radius is r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i. e. $f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta$.

Proof : Since $f(z)$ is analytic inside and on C and a is inside the circle C, then by the Cauchy's integral formula, $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz$ ... (1). Now the equation of the circle C is $|z - a| = r \Rightarrow z = a + r\, e^{i\theta}$ and $dz = i\, r\, e^{i\theta}\, d\theta$ where $0 \le \theta <$ 2π. Then (1) $\Rightarrow$ $f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta$. Hence the theorem is proved.

Maximum modulus theorem (Theorem — 139) : Let $f(z)$ be analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

D. U. H. '75, '86; J. U. H. '89, '91; D. U. M. SC. P. '8[illegible]
D. U. M. SC. P. T. '90.

First proof : If $f(z)$ is analytic inside and on a circle with centre at $z = a$ and radius r, then by Gauss's mean value theorem we have $f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})d\theta$ ... (1). The formula (1) shows that the value of an analytic function at the centre of a circle is equal to the arithmetic mean of its values on the circle subject to the condition $|z - a| \le r$ which is contained in the region of analyticity. Then by (1) we have $|f(a)| \le \int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta$ ... (2). Now we suppose that $|f(z)|$ is maximum and then we have $|f(a + re^{i\theta})| \le f(a)$ ... (3). Now if the strict inequality of (3) i. e. if $|f(a + re^{i\theta})| < f(a)$ held for a single value of θ it would hold, by continuity, on a whole arc. But in this case the mean value of $|f(a + re^{i\theta}|$ would be strictly less than $|f(a)|$ and then (2) implies the contradiction $|f(a)| < |f(a)|$. Thus it follows that in any neighbourhood of a i. e. for $|z - a| < \delta$, $f(z)$ must be a constant. Therefore, if $f(z)$ is not a constant, then the maximum value of $|f(z)|$ must occur on C. Thus the theorem is proved.

Second proof : Since $f(z)$ is analytic inside and on C, then it is continuous inside and on C and it follows that $|f(z)|$ has a maximum value on C. Suppose that the maximum value were taken at an interior point a. Then $|f(a)|$ must also be the maximum of $|f(z)|$ in a neighbourhood $|z - a| < \delta$ which lies inside C. But this is impossible unless $f(z)$ is constant in this neighbourhood and then $f(z)$ is constant at each point inside and on C. But $f(z)$ is not constant and therefore the maximum value can not occur at an interior point. Thus the maximum value of $|f(z)|$ must occur on C. Thus the theorem is proved.

Minimum modulus theorem (Theorem — 140) : If $f(z)$ is analytic inside and on a simple closed curve C and if $f(z) \neq 0$, then $|f(z)|$ must assume its minimum value on C.

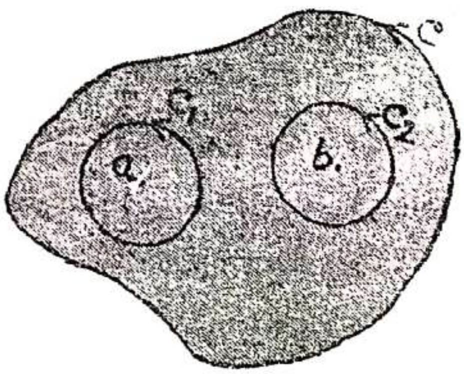
Proof : We have $f(z)$ is analytic inside and on C. Again $f(z) \neq 0$ inside C, then it follows that $1/f(z)$ is analytic inside C.

Now by the maximum modulus theorem, the maximum value of $\frac{1}{|f(z)|}$ cannot occur inside C. From this it follows that the minimum value of $|f(z)|$ cannot occur inside C. But $|f(z)|$ has a minimum value and this minimum value of $|f(z)|$ occurs on C. Thus the theorem is proved.

The argument theorem (Theorem — 141) : If $f(z)$ is analytic inside and on a simple closed curve C apart for a pole z = a of order p inside C and no zeros on C, then $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n - p$, where n is the order of the only one zero z = b of $f(z)$ inside C.

C. U. M. SC. P. '87; D.U.M. SC. P.'89; D.U. M. SC. P. T. '91.

Proof : Let C_1 and C_2 be two non-overlapping circles lying inside C enclosing z = a and z = b respectively.



Then $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz$

$= \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f'(z)}{f(z)}\,dz$ ... (1).

Since $f(z)$ has a pole of order p at z = a, then we have $f(z) = \frac{\phi(z)}{(z-a)^p}$ ...(2), where $\phi(z)$ is analytic and it is different from zero inside and on C_1. Now taking log both sides of (2), we have $\log f(z) = \log \phi(z) - p \log (z - a)$ and differentiating this with respect to z, we get

$\frac{f'(z)}{f(z)} = \frac{\phi'(z)}{\phi(z)} - \frac{p}{z-a}$. Then $\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\, dz$

$= \frac{1}{2\pi i} \oint_{C_1} \frac{\phi'(z)}{\phi(z)}\, dz - \frac{p}{2\pi i}\oint_{C_1} \frac{1}{z-a}\, dz$

$= 0 - \frac{p}{2\pi i}(2\pi i) = -p$ .. (3) since $\frac{\phi'(z)}{\phi(z)}$ is analytic inside and on C_1. Again, since $f(z)$ has a zero of order n at $z = b$, then we have $f(z) = (z-b)^n \psi(z)$ ... (4) where $\psi(z)$ is analytic and different from zero inside and on C_2. Now taking log bothsides of (4) and differentiating this with respect to z, we get

$\frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{\psi'(z)}{\psi(z)}$. Then $\frac{1}{2\pi i}\oint_{C_2} \frac{f'(z)}{f(z)}\, dz$

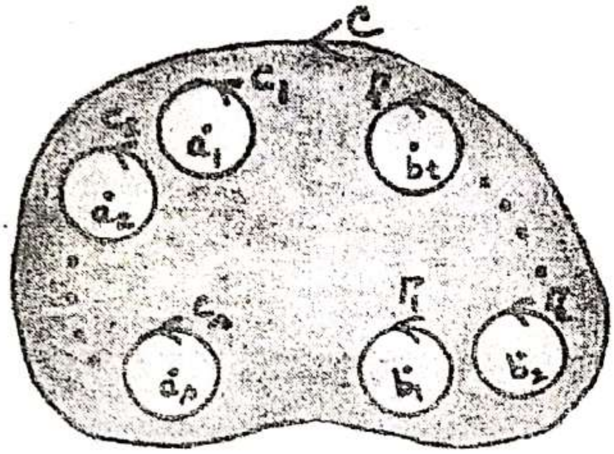
$= \frac{n}{2\pi i}\oint_{C_2} \frac{dz}{z-b} + \frac{1}{2\pi i}\oint_{C_2} \frac{\psi'(z)}{\psi(z)}\, dz = \frac{n}{2\pi i}(2\pi i) + 0$

$= n$ ... (5) since $\frac{\psi'(z)}{\psi(z)}$ is analytic inside and on C_2. Now by (1), (3) and (5), we have $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = n - p$. Hence the theorem is proved.

The general argument theorem (Theorem — 142) : Let $f(z)$ be analytic inside and on a simple closed curve C apart from a finite number of poles inside C and no zeros on C. Then $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = N - P$ where N is the number of zeros and P is the number of poles inside C counting multiplicities.

C.U.H.'81, '85; R. U. H..72, 81, 86, 88; D.U. M. SC. P. '89; C. U. M. SC. P.T. '91; C. U. M. SC. P. '87; R.U.M. SC. P. '88.

Proof : Let $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_t$ be the poles and zeros of $f(z)$ respectively lying inside C. Let $p_1, p_2, \ldots p_s$ and $n_1, n_2, \ldots, n_t$ be their multiplicities (i. e. orders) respectively. Now we enclose each pole and zero by non–overlapping circles $C_1, C_2, \ldots, C_s$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ and it can be done since these poles and zeros are isolated. Then by the argument theorem we have



$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{s} \frac{1}{2\pi i}\oint_{C_k} \frac{f'(z)}{f(z)}\,dz +$$

$$\sum_{k=1}^{t} \frac{1}{2\pi i}\oint_{\Gamma_k} \frac{f'(z)}{f(z)}\,dz = -\sum_{k=1}^{s} p_k + \sum_{k=1}^{t} n_k = -P + N$$

$= N - P$. Thus the theorem is proved.

Theorem-143 : If $\phi(z)$ is analytic inside and on a simple closed curve C and if $f(z)$ has zeros at $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ inside C, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\phi(z)\,dz = \sum_{k=1}^{m} \phi(a_k) - \sum_{k=1}^{n} \phi(b_k).$$

Proof : Try yourself.

Theorem-144 : If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C except that $f(z)$ has zeros at $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ of orders (multiplicities) $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots q_n$ respectively, then

$$\frac{1}{2\pi i}\oint_C g(z)\,\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} p_k\, g(a_k) - \sum_{k=1}^{n} q_k\, g(b_k).$$

Proof : Try yourself.

The principal of the argument (Theorem - 145) : If $f(z)$ is analytic inside and on a closed curve C and if $f(z) \neq 0$ on C, then $N = \frac{1}{2\pi}\Delta_C \arg\{f(z)\}$, where N is the number of zeros inside C. **R. U. H. '88; D. U. H. '90.**

Proof : Since $f(z)$ is analytic inside and on C, then there are no poles inside C and we have $P = 0$. Now by the general argument theorem we have $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N$ ... (1). Again we have $\frac{d}{dz}\log\{f(z)\} = \frac{f'(z)}{f(z)}$ and $\int_C \frac{f'(z)}{f(z)}\,dz = \Delta_C \log\{f(z)\}$ ... (2), where Δ_C denotes the variation of $\log\{f(z)\}$ round the closed curve. The value of the logarithm with which we start is clearly indifferent. Also $\log\{f(z)\} = \log|f(z)| + i\arg\{f(z)\}$ ... (3) and $\log|f(z)|$ is one-valued. Hence by (1), (2) and (3) we have $N = \frac{1}{2\pi}\Delta_C \arg\{f(z)\}$. Hence the theorem is proved.

Example 203 : If C is the circle $|z| = \pi$, then show that

$$\oint_C \frac{f'(z)}{f(z)}\, dz = \begin{cases} 14\pi i \text{ if } f(z) = \sin \pi z; \\ 12\pi i \text{ if } f(z) = \cos \pi z; \\ 2\pi i \text{ if } f(z) = \tan \pi z. \end{cases}$$

R. U. H. '85.

Solution : (First part) : The function $f(z)$ is analytic inside and on C. It has no poles inside C. The poles of $f(z)$ can be obtained by solving $f(z) = 0 \Rightarrow \sin \pi z = 0 \Rightarrow \pi z = n\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots \Rightarrow z = n$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Only the zeros 0, 1, -1, 2, -2, 3 and -3 lie within C and the ordes of each is 1. Then by the general argument theorem, $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = N - P = (1 + 1 + 1 + 1 + 1 + 1 + 1) - 0 = 7 \Rightarrow \oint_C \frac{f'(z)}{f(z)}\, dz = 14\pi i$, which is the required result.

Second part : The zeros of $f(z)$ can be obtained by solving $f(z) = 0 \Rightarrow \cos \pi z = 0 \Rightarrow \pi z = (2n + 1)\pi/2$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Only the zeros $1/2, -1/2, 3/2, -3/2, 5/2, -5/2$ lie within C and the order of each is 1. Then by the general argument theorem, we have $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = N - P = (1 + 1 + 1 + 1 + 1 + 1) - 0 = 6 \Rightarrow \oint_C \frac{f'(z)}{f(z)}\, dz = 12\pi i$, which is the required result.

Third part : We have $f(z) = \tan \pi z = \frac{\sin \pi z}{\cos \pi z}$, which have seven zeros $0, \pm 1, \pm 2, \pm 3$, each of order 1 and six poles $\pm \frac{1}{2}, \pm 3/2, \pm 5/2$, each of order1, Then by the general argument theorem $\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(N - P) = 2\pi i\,(7 - 6) =$ $2\pi i$, which is the required result.

Example 204 : Show that $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = -2$, where $f(z) = \frac{(z^2 + 1)^2}{(z^2 + 2z + 2)^3}$ and C is the circle $|z| = 4$.

C. U. H. '83; R. U. M. SC. P. '88.

Solution : The function $f(z)$ is analytic inside and on C. The zeros of $f(z)$ can be obtained by solving $(z^2 + 1)^2 = 0 \Rightarrow z = \pm i$, where each zero is of order 2 and both lies inside C. Again the poles of $f(z)$ can be obtained by solving $(z^2 + 2z + 2)^3 = 0 \Rightarrow z^2 + 2z + 2 = 0 \Rightarrow z = -1 \pm i$, where each pole is of order 3 and each lies inside C. Then by the general argument theorem, we have $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$

$$= (2 + 2) - (3 + 3) = -2.$$

Example 205 : If $f(z) = z^5 - 3\,i\,z^2 + 2z - 1 + i$, then show that $\oint_C \frac{f'(z)}{f(z)}\,dz = 10\pi i$, where C encloses all the zeros of $f(z)$.

Solution : The function $f(z) = z^5 - 3iz^2 + 2z - 1 + i$ is analytic inside and on C and it has no any poles inside C. Again $f(z)$ is a polynomial of degree 5 and hence by the fundamental theorem of algebra it has five roots. Hence by the general argument theorems, we have $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz =$ $N - P = 5 - 0 = 5 \Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz = 10\pi i.$

Example 206 : Show that $\oint_C \frac{z f'(z)}{f(z)} dz = 4\pi i$, where $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ and C is the circle $|z| = 5$.

J. U. H. '89.

Solution : We have $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$

$= (z - 2)^2 \{z - (-1 + 2i)\} \{z - (-1 - 2i)\}$, which is analytic inside and on C. It has 4 zeros $z = 2$ of order 2 and other two are simple and they are inside C. Again $f(z)$ has no any poles inside C and we have $g(z) = z$. Then

$$\oint_C g(z) \frac{f'(z)}{f(z)} dz = 2\pi i [2g(2) + g(-1 + 2i) + g(-1 - 2i)]$$

$= 2\pi i [2.2 + (-1 + 2i) + (-1 - 2i)] = 4\pi i$, which is our required result.

Example 207 : Show that $\frac{1}{2\pi i} \oint_C \frac{z^2}{z^2 + 4} dz = i$, where C is the square with vertices at $\pm 2, \pm 2 + 4i$.

Solution : Let $f(z) = z^2 + 4 \Rightarrow f'(z) = 2z$. Again $f(z) = 0 \Rightarrow z^2 + 4 = 0 \Rightarrow z = \pm 2i$. Of which only the simple zero $z = 2i$ lies inside the square C. Here $\frac{z^2}{z^2 + 4} = g(z) \frac{f'(z)}{f(z)}$ where $g(z) = z/2$. Then $\frac{1}{2\pi i} \oint_C g(z) \frac{f'(z)}{f(z)} dz = g(2i) = \frac{2i}{2} = i$. which is our required result.

Example 208 : If $f(z) = \sum_{p=0}^{n} a_p z^{n-p}$, then show that

$\frac{1}{2\pi i}\oint_C \frac{zf'(z)}{f(z)}\,dz = -\frac{a_1}{a_0}$, where $a_0 \neq 0$, $a_1, a_2, \ldots, a_n$ are complex constants and C encloses all the zeros of $f(z)$.

Solution : Try yourself.

Example 209 : If $f(z) = \sum_{p=0}^{n} a_p z^{n-p}$, then show that

$\frac{1}{2\pi i}\oint_C \frac{z^2 f'(z)}{f(z)}\,dz = (a_1^2 - 2a_0a_2)/a_0^2$, whers $a_0 \neq 0$, $a_1, a_2, \ldots, a_n$ are complex constants and C encloses all the zeros of $f(z)$.

Solution : Try yourself.

Rouche's theorem (Theorem — 146) : If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

D. U. H. T. '75, '78, '84, '87, '89; D. U. H. '90; R. U. H. '72, '75, '82; C. U. H. '87, '89; J. U. H. '86, '91; C. U. M. SC. P. '88; R. U. M. SC. P. '84, '86, '88; D. U. M. SC. P. T. '89 .

Proof : Let $F = \frac{g}{f}$ ... (1) where $F = F(z)$, $f = f(z)$ and $g = g(z)$. Then (1) $\Rightarrow g = fF$ and $g' = f'F + fF'$ ... (2). Now if N_1 and N_2 are the number of zeros inside C of $f + g$ and f respectively, then by the general argument theorem we have $N_1 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz$ ... (3) and $N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$ ... (4) using the fact that these two functions have no poles inside C. Now by (2) and (3) $\Rightarrow N_1 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz$

$$= \frac{1}{2\pi i}\oint_C \frac{f' + f'F + fF'}{f + fF}\,dz = \frac{1}{2\pi i}\oint_C \frac{f'(1+F) + fF'}{f(1+F)}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \left(\frac{f'}{f} + \frac{F'}{1+F}\right) dz = N_2 + \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz, \text{ by (4)}$$

$$\Rightarrow N_1 - N_2 = \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz$$

$$= \frac{1}{2\pi i}\oint_C F'(1 - F + F^2 - F^3 + \ldots)\,dz$$

$= 0$, by expanding in powers of F and integrating term by term and also using the fact that $|F| < 1$ on C, where the series is uniformly convergent on C. Thus $N_1 = N_2$ and the theorem is proved.

Theorem -147 : Show that every polynomial of degree n has exactly n zeros.

D. U. H. '75; R. U. H. '73; C. U. M.. SC. P. '88.

Proof : Let $f(z) + g(z) = \sum_{p=0}^{n} a_p z^p$ be a polynomial of degree n, where $a_n \neq 0$ and $f(z) = a_n z^n$ and $g(z) = \sum_{p=0}^{n-1} a_p z^p$.

Now if C is a circle having centre at the origin and radius r >1, then on C, we have $\left|\frac{g(z)}{f(z)}\right|$

$$= \frac{\left|\sum_{p=0}^{n-1} a_p z^p\right|}{|a_n z^n|} \le \frac{\sum_{p=0}^{n-1} |a_p| r^p}{|a_n| r^n} \le \frac{\left\{\sum_{p=0}^{n-1} |a_p|\right\} r^{n-1}}{|a_n| r^n} = \frac{\sum_{p=0}^{n-1} |a_p|}{|a_n| r} < 1,$$

by choosing r large enough. Thus $\left|\frac{g(z)}{f(z)}\right| < 1 \Rightarrow |g(z)|$

$< |f(z)|$. Thus by the Rouche's theorem the given polynomial $f(z) + g(z) = \sum_{p=0}^{n} a_p z^p$ has the same number of zeros as $f(z) = a_n z^n$ ... (1). But (1) has n zeros all located at $z = 0$. Therefore, $f(z) + g(z)$ has exactly n zeros and the theorem is proved.

Example 210 : Show that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

D. U. M. SC. P. T. '89; R. U. M. SC. P. '84; C. U. H. '87.

Solution : Let C_1 be the circle $|z| = 1$. Let $f(z) = 12$ and $g(z) = z^7 - 5z^3$. Then on, C_1 we have $|g(z)| = |z^7 - 5z^3| \le |z^7| + |5z^3| \le 6 < 12 = |f(z)|$. Since $f(z)$ and $g(z)$ is analytic inside and on C_1 and $|g(z)| < |f(z)|$. Then by the Rouche's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ and $f(z) = 12$ have the same number of zeros inside $|z| = 1$. But $f(z) = 12$ have no any zeros inside C_1. Therefore, the given equation $f(z) + g(z) = z^7 - 5z^3 + 12 = 0$ ... (1) . have no any zeros inside C_1. Again, let C_2 be the circle $|z| = 2$. Let $f(z) = z^7$ and $g(z) = -5z^3 + 12$. Then on C_2, we have $|g(z)| = |-5z^3 + 12| \le |-5z^3| + |12| \le |5z^3| + 12 \le 52 < 2^7 = |f(z)|$. Then by the Rouche's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ and $f(z) = z^7$ have the same number of zeros inside $|z| = 2$. But $f(z) = z^7$ have the all zeros inside C_2. Thus $f(z) + g(z) = z^7 - 5z^3 + 12 = 0$ ... (2). have the all zeros inside $|z| = 2$. Now by (1) and (2) the roots of the given equation lie between the circles $|z| = 1$ and $|z| = 2$.

Example 211 : If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$.

D. U. H. '86; R. U. M. SC. P. '88.

Solution : Let C be the circle $|z| = 1$. Now we consider $f(z) = az^n$ and $g(z) = -e^z$. On C we have $|f(z)| = |az^n|$

$= |a|\ |z^n| = a\ |z|^n$

$= a|1|^n = a > e \ldots (1).$

Again $|g(z)| = |-e^z| = |e^z|$

$$= \left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots\right|$$

$$\leq 1 + |z| + \frac{1}{2!}|z|^2 + \frac{1}{3!}|z|^3 + \ldots \qquad [\because |z^n| = |z|^n]$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = e. \qquad [\because |z| = 1]$$

i. e. $|g(z)| < e \ldots (2)$. Now by (1) and (2), we have $|g(z)| < |f(z)|$. Thus by the Rouche's theorem, $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C. But $f(z) = az^n$ has n zeros inside C. Thus the given equation has n zeros inside the circle $|z| = 1$. Hence the required result is proved.

Example 212 : If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the region $|z| < \frac{1}{2}$ **D. U. M. SC. P. '89.**

Solution : Try yourself.

Example 213 : If a is real and $a \neq 0$, then show that $ze^z =$ a has infinitely many roots. **D. U. H. '86.**

Solution : Try yourself.

Example 214 : Show that $z \tan z = a$, $a > 0$ has infinitely many real roots but no imaginary roots. **J. U. H. '91.**

Solution : Try yourself.

Poisson's integral formulas for a circle (Theorem —148) : If $f(z)$ is analytic inside and on the circle C which is defined by $|z| = r$, then $f(r_1e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - r_1^2) f(re^{i\phi})}{r^2 - 2rr_1 \cos(\theta - \phi) + r^2} d\phi$,

where $z = r_1e^{i\theta}$ is any point inside C.

Let $u(r_1, \theta)$ and $v(r_1, \theta)$ be the real and imaginary parts of $f(r_1e^{i\theta})$. Then show that

$$u(r_1, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - r_1^2)\, u(r, \phi)}{r^2 - 2r\, r_1 \cos(\theta - \phi) + r^2} d\phi \text{ and}$$

$$v(r_1, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - r_1^2)\, v(r, \phi)}{r^2 - 2r\, r_1 \cos(\theta - \phi) + r^2} d\phi, \text{ where}$$

$u(r, \phi)$ and $v(r, \phi)$ are the real and imaginary parts of $f(re^{i\phi})$.

Proof : The proof is not given here. Try yourself or see in any standard book of complex variables.

Poisson's integral formulas for a half plane (Theorem —149) : If $f(z)$ is analytic in the upper half $y > 0$ of the

Argand plane and if $\eta = \alpha + i\beta$ is any point in this upper half plane, then $f(\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta f(x)}{((x-\alpha)^2 + \beta^2)} dx.$

Let $u(\alpha, \beta)$ and $v(\alpha, \beta)$ be the real and imaginary parts of $f(\eta)$. Then show that

$$u(\alpha, \beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta\, u(x, 0)}{(x-\alpha)^2 + \beta^2} dx \text{ and}$$

$$v(\alpha, \beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta\, v(x, 0)}{(x-\alpha)^2 + \beta^2} dx,$$

where $f(x) = u(x, 0) + iv(x, 0)$.

Proof : The proof is not given here. Try yourself or see in any standard book of complex variables.

Schwarz's theorem (Theorem — 150) : If $f(z)$ is analytic for $|z| \le r$ and bounded i. e. $|f(z)| \le M$ and also $f(0) = 0$, then $|f(z)| \le \frac{M|z|}{r}$.

Proof : Since $f(0) = 0$, then the function $f(z)/z$ is analytic in $|z| \le r$. Now on $|z| = r$, by the maximum modulus theorem, $\left|\frac{f(z)}{z}\right| \le \frac{m}{r}$ ... (1) and this inequality also holds for points inside $|z| = r$. Thus for $|z| \le R$, (1) $\Rightarrow$ $|f(z)| \le \frac{M|z|}{r}$.

CHAPTER-6

SERIES, TAYLOR'S AND LAURENT'S SERIES

200. Sequence of functions : Let $< u_n(z) >$ be a sequence of functions of z which is defined and single-valued in some region of the Argand plane. The nth term of the sequence of the functions $< u_n(z) >$ is $u_n(z)$.

N. B. By the sequence $< u_n(z) >$, we will mean the sequence of the functions $u_1(z), u_2(z), \dots, u_n(z), \dots$ whose nth term is $u_n(z)$.

201. Limit of a sequence of functions : The function $l(z)$ is called the limit of the sequence $< u_n(z) >$ if for any positive number ε we can determine a positive number N (which in general depends on both ε and z) such that $|u_n(z) - l(z)| < \varepsilon$ for all $n > N$ and is denoted by $\lim_{n\to\infty} u_n(z) = l(z)$.

202. Convergent : If the limit of the sequence $< u_n(z) >$ exists, then the sequence is called convergent.

203. Divergent : If the limit of the sequence $<u_n(z)>$ does not exist, then the sequence is called divergent.

204. Region of convergence of sequence : If the sequence $<u_n(z)>$ converges for all values of z (i. e. for all points) in a region $\mathfrak{R}$, then $\mathfrak{R}$ is called the region of convergence of the sequence.

205. nth partial sum : Let $< u_n(z) >$ be a sequence of functions. Suppose $S_1(z) = u_1(z)$, $S_2(z) = u_1(z) + u_2(z)$, $S_3(z) =$

$u_1(z) + u_2(z) + u_3(z), \dots, S_n(z) = u_1(z) + u_2(z) + \dots + u_n(z)$. where $S_n(z)$ is called the nth partial sum of the first n terms of the sequence $< u_n(z)>$. The **sequence of partial sums** is $< S_n(z) >$ whose nth term is $S_n(z) = \sum_{n=1}^{n} u_n(z)$ and it is symbolized by $\sum_{n=1}^{\infty} u_n(z)$ which is called **an infinite series.** If

$\lim_{n\to\infty} S_n(z) = S(z)$, then the series $\sum_{n=1}^{\infty} u_n(z)$ is called **convergent series** and S(z) is called its **sum.** If it is not **convergent**, then it is called **divergent.**

206. Region of Convergence of series : If the series $\sum_{n=1}^{\infty} u_n(z)$ is convergent for all values of z (i. e. for all points) in a region $\mathfrak{R}$, then $\mathfrak{R}$ is called the region of convergence of the series.

207. Absolute Convergent : D. U. H. '90

A series $\sum_{n=1}^{\infty} u_n(z)$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n(z)|$ converges.

208. Conditionally convergent : If the series $\sum_{n=1}^{\infty} u_n(z)$ converges but $\sum_{n=1}^{\infty} |u_n(z)|$ does not converge, then the series $\sum_{n=1}^{\infty} u_n(z)$ is called conditionally convergent.

209. Uniform convergent sequence of functions : A sequence $<u_n(z)>$ is said to be uniformly convergent or converges uniformly to $l(z)$ in a region $\Re$ if corresponding to any $\varepsilon > 0$ we can find a positive number N (which is a function of ε only) such that $|u_n(z) - l(z)| < \varepsilon$ for all $n > N$, where the same number N holds for all z in the region $\Re$.

210. Uniform convergent series of functions :

D. U. H. '86, 90.

A sequence of partial sums $<S_n(z)>$ is called converges uniformly to $S(z)$ in a region $\Re$ if the series $\sum_{n=1}^{\infty} u_n(z)$ converges uniformly or is uniformly convergent to $S(z)$ in the region $\Re$.

211. Remainder of the series $\sum_{n=1}^{\infty} u_n(z)$: Let $<S_n(z)>$ be a sequence of partial sums whose n th term is $S_n(z) = \sum_{n=1}^{\infty} u_n(z)$. Then $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \dots \dots$

$= S(z) - S_n(z)$ is called the remainder of the infinite series $\sum_{n=1}^{\infty} u_n(z)$ after n terms.

212. Second definition of uniform convergent series of functions : The series $\sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent to $S(z)$ in a region $\Re$ if given any $\varepsilon > 0$ we can find a positive mumber N such that for all z in $\Re$, $|R_n(z)| = |S(z) - S_n(z)| < \varepsilon$ for all $n > N$.

213. Power series : An infinite series $\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots$ (1) is called a power series in $(z - a)$ where $a_1, a_2, a_3, \dots$ are complex constants. The series (1) converges for $z = a$. In general, the series (1) converges for other points as well. It can be proved that there exists a positve number r such that the series (1) converges for $|z - a| < r$, diverges for $|z - a| > r$ and for $|z - a| = r$ it may or may not converge.

214. Read the following important theorems : The following theorems are given without proofs. Many of these theorems will be famillar from their analogs for real variables.

Theorem -151 : Prove that every convergent sequence $<a_n>$ has a unique limit.

