SSY285 - Home Assignment M3

Diego Jauregui Lucas Rath Sondre Wiersdalen

December 14, 2018

1 Linear state estimation and control of DC-motor with flywheel

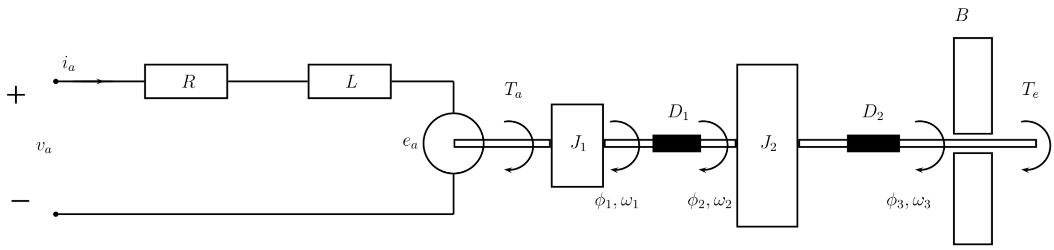

Figure 1: *DC motor with flywheel*

$$\begin{bmatrix} \phi_{1,2} \\ \omega_{1,2} \\ \phi_{2,2} \\ \omega_{2,2} \\ \phi_{3,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{D_1}{J_1} & -\frac{K_e\,K_t}{J_1\,R} & \frac{D_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{D_1}{J_2} & 0 & -\frac{D_1+D_2}{J_2} & 0 & \frac{D_2}{J_2} \\ 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{A} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{K_t}{J_1\,R} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{B} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{1}$$

$$y = \begin{bmatrix} \phi_{2,1} \\ \omega_{2,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{C1} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{2}$$

$$A_d = \begin{pmatrix} 0.4 & 4.5\,10^{-4} & 0.6 & 2.3\,10^{-4} & 2.6\,10^{-3} \\ -799.0 & -0.049 & 788.0 & 0.6 & 9.1 \\ 0.21 & 5.9\,10^{-5} & 0.77 & 9.2\,10^{-4} & 0.018 \\ 344.0 & 0.15 & -388.0 & 0.77 & 28.0 \\ 0.059 & 1.3\,10^{-5} & 0.57 & 3.5\,10^{-4} & 0.37 \end{pmatrix} \tag{3}$$

$$B_d = \begin{pmatrix} 3.2\,10^{-3} & 2.9\,10^{-4} \\ 4.5 & 1.3 \\ 1.6\,10^{-4} & 3.2\,10^{-3} \\ 0.59 & 8.8 \\ 2.9\,10^{-5} & 0.32 \end{pmatrix} \tag{4}$$

a) Assume that (discrete time) white noises are added to both inputs, to the external torque T_e and to the applied motor voltage v_a. Suppose they are zero mean and uncorrelated sequences. The voltage disturbances, mostly due to variations in the power supply unit, are upper bounded by $0.3\,V$ (normally distributed noise, account for 99.7% of the realization as bound). The torque disturbance (normally distributed noise, account for 99.7% of the realization as bound) is estimated to be less than 10% of the maximum applied external torque value (assumed to be $1\,Nm$). Propose a covariance matrix for the disturbance vector w (having the above two components). Which N-matrix should be used in $x(t+1) = Ax(t) + Bu(t) + Nw(t)$?

Since the noise added to the inputs is assumed to be white, zero mean, uncorrelated and normal distributed, we don't have to characterize the model disturbances and we can propose the covariance matrix R_w to be diagonal. Moreover, since $w \in \Re^{2\times1}$, R_w is a 2x2 matrix with diagonal corresponding to the variance of each noise input.

The variance can be determined by calculating which is the variance that gives a probability of 99.7% when a centered normal distribution is bounded between two values. This can be translated by solving the equation below, where "Percentage of realization"=99.7.

$$\frac{\text{Percentage of realization}}{100} = \int_{lb}^{ub} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx \tag{5}$$

As given by the exercise, the bounds corresponding to the 99.7% of realization is $lb = -0.3, ub = 0.3V$ for v_a and $lb = -0.1\,Nm, ub = 0.1\,Nm$ for T_e. We then calculate variances by solving equation (5) for σ^2 for each noise input. Finally:

$$R_w = \begin{bmatrix} \sigma_{v_a}^2 & 0 \\ 0 & \sigma_{T_e}^2 \end{bmatrix} = \begin{bmatrix} 0.0102 & 0 \\ 0 & 0.0011 \end{bmatrix} \tag{6}$$

