Number Theory Problem Set

1. Find all pairs (a, b) of positive integers satisfying $a^{b^2} = b^a$.

2. Find all pairs of positive integers (n, k) such that $n! = (n + 1)^k - 1$.

3. Solve in integers (x, y, z): $2x^4 + 2x^2y^2 + y^4 = z^2$.

4. Determine all positive integer triples (x, y, z) such that $(x + 1)^{y+1} + 1 = (x + 2)^{z+1}$.

5. Solve $x^a - 1 = y^b$, where $x, y, a, b > 1$ and $x \equiv 1 \pmod{y}$.

6. Find all positive integer pairs (p, n) such that $2^n = p + 3^p$.

7. Find all positive integer solutions to $2^x = 3^y + 5$.

8. Let p, q, r be distinct primes. Prove that $z^r + x^p = 2y^q$ has infinitely many solutions with $x, y, z > 1$ and $z^r \neq x^p$.

9. Prove that there are no positive integer solutions of $x^2y^2 = z^2(z^2 - x^2 - y^2)$.

10. Solve in positive integers: $3^x = 2^y + 1$.

11. Find all integer solutions of $x^5 = y^2 + 1$.

12. Prove that $x^3 + y^3 = 9$ has infinitely many rational solutions.

13. Find all positive integers x, y, z such that $xy^2 = z^3 + 1$, and x has no prime factor of the form $6k + 1$.

14. Find all integer solutions of $7^x = 3^y + 4$.

15. Prove that $x^3 + y^3 = z^4 - t^4$ has infinitely many solutions with $\gcd(x, y, z, t) = 1$.