Theorem- 152 : Prove that a convergent sequence $<a_n>$ is bounded.

Theorem-153 : A necessary and sufficient condition that $<u_n>$ converge is that $<a_n>$ and $<b_n>$ converge where

$u_n = a_n + ib_n$ and a_n and b_n are real.

Theorem-154 : Every bounded monotonic (increasing or decreasing) sequence has a limit.

Theorem-154 : A necessary and sufficient condition for a monotonic sequence to be convergent is that it is bounded.

Theorem-156 : A necessary and sufficient condition that the sequence $<u_n>$ converges is that given any $\varepsilon > 0$, we can find a number N such that $|u_p - u_q| < \varepsilon$ for all $p > N$, $q > N$.

Theorem-157 : A necessary condition that the series $\sum_{n=1}^{\infty} u_n$ converge is that $\lim_{n\to\infty} u_n = 0$. The condition of this theorem is not sufficient.

Theorem-158 : A convergent (divergent) series remains convergent (divergent) if after any or all of its first n terms are altered.

Theorem-159 : The sum of a convergent series is unique.

Theorem-160 : If a series is convergent (divergent), then multiplication of each term of this series by a constant differerent from zero does not affect the convergence (divergence).

Theorem-161 : If a series is convergent (divergent) then removal (or addition) of a finite number of terms from (or to) a series does not affect the convergence (divergence).

Theorem=162 : A necessary and sufficient condition that $\sum_{n=1}^{\infty} (a_n + i\, b_n)$ converge is that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge where a_n and b_n are real.

Theorem-163 : Every absolutely convergent series is convergent.

Theorem-164 : Every convergent positive series is absolutely convergent.

Theorem-165 : The sum, difference and product of absolutely convergent series is absolulely convergent.

Comparison tests (Theorem-166) :

(i) If $\sum_{n=1}^{\infty} |v_n|$ converges and $|u_n| \le |v_n|$, then $\sum_{n=1}^{\infty} u_n$ converges absolutely.

(ii) If $\sum_{n=1}^{\infty} |v_n|$ diverges and $|u_n| \ge |v_n|$, then $\sum_{n=1}^{\infty} |u_n|$ diverges but $\sum_{n=1}^{\infty} u_n$ may or may not converge.

Ratio test (Theorem-167) : A series $\sum_{n=1}^{\infty} u_n$ with mixed terms is absolutely convergent if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ and is divergent if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$. If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 1$, then the test fails.

nth Root test (Theorem —168) : If the sequence $\sqrt[n]{u_n}$ converges to the limit L, then the series $\sum_{n=1}^{\infty} u_n$ absolutely convergent if $L < 1$ and diverges if $L > 1$. If $L = 1$, then the test fails.

Integral test (Theorem — 169) : Let $f(n)$ denote the general term of the positive series $\sum_{n=1}^{\infty} u_n$. Now if $f(x) \ge 0$ for $x \ge a$, then $\sum_{n=1}^{\infty} f(n)$ converges or diverges according as

$\lim_{M\to\infty} \int_a^M f(x)\,dx$ converges or diverges.

Raabe's test (Theorem — 170) : If $\lim_{n\to\infty} n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right)$ $= L$, then the series $\sum_{n=1}^{\infty} u_n$ absolutely convergent if $L > 1$ and diverges if $L < 1$. If $L = 1$, then the test fails.

Gauss's test (Theorem —171) : If $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{\alpha}{n} + \frac{\beta_n}{n^2}$ where $|\beta_n| < M$ for all $n > M$, then the series $\sum_{n=1}^{\infty} u_n$ absolutely converges if $\alpha > 1$ and diverges if $\alpha \le 1$.

215. Alternating series : A series $u_1 - u_2 + u_3 - u_4 + \ldots \ldots$ $= \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is called alternating series where each u_i is positive.

Alternating series test (Theorem —172) : An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges if (i) $u_n > u_{n+1}$ for every value of $n \varepsilon N$ and (ii) $\lim_{n\to\infty} u_n = 0$.

Weierstrass M test (Theorem —173) : If for all z in a region $\Re$, the absolute values of the terms of the series $\sum_{n=0}^{\infty} u_n(z)$ ... (1) are respectively, less than or equal to the corresponding terms in a convergent series of constant terms $\sum_{n=0}^{\infty} M_n$, then the series (1) converges uniformly in $\Re$.

Theorem —174 : If $u_n(z)$ is continuons in a region $\Re$ for $n = 1, 2, 3, \ldots$ and $S(z) = \sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent in $\Re$, then $S(z)$ is continuous in $\Re$.

Theorem —175 : If $u_n(z)$ is continuous in $\Re$ for $n = 1, 2, 3, \ldots$ and $S(z) = \sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent in $\Re$, then $\int_C \{\sum u_n(z)\}\,dz = \sum \int_C u_n(z)dz$ where C is a curve in $\Re$.

Theorem —176 : If $u'_n(z)$ exists in $\Re$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} u'_n(z)$ converges uniformly in $\Re$ and also $\sum_{n=1}^{\infty} u_n(z)$ converges in $\Re$, then $\frac{d}{dz}\sum_{n=1}^{\infty} u_n(z) = \sum u'_n(z)$ where $u'_n(z)$ $= \frac{d}{dz} u_n(z)$.

Theorem-177 : If $u_n(z)$ is analytic in $\Re$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent in $\Re$, then $S(z) = \sum_{n=1}^{\infty} u_n(z)$ is analytic in $\Re$.

Theorem-178 : A power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges uniformly and absolutely in any region $\mathfrak{R}$ which lies entirely inside its circle of convergence : $|z - a| = r$, where r is its radius of convergence.

Theorem-179 : A power series $\sum_{n=0}^{\infty} a_n(z - a)^n$ can be differentiated term by term in any region $\mathfrak{R}$ which lies entirely inside its circle of convergence : $|z - a| = r$, where r is its radius of convergence.

Theorem-180 : A power series $\sum_{n=0}^{\infty} a_n(z - a)^n$ can be integrated term by term along any curve C which lies entirely inside its circle of convergence : $|z - a| = r$, where r is its radius of convergence.

Theorem-181 : The sum of a power series is continuous in any region $\mathfrak{R}$ which lies entirely inside its circle of convergence.

Abel's theorem (Theorem —182) : Let the power series $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence r and if z_0 is a point on the circle of convergence such that $\sum_{n=0}^{\infty} a_n z_0^n$ converges. Then $\sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} a_n z_0^n$ as $z \to z_0$, where $z \to z_0$ from within the circle of convergence.

Theorem —183 : If $\sum_{n=0}^{\infty} a_n z^n \to 0$ for all z such that $|z| < r$ where $r > 0$, then $a_n = 0$. Equivalently, if $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$ for all values of z such that $|z| < r$, then $a_n = b_n$.

Theorem-184 : If a power series $\sum_{n=1}^{\infty} a_n z^n$ converges for $z = z_0 \neq 0$, then show that it converges absolutely.

D. U. H. T. '86.

Proof : Since $\sum_{n=1}^{\infty} a_n z^n$ converges for $z = z_0 \neq 0$, then

$\lim_{n \to \infty} a_n z_0^n = 0$ and we can make $|a_n z_0^n| = |a_n|\,|z_0|^n < 1$ by choosing n large enough $\Rightarrow |a_n| \le \dfrac{1}{|z_0|^n}$ ... (1) for $n > N$. Then

$$\sum_{n=N+1}^{\infty} |a_n z^n| = \sum_{n=N+1}^{\infty} |a_n|\,|z|^n \le \sum_{n=N+1}^{\infty} \frac{|z|^n}{|z_0|^n} \quad \ldots (2)$$

using (1). In (2) the last series converges for $|z| < |z_0|$ and therefore by the comparison test the first series in (2) converges. Thus the given series is absolutely convergent. Hence the theorem is proved.

Theorem-185 : If a power series $\sum_{n=1}^{\infty} a_n z^n$ converges for $z = z_0 \neq 0$, then show that it converges uniformly for $|z| \le |z_1|$ where $|z_1| < |z_0|$.

D. U. H. '90.

Proof : Since $\sum_{n=1}^{\infty} a_n z^n$ converges for $z = z_0 \neq 0$, then

$\lim_{n \to \infty} a_n z_0^n = 0$ and we can make $|a_n z_0^n| = |a_n|\ |z_0|^n < 1$ by choosing n large enough $\Rightarrow |a_n| < \frac{1}{|z_0|^n}$ ... (1). Now we

suppose $M_n = \frac{|z_1|^n}{|z_0|^n}$ ... (2). Then $\sum_{n=1}^{\infty} M_n$ converges since we

have $|z_1| < |z_0|$. Now by (1) and (2) we have $|a_n z^n| = |a_n|\ |z|^n < \frac{|z|^n}{|z_0|^n} < \frac{|z_1|^n}{|z_0|^n} = M_n$ since $|z| \leq |z_1|$. Thus by the

Weierstrass M test the series $\sum_{n=1}^{\infty} a_n z^n$ is uniformly

convergent. Hence the theorem is proved.

Theorem-186 : A power series converges uniformly and absolutely in any region which lies entirely inside its circle of convergence.

Proof : See the above two theorems.

Theorem-187 : Show that both the power series $\sum_{n=0}^{\infty} a_n z^n$

and $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Proof : Try yourself.

Theorem-188 : Show that in any region which lies entirely within its circle of convergence, a power series $\sum_{n=0}^{\infty} a_n z^n$:

(i) represents a continuous function $f(z)$ (say);

(ii) can be integrated term by term to yeild the integral of $f(z)$;

(iii) can be differentiated term by term to yeild the derivative of $f(z)$.

Proof : Try yourself.

Example-215 : Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has a finite value at all points inside and on its circle of convergence but that this is not true for the series of derivatives.

Solution : Try yourself.

216. Circle of convergence : **D. U. H. '86, '90.**

Let $\sum_{n=0}^{\infty} a_n (z-a)^n$ ... (1) be a power series. If $\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = r$, then from the Ratio test it follows that the series (1) is absolutely convergent within the circle $|z - a| = r$ and the series (1) is absolutely divergent without the circle $|z - a| = r$. Then this circle is called the circle of convergence.

N. B. On the circle $|z - a| = r$, the series (1) may or may not convergence.

217. Radius of convergence : In the above definition, the radius r of the circle of convergence $|z - a| = r$ is called the radius of convergence.

Example-216 : The radius of the geometric series $1 + z + z^2 + z^3 + \ldots$ is unity.

The p-series (Theorem —189) : The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Proof : Try yourself.

Example-217 : Find the region of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$. D. U. H. '87.

Find the circle and radius of conuergence.

Solution : We have $u_n = \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$ and $u_{n+1} = \frac{(z+2)^n}{(n+2)^3 4^{n+1}}$.

Then $\frac{u_{n+1}}{u_n} = \frac{(z+2)(n+1)^3}{4(n+2)^3}$ and $\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \frac{|z+2|}{4}$

since $\lim_{n\to\infty} \frac{(n+1)^3}{(n+2)^3} = \lim_{n\to\infty} \frac{(1+1/n)^3}{(1+2/n)^3} = 1$. Thus the series converges absolutely for $\frac{|z+2|}{4} < 1$ i. e for $|z+2| < 4$ including the point $z = -2$. If $\frac{|z+2|}{4} = 1$ i. e. if $|z+2|$ $= 4$, the ratio test fails. In this case we have $|u_n| = \left| \frac{(z+2)^{n-1}}{(n+1)^3 4^n} \right|$ $= \frac{1}{4(n+1)^3} \le \frac{1}{n^3}$. Let $v_n = \frac{1}{n^3}$, then $|u_n| \le |v_n|$ for $|z+2| = 4$.

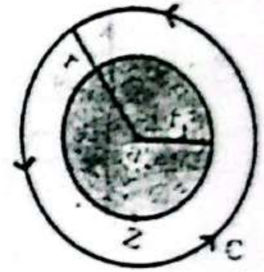
But $\sum_{n=1}^{\infty} v_n$ is a p-series with $p = 3$ and it is convergent by the comparison test, the series $\sum |u_n|$ converges i. e. $\sum u_n$ converges absolutely. Thus the series converges absolutely for $|z+2| \le 4$ and it is the region of convergence. Here the circle of convergence of the series is $|z+2| = 4$ with centre at $z = -2$ and radius $= 4$. The radius of convergence of the series is equal to 4.

Taylor's theorem (Theorem —190) : If $f(z)$ is analytic for all values of z inside a circle C with centre at a, then

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \ldots$$

D. U. M. SC. P. '88; D. U. M. SC. P. T. '89; D. U. H. T. '75; D. U. H '83; R. U. '76, 79; C. U. H. '81; R. U. H. '73, '82, '88; R. U. M. SC. P. '84.

Proof : Let C be a circle of centre a and radius r. Let z be any point inside C such that $|z - a| = r_1 < r$. Then by the Cauchy's integral formula,



$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw =$$

$$\frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-a)-(z-a)}\, dw$$

$$= \frac{1}{2\pi i} \oint_C \left\{ 1 - \frac{z-a}{w-a} \right\}^{-1} \frac{f(w)}{w-a}\, dw$$

$$= \frac{1}{2\pi i} \oint_C \Big[\Big\{ 1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \ldots$$

$$+ \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n \left\{ 1 - \frac{z-a}{w-a} \right\}^{-1} \Big] \frac{f(w)}{w-a}\, dw$$

$$= \frac{1}{2\pi i} \oint_C \Big[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \ldots$$

$$+ \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n \frac{w-a}{w-z} \Big] \frac{f(w)}{w-a}\, dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}dw + \ldots$$

$$+ \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)}{(w-a)^n}dw + U_n \ldots (1).$$

where $U_n = \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)}{(w-a)^n (w-z)}dw.$

Now $|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| \geq$

$r - r_1$ where $|w-a| = r$ for all points w on C. Then

$$|U_n| = \frac{1}{2\pi}\left|(z-a)^n \oint_C \frac{f(w)}{(w-a)^n (w-z)}dw\right|$$

$\leq \frac{r_1{}^n M(2\pi r)}{2\pi r^n (r-r_1)} = \frac{M}{1-\frac{r_1}{r}}\left(\frac{r_1}{r}\right)^n \ldots$ (2) where M is the maximum value of $f(w)$ on C and $2\pi r$ is the length of the circle C. In (2) $\frac{r_1}{r} < 1$, then $\lim_{n\to\infty} |U_n| \leq 0 \Rightarrow \lim_{n\to\infty} U_n = 0 \ldots$ (3). Now taking limit $n\to\infty$ in (1) and using (3)

$$\Rightarrow f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) +$$

$\frac{(z-a)^3}{3!}f'''(a) + \ldots$ (4) and the theorem is proved. The series (4) is called the Taylor's series or expansion.

Historical Note : Brook Taylor (1685 – 1731) was an English mathematician. He introduced this theorem for the function of real variable.

218. Other forms of the Taylor's theorem :

(i) $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ where $a_n = \frac{f^{(n)}(a)}{n!}$

or simply $f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a) \ldots (1)$

Now writing $z = a + h$ in (1) $\Rightarrow$

(ii). $f(a+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(a)$

$$= f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots$$

219. Region of convergence of the Taylor's series : The region of the convergence of the Taylor's series

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

is given by $|z-a| < r$, where r is the radius of convergence and it is the distance from a to the nearest singularity of the function $f(z)$. On the circle $|z-a| = r$, the series may or may not convergence.

If $|z-a| > r$, the series **diverges.**

If the nearest singularity of the function $f(z)$ is at infinity, the radius of convergence is infinite and in this case the series converges for all z.

220. Maclaurin series : We have the Taylor's series is

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) +$$

$\ldots \ldots$ (1). Now putting $a = 0$ in (1), we have

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \ldots \ldots,$$

which is called the **Maclaurin series.**

Historical Note : Colin Maclaurin (1698 – 1746) was a Scots mathematician.

Example-218 : Show that $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$ ..., using integration.

Solution : If $|z| < 1$, then we have

$$\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \ldots \ldots \quad (1)$$

Now integrating (1) with respect to z from o to z ⇒

$$\int_0^z \frac{1}{1+z} dz = \int_0^z (1 - z + z^2 - z^3 + z^4 - \ldots) dz$$

$$\Rightarrow \Big[\log(1+z)\Big]_0^z = \left[z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots\right]_0^z$$

$$\Rightarrow \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots$$

Example-219 : Show that $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} -$ ... ... where $|z| < 1$.

Solution : If $|z|<1$, then $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$ (1).

Now on replacing z by – z in (1) ⇒

$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots\ldots \text{ where } |z| < 1.$$

Example-220 : Expand $f(z) = \log(1+z)$ in a Taylor series about $z = 0$ and determine the region of convergence for the required series.

Solution : (First part) :

$f(z) = \log(1+z)$. then $f(0) = 0$

$f'(z) = \frac{1}{1+z}$. ∴ $f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2}$. ∴ $f''(0) = -1$

$f'''(z) = \frac{(-1)(-2)}{(1+z)^3}$. ∴ $f'''(0) = (-1)(-2) = 2!$

$$f^{(n)}(z) = \frac{(-1)(-2)\ldots\{-(n-1)\}}{(1+z)^n}$$

∴

$$f^{(n)}(0) = (-1)(-2)\ldots\{-(n-1)\} = (-1)^{n-1}(n-1)!$$

Thus $f(z) = \log(1+z)$

$$= f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \ldots\ldots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots$$

Second part : Here n th term $= u_n = \frac{(-1)^{n-1} z^n}{n}$ and $u_{n+1} = \frac{(-1)^n z^{n+1}}{n+1}$. Then we have $\frac{u_{n+1}}{u_n} = \frac{-nz}{n+1}$ and using the ratio test $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{nz}{n+1}\right| = |z|$.

Thus the series is convergent for $|z| < 1$ and also it can be shown that the series is convergent for $|z| = 1$ except for $z = -1$. From this result it follows that the series converges in a circle which extends to the nearest singularity (i. e. $z = -1$) of $f(z)$.

Example-221 : Expand cos z about $z = \pi/2$ in a Taylor series. R. U. '76.

Solution : We have $f(z) = \cos z$. Then $f'(z) = -\sin z$, $f''(z) = -\cos z$, $f'''(z) = \sin z$, $f^{iv}(z) = \cos z$, ... and $f(\pi/2) = 0$, $f'(\pi/2) = -1$, $f''(\pi/2) = 0$, $f'''(\pi/2) = 1$, $f^{iv}(\pi/2) = 0$, ...

Then about $z = a = \pi/2$, we have

$$f(z) = f(a) + (z-a)\,f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \dots\dots$$

$$\Rightarrow f(z) = -(z-\pi/2) + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \dots\dots$$

Example-222 : Expand $\sin z$ about $z = \pi/4$ in a Taylor series.

Solution : We have $f(z) = \sin z$. Then $f'(z) = \cos z$, $f''(z) = -\sin z$, $f'''(z) = -\cos z$, $f^{iv}(z) = \sin z$, ... and

$$f(\pi/4) = \frac{1}{\sqrt{2}},\ f'(\pi/4) = \frac{1}{\sqrt{2}},\ f''(\pi/4) = -\frac{1}{\sqrt{2}},\ f'''(\pi/4) = -\frac{1}{\sqrt{2}} \dots$$

Then about $z = a = \pi/4$, we have $f(z) =$

$$f(a) + (z-a)\,f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \dots\dots$$

$$= \frac{1}{\sqrt{2}} + (z-\pi/4)\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^2}{2!}\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^3}{3!}\frac{1}{\sqrt{2}} + \dots\dots$$

$$= \frac{1}{\sqrt{2}}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \dots\dots\right\}$$

Example-223 : Expand $\log\left(\frac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$. D. U. H. '88, '90. D. U. M. SC. P. T. '91.

Solution : If $|z| < 1$, then we have

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \dots\dots \ (1) \text{ and}$$

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$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \frac{z^5}{5} - \dots\dots \ (2)$$

Now (1)–(2) $\Rightarrow \log(1+z) - \log(1-z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\right)$

$\Rightarrow \log\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\dots\right)$ which converges for $|z| < 1$. It can be also proved that this series converge for $|z| = 1$ except for $z = -1$.

Example-224 : Using the Maclaurin series prove the following series :

(i). $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $|z| < \infty$;

(ii). $\sin z = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}$, $|z| < \infty$,

(iii). $\cos z = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!}$, $|z| < \infty$;

(iv). $\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{z^n}{n}$, $|z| < \infty$;

(v). $\tan^{-1} z = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{z^{2n-1}}{2n-1}$, $|z| < 1$;

(vi). $(1+z)^n = \sum_{p=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-p+1)}{p!}z^p$, $|z| < 1$

Here (vi). is called the binomial theorem or formula. If $(1+z)^n$ is multiple-valued, then the result is valid for that branch of the function which has the value 1 when $z = 0$.

Solution : Try yourself.

N. B. In the above example, in the case of multiple valued functions, the principal branch is used.

Laurent's theorem (Theorem —191) : If $f(z)$ is analytic inside and on the boundary of a ring-shaped region $\Re$ bounded by two concentric circles C_1 and C_2 of centre a and radii r_1 and r_2 $(r_2 < r_1)$ respectively' then for all z in $\Re$,

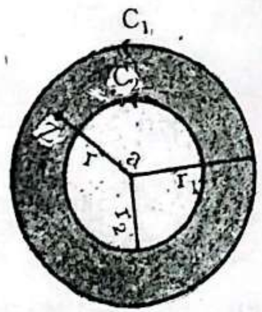
$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \quad \text{where}$$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw,\ n = 0, 1, 2, 3, \ldots$$

$$\text{and } a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw,\ n = 1, 2, 3, \ldots$$

D. U. H. '85, '88; R. U. M. SC. P. '85, '88; R. U. '79, 83; C. U. H. '82; D. U. '82; D. U. H. T. '82; R. U. H. '76.

Proof : Let the ring-shaped region $\Re = \{|z-a| = r : r_2 < r < r_1\}$. If $z \varepsilon \Re$, then by the Cauchy's integral formula for multiply-connected region, i. e. for the ring-shaped region $\Re$, we have $f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw$



$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)-(z-a)}\,dw + \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(z-a)-(w-a)}\,dw$$

$$= \frac{1}{2\pi i}\oint_{C_1}\left\{1-\frac{z-a}{w-a}\right\}^{-1}\frac{f(w)}{w-a}\,dw + \frac{1}{2\pi i}\oint_{C_2}\left\{1-\frac{w-a}{z-a}\right\}^{-1}\frac{f(w)}{z-a}\,dw$$

$$= \frac{1}{2\pi i}\oint_{C_1}\left\{1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\ldots + \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^{n}\frac{w-a}{w-z}\right\}\frac{f(w)}{w-a}\,dw$$

$$+ \frac{1}{2\pi i}\oint_{C_2}\left\{1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\ldots\ldots + \left(\frac{w-a}{z-a}\right)^{n-1} + \left(\frac{w-a}{z-a}\right)^{n}\frac{z-a}{z-w}\right\}\frac{f(w)}{z-a}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \ldots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n$$

$$+ \frac{1}{(z-a)2\pi i}\oint_{C_2} f(w)\,dw + \frac{1}{(z-a)^2 2\pi i}\oint_{C_2}(w-a)\,f(w)\,dw + \ldots + \frac{1}{(z-a)^n\,2\pi i}\oint_{C_2}(w-a)^{n-1}\,f(w)\,dw + V_n$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{n-1}(z-a)^{n-1} + U_n + \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \ldots + \frac{a_n}{(z-a)^n} + V_n \ \ldots (1)$$

where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw$, $n = 0, 1, 2, 3, \ldots$;

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw\ ,\ n = 1, 2, 3, \ldots ;$$

$$U_n = \frac{1}{2\pi i}\oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw \text{ and}$$

$$V_n = \frac{1}{2\pi i}\oint_{C_2} \left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw.$$

Now $|w - z| = |(w - a) - (z - a)| \geq |w - a| - |z - a|$ $\geq r_1 - r$ where $|w - a| = r_1$ for all points w on C_1. Then $|U_n|$ $= \frac{1}{2\pi}\left|(z-a)^n \oint_{C_1} \frac{f(w)}{(w-a)^n (w-z)}\,dw\right|$

$\leq \frac{r^2 M_1 (2\pi r_1)}{2\pi r_1^n (r_1 - r)} = \frac{M_1}{1 - r/r_1}\left(\frac{r}{r_1}\right)^n$ ... (2) where M_1 is the maximum value of $f(w)$ on C_1 and $2\pi r_1$ is the length of the circle C_1. In(2) $\frac{r}{r_1} < 1$, then $\lim_{n\to\infty} |U_n| \leq 0 \Rightarrow \lim_{n\to\infty} U_n = 0$ ... (3).

Again $|z - w| = |(z - a) - (w - a)| \geq |z - a| - |w - a|$ $\geq r - r_2$ where $|w - a| = r_2$ for all points w on C. Then $|V_n| = \frac{1}{2\pi}\left|\frac{1}{(z-a)^n}\oint_{C_2} (w-a)^n \frac{f(w)}{z-w}\,dw\right|$

$\leq \frac{r_2^n M_2 (2\pi r_2)}{2\pi r^n (r - r_2)} = \frac{M_2}{(r/r_2) - 1}\left(\frac{r_2}{r}\right)^n$ ... (4)

where M_2 is the maximum value of $f(w)$ on C_2 and $2\pi r_2$ is the length of the circle C_2. In (4) $\frac{r_2}{r} < 1$, then $\lim_{n\to\infty} |V_n| \leq 0 \Rightarrow \lim_{n\to\infty} V_n = 0$ ... (5) Now taking limit $n\to\infty$ in (1) and using (3) and (5) we have

$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots + a_n(z - a)^n + \ldots$ $+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots + \frac{a_{-n}}{(z-a)^n} + \ldots$ and the theorem is proved.

Historical Note : Pierre Alphonse Laurrent (1813 - 1854) was a French mathematician.

Other forms of the Laurent's theorem :

(i). $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz,\ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

(ii). Now writting $z = a + h$ in the Laurent series, we have

$$f(a+h) = \sum_{n=-\infty}^{\infty} a_n h^n = a_0 + a_1 h + a_2 h^2 + \ldots + a_n h^n + \ldots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \ldots + \frac{a_{-n}}{h^n} + \ldots\ldots$$

222. Laurent series : The series

$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots + a_n(z - a)^n + \ldots + \frac{a_{-1}}{z-a}$ $+ \frac{a_{-2}}{(z-a)^2} + \ldots + \frac{a_{-n}}{(z-a)^n} + \ldots$ is called the Laurent series or expansion with coefficients $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$.

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

223. Analytic and principal part of the Laurent series : We have the Laurent series is $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n +$

$$\sum_{n=1}^{\infty} \frac{a_n}{(z-a)^n} = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n \dots .$$
$$+ \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \dots + \frac{a_n}{(z-a)^n} + \dots \text{(1)}$$

Here the part $\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 +$ ... (2) is called the analytic part of the Laurent series. Again, the part $\sum_{n=1}^{\infty} \frac{a_n}{(z-a)^n} = \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \dots + \frac{a_n}{(z-a)^n} + \dots$ (3) is called the principal part of the Laurent series. If the principal part (3) is zero, then (1)

$\Rightarrow f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots$

which is the Taylor's series. Thus if the principal part of the Laurent series is zero, then the Laurent series reduces to a Taylor series.

D. U. M. SC. P. '89; D. U. H. '82.

B **Example-225** : Expand the function $f(z) = \frac{1}{z-3}$ in a Laurent series for the region :

(i). $|z| < 3$; (ii). $|z| > 3$. **D. U. H. T. '83**

Solution : (i) We have $|z| < 3 \Rightarrow 3 > |z|$ and using this condition, we have $f(z) = \frac{1}{z-3}$

$$= -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} = -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\frac{z^4}{81}+\dots\right)$$

$$= -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}-\dots$$

(ii). We have $|z| > 3$ and using this we have $f(z) = \frac{1}{z-3} =$

$$\frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} = \frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\dots\right)$$

$$= \frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\frac{27}{z^4}+\dots$$

B **Example 226** : Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series for the region. **(i).** $0 < |z| < 2$; **(ii).** $|z| > 2$.

D. U. M. SC. P. T. '91.

Solution : Resolving into partial fractions, we have

$$f(z) = \frac{1}{z(z-2)} = \frac{1}{2}\left[\frac{1}{z-2}-\frac{1}{z}\right] \dots \text{(1)}.$$

(i). We have $0 < |z| < 2$. Now using the condition $|z| < 2$,

$$\text{(1)} \Rightarrow \frac{1}{z(z-2)} = \frac{1}{2}\left[-\frac{1}{2}(1-z/2)^{-1}-\frac{1}{z}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\frac{z^4}{16}+\dots\right)-\frac{1}{z}\right]$$

$$= -\frac{1}{2z}-\frac{1}{4}-\frac{z}{8}-\frac{z^2}{16}-\frac{z^3}{32}-\frac{z^4}{64}-\dots$$

(ii). We have $|z| > 2$ and using this condition (1) $\Rightarrow$

$$\frac{1}{z(z-2)} = \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\right] = \frac{1}{2z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{2z}$$

$$= \frac{1}{2z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\frac{16}{z^4}+\dots\right)-\frac{1}{2z}$$

$$= \frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\frac{8}{z^5}+\dots$$

উপরে কম এর, নিচে বেশি → Naturally
উপরে নিচে সমান → আমরা 1 আর.