The N matrix will be exactly the same as B_d because the system is linear and the disturbances are directly added to the inputs ($v_a + v_{aw}$ and $T_e + T_{ew}$). Thus, we get the following system dynamics:

$$x(t+1) = Ax(t) + Bu(t) + \underbrace{\begin{pmatrix} 3.2\,10^{-3} & 2.9\,10^{-4} \\ 4.5 & 1.3 \\ 1.6\,10^{-4} & 3.2\,10^{-3} \\ 0.59 & 8.8 \\ 2.9\,10^{-5} & 0.32 \end{pmatrix}}_{N} \underbrace{\begin{bmatrix} w_{v_a} \\ w_{T_e} \end{bmatrix}}_{w(t)} \tag{7}$$

---

b) Measurement disturbances n_1; n_2 are added to the output. The disturbances are upper bounded by 0.02 radians and 0.01 radians per second, and are assumed to be discrete time, zero mean uncorrelated white noises (normally distributed, account for 99.7% of the realization for bounds). Propose a covariance matrix for the measurement disturbance vector n.

Assuming same conditions as in question a), we applied the same procedure to obtain the variance of the measurement disturbances, using the given realization bounds. Thus, the proposed covariance matrix for the output disturbance $n = [n_1 \quad n_2]^T = [n_{\phi_2} \quad n_{\omega_2}]^T$ is:

$$R_n = \begin{bmatrix} \sigma_{\phi_2}^2 & 0 \\ 0 & \sigma_{\omega_2}^2 \end{bmatrix} = \begin{bmatrix} 45.4160e-006 & 0 \\ 0 & 11.3540e-006 \end{bmatrix} \tag{8}$$

c) Provided the cross spectrum between w and n is zero, compute a (discrete time) Kalman filter to estimate (the "current") $\hat{x}(k|k)$ of the system $x(k+1) = Ax(k) + Bu(k) + Nw(k)$; $y(k) = Cx(k) + n(k)$. Find the the observer gain matrix, L! What is the covariance matrix of the state estimation error, P? What are the observer eigenvalues in this case?

The general form of system model including input and output noise is

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Nw(k) \\ y(k) &= Cx(k) + Du(k) + Hw(k) + n(k) \end{aligned}$$

We define our discrete system with the following matrices:

$$A_d = \begin{pmatrix} 0.4 & 4.5\,10^{-4} & 0.6 & 2.3\,10^{-4} & 2.6\,10^{-3} \\ -799.0 & -0.049 & 788.0 & 0.6 & 9.1 \\ 0.21 & 5.9\,10^{-5} & 0.77 & 9.2\,10^{-4} & 0.018 \\ 344.0 & 0.15 & -388.0 & 0.77 & 28.0 \\ 0.059 & 1.3\,10^{-5} & 0.57 & 3.5\,10^{-4} & 0.37 \end{pmatrix} \tag{9}$$

$$B_d = \begin{pmatrix} 3.2\,10^{-3} & 2.9\,10^{-4} \\ 4.5 & 1.3 \\ 1.6\,10^{-4} & 3.2\,10^{-3} \\ 0.59 & 8.8 \\ 2.9\,10^{-5} & 0.32 \end{pmatrix} \tag{10}$$

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad D = 0 \qquad N = \begin{pmatrix} 3.2\,10^{-3} & 2.9\,10^{-4} \\ 4.5 & 1.3 \\ 1.6\,10^{-4} & 3.2\,10^{-3} \\ 0.59 & 8.8 \\ 2.9\,10^{-5} & 0.32 \end{pmatrix} \qquad H = 0 \tag{11}$$

To compute a DT Kalman filter in Matlab we use the function:

```
[~,L,P] = kalman(sys,Qn,Rn,Nn)
```

Where the inputs parameters are:

- sys, a discrete state space model(A,B,C,D) with (Ad, $[Bd\ N]$, C, $[DH]$) with sample time equal to T_s
- Qn, the spectrum of input noise, equal to R_w
- Rn, the spectrum of measurement disturbance, equal to R_n
- Nn, zero-matrix as provided by the question (cross-spectrum between w and n)

The output of the kalman command will be:

- L_{kalman}, Kalman gain
- P, covariance matrix of the state estimation error

Thus, the code to generate the kalman filter is:

```
% Generate the system model
sysmodel = ss(Ad, [Bd N], C, [D H],Ts)
% Define Qm and Rm with the covariances matrices of the input and output noises
Qm = diag([sigma_va,sigma_Te]);
Rm = diag([sigma_phi2,sigma_w2]);
% Compute the kalman filter
[~,L_kalman,P] = kalman( sysmodel, Qm , Rm, 0*Qm*Rm )
% Observer eigenvalues
eig(Ad-L*C)
```

Which results in:

```
L_kalman =

   460.6475e-006   858.8209e-006
   -77.5651e-006   696.9576e-003
   487.2207e-006     1.8409e-003
   -29.2172e-003     1.6755e+000
   293.4529e-006    11.2787e-003


P =

   234.9022e-009    62.0796e-006    46.3078e-009    62.7307e-006   -69.6582e-009
    62.0796e-006   350.3862e-003     5.4658e-006    25.3089e-003     1.0863e-003
    46.3078e-009     5.4658e-006    38.5621e-009    41.9816e-006     1.1998e-006
    62.7307e-006    25.3089e-003    41.9816e-006   111.9521e-003     3.2064e-003
   -69.6582e-009     1.0863e-003     1.1998e-006     3.2064e-003   117.4099e-006
```

The eigenvalues of the observer are:

```
eig(Ad-L_kalman*C) =

  -688.2697e-003 +   0.0000e+000i
   999.5001e-003 +   0.0000e+000i
   143.8026e-003 +520.8596e-003i
   143.8026e-003 -520.8596e-003i
   -68.2936e-006 +   0.0000e+000i
```

As can be seen above, all the observer eigenvalues are inside the unit circle, so the observer is stable for all states and therefore also detectable.

---

d) Design a discrete time Linear Quadratic Gaussian controller and simulate the closed-loop answer to a step r_{ω_2} (jumping from an initial value 10 to 100) for the discrete time and noise corrupted system above. Use the previously computed Kalman filter gain, answered in c) to reconstruct system states. For the LQ controller, choose and use two appropriate dimensional weighting matrices Q_u and Q_x provided that you know the control input energy is "expensive". Hint: You may chose to use either integral states or a prefilter to ensure the reference is followed. However there may occur numerical problems when calculating the prefilter.

A discrete LQG controller with integral action was developed. The system matrix A, and input matrix B was extended in the following way to allow to integrate the reference error for velocity (ω_2)

$$A_e = \begin{bmatrix} A_d & 0 \\ -C_v & 1 \end{bmatrix} \qquad B_e = \begin{bmatrix} B_d \\ 0 \end{bmatrix} \tag{12}$$

Where C_v selects ω_2 from the new extended state vector. Therefore:

$$\begin{aligned} x &= \begin{bmatrix} \phi_{1,1} & \omega_{1,1} & \phi_{2,1} & \omega_{2,1} & \phi_{3,1} & x_I \end{bmatrix} \\ C_v &= [\,0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0\,] \end{aligned} \tag{13}$$

The optimal gain was found using the matlab function dlqr.m with the following weighing matrices

$$Q_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1000} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1000} \end{pmatrix} \qquad Q_u = \begin{pmatrix} 2000000 & 0 \\ 0 & 1000000 \end{pmatrix} \tag{14}$$

A short rationale for the weighing matrices

- It was found by some trial and error that the velocity should be weighed less because its difficult to attain an abrupt change in velocity without oscillations.
- Q_u: Since the input is expensive and should be used conservatively it is penalized a lot. The ratio between the entries is some way arbitrary.

A simulink model was developed to simulate the system and is shown in Figure 2.