B **Example 227** : Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in a Laurent series for the region $1 < |z| < 2$. **D. U. H. T. '87.**

Solution : We have $f(z) = \frac{z}{(z+1)(z+2)} = \frac{2}{z+2} - \frac{1}{z+1} \dots$ (1). But we have $1 < |z| < 2 \Rightarrow 2 > |z|$ and $|z| > 1$ ... (2). Then using the condition (2), (1) $\Rightarrow$

$$f(z) = \frac{2}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\dots\right) - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right)$$

$$= \left(1-\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}-\dots\right) - \left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\dots\right)$$

$$= \dots\dots -\frac{1}{z^3}+\frac{1}{z^2}-\frac{1}{z}+1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\dots\dots$$

Example 228 : Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series for the following given region : (i). $1 < |z| < 3$; (ii). $|z| > 3$; (iii). $0 < |z+1| < 2$; (iv). $|z| < 1$.

R. U. '83; R. U. M. SC. P. '84; D. U. H. '86, '90; C. U. H. '89.

Solution : Resolving into partial fractions, we have

$$\frac{1}{(z+1)(z+3)} = \frac{\frac{1}{-1+3}}{z+1} + \frac{\frac{1}{-3+1}}{z+3} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right) \quad \dots (1).$$

(i). We have $1 < |z| < 3 \Rightarrow |z| > 1$ ... (2) and $3 > |z|$ ... (3). In the first term of the right hand side of (1), we will use the condition (2) and in the second we will use the condition (3). Now using (2) and (3), the equation (1) $\Rightarrow$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\dots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\dots\right) - \left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\dots\right)$$

(ii). We have $|z| > 3$ ... (4). Both in the first and second term of the right hand side of (1), we will use the condition (4) and using this, we have $\frac{1}{(z+1)(z+3)}$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1} = \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right)$$

$$-\frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\dots\right) = \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\dots\right)$$

$$-\left(\frac{1}{2z}-\frac{3}{2z^2}+\frac{9}{2z^3}-\frac{27}{2z^4}+\dots\right) = \left(\frac{1}{2z}-\frac{1}{2z}\right) + \left(\frac{3}{2}-\frac{1}{2}\right)\frac{1}{z^2}$$

$$-\left(\frac{9}{2}-\frac{1}{2}\right)\frac{1}{z^3} + \left(\frac{27}{2}-\frac{1}{2}\right)\frac{1}{z^4} - \dots = \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\dots$$

(iii). We have $0 < |z+1| < 2 \Rightarrow 2 > |z+1|$. Now using this condition (1) $\Rightarrow \frac{1}{(z+1)(z+3)}$

$$= \frac{1}{2(z+1)} - \frac{1}{2}\left\{\frac{1}{(z+1)+2}\right\}$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left\{1+\left(\frac{z+1}{2}\right)\right\}^{-1} = \frac{1}{2(z+1)}$$

$$-\frac{1}{4}\left\{1-\frac{1}{2}(z+1)+\frac{1}{4}(z+1)^2-\frac{1}{8}(z+1)^3+\dots\right\}$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \dots$$

(iv). We have $|z| < 1 \Rightarrow |z| < 1 < 3$ and using these (1)

$$\Rightarrow \frac{1}{(z+1)(z+3)} = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}(1+z/3)^{-1}$$

$$= \frac{1}{2}(1-z+z^2-z^3+\dots) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\dots\right)$$

$$= \left(\frac{1}{2}-\frac{1}{6}\right) - \left(\frac{1}{2}-\frac{1}{18}\right)z + \left(\frac{1}{2}-\frac{1}{54}\right)z^2 - \left(\frac{1}{2}-\frac{1}{162}\right)z^3 + \dots$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \ldots$$

Example 229 : Expand $f(z) = \dfrac{1}{z^2(z-1)(z+2)}$ in a Laurent series in the region $|z| > 2$. **D. U. H. T. '78.**

Solution : We have $\dfrac{1}{z^2(z-1)(z+2)} = \dfrac{1}{z^2}\left[\dfrac{1}{3(z-1)} - \dfrac{1}{3(z+2)}\right]$

$$= \frac{1}{3z^2}\left[\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}\right] \quad [\because |z| > 2]$$

$$= \frac{1}{3z^3}\left[\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right) - \left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \ldots\right)\right]$$

$$= \frac{1}{3z^3}\left(\frac{3}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \frac{15}{z^4} + \frac{33}{z^5} \ldots\ldots\right)$$

$$= \frac{1}{z^4} - \frac{1}{z^5} + \frac{3}{z^6} - \frac{5}{z^7} + \frac{11}{z^8} - \ldots\ldots$$

Example 230 : Expand $f(z) = \dfrac{z^2}{(z-1)(z-2)}$ in a Laurent series for the region $1 < |z| < 2$. **D. U. H. T. '71, '85, '88.**

Solution : We have $f(z) = \dfrac{z^2}{(z-1)(z-2)} = 1 - \dfrac{1}{z-1} + \dfrac{4}{z-2}$ ... (1). But we have $1 < |z| < 2 \Rightarrow |z| > 1$ and $2 > |z|$ ... (2).

Now using the condition (2), (1) $\Rightarrow f(z) = \dfrac{z^2}{(z-1)(z-2)}$

$$= 1 - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{4}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right) - 2\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \ldots\right)$$

$$= \left(1 - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \ldots\right) - \left(2 + z + \frac{z^2}{2} + \frac{z^3}{2^2} + \ldots\right)$$

$$= \ldots\ldots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - 1 - z - \frac{z^2}{2} - \frac{z^3}{2^2} - \ldots\ldots$$

Example-231 : Expand $f(z) = \dfrac{z^3}{(z+1)(z-2)}$ in a Laurent series in the powers of $(z+1)$ in the region $0 < |z+1| < 3$. **D. U. H. T. '76.**

Solution : We have $f(z) = \dfrac{z^3}{(z+1)(z-2)} = \dfrac{(z^3-8)+8}{(z+1)(z-2)}$

$$= \frac{(z-2)(z^2+2z+4)+8}{(z+1)(z-2)} = \frac{z^2+2z+4}{z+1} + \frac{8}{(z+1)(z-2)}$$

$$= \frac{(z+1)^2+3}{z+1} - \frac{8}{3(z+1)} + \frac{8}{3(z-2)} \quad [\text{ by cover up rule }]$$

$$= (z+1) + \frac{3}{z+1} - \frac{8}{3(z+1)} + \frac{8}{3}\left\{\frac{1}{(z+1)-3}\right\}$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left\{1 - \left(\frac{z+1}{3}\right)\right\}^{-1} \quad [\because 3 > |z+1|]$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left\{1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \ldots\right\}$$

$$= \frac{1}{3(z+1)} - \frac{8}{9} + \frac{19}{27}(z+1) - \frac{8}{9}\left\{\frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \ldots\ldots\right\}$$

Example-232 : Expand $f(z) = \dfrac{1}{(z^2+1)(z+2)}$ in a Laurent series for the region $1 < |z| < 2$. **D. U. H. T. '78.**

Solution : We have $f(z) = \dfrac{1}{(z^2+1)(z+2)} = \dfrac{1}{5}\left[\dfrac{1}{z+2} - \dfrac{z-2}{z^2+1}\right]$

$$= \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{z-2}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}\right]$$

$$[\because 2 > |z| \text{ and } |z| > 1]$$

$$= \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1} - \frac{z-2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\dots\right) - \frac{z-2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\dots\right).$$

Example 233 : Expand the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ for the following regions :

(i). $2 < |z| < 3$; (ii). $|z| < 2$; (iii). $|z| > 3$.

D. U. M. SC. P. T. '89; D. U. M. SC. P. '88; R. U. M. SC. P. '84; R. U. H. '80, '85; D. U. H. '90.

Solution : We have $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$ ... (1)

(i). We have $2 < |z| < 3 \Rightarrow |z| > 2$ and $3 > |z|$ ... (2).

Now using (2), (1) $\Rightarrow f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\dots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\dots\right),$$

$$= \dots - \frac{24}{z^4} + \frac{12}{z^3} - \frac{6}{z^2} + \frac{3}{z} - \frac{5}{3} + \frac{8z}{3^2} - \frac{8z^2}{3^3} + \frac{8z^3}{3^4} - \dots$$

(ii). We have $|z| < 2$. Now using this (1) $\Rightarrow$

$$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} \quad [\because |z| < 2 < 3]$$

$$= 1 + \frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\dots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\dots\right)$$

$$= \left(1+\frac{3}{2}-\frac{8}{3}\right) - \left(\frac{3}{2^2}-\frac{8}{3^2}\right)z + \left(\frac{3}{2^3}-\frac{8}{3^3}\right)z^2 - \left(\frac{3}{2^4}-\frac{8}{3^4}\right)z^3 + \dots$$

$$= -\frac{1}{6} - \left(\frac{3}{2^2}-\frac{8}{3^2}\right)z + \left(\frac{3}{2^3}-\frac{8}{3^3}\right)z^2 - \left(\frac{3}{2^4}-\frac{8}{2^4}\right)z^3 + \dots$$

(iii). We have $|z| > 3$. Using this (1) $\Rightarrow$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\dots\right) - \frac{8}{z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\dots\right)$$

$$= 1 + (3-8)\frac{1}{z} - (6-24)\frac{1}{z^2} + (12-72)\frac{1}{z^3} - \dots \dots$$

$$= 1 - \frac{5}{z} + \frac{18}{z^2} - \frac{60}{z^3} + \dots \dots$$

Example 234 : Expand the function $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$ for the following regions :

(i). $1 < |z| < 4$; (ii). $|z| < 1$; (iii). $|z| > 4$.

R. U. M. SC. P. '86; D. U. M. SC. P. '88.

Solution : We have $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$

$$= 1 - \frac{1}{z+1} - \frac{4}{z+4} \dots (1).$$

(i). We have $1 < |z| < 4 \Rightarrow |z| > 1$ and $4 > |z|$ ... (2). Now using (2), (1) $\Rightarrow f(z) = 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{4}\left(1+\frac{z}{4}\right)^{-1}$

$$= 1 - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right) - \left(1-\frac{z}{4}+\frac{z^2}{4^2}-\frac{z^3}{4^3}+\dots\right)$$

$$= \left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}-\dots\right) - \left(1-\frac{z}{4}+\frac{z^2}{4^2}-\frac{z^3}{4^3}+\dots\right)$$

$$= \dots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + \frac{z}{4} - \frac{z^2}{4^2} + \frac{z^3}{4^3} - \dots$$

(ii). We have $|z| < 1$. Now using this condition

$$(1) \Rightarrow f(z) = 1 - (1+z)^{-1} - \frac{4}{4}\left(1+\frac{z}{4}\right)^{-1} \quad [\because |z| < 1 < 4]$$

$$= 1 - (1 - z + z^2 - z^3 + \ldots) - \left(1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \ldots\right)$$

$$= -1 + \left(1+\frac{1}{4}\right)z - \left(1+\frac{1}{4^2}\right)z^2 + \left(1+\frac{1}{4^3}\right)z^3 - \ldots\ldots$$

(iii). We have $|z| > 4$. Now using this condition

$$(1) \Rightarrow f(z) = 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right) - \frac{4}{z}\left(1 - \frac{4}{z} + \frac{4^2}{z^2} - \frac{4^3}{z^3} + \ldots\right)$$

$$= 1 - (1+4)\frac{1}{z} + (1+4^2)\frac{1}{z^2} - (1+4^3)\frac{1}{z^3} + \ldots\ldots$$

Example-235 : Show that $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=0}^{\infty} a_n (z^n + z^{-n})$ where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh (2\cos\theta)\, d\theta.$

R. U. H. '73; D. U. M. SC. P. 80; R. U. M. SC. P. '85.

Solution : Let $f(z) = \cosh(z + z^{-1})$ and it is analytic everywhere in the finite part of the z-plane except at $z = 0$. Therefore, $f(z)$ is analytic in the annular region $r \le |z| \le R$ where r is small and R is large. Then by the Laurent's theorem we have $f(z) = \cosh(z + z^{-1}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \ldots$ (1) where $a_n = \frac{1}{2\pi i}\oint_C \cosh(z + z^{-1})\frac{dz}{z^{n+1}}$ ... (2) and $b_n =$ $\frac{1}{2\pi i}\oint_C \cosh(z + z^{-1})\, z^{n-1}\, dz$ ... (3), where C is any circle with centre at the origin. Let C be a circle of radius one, then $|z| = 1 \Rightarrow z = e^{i\theta}$, $z + z^{-1} = 2\cos\theta$, $dz = i e^{i\theta} d\theta$ and $0 \le \theta < 2\pi$.

Now using these (2) $\Rightarrow$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \cosh(2\cos\theta)\frac{ie^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)(\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta$$

since $\int_0^{2\pi} \cosh(2\cos\theta)\sin n\theta\, d\theta = 0.$

Again (3) $\Rightarrow b_n = a_{-n} = \int_0^{2\pi} \cos h(2\cos\theta)\cos(-n\theta)\, d\theta$

$= \int_0^{2\pi} \cos h(2\cos\theta)\cos n\theta\; d\theta = a_n.$

Thus $f(z) = \cosh(z + z^{-1}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n} = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$ where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cos h(2\cos\theta)\, d\theta.$

224. Classification of singularities :

R. U. M. SC. P. '84; D. U. H. '88; R. U. H. '72, '76, '77.

Definitions of zeros, singular points and various types of singularities are given in chapter 3. Now we will classify singularities using Laurent series. Let $f(z)$ have an isolated singularity at $z = a$ and let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \quad \ldots (1)$$ be the Laurent expansion in $0 < |z - a| < r$. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called analytic part and the part $\sum_{n=1}^{\infty} \frac{a_n}{(z-a)^n} = \sum_{n=1}^{\infty} a_{-n} \cdot (z-a)^{-n}$ is called the principal part of the Laurent expansion or series.

(i). Poles : In (1), if $a_{-p} \neq 0$ for some positive integer p but $a_n = 0$ for all $n < -p$, then a is called a pole of order p for the function $f(z)$. In this case the principal part of (1) has only a finite number of p terms, then (1) $\Rightarrow \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \ldots + \frac{a_p}{(z-a)^p}$. If p = 1, then z = a is called a simple pole. If p = 2, then it is called a double pole or pole of order 2. If p = 3, 4, 5, ... , then a are called poles of order three, four, five, ... respectively. If the function $f(z)$ has a pole at the point z = a, then $\lim_{z \to a} f(z) = \infty$.

Example-236 : The Laurent series for a function $f(z)$ with a pole at z = a of order n takes the following form :

$$f(z) = \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \ldots + \frac{a_n}{(z-a)^n} + a_0 + a_1 (z-a) +$$

$$a_2 (z-a)^2 + \ldots = \sum_{p=1}^{n} \frac{a_p}{(z-a)^p} + \sum_{p=0}^{\infty} a_p (z-a)^p.$$

(ii). Removal singularities : In (1), if $a_n = 0$ for all n < 0, then a is called a removal singularity of the function $f(z)$.

Example-237 : If $f(z)$ has a removal singularity at the point z = a, then its Laurent series takes the following form : $f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \ldots = \sum_{n=0}^{\infty} a_n (z-a)^n$

where $0 < |z - a| < r$.

(iii). Essential singularities : In (1) if $a_n \neq 0$ for an infinite number of negative values of n, then a is called an essential singularity of $f(z)$. Thus if z = a is an essential singularity of the function $f(z)$, then the principal part of the Laurent expansion has infinitely many terms.

Example-238 : We have.

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

Here z = 0 is an essential singularity of $f(z)$.

(iv). Branch points : Let $f(z)$ be a multiple-valued function. Then a branch F(z) of the function $f(z)$ is any single valued function that is analytic in some region at each point z of which the value F(z) is one of the values of the function $f(z)$. A point $z = a$ is called a branch point of the above function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about the point $z = a$. The branch points are singular points. In each branch of a multiple-valued function Taylor's theorem can be applied.

Example-239 : Let $f(z) = z^{1/2}$. Then the branch of $f(z)$ which has the value 1 for $z = 1$ has a Taylor series of the following form :

$$a_0 + a_1 (z - 1) + a_2 (z - 1)^2 + \dots \dots$$

(v). Singularities at infinity : Replacing $z = 1/w$ in $f(z)$ we will obtain the function $f(1/w)$. The type of the singularity of the function $f(z)$ at $z = \infty$ is the same as that of the function $f(1/w)$ at $w = 0$.

Example - 240 : Let $f(z) = z^4$. Then $f(z)$ has a pole of order 4 at $z = \infty$ since $f(1/w) = 1/w^4$ has a pole of order 4 at $w = 0$.

Example - 241 : Expand the function $f(z) = \dfrac{1}{z^2(z-3)^2}$ about $z = 3$ in a Laurent series naming the singularity and the region of convergence.

Solution : (First part) : Let $z - 3 = p \Rightarrow z = p + 3$. Then

$$f(p + 3) = \frac{1}{(p+3)^2 p^2} = \frac{1}{9p^2} \frac{1}{(1+p/3)^2} = \frac{1}{9p^2} (1 + p/3)^{-2}$$

$$= \frac{1}{9p^2} \{ 1 - 2 (p/3) + 3(p/3)^2 - 4(p/3)^3 + \dots \}$$

[by binomial theorem]

$$= \frac{1}{9p^2} - \frac{2}{27p} + \frac{1}{27} - \frac{4p}{243} + \dots \dots$$

$$\Rightarrow f(z) = \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \dots \dots$$

(Second part) : Here $z = 3$ is a pole of order 2.

(Third part) : The series converges for all values of p such that $0 < | p/3 | < 1$ or $0 < |P| < 3$ i. e. for all values of z such that $0 < |z - 3| < 3$ where $p = z - 3$.

Example -242 : Expand the function $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $z = -2$ in a Laurent series naming the type of singularity and the region of convergence.

Solution : (First part) : Let $z + 2 = p \Rightarrow z = p - 2$. Then

$$f(z) = \frac{P-2}{(p-1)p} = \frac{2}{p} + \frac{1}{1-p} = \frac{2}{p} + (1 - p)^{-1}$$

$$= \frac{2}{p} + 1 + p + p^2 + p^3 + \dots = \frac{2}{z+2} + 1 + (z + 2) + (z + 2)^2 + \dots \dots$$

(Second part) : Here $z = -2$ is a pole of order 1 or simple pole.

(Third part) : The series converges for all values of p such that $0 < |p| < 1$ i. e. for all values of z such that

$0 < |z + 2| < 1$ where $p = z + 2$.

Example -243 : Expand the function $f(z) = \dfrac{z - \sin z}{z^3}$ about $z = 0$ in a Laurent series naming the type of singularity and the region of convergence.

Solution : (First part) : We have $f(z) = \dfrac{z - \sin z}{z^3}$

$$= \frac{1}{z^3}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)\right\}$$

$$= \frac{1}{z^3}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots\right) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots\dots$$

(Second part) : Here $z = 0$ is a removable singularity of the function.

(Third part) : The series converges for all values of z.

Example -244 : Find the Laurent series for the function $f(z) = (z - 3) \sin \frac{1}{z+2}$ about $z = -2$. Name the singularity and the region of convergence for the series.

Solution : (First part) : Let $z + 2 = p \Rightarrow z = p - 2$. Then $f(z)$

$$= (z-3) \sin \frac{1}{z+2} = (p-5) \sin \frac{1}{p} = (p-5) \left\{\frac{1}{p} - \frac{1}{3!\,p^3} + \frac{1}{5!\,p^5} - \dots\right\}$$

$$= 1 - \frac{5}{p} - \frac{1}{6p^2} + \frac{5}{6p^3} + \frac{1}{120\,p^4} - \dots\dots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \dots\dots$$

(Second part) : Here $z = -2$ is an essential singularity of the function.

(Third part) : The series converges for all values of z except for $z = -2$

Example - 245 : Find the Laurent series for the function $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$. Name the singularity and gives the region of convergence of the series. **D. U. H. T. '82.**

Solution : (First part) : Let $z - 1 = p \Rightarrow z = 1 + p$. Then

$$f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+p)}}{p^3} = \frac{e^2}{p^3} e^{2p}$$

$$= \frac{e^2}{p^3}\left\{1 + 2p + \frac{(2p)^2}{2!} + \frac{(2p)^3}{3!} + \frac{(2p)^4}{4!} + \dots\right\}$$

$$= \frac{e^2}{p^3} + \frac{2e^2}{p^2} + \frac{2e^2}{p} + \frac{4e^2}{3} + \frac{2e^2}{3} p + \dots\dots$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \dots\dots$$

(Second part) : Here $z = 1$ is a pole of order 3.

(Third part) : The series converges for all values of z except $z = 1$.

225. Zero of order n : Let $f(z)$ be analytic at the point $z = a$. Then the point a is called a zero of order n for the function $f(z)$ if $f(z)$ and its first $n - 1$ derivatives vanish at a, but $f^{(n)}(a) \neq 0$. Thus if a is a zero of order n, then $f(a) = f'(a)$ $f''(a) = \dots = f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$.

Theorem - 192 : Let $f(z)$ be analytic at the point $z = a$. Then $f(z)$ has a zero of order n at a if and only if $f(z)$ can be written as $f(z) = (z - a)^n g(z)$ where $g(z)$ is analytic at $z = a$ and $g(a) \neq 0$.

Proof : Try yourself.

Theorem -193 : If $f(z)$ has a pole at $z = a$ then $|f(z)| \to \infty$ as $z \to a$.

Proof : Try yourself.

194. Theorem : If the analytic function $f(z)$ has a pole of order n at the point $z = a$, then $\frac{1}{f(z)}$ has a zero of order n at $z = a$ and conversely. **R. U. H. '75.**

Proof : Try yourself.

Example -246 : The function $f(z) = \tan \frac{\pi z}{2}$ has a zero at each even integer and an isolated singularity at each odd integer.

Solution : Try yourself.

226. Meromorphic function : A function $f(z)$ is called meromorphic if it is analytic every where in the finite plane except at a finite number of poles.

Example -247 : Let $f(z) = \frac{z+2}{(z-3)(z+5)^4}$. Here $f(z)$ is analytic every where in the finite plane except at the poles $z = 3$ which is simple and $z = -5$ is a pole of order 4 and is therefore it is a meromorphic function.

Lagrange's expansion (Theorem — 195) : If z is a root of $z = a + \alpha \phi(z)$ which has the value $z = a$ when $\alpha = 0$, then if $\phi(z)$ is analytic inside and on a circle C containing $z = a$, we have $z = a + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^{n-1}}{da^{n-1}} \left\{ [\phi(a)]^n \right\}$.

General theorem : If F(z) is analytic inside and on C, then

$$F(z) = F(a) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^{n-1}}{da^{n-1}} \left\{ F'(a) [\phi(a)]^n \right\}.$$

Proof : The proof is not given here.

Theorem -196 : If $< f_n(z) >$ be a sequence of analytic functions in a region $\Re$ and $F(z) = \sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in $\Re$, then $F(z)$ is analytic in $\Re$.

Proof : Try yourself.

Theorem -197 : An analytic function can not be bounded in the neighbourhood of an isolated singularity.

Proof : Try yourself.

Theorem -198 : Show that if $z \neq 0$, then $e^{1/2\alpha(z - 1/z)}$

$$= \sum_{n=-\infty}^{\infty} J_n(\alpha) z^n \text{ where}$$

$$J_n(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - \alpha \sin\theta)\, d\theta,\ n = 0, 1, 2, 3, \ldots$$

Proof : Try yourself.

Schlaefli's formula (Theorem — 199) : If C is any simple closed curve enclosing the point $z = t$, then $P_n(t)$

$$= \frac{1}{2\pi i} \cdot \frac{1}{2^n n!} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz, \text{ where}$$

$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$ is the **Rodrigues formula** for the **Legendre polynomial** $P_n(z)$, $n = 1, 2, 3, \ldots$

Proof : Try yourself.

Example -248 : Show that

$$P_n(z) = \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2 - 1} \cos\theta)^n d\theta.$$

Proof : Try yourself.

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CHAPTER – 7

RESIDUES AND THE RESIDUE THEOREMS

227. Residue : **D. U. H. '90; D. U. H. T. '83, '89.**

If the function $f(z)$ has an isolated singularity at the point $z = a$, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laurent's expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$= a_0 + a_1 (z-a) + a_2 (z-a)^2 + \dots \dots$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \dots \dots$$

around $z = a$ is called the residue of $f(z)$ at $z = a$ and it is denoted by Res $[f(z), a]$ or Res (a) or a_{-1}.

N. B. In this book, by $[a_p]_{-1}$ we will mean the residue at the finite point $z = a_p$.

Theorem -200 : If $f(z)$ is analytic inside and on a simple closed curve C except at the point $z = a$ inside C, then

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Proof : The Laurent series about $z = a$ is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \dots (1) \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

$n = 0, \pm 1, \pm 2, \dots (2)$

Now putting $n = -1$ in (2), we get

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$ and the theorem is proved.

Theorem -201 : If $f(z)$ has a simple pole at $z = a$, then show that

$$a_{-1} = \lim_{z \to a} (z-a) f(z),$$ where a_{-1} is the residue of $f(z)$ at the point $z = a$.

Proof : If $f(z)$ has a simple pole at $z = a$, then the corresponding Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-1}}{z-a} + \sum_{n=0}^{\infty} a_n (z-a)^n \dots (1)$$ where $a_{-1} \neq 0$. Now

multiplying (1) by $z - a$, then we have

$$(z-a) f(z) = a_{-1} + (z-a) \sum_{n=0}^{\infty} a_n (z-a)^n \dots (2).$$

Now on letting $z = a$ in (2), we have $a_{-1} = \lim_{z \to a} (z-a) f(z)$

and the theorem is proved.

Theorem -202 : The residue at a finite point $z = a$ is given by the equation Res $[f(z), a] = a_{-1}$ where $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \dots (1)$ is a Laurent expansion of $f(z)$ at $z = a$.

Proof : We have Res $[f(z), a] = \frac{1}{2\pi i} \oint_C f(z)\, dz \dots (2)$

where a is finite. Let C be a circle $|z - a| = r$ in the region $\Re$ enclosing no singuarity other than a. Than the Laurent series converges uniformly on C' and (1) can be integrate term by term. But we know.

$$\oint_C \frac{1}{(z-a)^n}\,dz = \int_0^{2\pi} \frac{r\,i\,e^{i\theta}}{r^n e^{ni\theta}}\,d\theta = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases} \quad \ldots (3)$$

Now by (1), (2) and (3), we have Res $[f(z), a] = a_{-1}$ and the theorem is proved.

Theorem – 209 : If $f(z) = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are analytic at $z = a$ but $P(a) \neq 0$, then $a_{-1} = \dfrac{P(a)}{Q'(a)}$ if $Q(z)$ has a simple zero at $z = a$.

Proof : Since $Q(z)$ has a simple zero at $z = a$, then $f(z) = \dfrac{P(z)}{Q(z)}$ has a simple pole at $z = a$ where $Q(a) = 0$ and $P(a) \neq 0$. Now the residue of $f(z)$ at $z = a$ is $a_{-1} = \lim\limits_{z \to a} (z-a)\, f(z)$

$$= \lim_{z \to a} (z-a) \frac{P(z)}{Q(z)}$$

$$= \lim_{z \to a} \frac{P(z)}{\dfrac{Q(z) - Q(a)}{z-a}} = \frac{P(a)}{Q'(a)}$$ and the theorem is proved.

Theorem – 204 : If $f(z)$ has a double pole at $z = a$, then show that

$$a_{-1} = \lim_{z \to a} \frac{1}{1!} \frac{d}{dz} \{(z-a)^2 f(z)\}.$$

Proof : If $f(z)$ has a double pole at $z = a$, then the corresponding Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + \sum_{n=0}^{\infty} a_n (z-a)^n \ldots (1)$$

Now multiplying (1) by $(z-a)^2$, then we get

$$(z-a)^2 f(z) = a_{-2} + a_{-1}(z-a) + \sum_{n=0}^{\infty} a_n (z-a)^{n+2} \ldots (2)$$

Now differentiating (2) with respect to z we get

$$\frac{d}{dz}\{(z-a)^2 f(z)\} = a_{-1} + \sum_{n=0}^{\infty} (n+2)\, a_n (z-a)^{n+1} \ldots (3).$$

Now on letting $z \to a$ in (3) we get

$$a_{-1} = \lim_{z \to a} \frac{d}{dz} \{(z-a)^2 f(z)\}$$ and the theorem is proved.

Theorem – 205 : If $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are analytic at $z = a$ but $P(a) \neq 0$, then $a_{-1} \dfrac{6P'(a)\,Q''(a) - 2P(a)\,Q'''(a)}{3\,[Q''(a)]^2}$.

where $P(a)$, $P'(a)$, $Q''(a)$ and $Q'''(a)$ are exist and $Q(z)$ has a double zero at $z = a$.

Proof : Try yourself.

228. Residue at a multiple point :

Theorem – 206 : If $f(z)$ is analytic inside and on a simple closed curve C except at pole $z = a$ of order m inside C, then the residue of $f(z)$ at $z = a$ is

$$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}.$$

D. U. H. T. '84, '86; D. U. M. SC. P. T. '88; D. U. '79; D. U. H. '88, '90; J. U. H.

Proof : (Method 1) : Since $f(z)$ has a pole of order m at $z = a$, then $f(a) = \frac{F(z)}{(z-a)^m}$, where $F(z)$ is analytic inside and on C and also $F(a) \neq 0$. Then by the Cauchy's $(m-1)$th differential integral formula, we have

$$\frac{1}{2\pi i}\oint_C f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{F(z)}{(z-a)^m}\,dz = \frac{1}{2\pi i}\,\frac{F^{(m-1)}(a)}{(m-1)!}$$

$$= \lim_{z \to a} \frac{1}{(m-1)!}\,\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\} = a_{-1}$$ and the

answer line

theorem is proved.

(Method 2) : Since $f(z)$ has a pole at $z = a$ of order m, then by the Laurent series we have

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \ldots + \frac{a_{-1}}{z-a}$$

$$+ a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots \quad (1)$$

Now multiplying (1) by $(z-a)^m \Rightarrow$

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \ldots + a_{-1}(z-a)^{m-1}$$

$$+ a_0(z-a)^m + a_1(z-a)^{m+1} + \ldots \quad (2)$$

Now differentiating both sides of (2) $m-1$ times with respect to z, then (2) $\Rightarrow$

$$\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\} = (m-1)!\,a_{-1} + \frac{m!}{1!}a_0(z-a) +$$

$$\frac{(m+1)!}{2!}a_1(z-a)^2 + \ldots \quad (3)$$

Now letting $z \to a$ in (3)

$$\Rightarrow \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\,a_{-1} \Rightarrow$$

$$a_{-1} = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}$$ and the theorem is proved.

Theorem - 207 : If $f(z)$ is analytic at $z = a$, then the residue of $\frac{f(z)}{(z-a)^{n+1}}$ at $z = a$ is $\frac{f^{(n)}(a)}{n!}$, where n is a positive integer.

Proof : We have $f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$

$$= f(a) + \frac{z-a}{1!}f'(a) + \ldots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \ldots\ldots$$

$$\Rightarrow \frac{f(z)}{(z-a)^{n+1}} = \frac{f(a)}{(z-a)^{n+1}} + \frac{f'(a)}{1!}\frac{1}{(z-a)^n} + \ldots + \frac{f^{(n)}(a)}{n!}\frac{1}{z-a} + \ldots$$

Here the coefficient of $\frac{1}{z-a}$ in $\frac{f(z)}{(z-a)^{n+1}}$ is $\frac{f^{(n)}(a)}{n!}$

$$= \text{Res}\left[\frac{f(z)}{(z-a)^{n+1}}, a\right]$$

and the theorem is proved.

B **Example - 249 :** Evaluate the residues of $f(z) = \frac{z^2}{z^2+a^2}$ at the poles. $\rightarrow$ poles.

Solution : The poles of $f(z)$ are given by $z^2 + a^2 = 0$

$\Rightarrow z = \pm\, ai$ and they are simple poles.

Final

Method 1 : Now Res $(ai) = \lim_{z\to ai}\{(z-ai)\,f(z)\}$

$$= \lim_{z\to ai}\left\{(z-ai)\frac{z^2}{z^2+a^2}\right\} = \lim_{z\to ai}\left\{(z-ai)\frac{z^2}{(z-ai)(z+ai)}\right\}$$

$$= \lim_{z \to ai} \frac{z^2}{z + ai} = \frac{(ai)^2}{2ai} = \frac{1}{2} ai.$$

Similarly, Res $(-ai) = -\frac{1}{2} ai.$

Method 2 : Let $f(z) = \frac{P(z)}{Q(z)}$ where $P(z) = z^2$ and $Q(z) = z^2 + a^2$. Then $P(ai) = (ai)^2$, $Q'(z) = 2z$ and $Q'(ai) = 2ai$.

Now Res $(ai) = \frac{P(ai)}{Q'(ai)} = \frac{(ai)^2}{2ai} = \frac{1}{2}$ ai. Similarly.

Res $(-ai) = \frac{P(-ai)}{Q'(-ai)} = \frac{(-ai)^2}{-2ai} = -\frac{1}{2}$ ai.

Example 250 : Evalute the residues of $f(z) = \frac{z^4}{z^2 + a^2}$ at the poles.

Solution : The poles of $f(z)$ are obtained by solving $z^2 + a^2 = 0 \Rightarrow z = \pm ai$ and they are simple poles. Now Res (ai)

$$= \lim_{z \to ai} \left\{ (z - ai) \frac{z^4}{z^2 + a^2} \right\} = \frac{(ai)^4}{2ai} = -i \frac{a^3}{2}$$

$$\lim_{z \to ai} \left\{ (z-ai) \frac{z^4}{(z-ai)(z+ai)} \right\} = \lim_{z \to ai} \frac{z^4}{z+ai}$$

Similarly, Res $(-ai) = i \frac{a^3}{2}$

Method 2 : Let $f(z) = \frac{P(z)}{Q(z)}$. Then $P(z) = z^4$ and $Q(z) = z^2 + a^2$. Here $Q'(z) = 2z$. Now Res $(ai) = \frac{P(ai)}{Q'(ai)} = \frac{(ai)^4}{2(ai)} = -\frac{1}{2} ia^3$

and Res $(-ai) = \frac{P(-ai)}{Q'(-ai)} = \frac{(-ai)^4}{2(-ai)} = \frac{1}{2} ia^3.$

Example 251 : Find Res $[f(z), i]$ where $f(z) = \frac{e^{iz}}{(z^2 + 1)^4}$

Solution : Here $z = i$ is a pole of order 4. Now Res (i)

$$= \lim_{z \to i} \frac{1}{3!} \frac{d^3}{dz^3} \left[(z - i)^4 \frac{e^{iz}}{(z^2 + 1)^4} \right] = \lim_{z \to i} \frac{1}{6} \frac{d^3}{dz^3} \left[\frac{e^{iz}}{(z + i)^4} \right]$$

But $\frac{d^3}{dz^3} \left[\frac{e^{iz}}{(z + i)^4} \right] = i^3 e^{iz} (z + i)^{-4} + 3c_1 i^2 e^{iz} (-4) (z + i)^{-5}$

$+ 3c_2 \, i \, e^{iz} (-4) (-5) (z + i)^{-6} + e^{iz} (-4) (-5) (-6) (z + i)^{-7}$

Then $\lim_{z \to i} \frac{d^3}{dz^3} \left[\frac{e^{iz}}{(z + i)^4} \right] = \frac{e^{-1}}{(i + i)^4} (-i - 6i - 15i - 15i)$

$\Rightarrow$ Res $(i) = -\frac{37e^{-1}}{16} i.$

Example 252 : Find the residues of the function $f(z) = \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)}$ D. U. H. T. '89; D. U. M. SC. P. '88.

Solution : The poles of $f(z)$ are obtained by solving $(z + 1)^2 (z^2 + 4) = 0 \Rightarrow f(z)$ has a double pole at $z = -1$ and simple poles at $z = \pm 2i$. Now the residue at $z = -1$ is

$$\lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z + 1)^2 \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)} \right\} = \lim_{z \to -1} \frac{d}{dz} \left\{ \frac{z^2 - 2z}{z^2 + 4} \right\}$$

$$= \lim_{z \to -1} \frac{z^2 - 2z}{z^2 + 4} \left[\frac{2z - 2}{z^2 - 2z} - \frac{2z}{z^2 + 4} \right] = \frac{3}{5} \left(\frac{-4}{3} + \frac{2}{5} \right) = -\frac{14}{25}$$

Residue at $z = 2i$ is $\lim_{z \to 2i} \left\{ (z - 2i) \frac{z^2 - 2z}{(z + 1)^2 (z^2 + 4)} \right\}$

$$= \frac{(2i)^2 - 4i}{(2i + 1)^2 (4i)} = \frac{-4 - 4i}{(4i^2 + 4i + 1)(4i)} = \frac{1 + i}{4 + 3i} = \frac{7 + i}{25}$$

Similarly, residue at $z = -2i$ is $\frac{7 - i}{25}$.

Example 253 : Show that Res $\left[\frac{1}{z^2 (z - 1)}, 0 \right] = -1.$

Solution : We have

$$\frac{1}{z^2 (z - 1)} = -\frac{1}{z^2} (1 - z)^{-1} = -\frac{1}{z^2} (1 + z + z^2 + z^3 + \dots), \quad 0 < |z| < 1$$

$$= -\frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 - \dots\dots$$

Here the coefficient of $\frac{1}{z}$ is -1 and Res $\left[\frac{1}{z^2(z-1)}, 0\right] = -1$

Example – 254 : Show that Res $\left[\frac{1}{z^2(z^2-1)}, 0\right] = 0$.

Solution : Try yourself.

229. Another method of finding the residue at a multiple point : If $f(z)$ has a pole of order m, then the Laurent series of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots + \frac{a_{-m}}{(z-a)^m} \ldots (1)$$

Now putting $z = t + a$ in (1), where t small,

$$f(t+a) = \sum_{n=0}^{\infty} a_n t^n + \frac{a_{-1}}{t} + \frac{a_{-2}}{t^2} + \ldots + \frac{a_{-m}}{t^m} \ldots (2)$$

In (1) a_{-1} is the coefficient of $\frac{1}{z-a}$ and in (2) it is the coefficient $\frac{1}{t}$ and it is called the residue of $f(z)$ at the point $z = a$.

See Rough khata

Example – 255 : Find Res $[f(z), 1]$ where $f(z) = \frac{z^2}{(z-1)^3(z-2)} \ldots (1)$.

Solution : Here $z = 1$ is a pole of order 3 of the function $f(z)$. Now let $z = t + 1$ in (1), we get $f(t+1) = \frac{(t+1)^2}{t^3(t-1)}$

$$= -\frac{t^2+2t+1}{t^3}(1-t)^{-1}$$

$$= -\frac{(t^2+2t+1)}{t^3}(1+t+t^2+t^3+\ldots)$$

$$= -\left(\frac{1}{t}+\frac{2}{t^2}+\frac{1}{t^3}\right)(1+t+t^2+t^3+\ldots)$$

Here coefficient $\frac{1}{t} = -1-2-1 = -4 = \text{Res}\,[f(z), 1]$.

Example – 256 : Find the residues at the poles of the following functions :

(i). $\frac{\sin z}{z^2}$; **(ii).** $\frac{\tan z}{z^2}$; **(iii).** $\frac{\cos z}{z}$; **(iv).** $\cot z$;

(v). $\operatorname{cosec} z$; **(vi).** $\operatorname{sech} z$; **(vii).** $\left(\frac{z+1}{z-1}\right)^2$.

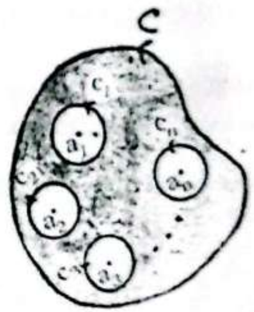
Solution : Try yourself.

Cauchy's residue theorem – 208 : If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n singular points $a_1, a_2, \ldots, a_n$ inside C, then

$$\oint_C f(z)\,dz = 2\pi i\,[\text{Res}\,(a_1) + \text{Res}\,(a_2) + \ldots + \text{Res}\,(a_n)]$$ i. e. $2\pi i$ times the sum of the residues at the singularities within C.

R. U. M. SC. P. '85; D. U. M. SC. P. '88; R. U. '76, '82; C. U. H. '81, '88; J. U. H. '87, '91; C. U. M. SC. P. '87; D. U. H. T. '82, '84; D. U. '72.

Proof : Let $C_1, C_2, \ldots, C_n$ be n circles with centres at the points $a_1, a_2, \ldots, a_n$. Let the radii of these circles are so small that they lie entirely inside C and do not overlap. Then $f(z)$ is analytic in the region between C and these circles and so by a corrolary of the Cauchy's theorem, we have



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz \ldots$$

(1)

But we have

$$\oint_{C_1} f(z)\,dz = \boxed{2\pi i \text{ Res } (a_1)},\ \oint_{C_2} f(z)\,dz = 2\pi i \text{ Res } (a_2), \ldots,$$

$$\oint_{C_n} f(z)\,dz = 2\pi i \text{ Res } (a_n). \ldots (2)$$

Then, by (1) and (2) we have

$$\oint_C f(z)\,dz = 2\pi i\,[\text{ Res } (a_1) + \text{Res } (a_2) + \ldots + \text{Res } (a_n)\,]$$

$= 2\pi i$ [sum of the residues within C] and the theorem is proved.

Theorem -209 : If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n poles $a_1, a_2, \ldots, a_n$ inside C, then

$$\oint_C f(z)\,dz = 2\pi i\,[\text{ Res } (a_1) + \text{Res } (a_2) + \ldots + \text{Res } (a_n)\,].$$

Proof : Use the above proof. In some books, this theorem is also known as the Cauchy's residue theorem.

Example -257 : Show that $I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$

$= \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$ where C is the circle with equation $|z| = 3$. **C. U. H. '88; R. U. M. SC. P. '84.**

Solution : The poles of $\frac{e^{zt}}{z^2(z^2+2z+2)}$ are obtained by solving $z^2(z^2+2z+2) = 0 \Rightarrow$ the integrand has a double pole at $z = 0$ and two simple poles at $z = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ and also all these poles are inside C.

Now the residue at $z = 0$ is $\lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left\{z^2 \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$

$$= \lim_{z\to 0} \frac{e^{zt}}{z^2+2z+2}\left[t - \frac{2z+2}{z^2+2z+2}\right] = \frac{t-1}{2}.$$

Residue at $z = -1+i = \lim_{z\to -1+i}\left\{(z+1-i)\frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$

$(z+1)^2+1$

$= \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)} = \frac{-e^{-t}}{4}e^{it}$. Similarly the residue at $z = -1-i$ is $\frac{e^{-t}}{4}e^{-ti}$

$(z+1)^2 - i^2$

$(z+1+i)(z+1-i)$

Thus by the Cauchy's residue theorem

$$I = 2\pi i \text{ (sum of the residues)} = 2\pi i\left[\frac{t-1}{2} + \frac{e^{-t}}{4}(e^{it} + e^{-ti})\right]$$

$$= 2\pi i\left[\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right]$$

$[\because \frac{1}{2}(e^{ti} + e^{-ti}) = \cos t]$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t.$$

Example -258 : Show that $\oint_C \frac{e^{tz}}{(z^2+1)^2}dz$

$= \pi i\,(\sin t - t\cos t)$

where C is the circle $|z| = 3$ and $t > 0$. **R. U. H. '80, '82.**

Solution : Here the poles of $\frac{e^{tz}}{(z^2+1)^2} = \frac{e^{tz}}{(z-i)^2(z+i)^2}$ are at $z = \pm i$ inside C and both is of order two.

Now the residue at $z = i$ is $\lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\left\{\frac{(z-i)^2 e^{tz}}{(z-i)^2(z+i)^2}\right\}$

Final

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{e^{tz}}{(z+i)^2}\right\} = \lim_{z\to i} \frac{t(z+i)^2 e^{tz} - 2(z+i)\, e^{tz}}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{t(z+i)e^{tz} - 2e^{tz}}{(z+i)^3} = \frac{2(it-1)e^{it}}{(2i)^3} = \frac{-2(t+i)e^{it}}{4}$$

Similarly, the residue at $z = -i$ is $\frac{-2(t-i)e^{-it}}{4}$

Then by the Cauchy's residue theorem, $\oint_C \frac{e^{tz}}{(z^2+1)^2}\,dz$

$$= 2\pi i \text{ (sum of the residues)} = 2\pi i \left\{\frac{-2(t+i)e^{it}}{4} + \frac{-2(t-i)e^{-it}}{4}\right\}$$

$$= -\frac{\pi i}{2}\left\{t(e^{it} + e^{-it}) + i(e^{it} - e^{-it})\right\} = \boxed{\pi i(\sin t - t\cos t).}$$

Problem

Example-259 : Show that $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{i}{\pi}$,

where C is the circle $|z| = 4$. **C. U. M. SC. P. '87.**

Solution : Here $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}$ has double pole at $z = \pi i$ and also at $z = -\pi i$.

Now the residue at $z = \pi i$ is $\lim_{z\to\pi i} \frac{1}{1!}\frac{d}{dz}\left\{\frac{(z-\pi i)^2 e^z}{(z-\pi i)^2(z+\pi i)^2}\right\}$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\} = \lim_{z\to\pi i} \frac{(z+\pi i)^2 e^z - 2e^z(z+\pi i)}{(z+\pi i)^4}$$

$= \frac{(2\pi i - 2)e^{\pi i}}{(\pi i + \pi i)^3} = \frac{\pi + i}{4\pi^3}$ and the residue at $z = -\pi i$ is

$$\lim_{z\to -\pi i} \frac{d}{dz}\left\{\frac{e^z}{(z-\pi i)^2}\right\} = \lim_{z\to-\pi i} \frac{(z-\pi i)e^z - 2e^z}{(z-\pi i)^3} = \frac{\pi - i}{4\pi^3}$$

by the residue theorem

$$\frac{\pi + i}{4\pi^3}, \quad \frac{-2(t+i)e^{ti}}{4}$$

$$\Rightarrow \oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\left(\frac{\pi+i}{4\pi^3} + \frac{\pi - i}{4\pi^3}\right) = \frac{i}{\pi}.$$

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Example-260 : Show that

$\oint_C \frac{ze^z}{z^2-1}\,dz = 2\pi i \cos h\ 1$, where C is the circle $|z| = 2$.

Solution : Since $f(z) = \frac{ze^z}{z^2-1}$ has poles at $z = \pm 1$ and they are simple, then by the Cauchy's residue theorem, we have $\oint_C \frac{ze^z}{z^2-1}\,dz = 2\pi i\,[\text{Res}(1) + \text{Res}(-1)]$ ... (1) since both the poles lies inside C. Now

$$\text{Res}(1) = \lim_{z\to 1}\{(z-1)f(z)\} = \lim_{z\to 1}\frac{ze^z}{z+1} = \frac{e}{2} \text{ ... (2).}$$

$$\text{Res}(-1) = \lim_{z\to -1}\{(z+1)f(z)\} = \lim_{z\to -1}\frac{ze^z}{z-1} = \frac{e^{-1}}{2} \text{ ... (3)}$$

Then by (2) and (3), (1) $\Rightarrow$

$$\oint_C \frac{ze^z}{z^2-1}\,dz = 2\pi i\left(\frac{e}{2} + \frac{e^{-1}}{2}\right) = 2\pi i \cos h\ 1 \text{ (proved).}$$

$\left[\frac{e}{2} + \frac{e^{-1}}{2} = \cosh\right.$

Example-261 : Show that

$\oint_C \frac{e^z}{z(z-1)^2}\,dz = 2\pi i$, where C is the circle $|z| = 2$.

Solution : The poles of $f(z) = \frac{e^z}{z(z-1)^2}$ are obtained by solving $z(z-1)^2 = 0 \Rightarrow z = 0,\ 1$ and both the poles lies in side C.

Here $z = 0$ is a simple pole and $z = 1$ is a double pole

$$\text{Res}(0) = \lim_{z\to 0}\left\{z\frac{e^z}{z(z-1)^2}\right\} = \lim_{z\to 0}\frac{e^z}{(z-1)^2} = 1;$$

$$\text{Res}\ (1) = \lim_{z\to 1} \frac{d}{dz}\left\{(z-1)^2 \frac{e^z}{z(z-1)^2}\right\} = \lim_{z\to 1} \frac{d}{dz}\left(\frac{e^z}{z}\right)$$

$$= \lim_{z\to 1} \frac{e^z(z-1)}{z^2} = 0.$$

Now by the residue theorem, we have $\oint_C \frac{e^z}{z(z-1)^2} dz$

$= 2\pi i\ [\text{Res}\ (0) + \text{Res}\ (1)\ ] = 2\pi i\ (1 + 0) = 2\pi i.$

Example -262 : Show that $\oint_C \frac{z}{z^4 - 1} dz = 0,$

where C is he circle $|z| = 2.$

Salution : ry yourself.

Ezample - 260 : Show that $\oint_C \frac{\tan z}{z} dz = 0,$

where C is th circle $|z| = 2.$

Solution : Tr, urself

230. Laurent series at ∞.

A function $f(z)$ is said to be analytic in a deleted neighbourhood of ∞ if the function $f(z)$ is analytic for $|z| > r_2$ for some r_2. In this case the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,\ r_2 < |z-a| < r_1$$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,\ |z| > r_2 \ldots (1)$$

if we take $r_1 = \infty$ and $a = 0$.

If there is no positive power of z in (1), the $f(z)$ is said to have a removal singularity at $z = \infty$ and we can make $f(z)$ analytic at $z = \infty$ by defining $f(\infty) = a_0$. In this case (1)

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n,\ |z| > r_2 \Big\} \ldots (2)$$

$f(\infty) = a_0$

If $f(z)$ is analytic at $z = \infty$ as in (2) and $f(\infty) = a_0 = 0$, then $f(z)$ is said to have **zero** at the point $z = \infty$.

231. Zero at infinity : If $f(z)$ is analytic at $z = \infty$ and if $\frac{1}{f(z)}$ has a pole of order n at $z = \infty$, then $f(z)$ has a **zero of order n** at $z = \infty$ and conversely.

If $n = 1$, then $f(z)$ has a **first order zero or zero of order one** at $z = \infty$.

232. Residue at infinity : If $f(z)$ is analytic for $|z| > r$ for some r, then the residue of $f(z)$ at $z = \infty$ is defined by the following :

$$\text{Res}\ [\ f(z), \infty\ ] = \frac{1}{2\pi i} \oint_C f(z)\ dz$$

where the integral is taken in the negative direction on a simple closed curve C, in the region of analyticity of $f(z)$ and outside of which $f(z)$ has no singularity other than infinity.

Theorem -210 : The residue of $f(z)$ at $z = \infty$ is given by the equation $\text{Res}\ [f(z), \infty\ ] = -a_{-1}$, where $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \ldots$ (1) is a Laurent expansion of $f(z)$ at $z = \infty$ and a_{-1} is the coefficient of z^{-1} in (1).

Prac

Proof : Try yourself.

Example – 264 : The function $f(z) = e^{1/z}$

$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\dots$ is analytic at $z = \infty$. Here the coefficient of z^{-1} is $1 = a_{-1}$ and the residue at $z = \infty$ is -1 i. e.

Res $[f(z), \infty] = -a_{-1} = -1$.

Theorem – 211 : If $f(z)$ is analytic at $z = \infty$, then Res $[f(z), \infty] = \lim_{z\to\infty} \{-zf(z)\}$.

Proof : Try yourself.

Theorem – 212 : Show that

$$\text{Res}\left[f(z), \infty\right] = -\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right].$$

Proof : The Laurent expansion of $f(z)$ at $z = \infty$ is $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \dots$ (1) where $|z| > r$ for some r. Then $f\left(\frac{1}{z}\right)$

$$= \sum_{n=-\infty}^{\infty} \frac{a_n}{z^n} \dots (2) \text{ where } 0 < |z| < \frac{1}{r}.$$

Now multiplying (2) by $\frac{1}{z^2}$, then we get

$$\frac{1}{z^2} f\left(\frac{1}{z^2}\right) = \dots + \frac{a_0}{z^2} + \frac{a_{-1}}{z} + a_{-2} + \dots (3).$$

Now letting $z\to 0$ in (3), we have

$$\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right] = a_{-1} = -\text{Res}\,[f(z), \infty]$$

$\Rightarrow$ Res $[f(z), \infty] = -$ Res $\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right]$ and the theorem is proved.

Theorem – 213 : If $f(z)$ has a zero of second or higher order at $z = \infty$, then Res $[f(z), \infty] = 0$.

Proof : Try yourself.

Theorem – 214 : If a single valued function $f(z)$ has only a finite number of singularities, then the sum of all the residues at these singularities of $f(z)$ including the residue at $z = \infty$ is zero.

Proof : Let C be a closed curve enclosing all the singularities of $f(z)$ except at ∞. Then the sum of the residues at these singularities is $\frac{1}{2\pi i}\oint_C f(z)\,dz \dots$ (1). But the residue at ∞ is $-\frac{1}{2\pi i}\oint_C f(z)\,dz \dots$ (2)

Adding (1) and (2), we get our required result. Thus the theorem is proved.

Example – 265 : Find Res $[f(z), 1]$ where

$$f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}.$$

Solution : (Method 1). Here $z = 1$ is a pole of order 4.

Now Res $(1) = \lim_{z\to 1} \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 f(z)\right\}$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right]$$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[z + A - \frac{8}{z-2} + \frac{27}{z-3}\right]$$ [by partial fractions where A is a constant]

$= \lim_{z\to 1} \frac{1}{6}\left[-8(-1)^3\, 3!\,(z-2)^{-4} + 27(-1)^3\, 3!\,(z-3)^{-4}\right]$

$= 8 - \frac{27}{16} = \frac{101}{16}.$

(Method 2). We have

$\text{Res}\,(1) + \text{Res}\,(2) + \text{Res}\,(3) + R\,(\infty) = 0 \;\ldots\; (1)$

Here $\text{Res}\,(2) = \lim_{z\to 2} \{(z-2)\, f(z)\}$

$= \lim_{z\to 2} \frac{z^3}{(z-1)^4 (z-3)} = -8 \;\ldots\; (2)$

$\text{Res}\,(3) = \lim_{z\to 3} \{(z-3) f(z)\}$

$= \lim_{z\to 3} \frac{z^3}{(z-1)^4 (z-2)} = \frac{27}{16} \;\ldots\ldots\; (3)$

$\text{Res}\,(\infty) = 0. \;\ldots\; (4)$ since at $z = \infty$, the function has a zero of order 3.

Now by (1), (2), (3) and (4) we have

$\text{Res}\,(1) - 8 + \frac{27}{16} + 0 = 0 \Rightarrow \text{Res}\,(1) = \frac{101}{16}.$

Example – 266 : Find Res $[f(z), \infty]$, where $f(z) = \frac{z^3}{z^2-1}$.

Solution : The finite poles of $f(z)$ are $z = \pm 1$ and they are simple.

Here $\text{Res}\,(1) = \lim_{z\to 1} \left\{(z-1)\frac{z^3}{z^2-1}\right\} = \frac{1}{2}$ and

$\text{Res}\,(-1) = \lim_{z\to -1} \left\{(z+1)\frac{z^3}{z^2-1}\right\} = \frac{1}{2}$

Now $\text{Res}\,(1) + \text{Res}\,(-1) + \text{Res}\,(\infty) = 0$

$\Rightarrow \frac{1}{2} + \frac{1}{2} + \text{Res}\,(\infty) = 0 \Rightarrow \text{Res}\,(\infty) = -1.$

Example –267 : Compute the residues at the singularities of $f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$ and also at ∞. Also show that their sum is zero including the residue at ∞.

Solution : The poles of $f(z)$ are $z = 1, 2, 3$ and they are simple poles. Now :

$\text{Res}\,(1) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{z^3}{(z-2)(z-3)} = \frac{1}{2}.$

$\text{Res}\,(2) = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \frac{z^3}{(z-1)(z-3)} = -8.$

$\text{Res}\,(3) = \lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} \frac{z^3}{(z-1)(z-2)} = \frac{27}{2}.$

Again we have $f(z) = 1 + \frac{1/2}{z-1} - \frac{8}{z-2} + \frac{27/2}{z-3}$

$= 1 + \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} - \frac{8}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{27}{2z}\left(1-\frac{3}{z}\right)^{-1} \;\ldots\; (1)$

In (1), the coefficient of $\frac{1}{z}$ is $\frac{1}{2} - 8 + \frac{27}{2} = 6$ and the residue at ∞ is -6.

Now the sum of the residues including the residue at ∞ = $\frac{1}{2} - 8 + \frac{27}{2} - 6 = 0$.

233. Cauchy's residue theorem including the point infinity.

Theorem –215 : Let $f(z)$ be analytic in a region $\Re$ which includes a deleted neighbourhood of ∞. If C is a closed path in $\Re$ out side of which $f(z)$ is analytic except for isolated singularities at the points $a_1, a_2, \ldots, a_n$, then $\oint_C f(z)\, dz = 2\pi i\left\{\sum_{p=1}^{n} \text{Res}\,[f(z), a_p] + \text{Res}\,[f(z), \infty]\right\}$ where the integral is taken on C in the negative direction and the residue at ∞ must be included on the right.

Proof : Try yourself.

Example – 268 : Show that $\int_C \frac{1}{(z-1)^3(z-7)}\,dz = -\frac{\pi i}{108}$

where C is the circle $|z| = 2$.

Solution : Using the Cauchy's residue theorem including the point at infinity we will prove the result. Outside the circle $|z| = 2$, the function has a first Order pole at $z = 7$ and a zero of order 4 at $z = \infty$. Then by the residue theorem including the point at infinity, we have

$$\oint_C \frac{1}{(z-1)^3(z-7)}\,dz = -2\pi i\left[\text{Res}(7) + \text{Res}(\infty)\right] \quad \dots (1)$$

$$\text{Here Res}(7) = \lim_{z\to 7}\left\{(z-7)\frac{1}{(z-1)^3(z-7)}\right\} = \frac{1}{216} \quad \dots (2)$$

and $\text{Res}(\infty) = 0 \dots (3)$ since it is a zero of order 4.

Now by (2) and (3), (1) ⇒

$$\oint_C \frac{1}{(z-1)^3(z-7)}\,dz = -2\pi i\left(\frac{1}{216} + 0\right) = -\frac{\pi i}{108}$$ and the required result is obtained.

Example – 269 : Show that $\oint_C \frac{z}{z^4-1}\,dz = 0$,

where C is the circle $|z| = 2$.

Solution : The function $f(z) = \frac{z}{z^4-1}$ has no singularity outside the circle $|z| = 2$ other than ∞. At the point $z = \infty$ the function $f(z)$ has a zero of order 3, then $\text{Res}(\infty) = 0$. Now by the Cauchy's residue theorem including the point at infinity, we have

$$\oint_C \frac{z}{z^4-1}\,dz = -2\pi i\,\text{Res}(\infty) = 0.$$

CHAPTER – 8

CONTOUR INTEGRATION

In this chapter we will evalute a varity types of real definite integrals with the help of the Cauchy's residue theorem using a suitable types of closed path or contour. For this reason, the process is called contour integration. Now we will discuss it dividing several forms.

234. (Form 1). Integrals of the form :

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$, **where** $f(\cos\theta, \sin\theta)$ **is a rational function of** $\cos\theta$ **and** $\sin\theta$ **:**

Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta \dots (1)$, where $f(\cos\theta, \sin\theta)$ is a rational function of cos θ and sin θ. Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$

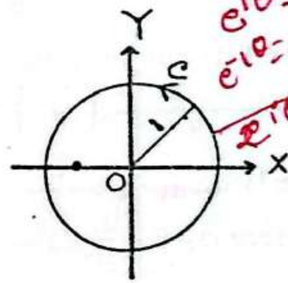
$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$ and $dz = ie^{i\theta}\,d\theta$

$= iz\,d\theta$ or $d\theta = \frac{dz}{iz}$. Now using these (1) ⇒ $I = \oint_C g(z)\,dz$,



where C is the unit circle $|z| = 1$, whose centre is at the origin and radius is equal to 1.

Example – 270 : Show that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$.

D. U. H. T. '75, 77, 87; D. U. H. '87.

$e^{-i\theta} = \frac{1}{z}$

CHAPTER – 8

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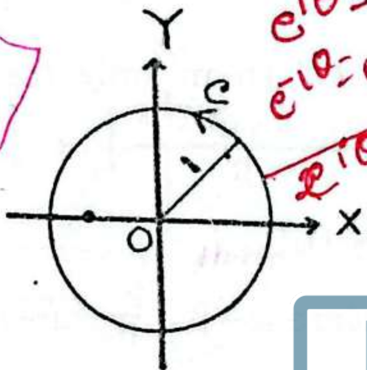
Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ ... (1), where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$. Let $z = e^{i\theta}$, then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z^2+1}{2z}$,

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z^2-1}{2iz}$ and $dz = ie^{i\theta}\, d\theta$

$= iz\, d\theta$ or $d\theta = \dfrac{dz}{iz}$. Now using these (1) $\Rightarrow I = \oint_C g(z)\, dz$,



$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

where C is the unit circle $|z| = 1$, whose centre is at the origin and radius is equal to 1.

Example 270 : Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ if $a > |b|$.

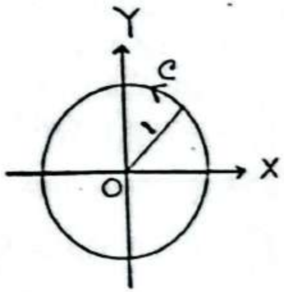
D. U. H. T. '75, 77, 87; D. U. H. '87.

$e^{-i\theta} = \dfrac{1}{e^{i\theta}} = \dfrac{1}{z}$

Solution : Suppose $I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ ... (1)

Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z}$

and $dz = ie^{iz}\, d\theta = iz\, d\theta$ or $d\theta = \frac{dz}{iz}$. Now using these



$$(1) \Rightarrow \oint_C \frac{1}{a + b\frac{z^2+1}{2z}} \frac{dz}{iz}$$

$$= \frac{2}{i}\oint_C \frac{dz}{bz^2 + 2az + b} \text{ ... (2).}$$

where C is the unit circle $|z| = 1$ whose radius is 1 and centre at the origin. The poles of $\frac{1}{bz^2 + 2az + b}$ are obtained by solving $bz^2 + 2az + b = 0$. If α and β are the poles, then we suppose $z = \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$ and $z = \beta = \frac{-a - \sqrt{a^2-b^2}}{b}$.

Of them only the pole α lies inside C since $|\alpha|$

$$= \left|\frac{-a + \sqrt{a^2-b^2}}{b}\right| = \left|\frac{a - \sqrt{a^2-b^2}}{b}\right| = \left|\frac{b}{a + \sqrt{a^2-b^2}}\right| < 1$$

if $a > |b|$.

Here $\alpha - \beta = \frac{2}{b}\sqrt{a^2-b^2}$ ... (3) Now

$$\text{Res}\,(\alpha) = \lim_{z\to\alpha} \left\{(z-\alpha)\frac{1}{bz^2 + 2az + b}\right\}$$

$$= \lim_{z\to\alpha} \frac{(z-\alpha)}{b(z-\alpha)(z-\beta)} = \frac{1}{b(\alpha-\beta)} = \frac{1}{2\sqrt{a^2-b^2}}, \text{ by ... (3)}$$

Now by the Cauchy's residue theorem

(2) $\Rightarrow I = \frac{2}{i}.\, 2\pi i.\, \text{Res}\,(\alpha) = \frac{2\pi}{\sqrt{a^2-b^2}}$ and the required result is obtained.

Example - 271 : Show that $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$

$= \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$.

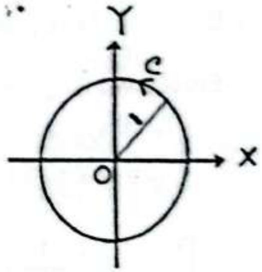
Solution : Try yourself.

Example - 272 : Show that $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \frac{\pi}{2}$.

Solution : Try yourself.

Example - 273 : Show that $\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1-a^2}$, $0 < a < 1$. **D. U. H. '84; R. U. '83; R. U. H. '73, 75, 81.**

Solution : Suppose $I = \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta}$ ... (1).



Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z}$

and $dz = ie^{i\theta}\, d\theta = iz d\theta$ or $d\theta = \frac{dz}{iz}$.

Using these (1) $\Rightarrow I = \oint_C \frac{1}{1 + a^2 - a\frac{z^2+1}{z}} \frac{dz}{iz}$

$$= -\frac{1}{ai}\oint_C \frac{dz}{z^2 - (a + a^{-1})z + 1}$$

$$= -\frac{1}{ai}\oint_C \frac{dz}{(z-a)(z-a^{-1})} \quad \dots (2)$$

where C is the circle $|z| = 1$.

The poles of $\frac{1}{(z-a)(z-a^{-1})}$ are the simple poles $z = a$, a^{-1}. Of them only $z = a$ lies inside C since $|z| < 1$ as $a < 1$.

Now Res (a) $= \lim\limits_{z \to a} \left\{ (z-a) \frac{1}{(z-a)(z-a^{-1})} \right\}$

$$= \frac{1}{a-a^{-1}} = \frac{a}{a^2-1}$$

Now by the Cauchy's residue theorem

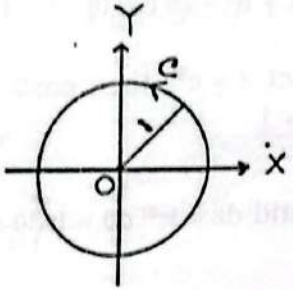
(2) $\Rightarrow I = -\frac{1}{ai} \cdot 2\pi i.$ Res (a) $= \frac{2\pi}{1-a^2}$ and the required result is proved.

Example - 274 : Show that $\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ where $a > 0$.

D. U. '84; D. U. H. T. '86; D. U. H. '86; R. U. '83.

Solution : Suppose I =

$$\int_0^{\pi} \frac{2a\,d\theta}{2a^2 + 2\sin^2\theta}$$



$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2 + 1 - \cos 2\theta} \dots (1).$$

Let $t = 2\theta$,

then $dt = 2d\theta$ and the limit of t is from o to 2π. Using these (1) $\Rightarrow I = \int_0^{2\pi} \frac{a\,dt}{2a^2 + 1 - \cos t} \dots (2).$

Now putting $z = e^{it}$, then $dz = ie^{it}\,dt = izdt$ or $dt = \frac{dz}{iz}$

and $\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z^2+1}{2z}$.

Now using these (2) $\Rightarrow I = \oint_C \frac{a}{2a^2 + 1 - \frac{z^2+1}{2z}} \frac{dz}{iz}$

$$= \frac{-2a}{i} \oint_C \frac{dz}{z^2 - 2(2a^2+1)z + 1} \dots (3),$$

where C is the circle $|z| = 1$. The poles of

$\frac{1}{z^2 - 2(2a^2+1)z + 1}$ are obtained by solving

$z^2 - 2(2a^2+1)z + 1 = 0$. If α and β are these poles, then

$$z = \frac{2(2a^2+1) \pm \sqrt{4(2a^2+1)^2 - 4}}{2} = (2a^2+1) \pm 2a\sqrt{a^2+1}$$

where $\alpha = 2a^2 + 1 + 2a\sqrt{a^2+1} = (\sqrt{a^2+1} + a)^2$ and

$\beta = 2a^2 + 1 - 2a\sqrt{a^2+1} = (\sqrt{a^2+1} - a)^2$. Here $\alpha\beta = 1$

where $\alpha > 1$, $\beta < 1$ and $\beta - \alpha = -4a\sqrt{1+a^2} \dots (4)$

Only the pole β lies inside C and it is a simple pole.

Now Res $(\beta) = \lim\limits_{z \to \beta} \left\{ (z-\beta) \frac{1}{z^2 - 2(a^2+1)z + 1} \right\}$

$$= \lim_{z \to \beta} \frac{z-\beta}{(z-\alpha)(z-\beta)} = \frac{1}{\beta - \alpha} = \frac{-1}{4a\sqrt{1+a^2}}, \text{ by (4)}$$

Hence by the Cauchy's residue theorem (3) $\Rightarrow$

$$I = \frac{-2a}{i} \cdot 2\pi i. \text{ Res } (\beta) = -4a\pi \cdot \frac{-1}{4a\sqrt{1+a^2}} = \frac{\pi}{\sqrt{1+a^2}}$$

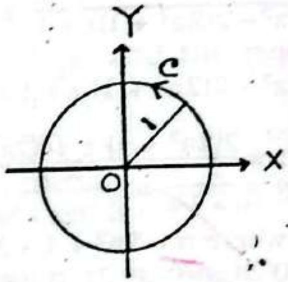
and we obtained the required result.

Example -275 : Show that $\int_0^{\pi} \frac{a\,d\theta}{a^2+\cos^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ where $a > 0$.

Solution : Try yourself.

Example - 276 : Show that $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \pi$.

Solution : Suppose $I = \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$ ... (1).



Let $z = e^{i\theta}$, then $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$

$= \frac{z^2-1}{2iz}$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$= \frac{z^2+1}{2z}$ and $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or

$d\theta = \frac{dz}{iz}$. Now using these (1) $\Rightarrow I =$

$\oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i}$ ...(2), where C is the circle $|z| = 1$.

The poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$ are obtained by solving $(1-2i)z^2+6iz-1-2i = 0 \Rightarrow z = \frac{-6i \pm 4i}{2(1-2i)}$

$= \frac{-2i}{10}(1+2i), \frac{-10i}{10}(1+2i) \Rightarrow z = \frac{1}{5}(2-i),\ 2-i$ and they are simple. Of them only the pole $\frac{1}{5}(2-i)$ lies inside C. Now

$\text{Res}\left\{\frac{1}{5}(2-i)\right\}$

$= \lim_{z \to \frac{1}{5}(2-i)} \left[\left\{z - \frac{1}{5}(2-i)\right\} \frac{2}{(1-2i)z^2+6iz-1-2i}\right]$

[by L. Hospital's rule]

$= \lim_{z \to \frac{1}{5}(2-i)} \frac{2}{2(1-2i)z+6i}$

$= \frac{5}{2-2-i-4i+15i} = \frac{1}{2i}$

Now by the Cauchy's residue theorem (2) $\Rightarrow$

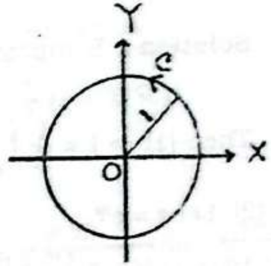
$I = 2\pi i.\ \text{Res}\left\{\frac{1}{5}(2-i)\right\} = 2\pi i.\ \frac{1}{2i} = \pi$ and the required value is obtained.

Example - 277 : Show that $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \frac{\pi}{6}$.

Solution : Suppose $I = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta}$

$= \text{Real part of} \int_0^{2\pi} \frac{e^{2i\theta}\,d\theta}{5+4\cos\theta}$ ...

(1).



Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$= \frac{z^2+1}{2z}$ and $dz = ie^{i\theta}d\theta = iz\,d\theta$ or $d\theta$

$= \frac{dz}{iz}$.

Now using these (1) $\Rightarrow I = \text{Real part of } \frac{1}{i}\oint_C \frac{z^2dz}{2z^2+5z+2}$

... (2) where C is the unit circle $|z| = 1$. The poles of $\frac{z^2}{2z^2+5z+2}$ are obtained by solving $2z^2+5z+z = 0$

$\Rightarrow (z+2)(2z+1) = 0 \Rightarrow z = -2, \frac{-1}{2}$ and they are simple poles. Of them only $z = -\frac{1}{2}$ lies inside C. Now

$$\text{Res}\left(\frac{-1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left\{\left(z+\frac{1}{2}\right) \frac{z^2}{(z+2)(2z+1)}\right\}$$

$$= \frac{\left(\frac{-1}{2}\right)^2}{2\left(\frac{-1}{2}+2\right)} = \frac{1}{12}$$

Now by the Cauchy's residue theorem,

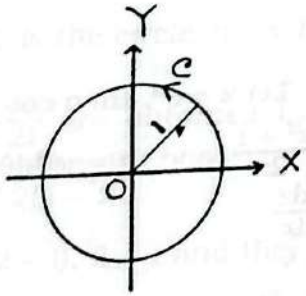
$$(2) \Rightarrow I = \text{Real part of } \frac{1}{i}\left\{2\pi i.\ \text{Res}\left(\frac{-1}{2}\right)\right\}$$

$$\Rightarrow I = 2\pi.\left(\frac{1}{12}\right) = \frac{\pi}{6}$$ and the result is proved.

Example – 278 : Show that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$

Solution : Suppose $I = \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$ ..... (1)

Then (1) $\Rightarrow I = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$



... (2). Let $z = e^{i\theta}$,

then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$ and

$dz = ie^{i\theta}d\theta$

$= izd\theta$ or $d\theta = \frac{dz}{iz}$. Now using

these (2) $\Rightarrow$

$$I = \frac{1}{2i}\oint_C \frac{z^2+z+1}{z(2z^2+5z+2)} = \frac{1}{4i}\oint_C \frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\, dz \ \ldots (3)$$

where C is the circle $|z| = 1$.

The poles of $\frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}$ are obtained by solving

$z\left(z+\frac{1}{2}\right)(z+2) = 0 \Rightarrow z = 0, -\frac{1}{2}, -2$

and they are simple. Of them the poles $z = 0, -\frac{1}{2}$ lie inside C. Now

$$\text{Res}(0) = \lim_{z \to 0}\left\{z\frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\right\} = 1 \text{ and } \text{Res}\left(-\frac{1}{2}\right)$$

$$= \lim_{z \to -\frac{1}{2}}\left\{\left(z+\frac{1}{2}\right)\frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\right\} = \frac{\frac{1}{4}-\frac{1}{2}+1}{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}+2\right)}$$

$= -1$

Hence by the Cauchy's residue theorem (3) $\Rightarrow$

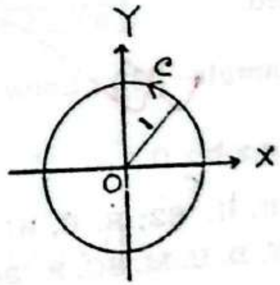
$$I = \frac{1}{4i}.2\pi i\left[\text{Res}(0) + \text{Res}\left(-\frac{1}{2}\right)\right] = \frac{\pi}{2}(1-1) = 0$$

Hence the result is proved.

Example – 279 : Show that $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta = \frac{\pi a^2}{1-a^2}$

if $a^2 < 1$. **D. U. H. T. '76, '86; C. U. H. '90.**

Solution : Suppose I =



$$\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta$$

$= \text{Real part of } \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{e^{2i\theta}\, d\theta}{1 - a(e^{i\theta} + e^{-i\theta}) + a^2}$ ... (1). Let $z = e^{i\theta}$

then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2 + 1}{2z} \text{ and } dz = ie^{i\theta}\, d\theta = iz d\theta \text{ or } d\theta = \frac{dz}{iz}$$

Now using these (1) $\Rightarrow$

$$I = \text{Real part of } \frac{1}{2} \oint_C \frac{z^2}{1 - a\dfrac{z^2+1}{z} + a^2} \frac{dz}{iz}$$

$$= \text{Real part } \frac{-1}{2ai} \oint_C \frac{z^2\, dz}{z^2 - (a + a^{-1})z + 1} \text{ ... (2)}$$

where C is the circle $|z| = 1$. Here the poles of $\dfrac{z^2}{z^2 - (a + a^{-1})z + 1} = \dfrac{z^2}{(z - a)(z - a^{-1})}$ are the simple poles $z = a, a^{-1}$. Of which only $z = a$ lies inside C since $|z| < 1$ as $a < 1$. Now

$$\text{Res } (a) = \lim_{z \to a} \left\{ (z - a) \frac{z^2}{(z - a)(z - a^{-1})} \right\} = \frac{a^3}{a^2 - 1}.$$

Hence by the Cauchy's residue theorem.

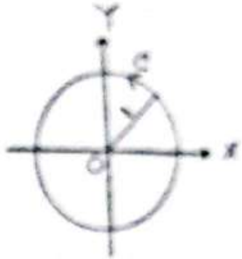
(2) $\Rightarrow I = \text{Real part of } \dfrac{-1}{2ai}\left\{2\pi i.\ \text{Res } (a)\right\}.$

$\Rightarrow I = \dfrac{-1}{a}.\ \pi.\ \dfrac{a^3}{a^2 - 1} = \dfrac{\pi a^2}{1 - a^2}$ and the required result is obtained.

Example- 280 : Show that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}(a - \sqrt{a^2 - b^2})$ where $a > b > 0$.

C. U. H. '82; R. U. H. '74; D. U. H. '90; D. U. H. T. '77, '83, '87; D. U. M. SC. P. '84.

Solution : Suppose $I = \displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta$



$$= \int_0^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b\cos\theta} d\theta = \text{Real part}$$

$$\text{of} \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + b(e^{i\theta} + e^{-i\theta})} d\theta \text{ ... (1).}$$

Let $z = e^{i\theta}$, then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z^2 + 1}{2z}$ and $dz = ie^{i\theta}\, d\theta = iz d\theta$ or $d\theta = \dfrac{dz}{iz}$.

Using these (1) $\Rightarrow I = \text{Real part of } \displaystyle\int_C \frac{(1 - z^2)}{2a + b\dfrac{(z^2 + 1)}{z}} \frac{dz}{iz}$

$= \text{Real part of } \dfrac{1}{i} \displaystyle\oint_C \frac{(1 - z^2)dz}{bz^2 + 2az + b}$ ... (2), where C is the circle $|z| = 1$. The poles of $\dfrac{1 - z^2}{bz^2 + 2az + b}$ are obtained by solving $bz^2 + 2az + b = 0$.

Now if α and β are the poles, then $z = \alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b}$ and $z = \beta = \dfrac{-a - \sqrt{a^2 - b^2}}{b}$.

Of them α lies inside C, which is simple.

$$\text{Now Res } (\alpha) = \lim_{z \to \alpha} \left\{ (z - \alpha) \frac{(1 - z^2)}{bz^2 + 2az + b} \right\}$$

$$= \lim_{z \to \alpha} \left\{ (z - \alpha) \frac{1 - z^2}{b(z - \alpha)(z - \beta)} \right\}$$

$$= \frac{1 - \alpha^2}{b(\alpha - \beta)} = \frac{\alpha\beta - \alpha^2}{b(\alpha - \beta)} = -\frac{\alpha}{b} \qquad [\because \alpha\beta = 1]$$

$$\Rightarrow \text{Res}(\alpha) = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

Now by the Cauchy's residue theorem

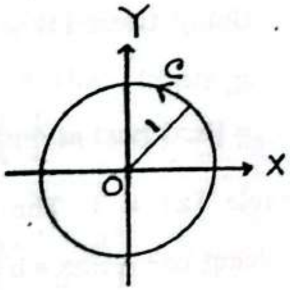
$$(2) \Rightarrow I = \text{Real Part of } \frac{1}{i}\left\{2\pi i.\ \text{Res}(\alpha)\right\}$$

$\Rightarrow I = 2\pi.\ \text{Res}(\alpha) = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$ and the required result is obtained.

Example − 281 : Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$.

D. U. M. SC. P. T. '90.

Solution : Suppose $I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$ ... (1)



Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$.

$\therefore \cos 3\theta = \frac{e^{3i\theta}+e^{-3i\theta}}{2} = \frac{z^6+1}{2z^3}$ and

$dz = ie^{i\theta}\, d\theta = iz\, d\theta$

or $d\theta = \frac{dz}{iz}$. Using these (1) $\Rightarrow$

$$I = \oint_C \frac{(z^6+1)/2z^3}{5-4(z^2+1)/2z}\ \frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

... (2), where C is the circle $|z| = 1$.

The poles of $\frac{z^6+1}{z^3(2z-1)(z-2)}$ are obtained by solving

$z^3(2z-1)(z-2) = 0 \Rightarrow z = 0$ is a pole of order 3 and $z = \frac{1}{2}, 2$ are simple poles.

Of them $z = 0$ and $z = \frac{1}{2}$ lie inside C. Now

$$\text{Res}(0) = \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3 \frac{(z^6+1)}{z^3(2z-1)(z-2)}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0} \frac{d^2}{dz^2}\left\{\frac{z^6+1}{(2z-1)(z-2)}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0} \frac{d^2}{dz^2}\left\{\frac{z^6}{(2z-1)(z-2)}\right\} + \frac{1}{2}\lim_{z\to 0} \frac{d^2}{dz^2}\left\{\frac{1}{(2z-1)(z-2)}\right\}$$

$$= 0 + \frac{1}{2}\lim_{z\to 0} \frac{d^2}{dz^2}\left\{\frac{1/3}{z-2} - \frac{2/3}{2z-1}\right\} \quad \text{[by partial fractions]}$$

$$= \frac{1}{2}\lim_{z\to 0}\left\{\frac{1}{3}(-1)^2\, 2!\,(z-2)^{-3} - \frac{2}{3}(-1)^2\, 2!\,(2z-1)^{-3}\, 2^2\right\}$$

$$= -\frac{1}{24} + \frac{8}{3} = \frac{63}{24} = \frac{21}{8}$$

$$\text{Again, Res}\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left\{\left(z-\frac{1}{2}\right)\frac{z^6+1}{z^3(2z-1)(z-2)}\right\}$$

$$= \frac{\frac{1}{64}+1}{\left(\frac{1}{8}\right)(2)\left(\frac{-3}{2}\right)} = -\frac{65}{24}$$

Now by the Cauchy's residue theorem, (2) $\Rightarrow$

$$I = -\frac{1}{2i}.\ 2\pi i\left[\text{Res}(0) + \text{Res}\left(\frac{1}{2}\right)\right]$$

$= -\pi\left[\frac{21}{8} - \frac{65}{24}\right] = \frac{\pi}{12}$ and the required result is proved.

Of them only $z = \frac{i}{3}$ lies inside C. Now

$$\text{Res}\,(i/3) = \lim_{z \to i/3} \frac{d}{dz}\left\{(z - i/3)^2 \frac{z}{(3z^2 - 10iz - 3)^2}\right\}$$

$$= \lim_{z \to i/3} \frac{d}{dz}\left\{\frac{z}{9(z - 3i)^2}\right\}$$

$$= \lim_{z \to i/3} \frac{(z - 3i)^2 - 2z\,(z - 3i)}{9(z - 3i)^4}$$

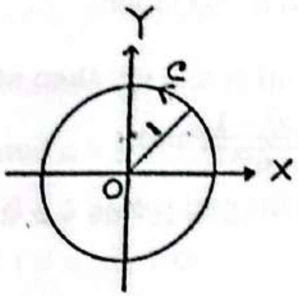
$$= \frac{(i/3 - 3i) - 2.i/3}{9(i/3 - 3i)^3} = -\frac{5}{256}$$

Now by the Cauchy's residue theorem

$(2) \Rightarrow I = -\frac{4}{i}.\, 2\pi i.\ \text{Res}\,(i/3) = (-8\pi)\left(-\frac{5}{256}\right) = \frac{5\pi}{32}$ and the required result is obtained.

Example – 285 : Show that $\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2}$

$= \frac{\pi(1 - p + p^2)}{1 - p}$ where $0 < p < 1$.



Solution : Suppose $I = \int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2\,p\cos 2\theta + p^2} d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1 + \cos 6\theta}{1 - 2p\cos 2\theta + p^2} d\theta$$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1 + e^{6i\theta}}{1 - p\ (e^{2i\theta} + e^{-2i\theta}) + p^2} d\theta \ \dots (1).$$

Let $z = e^{i\theta}$, then $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or $d\theta = \frac{dz}{iz}$.

Using these (1) $\Rightarrow I = \text{Real part of } \frac{-1}{2ip}\int_C \frac{z(1 + z^6)\,dz}{z^4 - (p + p^{-1})\,z^2 + 1}$ ... (2) where C is the circle $|z| = 1$. The poles of $\frac{z(1 + z^6)}{z^4 - (p + p{-}1)\,z^2 + 1} = \frac{z(1 + z^6)}{(z^2 - p)\,(z^2 - p^{-1})}$ are obtained by solving $(z^2 - p)\,(z^2 - p^{-1}) = 0$

$\Rightarrow z^2 = p,\ p^{-1} \Rightarrow z = \pm\sqrt{p},\ \pm\sqrt{p^{-1}}$ and they are simple poles.

Of them $z = \pm\sqrt{p}$ lies inside C since $p < 1$. Now

$$\text{Res}\,(\sqrt{p}) = \lim_{z \to \sqrt{p}} \left\{(z - \sqrt{p})\,\frac{z(1 + z^6)}{(z^2 - p)\,(z^2 - p^{-1})}\right\}$$

$$= \frac{\sqrt{p}\,(1 + p^3)}{2\sqrt{p}\,(p - p^{-1})} = \frac{p(1 + p^3)}{2(p^2 - 1)} = \frac{-p(1 - p + p^2)}{2(1 - p)}$$

Similarly, $\text{Res}\,(-\sqrt{p}) = \frac{-p(1 - p + p^2)}{2(1 - p)}$.

Hence by the Cauchy's residue theorem, (2) $\Rightarrow$

$$I = \text{Real part of } \frac{-1}{2ip}\left[2\pi i\left\{\text{Res}\,(\sqrt{p}) + \text{Res}\,(-\sqrt{p})\right\}\right]$$

$$\Rightarrow I = \frac{-\pi}{p}\left[\frac{-p(1 - p + p^2)}{2(1 - p)} + \frac{-p(1 - p + p^2)}{2(1 - p)}\right] = \frac{\pi(1 - p + p^2)}{1 - p}$$

and the required result is obtained.

Example – 286 : Show that $\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta}{3 + 2\cos\theta} d\theta$

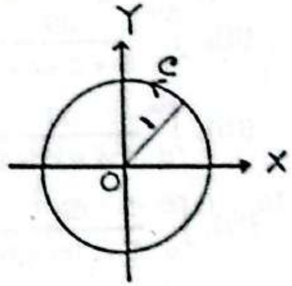
$= \frac{2\pi}{\sqrt{5}}(3 - \sqrt{5})^n$.

Solution : Suppose $I = \int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta}{3 + 2\cos\theta} d\theta$

$$= \text{Real part of } \int_0^{2\pi} \frac{(1 + e^{i\theta} + e^{-i\theta})^n\, e^{in\theta}}{3 + e^{i\theta} + e^{-i\theta}} d\theta \ \dots (1)$$

Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2 + 1}{2z}$ and $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or $d\theta = \frac{dz}{iz}$. Using these (1) $\Rightarrow$

$$I = \text{Real part of } \oint_C \frac{\left(1 + \frac{z^2 + 1}{z}\right)^n z^n}{3 + \frac{z^2 + 1}{z}} \frac{dz}{iz}$$



$$= \text{Real part of } \frac{1}{i}\oint_C \frac{(1 + z + z^2)^n}{1 + 3z + z^2}\,dz$$

... (2)

where C is the unit circle $|z| = 1$. The poles of $\frac{(1 + z + z^2)^n}{1 + 3z + z^2}$ are obtained by solving $1 + 3z + z^2 = 0$

$$\Rightarrow z = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$ and they are simple.

Let $\alpha = \frac{-3 + \sqrt{5}}{2}$ and $\beta = \frac{-3 - \sqrt{5}}{2}$. Of them α lies inside C.

$$\text{Res}(\alpha) = \lim_{z \to \alpha} \left\{(z - \alpha) \frac{(1 + z + z^2)^n}{1 + 3z + z^2}\right\}$$

$$= \lim_{z \to \alpha} \left\{(z - \alpha) \frac{(1 + z + z^2)^n}{(z - \alpha)(z - \beta)}\right\} = \frac{(1 + \alpha + \alpha^2)^n}{\alpha - \beta} \quad \dots (3)$$

We have $\alpha - \beta = \sqrt{5}$ and $1 + \alpha + \alpha^2 = -2\alpha = 3 - \sqrt{5}$ since α is a root of $1 + 3z + z^2 = 0$. i. e. $1 + 3\alpha + \alpha^2 = 0$

Now using these (3) $\Rightarrow \text{Res}(\alpha) = \frac{(3 - \sqrt{5})^n}{\sqrt{5}}$.

Hence by the Cauchy's residue theorem, (2) $\Rightarrow$

$$I = \text{Real part of } \frac{1}{i}\left\{2\pi i.\ \text{Res}(\alpha)\right\}$$

$$\Rightarrow I = \frac{2\pi}{\sqrt{5}}(3 - \sqrt{5})^n$$ and the required result is obtained.

Example – 287 : Show that :

(i). $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$;

(ii). $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \frac{\pi}{2}$;

(iii). $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}}$ if $a^2 < 1$;

(iv). $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = \frac{2\pi}{3}$.

Solution : Try yourself.

235. (Form 2). Integrals of the form : $\int_{-\infty}^{\infty} f(x)\,dx$, where $f(x)$ is a rational function of x : Consider $\oint_C f(z)\,dz$, where C is the contour consisting of :

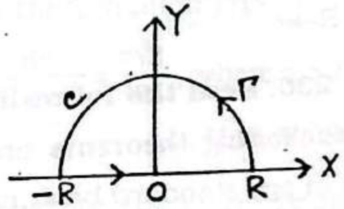
(i). the x- axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

Then let $R \to \infty$ and if $f(x)$ is an even function this can be used to evalute the integral $\int_0^{\infty} f(x)\,dx$.

Theorem – 216 : If $|f(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M is a constant, then show that $\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0$, where Γ is the upper semi-circle of radius R.

Proof : We have $\frac{M}{R^k}$ is the upper bound of $|f(z)|$ and πR is the upper semi circular arc length of Γ, then



$$\left|\int_\Gamma f(z)\,dz\right| \le \left(\frac{M}{R^k}\right)(\pi R)$$

$$= \frac{\pi M}{R^{k-1}} \dots (1).$$

Now taking $R \to \infty$ in both sides of (1)

$$\Rightarrow \lim_{R \to \infty} \left|\int_\Gamma f(z)\,dz\right| = 0 \Rightarrow \lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0$$ and the theorem is proved.

N. B. The above theorem is valid if the integration along the upper-semi circle Γ is replaced by the integration along the lower semi-circle Γ.

Theorem – 217 : Let $f(z) = \dfrac{P(z)}{Q(z)}$ be the quotient of two polynomials. If degree $Q(z) \ge 2 +$ degree $P(z)$, then

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0,$$ where Γ is the upper semi-circle of radius R.

Proof : We have degree $Q(z) \ge 2 +$ degree $P(z)$ ... (1). For large $|z|$, (1) $\Rightarrow$ $|f(z)| \le \dfrac{K}{|z|^2}$ where K is some constant.

Then $\left|\int_\Gamma f(z)\, dz\right| \le \dfrac{K}{R^2} \cdot \pi R = \dfrac{K\pi}{R}$ ... (2) where $|z| = R$ and length of $\Gamma = \pi R$.

Now letting $R\to\infty$ in (2) we have $\lim_{R\to\infty} \left|\int_\Gamma f(z)\, dz\right| \le 0$

$$\Rightarrow \lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0.$$

236. Read the following three theorems : The proofs of the following theorems are not given here. Try yourself or see in any standard book of complex variables and these are :

Theorem – 218 : Let $f(z)$ be analytic in the upper half of the z-plane or Argand plane except at a finite number of poles in it and having no poles on the real axis. If $zf(z) \to 0$ as $z \to \infty$, then $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i$ [sum of the residues at the poles in the upper plane].

Theorem –219 : If $\lim_{z\to\infty} (z-a) f(z) = K$ and if Γ is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z-a| = r$, then

$$\lim_{r\to 0} \int_\Gamma f(z)\, dz = iK(\theta_2 - \theta_1).$$

N. B. In the above theorem if $z = a$ is a simple pole of $f(z)$, then $K = \text{Res}\,[f(z), a]$ and we have

$$\lim_{r\to 0} \int_\Gamma f(z)\, dz = i(\theta_2 - \theta_1)\,\text{Res}\,[f(z), a].$$

Theorem –220 : If Γ is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$ and if $\lim_{R\to\infty} zf(z) = K$, then $\lim_{R\to\infty} \int_\Gamma f(z)\, dz = iK(\theta_2 - \theta_1)$.

N. B. In the above theorem if $\lim_{R\to\infty} zf(z) = 0$ ... (1), then

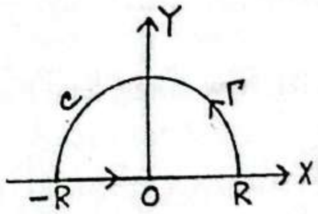
$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0 \text{ ... (2).}$$

In the following some examples we will use the condition (2) directly if it satisfies the condition (1).

Example –288 : Show that $\int_0^\infty \dfrac{dx}{x^4 + a^4} = \dfrac{\pi\sqrt{2}}{4a^3}$, where $a > 0$.

D. U. H. T. '88.

Solation : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{1}{z^4 + a^4}$ and C is the contour consisting of :



(i). the x-axis from –R to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

Now the poles of $f(z)$ are given by $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4$

$= a^4 e^{(2n+1)\pi i} \Rightarrow z = ae^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3$.

$\Rightarrow z = ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$ and they are simple poles. Only the poles $z = \alpha = ae^{\pi i/4}$ and $z = \alpha = ae^{3\pi i/4}$ lie within C. Now using L. Hospital's rule, the residues at these poles $z = \alpha = \lim\limits_{z\to\alpha} (z-\alpha) f(z) = \lim\limits_{z\to\alpha} \dfrac{z-\alpha}{z^4+a^4}$

$$= \lim_{z\to\alpha} \frac{1}{4z^3} = \frac{1}{4\alpha^3} = \begin{cases} \dfrac{1}{4a^3} e^{-3\pi i/4} \text{ at } z = \alpha = a\,e^{\pi i/4} \\ \dfrac{1}{4a^3} e^{-9\pi i/4} \text{ at } z = \alpha = ae^{3\pi i/4} \end{cases}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$= 2\pi i$ (sum of residues)

$$= 2\pi i \left(\frac{1}{4a^3} e^{-3\pi i/4} + \frac{1}{4a^3} e^{-9\pi i/4}\right)$$

$$= \frac{\pi i}{2a^3} \left[e^{-\pi i} e^{\pi i/4} + e^{-2\pi i} e^{-\pi i/4}\right]$$

$$= \frac{\pi i}{2a^3} \left[-e^{\pi i/4} + e^{-\pi i/4}\right] = \frac{\pi i}{2a^3} \left(-2i \sin\frac{\pi}{4}\right) = \frac{\pi}{a^3\sqrt{2}} \ldots (1).$$

Here $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \dfrac{z}{z^4+a^4} = 0$, then $\lim\limits_{R\to\infty} \int_\Gamma f(z)\,dz = 0 \ldots (2)$. Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{a^3\sqrt{2}} \Rightarrow 2\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi}{a^3\sqrt{2}} \Rightarrow \int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}.$$

Example - 289 : Show that $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

D. U. '84; D. U. H. S. T. '85; D. U. M. SC. P. 88.

Solution : Try yourself or, see the above example.

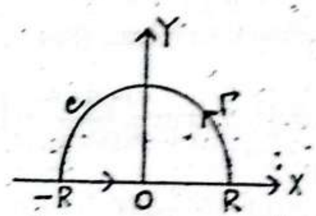
Example - 290 : Show that $\displaystyle\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$.

D. U. H. T. '86, '88.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{1}{z^6+1}$ and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $z^6 + 1 = 0 \Rightarrow z^6 = -1 = e^{(2n+1)\pi i} \Rightarrow z = e^{(2n+1)\pi i/6}$, where $n = 0, 1, 2, 3, 4, 5$

$\Rightarrow z = e^{\pi i/6}, e^{\pi i/2}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$ and they are simple. Of them only the poles $z = e^{\pi i/6}, e^{\pi i/2}$ and $e^{5\pi i/6}$ lie within C.

Now using L' Hospital's rule, the residue at the pole $z = \alpha = \lim\limits_{z\to\alpha} (z-\alpha) f(z) = \lim\limits_{z\to\alpha} \dfrac{z-\alpha}{z^6+1} = \lim\limits_{z\to\alpha} \dfrac{1}{6z^5}$

$$= \frac{1}{6\alpha^5} = \begin{cases} \dfrac{1}{6} e^{-5\pi i/6} = \dfrac{1}{6} e^{-\pi i} e^{\pi i/6} = -\dfrac{1}{6} e^{\pi i/6} \text{ at } z = e^{\pi i/6} \\ \dfrac{1}{6} e^{-5\pi i/2} = \dfrac{1}{6} e^{-2\pi i} e^{-\pi i/2} = -\dfrac{i}{6} \quad \text{at } z = e^{\pi i/2} \\ \dfrac{1}{6} e^{-25\pi i/6} = \dfrac{1}{6} e^{-4\pi i} e^{-\pi i/6} = \dfrac{1}{6} e^{-\pi i/6} \text{ at } z = e^{5\pi i/6} \end{cases}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$= 2\pi i$ (sum of residues)

$$= 2\pi i\left(-\frac{1}{6}e^{\pi i/6} - \frac{1}{6} + \frac{1}{6}e^{-\pi i/6}\right) = \frac{-\pi i}{3}\left(e^{\pi i/6} - e^{-\pi i/6} + 1\right)$$

$$= \left(-\frac{\pi i}{3}\right)\left(2i \sin\frac{\pi}{6} + i\right) = \frac{\pi}{3}(1+1) = \frac{2\pi}{3} \;\ldots (1)$$. Here

$$\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} \frac{z}{z^6+1} = 0, \text{ then } \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0 \;\ldots$$

(2). Now taking limit $R\to\infty$ in (1) and using (2)

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{3} \Rightarrow 2\int_0^\infty \frac{dx}{x^6+1} = \frac{2\pi}{3} \Rightarrow \int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$$ and

the result is proved.

Example-291 : Show that $\displaystyle\int_0^\infty \frac{x^6}{(a^4+x^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16a}$, $(a > 0)$.

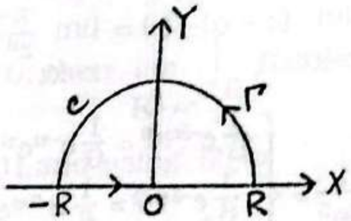
D. U. M. SC. P. '80; D. U. H. T. '77, '85.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{z^6}{(a^4+z^4)^2}$ and

C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ, of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by

$(z^4+a^4)^2 = 0 \Rightarrow z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = a^4 e^{(2n+1)\pi i} \Rightarrow$

$z = ae^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3$. Of them only $z = ae^{\pi i/4}$ and $z = ae^{3\pi i/4}$ lie inside C and each of order 2. Now we will find the residue at $z = ae^{\pi i/4} = \alpha$ (say). Now putting

$z = ae^{\pi i/4} + t = \alpha + t$, where t being very small, then we get

$$f(t+\alpha) = \frac{(\alpha+t)^6}{(a^4+(\alpha+t)^4)^2} = \frac{\alpha^6 + 6\alpha^5 t + \ldots\ldots}{(a^4+\alpha^4+4\alpha^3 t + 6\alpha^2t^2 + 4\alpha t^3 + t^4)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \ldots\ldots}{(a^4 - a^4 + 4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2}\left[\because \alpha^4 = (a\, e^{\pi i/4})^4 = a^4 e^{\pi i} = -a^4\right]$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \ldots\ldots}{(4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2} = \frac{\alpha^6 + 6\alpha^5 t + \ldots\ldots}{(4\alpha^3 t + 6\alpha^2 t^2 + \ldots)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \ldots\ldots}{16\alpha^6 t^2 \left(1 + \frac{6t}{4\alpha} + \ldots\right)^2} = \frac{\alpha^6 + 6\alpha^5 t + \ldots}{16\alpha^6 t^2} \times \left(1 + \frac{6t}{4\alpha} + \ldots\ldots\right)^{-2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \ldots\ldots}{16\alpha^6 t^2} \times \left(1 - 2.\frac{6t}{4\alpha} + \ldots\ldots\right)$$

Now the residue at $z = \alpha$ is the coefficient of

$$\frac{1}{t} \text{ in } f(t+\alpha) = \frac{6\alpha^5}{16\alpha^6} - \frac{12\alpha^6}{16.4\,\alpha^7} = \frac{3}{8\alpha} - \frac{3}{16\alpha}$$

$$= \frac{6-3}{16\alpha} = \frac{3}{16\alpha} = \frac{3}{16a}e^{-\pi i/4}.$$

Similarly, the residue at $z = \beta = ae^{3\pi i/4}$

$$= \frac{3}{16\beta} = \frac{3}{16a}e^{-3\pi i/4} = -\frac{3}{16a}e^{\pi i/4}.$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\left[\text{Res}(ae^{\pi i/4}) + \text{Res}(ae^{3\pi i/3})\right]$$

$$= 2\pi i\left[\frac{3}{16a}e^{-\pi i/4} - \frac{3}{16a}e^{\pi i/4}\right]$$

$$= -\frac{3\pi i}{8a}(e^{\pi i/4} - e^{-\pi i/4}) = \left(-\frac{3\pi i}{8a}\right)\left(2i\sin\frac{\pi}{4}\right)$$

$= \dfrac{3\pi}{4\sqrt{2}a}$ ... (1). Here $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty}\dfrac{z^7}{(a^4+z^4)^2}$

$= 0$, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2).

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{3\pi}{4\sqrt{2}\,a} \Rightarrow$$

$$2\int_0^{\infty} f(x)\,dx = \frac{3\pi}{4\sqrt{2}\,a} \Rightarrow \int_0^{\infty} f(x)\,dx = \frac{3\pi}{8\sqrt{2}\,a}$$

$\Rightarrow \displaystyle\int_0^{\infty}\frac{x^6}{(a^4+x^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16a}$ where $a > 0$ and the result is proved.

Example -292 : Show that $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+c^2)^2}$

$= \dfrac{(b+2c)\pi}{2bc^3(b+c)^2}$ where $b > 0$, $c > 0$.

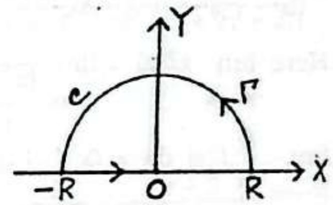
D. U. H. '80, '82, '88; R. U. H. '79; D. U. M. SC. P. '84.

Solution : Consider $\oint_C f(z)\,dz$

where $f(z) = \dfrac{1}{(z^2+b^2)(z^2+c^2)^2}$ and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ, of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $(z^2+b^2)(z^2+c^2)^2 = 0$

$\Rightarrow z = \pm bi$ are simple poles and $z = \pm ci$ are poles of order two. Here only $z = bi$ and $z = ci$ lie within C.

$$\text{Now Res }(bi) = \lim_{z\to bi}\left\{(z-bi)\frac{1}{(z^2+b^2)(z^2+c^2)^2}\right\}$$

$$= \lim_{z\to bi}\left\{(z-bi)\frac{1}{(z-bi)(z+bi)(z^2+c^2)^2}\right\} = \frac{-i}{2b(b^2-c^2)^2}$$

$$\text{Again Res }(ci) = \lim_{z\to ci}\frac{d}{dz}\left\{(z-ci)^2\frac{1}{(z^2+b^2)(z^2+c^2)^2}\right\}$$

$$= \lim_{z\to ci}\frac{d}{dz}\left\{(z-ci)^2\frac{1}{(z^2+b^2)(z-ci)^2(z+ci)^2}\right\}$$

$$= \lim_{z\to ci}\frac{d}{dz}\left\{\frac{1}{(z^2+b^2)(z+ci)^2}\right\}$$

$$= \lim_{z\to ci}\frac{-1}{(z^2+b^2)(z+ci)^2}\left[\frac{2z}{z^2+b^2}+\frac{2}{z+ci}\right]$$

$$= \frac{1}{4(b^2-c^2)c^2}\left[\frac{2ci}{b^2-c^2}-\frac{i}{c}\right] = \frac{i(3c^2-b^2)}{4c^3(b^2-c^2)^2}$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\,[\text{Res}\,(bi) + \text{Res}\,(ci)]$$

$$= 2\pi i\left[\frac{-i}{2b(b^2-c^2)^2}+\frac{i(3c^2-b^2)}{4c^3(b^2-c^2)^2}\right]$$

$$= \frac{\pi}{(b^2-c^2)^2}\,\frac{b^3-3bc^2+2c^3}{2bc^3}$$

$$= \frac{\pi}{(b^2-c^2)^2}\cdot\frac{(b+2c)(b-c)^2}{2bc^3} = \frac{(b+2c)\pi}{2bc^3(b+c)^2} \ \dots (1)$$

Here $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty}\frac{z}{(z^2+b^2)(z^2+c^2)^2}=0$, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{(b+3c)\pi}{2bc^3(b+c)^2}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dz}{(x^2+b^2)(x^2+c^2)^2} = \frac{(b+2c)\pi}{2bc^3(b+c)^2}.$$ Hence the result.

Example -293 : $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

R. U. M. SC. P. '84.

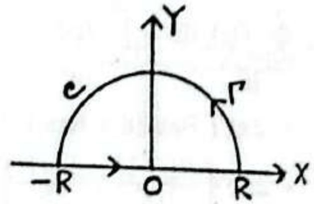
Solution : Consider $\oint_C f(z)\,dz$ where

$$f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$$

and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $(z^2+1)^2(z^2+2z+2) = 0 \Rightarrow z = \pm i$ of order 2 and $z = -1 \pm i$ of order 1. Of them only $z = i$ and $z = -1+i$ lie inside C. Now

$$\text{Res}\,(i) = \lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\,\frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to i}\frac{d}{dz}\,\frac{z^2}{(z+i)^2(z^2+2z+2)}$$

$$= \lim_{z\to i}\frac{z^2}{(z+i)^2(z^2+2z+2)}\left[\frac{2}{z}-\frac{2}{z+i}-\frac{2z+2}{z^2+2z+2}\right]$$

$$= \frac{i^2}{(2i)^2(i^2+2i+2)}\left[\frac{2}{i}-\frac{2}{2i}-\frac{2i+2}{i^2+2i+2}\right]$$

$$= \frac{1}{4(1+2i)}\left(\frac{1}{i}-\frac{2i+2}{1+2i}\right)$$

$$= \frac{1+2i-2i^2-2i}{4(1+2i)i(1+2i)} = \frac{3}{4i(1+4i+4i^2)} = \frac{3}{4i(-3+4i)}$$

$$= \frac{-3}{4(4+3i)} = \frac{9i-12}{100}$$

Again Res $(-1+i)$

$$= \lim_{z\to-1+i}\left\{(z+1-i)\,\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)}\right\}$$

$$= \frac{(-1+i)^2}{\{(-1+i)^2+1\}^2(2i)} = \frac{-2i}{(-2i+1)^2(2i)} = \frac{1}{3+4i} = \frac{3-4i}{25}.$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\,[\,\text{Res}\,(i) + \text{Res}\,(-1+i)]$$

$$= 2\pi i\left(\frac{9i-12}{100}+\frac{3-4i}{25}\right) = 2\pi i\left(\frac{-7i}{100}\right) = \frac{7\pi}{50}$$ ... (1). Here $\lim\limits_{z\to\infty} zf(z) = 0$, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{7\pi}{50} \Rightarrow$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2\,(x^2+2x+2)} = \frac{7\pi}{50}$$ and the result is proved.

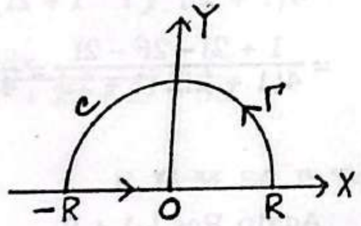
Example 294 : Show that $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx = \pi\log 2.$

C. U. '81; D. U. H. 87; D. U. M. SC. P. T. '89; C. U. H. '89; R. U. H. '82.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{\log(z+i)}{z^2+1}$ and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ, of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z)$ are given by $z^2 + 1 = 0 \Rightarrow z = \pm i$ which are simple poles and of which only the pole $z = i$ lies inside C. Now

$$\text{Res}\,(i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{\log(z+i)}{(z+i)(z-i)}.$$

$$= \lim_{z\to i}\frac{\log(z+i)}{z+i} = \frac{\log(2i)}{2i} = \frac{\log 2 + \log i}{2i} = \frac{\log 2 + \pi i/2}{2i}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\,.\,\text{Res}\,(i) = 2\pi i\left(\frac{\log 2 + \frac{\pi i}{2}}{2i}\right)$$

$$= \pi\left(\log 2 + \frac{\pi i}{2}\right)$$ ... (1). Now $\lim\limits_{z\to\infty} zf(z)$

$$= \lim_{z\to\infty}\frac{z\log(z+i)}{(z+i)(z-i)} = \lim_{z\to\infty}\frac{z}{z-i}\,.\,\lim_{z\to\infty}\frac{\log(z+i)}{z+i}$$

$= 1.\,0 = 0$, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2)

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi\left(\log 2 + \frac{\pi i}{2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\log(x+i)}{x^2+1}\,dx = \pi\left(\log 2 + \frac{\pi i}{2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\frac{1}{2}\log(x^2+1) + i\tan^{-1}1/x}{x^2+1}\,dx = \pi\left(\log 2 + \frac{\pi i}{2}\right)$$

Now equating real part, we have

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2 \Rightarrow \frac{2}{2}\int_{0}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx$$

$$= \pi \log 2 \Rightarrow \int_{0}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2.$$ Hence the result.

Example : Show that $\int_{0}^{\pi/2} \log \sin x\,dx = \int_{0}^{\pi/2} \log \cos x\,dx = -\frac{\pi}{2}\log 2.$

Solution : Using $\int_{0}^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2$, try yurself.

Example – 295 : Show that :

(i). $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$;

(ii). $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$;

(iii). $\int_{0}^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}$;

(iv). $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} = \frac{\pi}{3}$;

(v). $\int_{0}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{24}$;

(vi). $\int_{0}^{\infty} \frac{dx}{(a+bx^2)^n} = \frac{1.\,3.\,5\ldots(2n-3)\pi}{1.\,2.\,3.\ldots(2n-1)\,2^n\,b^{1/2}\,a^{(2n-1)/2}}$

(vii). $\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^{n+1}} = \frac{(2n)!\pi}{(n\,!)^2\,2^{2n+1}\,a^{2n+1}}$.

237. (Form 3). Integrals of the form : $\int_{-\infty}^{\infty} f(x)\cos mx\,dx$ **or** $\int_{-\infty}^{\infty} f(x)\sin mx\,dx$ **where** $f(x)$ **is a rational function of x :** Consider $\oint_C e^{imz} f(z)\,dz$, $m > 0$ where C is the contour consisting of :

(i). the x-axis from –R to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

Then let $R \to \infty$ and if $f(x)$ is an even function, this can be used to evalute the integral $\int_{-\infty}^{\infty} e^{imx} f(x)\,dx$, $m > 0$.

238. Jordan's inequality : The inequality $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ or $\frac{2\theta}{\pi} \le \sin\theta \le \theta$ is called the Jordan's inequality, where $0 \le \theta \le \frac{\pi}{2}$.

Jordan lemma (Theorem —221) : If $|f(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 0$ and M is a constant, then $\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\,dz = 0$, where Γ is the upper semi-circle of radius R and m is a positive constant.

Proof : If $z = Re^{i\theta}$, then $\int_{\Gamma} e^{imz} f(z)\,dz$

$$= \int_{0}^{\pi} e^{imRe^{i\theta}} f(Re^{i\theta})\,d(Re^{i\theta}) = i\int_{0}^{\pi} e^{imRe^{i\theta}} f(Re^{i\theta})\,Re^{i\theta}\,d\theta$$

$$\Rightarrow \left|\int_{\Gamma} e^{imz} f(z)\,dz\right| = \left|\int_{0}^{\pi} e^{imRe^{i\theta}} f(Re^{i\theta})\,Re^{i\theta}\,d\theta\right|$$

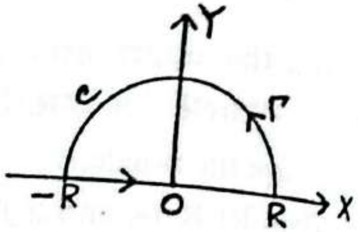
$$\le \int_0^{\pi} \left| e^{imRe^{i\theta}} f(Re^{i\theta})\, Re^{i\theta}\, d\theta \right| = \int_0^{\pi} e^{-mR\sin\theta} \left| f(Re^{i\theta}) \right| R\,d\theta$$

$$\le \frac{M}{R^{k-1}} \int_0^{\pi} e^{-mR\sin\theta}\, d\theta$$

$$= \frac{2M}{R^{k-1}} \int_0^{2\pi} e^{-mR\sin\theta}\, d\theta \ \ldots (1).$$

But we have $\sin\theta \ge 2\theta/\pi$. (2)

for $0 \le \theta \le \pi/2$. Using (2),



$$(1) \Rightarrow \left| \int_{\Gamma} e^{imz} f(z) dz \right|$$

$$\le \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta$$

$$= \frac{2M}{R^{k-1}} \left[\frac{\pi e^{-2mR\theta/\pi}}{-2MR} \right]_0^{\pi/2} = \frac{\pi M}{mR^k} (1 - e^{-mR}) \ \ldots (3)$$

Now taking $R \to \infty$ in both sides of (3) $\Rightarrow$

$$\lim_{R\to\infty} \left| \int_{\Gamma} e^{imz} f(z)\, dz \right| = 0 \Rightarrow \lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$$

and the theorem is proved.

N. B. If $m<0$, then also $\lim\limits_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$ where Γ is the lower semi-circle of radius R and it can be proved by changing $z = -w$ in the above theorem.

Theorem -222 : Let $f(z)$ be analytic in the upper half plane of the z-plane or Argand plane except at a finite number of poles in it and having no poles on the real axis. If $f(z) \to 0$ as $z \to \infty$, then

$$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx = 2\pi i \text{ [sum of the residues at the poles in the upper half plane] if } m>0.$$

Proof : Try yourself.

239. Other forms of Jordan lemma :

(i). If $f(z) = \dfrac{P(z)}{Q(z)}$ is the quotient of two polynomials $P(z)$ and $Q(z)$ such that degree $Q(z) \ge 1 +$ degree $P(z)$, then

$\lim\limits_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$, $m > 0$ where Γ is the upper half-circle of radius R.

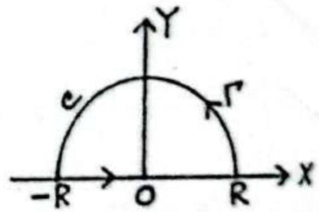
(ii). If $f(z)$ is analytic at finite number of singularities and if $\lim\limits_{z\to\infty} f(z) = 0$ ... (1) uniformly, then $\lim\limits_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$, $m > 0$ ... (2). where Γ denotes the upper half-circle of radius R. In the following some examples we will use condition (2) if it satisfies the condition (1).

Example -296 : Show that $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{2a} e^{-ma}$.

$m \ge 0$ and $a > 0$. **D. U. H. '78, '88; R. U. H. '76.**

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$ and C is the contour consisting of :

(i). the x-axis from −R to R, where R is large.



(ii). the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.

The poles of $f(z) = \dfrac{e^{imz}}{z^2+a^2}$ are obtained by solving $z^2 + a^2 = 0 \Rightarrow z = \pm ai$ and they are simple poles. Of which only $z = ai$ lies inside C.

$$\text{Res } [f(z), ai] = \lim_{z \to ai} \left\{ (z - ai) \frac{e^{imz}}{(z - ai)(z + ai)} \right\} = \frac{e^{-ma}}{2ai}.$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dz + \int_\Gamma f(z)\, dz$$

$$= 2\pi i \text{ Res } [f(z), ai] = 2\pi i \left(\frac{e^{-ma}}{2ai} \right) = \frac{\pi e^{-ma}}{a} \ldots (1).$$

Here $\lim\limits_{z \to \infty} \dfrac{1}{z^2+a^2} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where $F(z) = \dfrac{1}{z^2+a^2}$ and then by the Jordan's lemma, $\lim\limits_{R \to \infty} \int_\Gamma f(z)\, dz = 0 \ldots (2)$. Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \dfrac{\pi e^{-ma}}{a} \Rightarrow 2\int_0^\infty f(x)\, dx = \dfrac{\pi e^{-ma}}{a}$.

$\Rightarrow \displaystyle\int_0^\infty \frac{\cos m x}{x^2+a^2} = \frac{\pi e^{-ma}}{2a}$ and the required result is obtained.

Example – 297 : Show that (a) $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{2e}$;

(b) $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+1} = \frac{\pi}{2}$; **R. U. M. SC. P. '84.**

(c) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+a^2} = \frac{\pi e^{-a}}{2a}$, $a > 0$.

Solution : Use the above example.

Example – 298 : Using the above example, show that $\displaystyle\int_0^\infty \frac{x \sin mx}{x^2+a^2}\, dz = \frac{\pi}{2} e^{-ma}$, where $m \geq 0$ and $a > 0$.

Solution : By the above example, we have $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+a^2}\, dx = \frac{\pi}{2a} e^{-ma} \ldots (1)$ where $m \geq 0$ and $a > 0$.

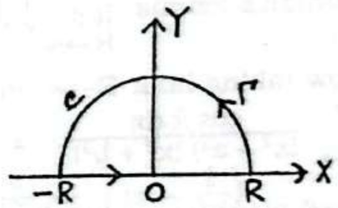
Now differentiating both sides of (1), with respect to m, we have $\displaystyle\int_0^\infty \frac{-x \sin mx}{x^2+a^2}\, dx = -\frac{a\pi}{2a} e^{-ma}$

$\Rightarrow \displaystyle\int_0^\infty \frac{x \sin mx}{x^2+a^2}\, dx = \frac{\pi}{2} e^{-ma}$ and the required result is obtained.

Example 299 : Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$ where $a > b > 0$.

D. U. H. '89, '90; D. U. H. T. '77, 82, 87.

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$ and C is the contour consisting of : (i) the x-axis from –R to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$ are obtained by solving $(z^2 + a^2)(z^2 + b^2) = 0 \Rightarrow z = \pm ai, \pm bi$ which are simple poles. Only the poles $z = ai$ and $z = bi$ lie inside C.

Now $\text{Res}\,(ai) = \lim\limits_{z \to ai} \left\{ (z - ai) \dfrac{e^{iz}}{(z - ai)(z + ai)(z^2 + b^2)} \right\}$

$= \lim\limits_{z \to ai} \dfrac{e^{iz}}{(z + ai)(z^2 + b^2)} = \dfrac{e^{-a}}{2ai(b^2 - a^2)} = - \dfrac{e^{-a}}{2ai(a^2 - b^2)}$.

Again $\text{Res}\,(bi) = \dfrac{e^{-b}}{2bi\,(a^2 - b^2)}$.

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

$= 2\pi i\,[\text{Res}\,(ai) + \text{Res}\,(bi)\,]$

$= 2\pi i \left[-\dfrac{e^{-a}}{2ai(a^2 - b^2)} + \dfrac{e^{-b}}{2bi(a^2 - b^2)} \right] = \dfrac{\pi}{a^2 - b^2} \left(\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a} \right)$

... (1)

Here $\lim\limits_{z \to \infty} \dfrac{1}{(z^2 + a^2)(z^2 + b^2)} = 0$ and $f(z)$ is a function of the form $e^{imz}\,F(z)$, where $F(z) = \dfrac{1}{(z^2 + a^2)(z^2 + b^2)}$ and then by the Jordan's lemma $\lim\limits_{R \to \infty} \int_{\Gamma} f(z)\,dz = 0$ ... (2).

Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow$

$\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$ and the required result is obtained.

Example – 300 : Show that (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 8)(x^2 + 12)}\,dx = \frac{\pi}{8}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}} \right)$; **D. U. H. '77.**

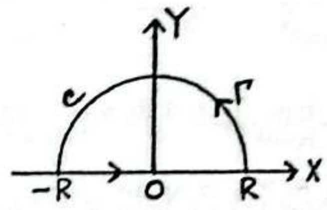
(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)(x^2 + 9)}\,dx = \frac{\pi}{8}\left(\frac{e^{-1}}{1} - \frac{e^{-3}}{3} \right)$. **R. U. H. '77.**

Solution : Try yourself.

Example –301 : Show that $\displaystyle\int_0^{\infty} \frac{\cos mx}{(x^2 + a^2)^2}\,dx = \frac{\pi(1 + ma)e^{-ma}}{4a^3}$.

where $a > 0$ and $m \geq 0$. **D. U. H. T. '86; C. U. '81.**

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{imz}}{(z^2 + a^2)^2}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \dfrac{e^{imz}}{(z^2 + a^2)^2}$ are obtained by solving $(z^2 + a^2)^2 = 0 \Rightarrow z = \pm\, ai$ which are of order 2. Only the pole $z = ai$ lies inside C. Now $\text{Res}\,(ai) =$

$\lim\limits_{z \to ai} \dfrac{d}{dz} \left\{ (z - ai)^2 \dfrac{e^{imz}}{(z - ai)^2 (z + ai)^2} \right\}$

$= \lim\limits_{z \to ai} \dfrac{d}{dz} \left\{ \dfrac{e^{imz}}{(z + ai)^2} \right\}$

$= \lim\limits_{z \to ai} \left[\dfrac{e^{imz}}{(z + ai)^2} \left\{ im - \dfrac{2}{z + ai} \right\} \right] = \dfrac{e^{-ma}}{(2ai)^2} \left\{ im - \dfrac{2}{2ai} \right\}$

$= - \dfrac{e^{-ma}}{4a^2} \left(\dfrac{-am - 1}{ai} \right) = \dfrac{e^{-ma}\,(ma + 1)}{4a^3 i}$. Then by the Cauchy's residue theorem, we have $\oint_C f(z)\,dz$

$= \displaystyle\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i\,\text{Res}\,(ai)$

$= 2\pi i \left\{ \dfrac{e^{-ma}(ma+1)}{4a^3 i} \right\} = \dfrac{\pi(1+ma)e^{-ma}}{2a^3}$ ... (1). Here

$\lim\limits_{z \to \infty} \dfrac{1}{(z^2+a^2)^2} = 0$ and $f(z)$ is a function of the form

$e^{imz} F(z)$, where $F(z) = \dfrac{1}{(z^2+a^2)^2}$ and then by the Jordan's

lemma

$\lim\limits_{R \to \infty} \int_\Gamma f(z)\, dz = 0$ ... (2).

Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx$

$= \dfrac{\pi(1+ma)e^{-ma}}{2a^3} \Rightarrow \int_{-\infty}^{\infty} \dfrac{\cos mx}{(x^2+a^2)^2}\, dx = \dfrac{\pi(1+ma)e^{-ma}}{2a^3}$

$\Rightarrow 2\int_0^{\infty} \dfrac{\cos mx}{(x^2+a^2)^2}\, dx = \dfrac{\pi(1+ma)e^{-ma}}{2a^3} \Rightarrow \int_0^{\infty} \dfrac{\cos mx}{(x^2+a^2)^2}\, dx$

$= \dfrac{\pi(1+ma)e^{-ma}}{4a^3}$ and the required result is proved.

Example –302 : Show that $\int_0^{\infty} \dfrac{\cos x}{(1+x^2)^2}\, dx = \dfrac{\pi}{2e}$.

R. U. '84; R. U. H. '72.

Solution : Try yourself.

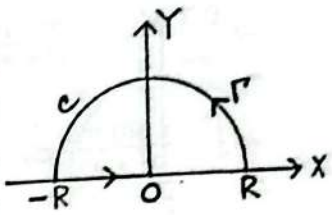
Example 303 : Show that $\int_0^{\infty} \dfrac{x \sin mx}{x^2+a^2}\, dx = \dfrac{\pi}{2} e^{-ma}$,

where $m \geq 0$ and $a > 0$.

D. U. H. T. '76, '78; R. U. '83; R. U. H. '73, '82; D. U. H. '87; D. U. M. SC. P. '78, '88.

Solution : Consider $\oint_C f(z)\, dz$

where $f(z) = \dfrac{z e^{imz}}{z^2+a^2}$ and C is the contour consisting of : (i) the x-axis from –R to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \dfrac{z e^{imz}}{z^2+a^2}$ are obtained by solving $z^2 + a^2 = 0 \Rightarrow z = \pm ai$, which are simple poles but only $z = ai$ lies inside C. Now

$\text{Res}(ai) = \lim\limits_{z \to ai} \left\{ (z-ai) \dfrac{z e^{imz}}{(z-ai)(z+ai)} \right\}$

$= \lim\limits_{z \to ai} \dfrac{z e^{imz}}{z+ai} = \dfrac{ai\, e^{-ma}}{2ai} = \dfrac{e^{-ma}}{2}$.

Then by the Cauchy's residue theorem, we have

$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz$

$= 2\pi i\, \text{Res}(ai) = 2\pi i \cdot \left(\dfrac{e^{-ma}}{2} \right) = \pi i e^{-ma}$ ... (1).

Here $\lim\limits_{z \to \infty} \dfrac{1}{z^2+a^2} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where $F(z) = \dfrac{1}{z^2+a^2}$ and then by the Jordan's lemma $\lim\limits_{R \to \infty} \int_\Gamma f(z)\, dz = 0$ ... (2). Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \pi i e^{-ma} \Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i e^{-ma} \Rightarrow \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2+a^2}\,dx = \pi i e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos mx + i\sin mx)}{x^2+a^2}\,dx = \pi i e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \pi e^{-ma} \qquad [\text{ equating imaginary parts}]$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \pi e^{-ma}$$

$\Rightarrow \int_{0}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}$ and the required result is obtained.

Example -304 : Show that (a) $\int_{0}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2}$;

C. U. H. '89.

(b) $\int_{0}^{\infty} \frac{x\sin x}{x^2+4}\,dx = \frac{\pi}{2} e^{-2}$. **R. U. '82; R. U. H. '75.**

Solution : Try yourself.

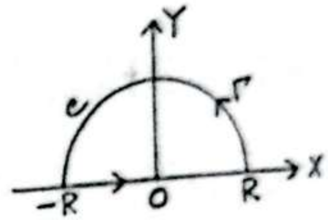
Example -305 : Show that (a) $\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+2x+5}\,dx = \frac{\pi}{2} e^{-2\pi}$;

(b) $\int_{-\infty}^{\infty} \frac{x\sin \pi x}{x^2+2x+5}\,dx = -\pi e^{-2\pi}$. **R. U. M. SC. P. '84.**

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{ze^{i\pi z}}{z^2+2z+5}$ and C is the contour consisting of :

(i) the x-axis from $-R$ to R, where R is large;

(ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \frac{ze^{i\pi z}}{z^2+2z+5}$ are obtained by solving

$z^2+2z+5 = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ and they are simple poles. But only $z = -1+2i$ lies inside C.

Now Res $(-1+2i)$

$$= \lim_{z\to -1+2i} \left\{(z+1-2i)\cdot \frac{z\,e^{i\pi z}}{(z+1-2i)(z+1+2i)}\right\}$$

$= \frac{(-1+2i)e^{-\pi i - 2\pi}}{4i}$. Then by the Caucly's residue theorem. we have $\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$

$$= 2\pi i \text{ Res }(-1+2i) = 2\pi i.\left\{\frac{(-1+2i)\,e^{-\pi i-2\pi}}{4i}\right\}$$

$= \frac{\pi}{2}(1-2i)e^{-2\pi}$ ... (1). Here $\lim_{z\to\infty} \frac{z}{z^2+2z+5} = 0$ and $f(z)$ is a function of the form $e^{imz}F(z)$, where $F(z) = \frac{z}{z^2+2z+5}$ and then by the Jordan's lemma. $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2).

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx =$

$\frac{\pi}{2}(1-2i)e^{-2\pi}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\, e^{i\pi x}}{x^2+2x+5}\, dx = \frac{\pi}{2}(1-2i)\, e^{-2\pi} \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\cos\pi x + ix\sin\pi x}{x^2+2x+5}\, dx = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+2x+5}\, dx = \frac{\pi}{2}e^{-2\pi} \text{ and}$$

$\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}\, dx = -\pi e^{-2\pi}$ and the required result of (a) and (b) are obtained.

Example -306 : Show that $\int_0^{\infty} \frac{x^3\sin x}{(x^2+a^2)(x^2+b^2)}\, dx$

$= \frac{\pi}{2(a^2-b^2)}(e^{-a}a^2 - e^{-b}b^2)$ where a, b>0.

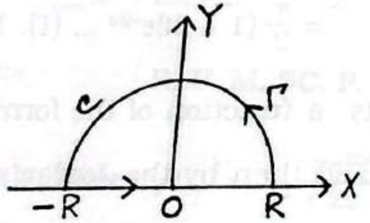
Solution : Try yourself.

Example -307 : Show that $\int_0^{\infty} \frac{x^3\sin x}{(x^2+a^2)^2}\, dx = -\frac{\pi}{4}(a-2)e^{-a}$

where $a > 2$.

Solution : Consider $\oint_C f(z)\, dz$

where $f(z) = \frac{z^3 e^{iz}}{(z^2+a^2)^2}$ and C is the contour consisting of : **(i)** the x-axis from $-R$ to R, where R is large; **(ii)** the upper semi-circle Γ of the circle $|z| = R$



which lies above the x-axis. The poles of $f(z)$ inside C is $z = ai$, which is of order two.

Now Res (ai) $= \lim_{z\to ai} \frac{d}{dz}\left\{(z-ai)^2 \frac{z^3e^{iz}}{(z-ai)^2(z+ai)^2}\right\}$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{\frac{z^3e^{iz}}{(z+ai)^2}\right\} = \lim_{z\to ai}\left[\frac{z^3e^{iz}}{(z+ai)^2}\left\{\frac{3}{z}+i-\frac{2}{z+ai}\right\}\right]$$

$= \frac{a^3e^{-a}(2-a)}{4a^3 i} = \frac{e^{-a}(2-a)}{4i}$. Then by the Cauchy's residue theorem, we have $\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz$

$= 2\pi i$ Res (ai)

$= 2\pi i\left\{\frac{e^{-a}(2-a)}{4i}\right\} = \frac{\pi e^{-a}(2-a)}{2}$ ... (1). Here

$\lim_{z\to\infty} \frac{z^3}{(z^2+a^2)^2} = 0$ and $f(z)$ is a function of the form $e^{imz}F(z)$, where $F(z) = \frac{z^3}{(z^2+a^2)^2}$ and then by the Jordan's lemma $\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0$ ... (2)

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi e^{-a}(2-a)}{2} \Rightarrow \int_{-\infty}^{\infty} \frac{x^3\sin x}{(x^2+a^2)^2}\, dx = \frac{\pi e^{-a}(2-a)}{2}$$

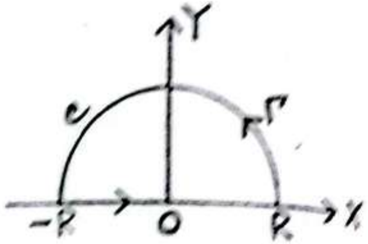
$$\Rightarrow 2\int_0^{\infty} \frac{x^3\sin x}{(x^2+a^2)^2}\, dx = \frac{\pi e^{-a}(2-a)}{2}$$

$\Rightarrow \int_0^{\infty} \frac{x^3\sin x}{(x^2+a^2)^2}\, dx = -\frac{\pi e^{-a}(a-2)}{4}$ where $a > 2$ and the required result is obtained.

Example -308 : Show that $\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\, dx = 2\pi e^{-a}$

where $a > 0$.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{iz}}{z-ai}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The pole of $f(z) = \dfrac{e^{iz}}{z-ai}$ is ai which is simple. Now

$$\text{Res}\,(ai) = \lim_{z\to ai}\left\{(z-ai)\frac{e^{iz}}{z-ai}\right\}$$

$= \lim\limits_{z\to ai} e^{iz} = e^{-a}$. Then by the residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\ \text{Res}\,(ai) = 2\pi i e^{-a}$$

... (1). Here $\lim\limits_{z\to\infty}\dfrac{1}{z-ai} = 0$ and $f(z)$ is a function of the form $e^{imz}F(z)$ where $F(z) = \dfrac{1}{z-ai}$ and then by the Jordan's lemma $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty}\dfrac{e^{ix}}{x-ai}dx = 2\pi i\,e^{-a} \Rightarrow \int_{-\infty}^{\infty}\dfrac{(x+ai)e^{ix}}{x^2+a^2}dx$

$$= 2\pi i e^{-a} \Rightarrow \int_{-\infty}^{\infty}\frac{(x+ai)\,(\cos x + i\sin x)}{x^2+a^2}\,dx = 2\pi i\,e^{-a} \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{(x\cos x - a\sin x) + i\,(a\cos x + x\sin x)}{x^2+a^2}\,dx = 2\pi i\,e^{-a}.$$

Equating imaginary parts, we have

$\int_{-\infty}^{\infty}\dfrac{a\cos x + x\sin x}{x^2+a^2}dx = 2\pi e^{-a}$ and the required result is obtained.

Example - 309 : Show that $\int_{-\infty}^{\infty}\dfrac{x\cos x - a\sin x}{x^2+a^2}dx = 0$

Solution : Try yourself.

Example-310 : Show that $\int_{-\infty}^{\infty}\dfrac{-a\cos x + x\sin x}{x^2+a^2}dx = 0$ where $a>0$.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{iz}}{z+ai}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis. Then : try yourself or use the above example.

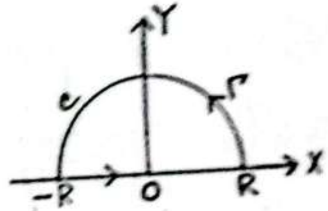
Example -311 : Show that $\int_0^{\infty}\dfrac{\cos mx}{x^4+a^4}dx$

$= \dfrac{\pi}{2a^3}e^{-ma/\sqrt{2}}\sin\left(\dfrac{ma}{\sqrt{2}} + \dfrac{\pi}{4}\right)$ where $m, a > 0$.

D. U. H. T. '75; D. U. H. '86.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{imz}}{z^4+a^4}$ and C is the contour consisting of :

(i) the x-axis from $-R$ to R; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z)$ are obtained by solving $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4$

$= e^{(2n+1)\pi i} a^4 \Rightarrow z = a\, e^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3 \Rightarrow$

$z = ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$. Only $z = \alpha = ae^{\pi i/4}$

$= a(1+i)/\sqrt{2}$ and $z = \alpha = ae^{3\pi i/4} = a(-1+i)/\sqrt{2}$ lie inside C

Now the residue at $z = \alpha = \lim_{z\to\alpha} \left\{(z-\alpha)\dfrac{e^{imz}}{z^4+a^4}\right\}$

$= \lim_{z\to\alpha} \dfrac{e^{imz}}{4z^3} = \dfrac{e^{im\alpha}}{4\alpha^3}$ [by L' Hospital's rule]

$$= \begin{cases} \dfrac{-1}{4a^3} e^{ima(1+i)/\sqrt{2}+\pi i/4} \text{ at } z = a(1+i)/\sqrt{2}\ ; [\alpha^3 = -a^3e^{-\pi i/4}] \\ \dfrac{1}{4a^3} e^{ima(-1+i)/\sqrt{2}-\pi i/4} \text{ at } z = a(-1+i)/\sqrt{2}\ ; [\alpha^3 = a^3 e^{\pi i/4}] \end{cases}$$

$$= \begin{cases} -\dfrac{1}{4a^3} e^{-ma/\sqrt{2}}\, e^{i\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)} & \text{at } z = a(1+i)/\sqrt{2} \\ \dfrac{1}{4a^3} e^{-ma/\sqrt{2}}\, e^{-i\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)} & \text{at } z = a(-1+i)/\sqrt{2} \end{cases}$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$= 2\pi i$ (sum of the residues inside C)

$$= 2\pi i \left[-\frac{1}{4a^3} e^{-ma/\sqrt{2}}\left\{e^{i\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)}\right\}\right]$$

$$= \frac{-\pi i}{2a^3} e^{-ma/\sqrt{2}} \cdot 2i \sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)$$

$$= \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right) \quad \ldots (1)\text{ . Here}$$

$\lim_{z\to\infty} \dfrac{1}{z^4+a^4} = 0$ and $f(z)$ is a function of the form

$e^{imz}\ F(z)$ where $F(z) = \dfrac{1}{z^4+a^4}$ and then by the Jordan's

lemma $\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in

(1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \dfrac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\dfrac{ma}{\sqrt{2}}+\dfrac{\pi}{4}\right)$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+a^4}\,dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x^4+a^4}\,dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)$$

Now equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4+a^4}\,dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{x^4+a^4}\,dx = \frac{\pi}{2a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{4}\right)$$

and the required result is obtained.

Example -312 : Show that $\displaystyle\int_0^{\infty} \frac{x\sin mx}{x^4+a^4}\,dx$

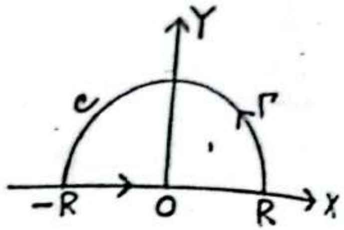
$= \dfrac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin\dfrac{ma}{\sqrt{2}}$ where $m, a > 0$. D. U. H. '86.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{ze^{imz}}{z^4+a^4}$ and C

is the contour consisting of :

(i) the x-axis from $-R$ to R.

(ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis. The poles of $f(z)$ are obtained by solving $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = e^{(2n+1)\pi i} a^4$



$\Rightarrow z = ae^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3 \Rightarrow z = ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$. Only $z = \alpha = a\, e^{\pi i/4} = a(\cos \pi/4 + i \sin \pi/4)$

$= \frac{a}{\sqrt{2}}(1 + i)$ and $z = \alpha = ae^{3\pi i/4} = a\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) = \frac{a}{\sqrt{2}}(-1 + i)$ lie inside C. Now the residue at $z = \alpha =$

$$\lim_{z\to\alpha} \left\{(z - \alpha)\frac{ze^{imz}}{z^4 + a^4}\right\} = \lim_{z\to\alpha} \frac{e^{im\alpha}}{4z^3} = \lim_{z\to\alpha} \frac{e^{im\alpha}}{4z^2}$$

[by L' Hospital's rule]

$$= \frac{e^{im\alpha}}{4\alpha^2} = \begin{cases} \dfrac{e^{ima(1+i)/\sqrt{2}}}{2a^2(1+i)^2} & \text{at } \alpha = a(1+i)/\sqrt{2} \\ \dfrac{e^{ima(-1+i)/\sqrt{2}}}{2a^2(-1+i)^2} & \text{at } a = a(-1+i)/\sqrt{2} \end{cases}$$

$$= \begin{cases} \dfrac{e^{-ma/\sqrt{2}} \cdot e^{ima/\sqrt{2}}}{4a^2 i} & \text{at } \alpha = a(1+i)/\sqrt{2} \\ \dfrac{e^{-ma/\sqrt{2}} \cdot e^{-im\,a/\sqrt{2}}}{-4a^2 i} & \text{at } \alpha = a(-1+i)/\sqrt{2} \end{cases}$$

Now by the Cauchy's residue theorem, we have

$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i$ (sum of the residues within C) $= 2\pi i \cdot \frac{e^{-ma/\sqrt{2}}}{4a^2 i}\left[e^{ima/\sqrt{2}} - e^{-ima/\sqrt{2}}\right]$

$$= \frac{\pi e^{-ma/\sqrt{2}}}{2a^2} \cdot 2i \sin \frac{ma}{\sqrt{2}} = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}} \quad \ldots (1)$$

Here $\lim_{z\to\infty} \frac{z}{z^4 + a^4} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$ where $F(z) = \frac{z}{z^4 + a^4}$ and then by the Jordan's lemma, $\lim_{R\to\infty} \oint_\Gamma f(z)\, dz = 0 \quad \ldots (2)$. Now taking limit $R\to\infty$ in (1) and using (2)

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\, e^{imx}}{x^4 + a^4}\, dx = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos mx + i \sin mx)}{x^4 + a^4}\, dx = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

Now equating imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^4 + a^4}\, dx = \frac{\pi}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

$\Rightarrow \int_0^{\infty} \frac{x \sin mx}{x^4 + a^4}\, dx = \frac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$ and the required result is obtained.

Example – 313 : Show that $\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\, dx$

$= \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}$ where $m, a > 0$.

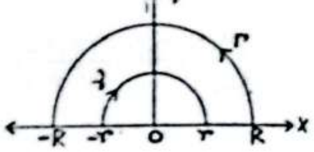
Solution : Try yourself.

239. (Form 4). Case of poles on the real axis : If $|f(x)| \to \infty$ as x tends to certain finite points on the x– axis, then we will use the **indented contours.**

Definition : A contour is said to be **indented** at the singularity when a part of a small semi—circle is described to avoid the singularity of the integrand keeping the singular point as centre. In the following examples we will use some **indented contour.**

Example – 314 : Show that $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2.$

D. U. H. S. 85; D. U. H. T. 85, 88; D. U. M. Sc. P. 80, 88; C. U. H. 90.



Solution : Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{iz}}{z}$.

The function $f(z)$ has a singularity at the point $z = 0$ on the real axis where integration is not possible and also there is no any other singularity in the upper half plane. Now we consider the contour C consisting of a large semicircle Γ of radius R in the upper half plane and the real axis indented by a small semicircle γ of radius r at $z = 0$. Thus C have four parts : **(i)** x – axis from—r to —R; **(ii)** the small semi circle γ of radius r where $r \to 0$; **(iii)** x — axis from r to R and **(iv)** the great semi circle Γ of radius R where $R \to \infty$. Now by the Cauchy's integral theorem,

$$\oint_C f(z)\, dz = \int_{-R}^{-r} f(x)\, dx + \int_\gamma f(z)\, dz + \int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz = 0 \quad \ldots\ldots (1)$$

If $r \to 0$ and $R \to \infty$, then $\int_{-R}^{-r} f(x)\, dx + \int_r^R f(x)\, dx$

$$\Rightarrow \int_{-\infty}^0 f(x)\, dx + \int_0^\infty f(x)\, dx = \int_{-\infty}^\infty f(x)\, dx$$

$= \int_{-\infty}^\infty \frac{e^{ix}}{x}\, dx$ .... (2). Here $\lim_{z \to \infty} \frac{1}{z} = 0$ and $f(z)$ is a function of the form $e^{iz} F(z)$, where $F(z) = \frac{1}{z}$. Hence by the Jordan's lemma, $\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0$ ..... (3)

Again we have $\lim_{z \to 0} z\, f(z) = \lim_{z \to 0} z \frac{e^{iz}}{z} = e^0 = 1.$

Then $\lim_{r \to 0} \int_\gamma f(z)\, dz = -\, i.\, 1.\, (\pi - 0) = -\pi i$ ..... (4),

where the negative sign being taken as the contour γ is in the clock wise direction. Now taking $r \to 0$ ansd $R \to \infty$ and using (2), (3) and (4), the equation (1)

$$\Rightarrow \int_{-\infty}^\infty \frac{e^{ix}}{x} dx - \pi i = 0$$

$$\Rightarrow \int_{-\infty}^\infty \frac{\cos x + i \sin x}{x} dx = \pi i \Rightarrow \int_{-\infty}^\infty \frac{\cos x}{x}\, dx = 0 \text{ and}$$

$\int_{-\infty}^\infty \frac{\sin x}{x} dx = \pi$. But $\int_{-\infty}^\infty \frac{\sin x}{x} dx = \pi \Rightarrow 2\int_0^\infty \frac{\sin x}{x} dx = \pi$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Example 315 : show that (i) $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi$;

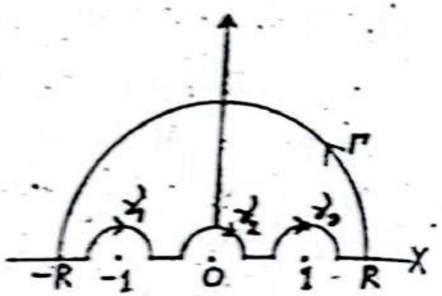
(ii) $\int_{-\infty}^{\infty} \cos x\,dx = 0$; (iii) $\int_{-\infty}^{\infty} \frac{\sin mx}{x}\,dx = \pi/2$.

Solution : Try yourself.

Example 316 : Show that. $\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi$.

R. U. H. 73, 74, 81.

Solution : Consider the integral $\oint_C f(z)\,dz$ where $f(z) = \frac{e^{\pi z i}}{z(1-z^2)}$. **The function** $f(z)$ **have** the singularities at the points $z = -1, 0, 1$ on the x-axis where integrations are not possible and also there are no other singularities in the upper half plane. Now we consider the contour C consisting of a large semicircle Γ of radius R in the upper half plane and the real axis indented at $z = -1, 0, 1$ by small semicircles γ_1, γ_2 and γ_3 of radius r_1, r_2 and r_3 respectively. Here C have eight parts : on the x— axis from $-R$ to $-1-r_1$, $-1+r_1$ to $-r_2$, r_2 to $1-r_3$ and $1+r_3$ to R; the small semicircles γ_1, γ_2 and γ_3 of radi r_1, r_2 and r_3 respectively where $r_1, r_2, r_3 \to 0$ and the large semi circle Γ of radius R where $R \to \infty$. Since $f(z)$ is analytic within and on C, then by the Cauchy's integral theorem, we have



$$\oint_C f(z)\,dz = \int_{-R}^{-1-r_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-1+r_1}^{-r_2} f(x)\,dx$$

$$+ \int_{\gamma_2} f(z)\,dz + \int_{r_2}^{1-r_3} f(x)\,dx + \int_{\gamma_3} f(z)\,dz + \int_{1+r_3}^{R} f(x)\,dx +$$

$\int_{\Gamma} f(z)\,dz = 0$ ..... (1). If $r_1, r_2, r_3 \to 0$ and $R \to \infty$, then (1) $\Rightarrow$

$$\int_{-\infty}^{-1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-1}^{0} f(x)\,dx + \int_{\gamma_2} f(z)\,dz +$$

$$\int_{0}^{1} f(x)\,dx + \int_{\gamma_3} f(z)\,dz + \int_{1}^{\infty} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 0 \text{ .... (2)}$$

Now we have : $\lim_{z\to -1} (z+1) f(z) = -\frac{1}{2} e^{-\pi i} = \frac{1}{2}$.

$\lim_{z\to 0} z f(z) = 1$ and $\lim_{z\to 1} (z-1) f(z) = \frac{-1}{2} e^{\pi i} = \frac{1}{2}$.

Using these again we have $\lim_{r_1\to 0} \int_{\gamma_1} f(z)\,dz = \frac{1}{2}(0-\pi) = -\frac{i\pi}{2}$. $\lim_{r_2\to 0} \int_{\gamma_2} f(z)\,dz = i(0-\pi) = -\pi i$ and $\lim_{r_3\to 0} \int_{\gamma_3} f(z)\,dz = \frac{i}{2}(0-\pi) = -\frac{\pi i}{2}$ where γ_1, γ_2 and γ_3 are described in the clockwise direction. Here $\lim_{z\to\infty} \frac{1}{z(1-z^2)} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where $F(z) = \frac{1}{z(1-z^2)}$. Hence by the Jordan's lemma, $\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$.

Then (2) $\Rightarrow \int_{-\infty}^{-1} f(x)\,dx - \frac{\pi i}{2} + \int_{-1}^{0} f(x)\,dx - \pi i$

$+ \int_{0}^{1} f(x)\,dx - \frac{\pi i}{2} + \int_{1}^{\infty} f(x)\,dx + 0 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$

$$\Rightarrow 2\int_0^{\infty} \frac{e^{\pi x i}}{x(1-x^2)}\,dx = 2\pi i$$

$$\Rightarrow \int_0^{\infty} \frac{\cos \pi x + i \sin \pi x}{x(1-x^2)}\,dx = \pi i \Rightarrow \int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi \text{ and}$$

the required result is proved.

Example –317 : Show that :

(i) $\int_0^{\infty} \frac{\cos \pi x}{x(1-x^2)}\,dx = 0;$

(ii) $\int_{0}^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}\,dx = \pi(b-a),\ a \geq b \geq 0\ ;$

(iii) $\int_0^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi/2\ ;$

(iv) $\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{2a^4} - \frac{\pi}{4a^3}\,e^{-ma}\left(m + \frac{2}{a}\right)$

$m > 0,\ a > 0;$

(vi) $\int_0^{\infty} \frac{\sin^2 mx}{x^2(x^2+a^2)} = \frac{\pi}{4a^3}(e^{-2ma} - 1 + 2ma),\ m > 0,\ a > 0;$

(vii) $\int_0^{\infty} \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi}{2a}\sin a,\ a > 0;$

(ix) $\int_0^{\infty} \frac{x^4}{x^6 - 1}\,dx = \frac{\pi}{6}\sqrt{3}\ ;$

(x) $\int_0^{\infty} \frac{x - \sin x}{x^3(a^2+x^2)}\,dx = \frac{\pi}{2a^4}[a^2/2 - a + 1 - e^{-a}],\ a > 0\ ;$

(xi) $\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8};$

(xii) $\int_0^{\infty} \frac{\log x}{1+x^2}\,dx = 0\ ;$ **D. U. H. T. 75.**

(xiii) $\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{4};$ **R. U. H. 80; D. U. H. T. 78.**

(xiv) $\int_0^{\infty} \frac{x^a}{1+x^2}\,dx = \frac{\pi}{2}\sec\frac{\pi a}{2},\ -1 < a < 1\ ;$

(xv) $\int_0^{\infty} \frac{\log(1+x^2)}{x^{1+a}}\,dx = 0,\ 0 < a < 1\ ;$

(xvi) $\int_0^{\infty} \frac{x^{a-1}}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec}\ \pi a/2\ ,\ 0 < a < 2\ ;$

(xvii) $\int_0^{\infty} \frac{x^a}{x^2-x+1}\,dx = \frac{2\pi}{\sqrt{3}}\sin\left(\frac{2\pi a}{3}\right)\operatorname{cosec} a\pi, -1 < a < 1;$

(xviii) $\int_0^{\infty} \frac{x^a}{(1+x^2)^2}\,dx = \frac{\pi(1-a)}{4}\sec\frac{\pi a}{2},\ -1 < a < 3;$

(xix) $\int_0^{\infty} \frac{x^{a-1}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}\operatorname{cosec} \pi a \cos\frac{2\pi a + \pi}{6},\ 0 < a < 2;$ **D. U. H. T. '82**

(xxx) $\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi},\ 0 < < a < 1\ ;$

R. U. M. Sc. P. 86 ; D. U. H. T. 86, 89; R. U. H. 76, 88;

D. U. M. Sc. P. 78.

(xxxi) $\int_0^{\infty} \frac{x^{a-1}}{1-x}\,dx = \pi \cot a\pi,\ 0 < a < 1\ .$ **R. U. M. Sc. P. 84.**

Solution : Try yourself.

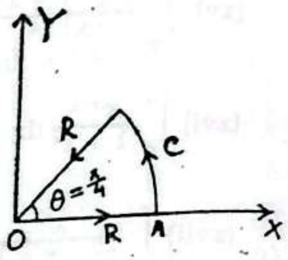
(Form 5) : Integral of the form $\int_0^\infty \sin x^2\,dx$ or $\int_0^\infty \cos x^2\,dx$

These two integals are called the **Fresnel integrals.** See the following example.

Example – 318 : Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

R. U. H. 81; D. U. M. Sc. P. 88.

Solution : consider $\oint_C f(z)\,dz$



where $f(z) = e^{iz^2}$ and C or OABO is the contour shown in the Fig. consisting of : (i) OA which lies on the x—axis from $x = 0$ to $x = R$; (ii) AB the arc of the circle $|z| = R$ i. e. $z = R\,e^{i\theta}$ from $\theta = 0$ to $\theta = \pi/4$; **(iii)** BO, the line $z = r\,e^{\pi i/4}$ from $r = R$ to $r = 0$. There are no singularties inside C and hence by the Cauchy residue theorem, we have

$$\oint_C f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BO} f(z)\,dz = 0 \;....\; (1)$$

Now on OA : We have $z = x$ and limit from $x = 0$ to $x = R$,

then $\int_{OA} f(z)\,dz = \int_0^R f(x)\,dx = \int_0^R e^{ix^2}\,dx$

$= \int_0^R (\cos x^2 + i \sin x^2)\,dx \Rightarrow \int_0^\infty (\cos x^2 + i \sin x^2)\,dx \;.....\; (2)$

as $R \to \infty$.

On AB : we have $z = Re^{i\theta}$ and limit from $\theta = 0$ to $\theta = \pi/4$.

then $\int_{AB} f(z)\,dz = \int_0^{\pi/4} e^{iR^2 e^{2i\theta}}\,d(Re^{i\theta})$

$= \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\, e^{iR^2 \cos 2\theta}\, iRe^{i\theta}\,d\theta$

$\Rightarrow \left|\int_{AB} f(z)\,dz\right| \le \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\, R\,d\theta = R/2 \int_0^{\pi/4} e^{-R^2 \sin\phi}\,d\phi$ [putting $2\theta = \phi$

$\le \frac{R}{2}\int_0^{\pi/2} e^{-2R^2\phi/\pi}\,d\phi = \frac{\pi}{4R}(1 - e^{-R^2}) \;....\; (3)$ [Using Jordan's inequality $\sin\phi \ge 2\phi/\pi$, $0 \le \phi \le \pi/2$

If $R\to\infty$, then (3) $\Rightarrow \lim_{R\to\infty} \left|\int_{AB} f(z)dz\right| \le 0$

$= \lim_{R\to\infty} \int_{AB} f(z)dz = 0 \;...\; (4)$

On BO : we have $z = r\,e^{\pi i/4}$ and limit from $r = R$ to $r = 0$.

then $\int_{BO} f(z)\,dz = \int_R^0 e^{ir^2 e^{\pi i/2}}\,d(re^{\pi i/4})$

$= -e^{\pi i/4}\int_0^R e^{-r^2}\,dr \;....\; (5)$. Now if $R\to\infty$, then (5)

$\Rightarrow -e^{\pi i/4}\int_0^\infty e^{-r^2}\,dr = -e^{\pi i/4}\,\frac{\Gamma(1/2)}{2}$

$= -(\cos \pi/4 + i \sin \pi/4)\,\frac{\sqrt{\pi}}{2}$

$= -\frac{\sqrt{\pi}}{2\sqrt{2}} - i\,\frac{\sqrt{\pi}}{2\sqrt{2}} \;....\; (6)$. Now taking limit $R\to\infty$ in (1)

and using (2) (4) and (6)

$$\Rightarrow \int_0^\infty (\cos x^2 + i \sin x^2) = -\frac{1}{2}\sqrt{\frac{\pi}{2}} - i\,\frac{1}{2}\sqrt{\frac{\pi}{2}} = 0$$

$$\Rightarrow \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and } \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and}$$

our required result is obtained.

240. (Form 6). Integrals involving many valued functions i. e. the integrals of the form $\int_0^\infty x^{p-1} f(x)\, dx$, where p is not an integer.

In the following examples we will integrate integrals involving many valued functions.

Example -319 : *If* $0 < p < 1$, then show that $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$.

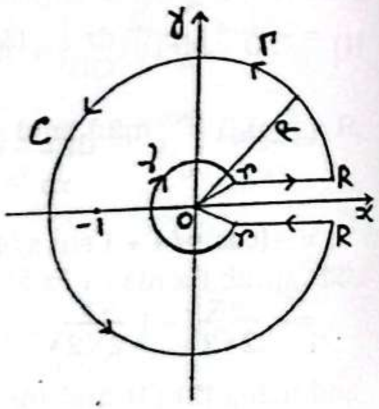
R. U. 84; D. U. H. 86; R. U. H. 72, 76, 81, 88; R. U. M. Sc. P. 86; D. U. M. Sc. P. T. 91; D. U. H. T. 86, 89; D. U. M. Sc. P. 78.

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \frac{z^{p-1}}{1+z}$.

Since $z = 0$ is a branch point and we choose C as the contour consisting of :

(i) the x—axis from r to R;

(ii) the large circle Γ :



$|z| = R$ where Γ is in the positive direction and $R \to \infty$; (iii) the x—axis from R to r; (iv) the small circle γ : $|z| = r$ where γ is in the negative direction and $r \to 0$. Here C is a closed contour which excludes the the origin and within this contour $f(z)$ is single—valued. The integrand $f(z) = \frac{z^{p-1}}{1+z}$ has a simple pole at $z = -1$ inside C.

Now we consider amp$z = 0$ on (i). Then on (iii) we have amp$z = 2\pi$ since the amplitude of z is increased by 2π in going around the circle Γ.

Now at the point $z = -1 \Rightarrow z = e^{\pi i}$ since amp $z = \pi$ and the residue at this point is $\lim_{z \to -1} \left\{ (z+1)\, \frac{z^{p-1}}{1+z} \right\} = (e^{\pi i})^{p-1}$

$= e^{(p-1)\pi i}$

Hence we have $\oint_C f(z)\, dz = 2\pi i\, e^{(p-1)\pi i}$

$$\Rightarrow \int_r^R \frac{x^{p-1}}{1+x}\, dx + \int_\Gamma \frac{z^{p-1}}{1+z}\, dz + \int_R^r \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\, dx + \int_\gamma \frac{z^{p-1}}{1+z}\, dz$$

$= 2\pi i\, e^{(p-1)\pi i}$ ... (1)

Here $\int_r^R \frac{x^{p-1}}{1+x}\, dx + \int_R^r \frac{(x\, e^{2\pi i})^{p-1}}{1+x\, e^{2\pi i}}\, dx$

$= \{1 - e^{2(p-1)\pi i}\} \int_r^R \frac{x^{p-1}}{1+x}\, dx$ .... (2) $[\because e^{2\pi i} = 1]$

Now on Γ we have $\left| \frac{z^{p-1}}{1+z} \right| \le \frac{R^{p-1}}{R-1}$ so that

$\left| \int_\Gamma \frac{z^{p-1}}{1+z}\, dz \right| \le \frac{R^{p-1}}{R-1} . 2\pi R = \frac{2\pi R^p}{R-1} \to 0$ as $R \to \infty$ since $0 < p < 1$. Thus $\lim_{R \to \infty} \int_\Gamma \frac{z^{p-1}}{1+z}\, dz = 0$ .... (3)

Similarly $\left| \int_\gamma \frac{z^{p-1}}{1+z} dz \right| \leq \frac{2\pi r^p}{1-r} \to 0$ as $r \to 0$ since $p > 0$.

Thus $\lim\limits_{r \to 0} \int_\gamma \frac{z^{p-1}}{1+z} dz = 0$ .... (4)

Now taking limit $r \to 0$ and $R \to \infty$ in (1) and (2) and using (3) and (4), we have

$$\{1 - e^{2(p-1)\pi i}\} \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{(p-1)\pi i}}{1 - e^{2(p-1)\pi i}} = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$$ and the result is Proved.

N. B. The above example can also be solved. by indenting at the points $z = -1, 0$ on the x—axis. It can be solved in other ways also.

Example -320 : Using the above example.

show that $\int_{-\infty}^\infty \frac{e^{pt}}{1+e^t} dt = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$.

Solution : In the above example we have

$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$ .... (1) where $0 < p < 1$.

Let $x = e^t$, then $dx = e^t\, dt$ and when $x = 0 \Rightarrow t = -\infty$ and when $x = \infty \Rightarrow t = \infty$ Now Using these (1) $\Rightarrow$

$$\int_{-\infty}^\infty \frac{(e^t)^{p-1}}{1+e^t} e^t\, dt = \frac{\pi}{\sin p\pi}$$

$\Rightarrow \int_{-\infty}^\infty \frac{e^{pt}}{1+e^t} dt = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$ and the result is proved.

Example -321 : Show that $\int_0^\infty \frac{x^p}{1 + 2x\cos\theta + x^2} dx$

$= \frac{\pi \sin(p\theta)}{\sin(\pi p)\sin\theta}$ where $-1 < p < 1,\ p \neq 0,\ -\pi < \theta < \pi,\ \theta \neq 0$

Solution : Try yourself.

Example -322 : Show that

$\int_0^\infty \frac{x^p}{(x+9)^2} dx = \frac{9^{p-1}\pi p}{\sin \pi p}$ where $-1 < p < 1,\ p \neq 0$

Solution : Try yourself.

Example -323 : Show that $\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$.

D. U. H. T. 87, 88, 89; C. U. 86; D. U. S. 86; D. U. 73, 75, 86, 88, 89, 91.

Solution : We know if $0 < p < 1$, then $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$ .... (1)

Again we know $\beta(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$ .... (2)

Now let $p + q = 1$, then $q = 1 - p$ and using these (2) $\Rightarrow$

$\frac{\Gamma(p)\,\Gamma(1-p)}{\Gamma(1)} = \int_0^\infty \frac{x^{p-1}}{1+x} dx \Rightarrow \Gamma(p)\,\Gamma(1-p) = \int_0^\infty \frac{x^{p-1}}{1+x} dx$ .... (3) $[\because \Gamma(1) = 1]$

Now by (1) and (3), we have

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad \text{where } 0 < p < 1$$

and the result is proved.

241. (Form 7). Rectangular contours :

In the following examples we will use the rectangular contours.

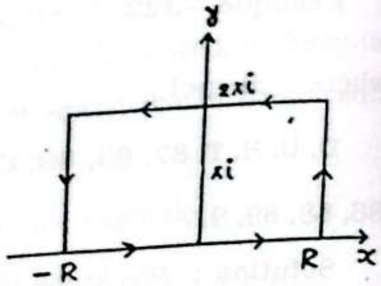
Example 324 : Show that $\displaystyle\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^{x}}\,dx = \frac{\pi}{\sin p\pi}$

where $0 < p < 1$.

R. U. 82; R. U. H. 75, 77, 87; R. U. M. Sc. P. 88; D. U. M. Sc. P. 78; D. U. H. T. 76, 78, 86, 87; D. U. H. 90.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{pz}}{1+e^{z}}$

and C is a rectangle having sides consisting of the x–axis and the lines $x = \pm R$, $y = 2\pi$.



The poles of $f(z) = \dfrac{e^{pz}}{1+e^{z}}$ are obtained by solving $1 + e^{z} = 0$

$\Rightarrow e^{z} = -1$

$= e^{(2n+1)\pi i}$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ The only pole enclosed by C is πi and it is simple. Now Res $[f(z), \pi i]$

$$\lim_{z\to\pi i}\left\{(z-\pi i)\,\frac{e^{pz}}{1+e^{z}}\right\}$$

$$= \lim_{z\to\pi i} e^{pz} \lim_{z\to\pi i}\frac{z-\pi i}{1+e^{z}} = e^{p\pi i}\lim_{z\to\pi i}\frac{1}{e^{z}} \text{ [by L' Hospital rule]}$$

$$= \frac{e^{p\pi i}}{e^{\pi i}} = -e^{p\pi i} \text{ since } e^{\pi i} = -1.$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{0}^{2\pi} f(R+iy)\,i\,dy + \int_{R}^{-R} f(x+2\pi i)\,dx$$

$$+ \int_{2\pi}^{0} f(-R+iy)\,i\,dy = -2\pi i\, e^{p\pi i} \quad \ldots (1)$$

Here $\displaystyle\int_{-R}^{R} f(x)\,dx + \int_{R}^{-R} f(x+2\pi i)\,dx$

$$= \int_{-R}^{R}\frac{e^{px}}{1+e^{x}}\,dx - \int_{-R}^{R}\frac{e^{p(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx$$

$$= (1-e^{2p\pi i})\int_{-R}^{R}\frac{e^{px}}{1+e^{x}}\,dx \ \ldots (2) \left[\because \begin{matrix} e^{2p\pi} = 1 \\ e^{2\pi} = 1\end{matrix}\right]$$

Now we have $\displaystyle |f(R+yi)| = \left|\frac{e^{p(R+yi)}}{1+e^{R+yi}}\right| \le \frac{e^{pR}}{e^{R}-1}$

$$\therefore \left|\int_{0}^{2\pi} f(R+yi)\,i\,dy\right| \le \frac{e^{pR}\,2\pi}{e^{R}-1} \to 0 \text{ as } R\to\infty$$

$$\text{Thus } \lim_{R\to\infty}\int_{0}^{2\pi} f(R+yi)\,i\,dy = 0 \quad \ldots (3)$$

Similarly we can show that $\displaystyle\lim_{R\to\infty}\int_{2\pi}^{0} f(-R+yi)\,i\,dy = 0 \ldots (4)$

Now taking limit $R\to\infty$ in (1) an (2) and using (3) and (4) we have $\displaystyle (1-e^{2p\pi i})\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^{x}}\,dx = -2\pi i\, e^{p\pi i}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{px}}{1+e^{x}}\,dx = \frac{-2\pi i e^{p\pi i}}{1-e^{2p\pi i}} = \frac{2\pi i}{e^{p\pi i}-e^{-p\pi i}}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{px}}{1+e^{x}}\,dx = \frac{\pi}{\sin p\pi} \quad \text{where } 0<p<1 \text{ and the result is}$$

Proved.

Example -325 : Using the above example, show that $\int_{-\infty}^{\infty} \frac{t^{p-1}}{1+t}\, dt = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$.

Solution : In the above example we have

$$\int_0^{\infty} \frac{e^{px}}{1+e^x}\, dx = \frac{\pi}{\sin px} \text{ ..... (1) where } 0 < p < 1.$$

Let $e^x = t$, then $dx = \frac{dt}{t}$ and when $x = \infty \Rightarrow t = \infty$ and when $x = \infty \Rightarrow t = 0$

Now using these (1) $\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{t^p}{1+t} \frac{dt}{t} = \frac{\pi}{\sin p\pi}$$

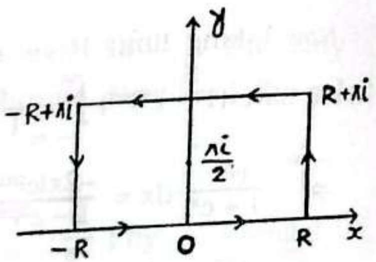
$$\Rightarrow \int_{-\infty}^{\infty} \frac{t^{p-1}}{1+t}\, dt = \frac{\pi}{\sin p\pi} \quad \text{where } 0 < p < 1$$

and the require result is proved.

Example -326 : Show that $\int_0^{\infty} \frac{\cosh ax}{\cosh x}\, dx = \frac{\pi}{2 \cos \pi a/2}$ where $|a| < 1$.

D. U. H. T. 75; D. U. H. 89.

Solution: Consider $\oint_C f(z)\, dz$ where $f(z) = \frac{e^{az}}{\cosh z}$ and C is a rectangle having vertices at $-R$, R, $R + \pi i$, $-R + \pi i$. The poles of $f(z) = \frac{e^{az}}{\cosh z}$ are obtained by solving $\cosh z = 0$ $\Rightarrow e^z + e^{-z} = 0 \Rightarrow e^{2z} + 1 \Rightarrow e^{2z} =$



$-1 = e^{(2n+1)\pi i} \Rightarrow z = (2n+1)\pi i/2$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ The only pole enclosed by C is $\pi i/2$ and it is simple.

Now $\text{Res}\left[f(z), \frac{\pi i}{2}\right] = \lim\limits_{z \to \pi i/2} \left\{ \left(z - \frac{\pi i}{2}\right) \frac{e^{az}}{\cosh z} \right\}$

$$= \lim_{z \to \pi i/2} e^{az} . \lim_{z \to \pi i/2} \frac{z - \pi i/2}{\cosh z}$$

$$= e^{a\pi i/2} \lim_{z \to \pi i/2} \frac{1}{\sinh z} \text{ [by L' Hospital rule]}$$

$$= \frac{e^{a\pi i/2}}{\sinh \pi i/2} = \frac{e^{a\pi i/2}}{i \sin \pi/2} = -i\, e^{a\pi i/2}.$$

Then by the residue theorem, we have $\oint_C f(z)\, dz =$

$\int_{-R}^{R} f(x)\, dx + \int_0^{\pi} f(R + iy)\, idy + \int_R^{-R} f(x + \pi i)\, dx +$

$\int_{\pi}^{0} f(-R + iy)\, idy = 2\pi i\,(-ie^{a\pi i/2})$ ...... (1). In (1) we have

$\int_{-R}^{R} f(x)\, dx + \int_R^{-R} f(x + i\pi)\, dx$

$$= \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\, dx - \int_{-R}^{R} \frac{2e^{ax}\, e^{a\pi i}}{e^{x+i\pi} + e^{-x-i\pi}}\, dx$$

$$= \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\, dx + \int_{-R}^{R} \frac{2e^{ax}\, e^{a\pi i}}{e^{x} + e^{-x}}\, dx \quad [\, e^{\pi i} = e^{-\pi i} = -1\,]$$

$$= (1 + e^{a\pi i}) \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\, dx \text{ ..... (2)}$$

Now $f(R + iy) = \frac{e^{a(R+iy)}}{\cosh (R+iy)} = \frac{2e^{aR}e^{aiy}}{e^R e^{iy} - e^{-R} e^{-iy}}$

Then $\left| f((R + iy) \right| \le \dfrac{2\,|e^{aR}|\,|e^{aiy}|}{|e^{R}|\,|e^{iy}| - |e^{-R}|\,|e^{-iy}|} = \dfrac{2e^{aR}}{e^{R} - e^{-R}}$

$\Rightarrow |f(R + idy| \le \dfrac{4\,e^{aR}}{e^{R}}$ since $e^{R} - e^{-R} \ge \dfrac{1}{2}\,e^{R}$

$\Rightarrow \left| \int_0^{\pi} f(R + idy)\, idy \right| \le \int_0^{\pi} \dfrac{4e^{aR}}{e^{R}}\, dy = 4\pi\, e^{(a-1)R} \to 0$ as $R \to \infty$

since $|a| < 1$.

$$\therefore \lim_{R \to \infty} \int_0^{\pi} f(R + iy)\, idy = 0 \text{ ..... (3)}$$

Similarly we can show that

$$\lim_{R \to \infty} \int_{\pi}^{0} f(-R + iy)\, idy = 0 \text{ .... (4)}$$

Now taking limit $R \to \infty$ in (1) and (2) and using (3) and (4) we have

$$(1 + e^{a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\, dx = 2\pi\, e^{a\pi i/2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\, dx = \frac{2\pi\, e^{a\pi i/2}}{1 + e^{a\pi i}} = \frac{2\pi}{e^{a\pi i/2} + e^{-a\pi i/2}} = \frac{2\pi}{2\cos\frac{a\pi}{2}}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{e^{ax}}{\cosh x}\, dx + \int_0^{\infty} \frac{e^{ax}}{\cosh x}\, dx = \frac{\pi}{\cos a\pi/2} \text{ .... (5)}$$

Now replacing x by $-x$ in the first integral of (5), then we get.

$$\int_0^{\infty} \frac{e^{-ax}}{\cosh x}\, dx + \int_0^{\infty} \frac{e^{ax}}{\cosh x}\, dx = \frac{\pi}{\cos a\pi/2}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{ax} + e^{-ax}}{\cosh x}\, dx = \frac{\pi}{\cos a\pi/2}$$

$$\Rightarrow \int_0^{\infty} \frac{2\cosh ax}{\cosh x}\, dx = \frac{\pi}{\cos a\pi/2}$$

$\Rightarrow \displaystyle\int_0^{\infty} \frac{\cosh ax}{\cosh x}\, dx = \frac{\pi}{2\cos a\pi/2}$ and the required result is obtained.

Example -327 : Show that

(i) $\displaystyle\int_0^{\infty} \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2}\sec a/2$ where $-\pi < a < \pi$;

D. U. H. T. 75

(ii) $\displaystyle\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x}\, dx = \frac{1}{2}\tan\frac{a}{2}$ where $-\pi < a < \pi$;

R. U. H. 73.

(iii) $\displaystyle\int_0^{\infty} \frac{x \sin x}{1 + a^2 - 2a\cos x}\, dx = \frac{\pi}{a}\log(1+a)$ where $0 < a < 1$;

D. U. H. T. 75

(iv) $\displaystyle\int_0^{\infty} e^{-x^2} \cos 2ax\, dx = \frac{1}{2}\sqrt{\pi}\, e^{-a^2}$.

D. U. H. T. 75; R. U. H. 76.

Solution : Try yourself.