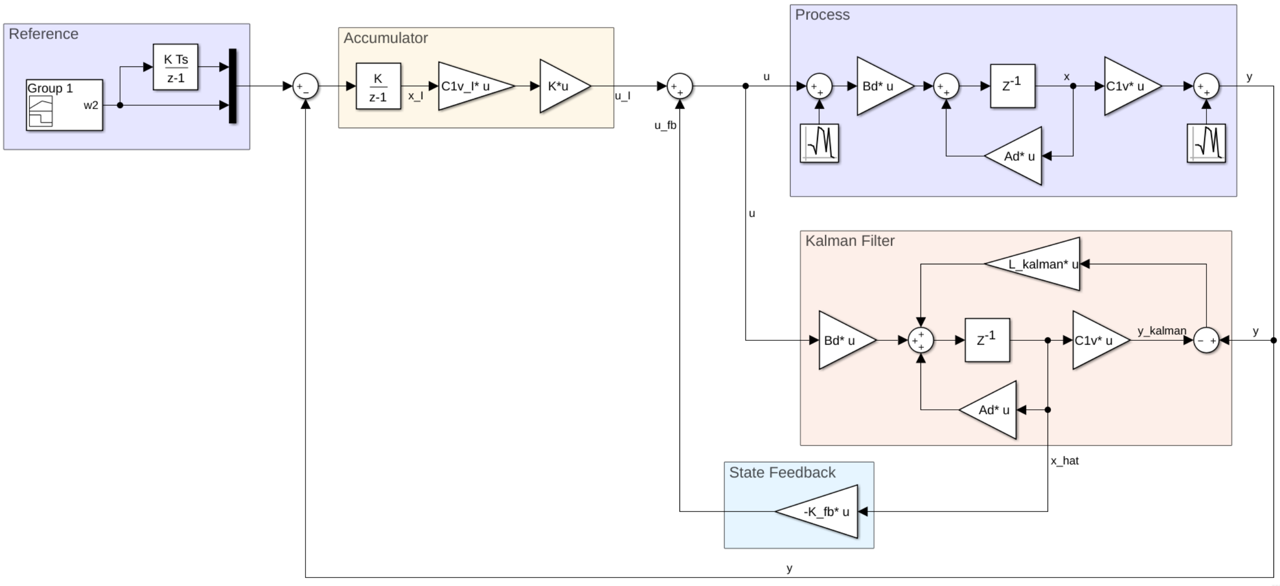

Figure 2: *Simulink model: Observer state feedback w. integral action*

The step response from 10 to 100 rad/s is shown in Figure 3 including the input in Figure 4 below

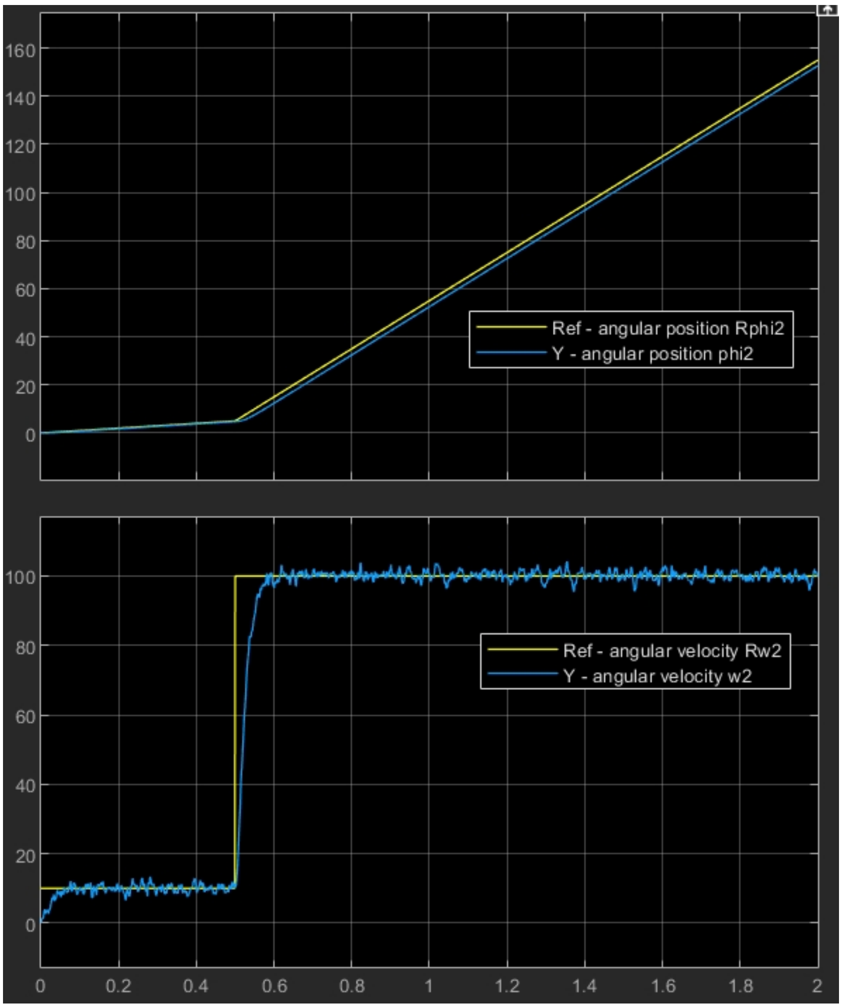

Figure 3: *Step response in angular velocity. yellow indicates reference and blue is output*

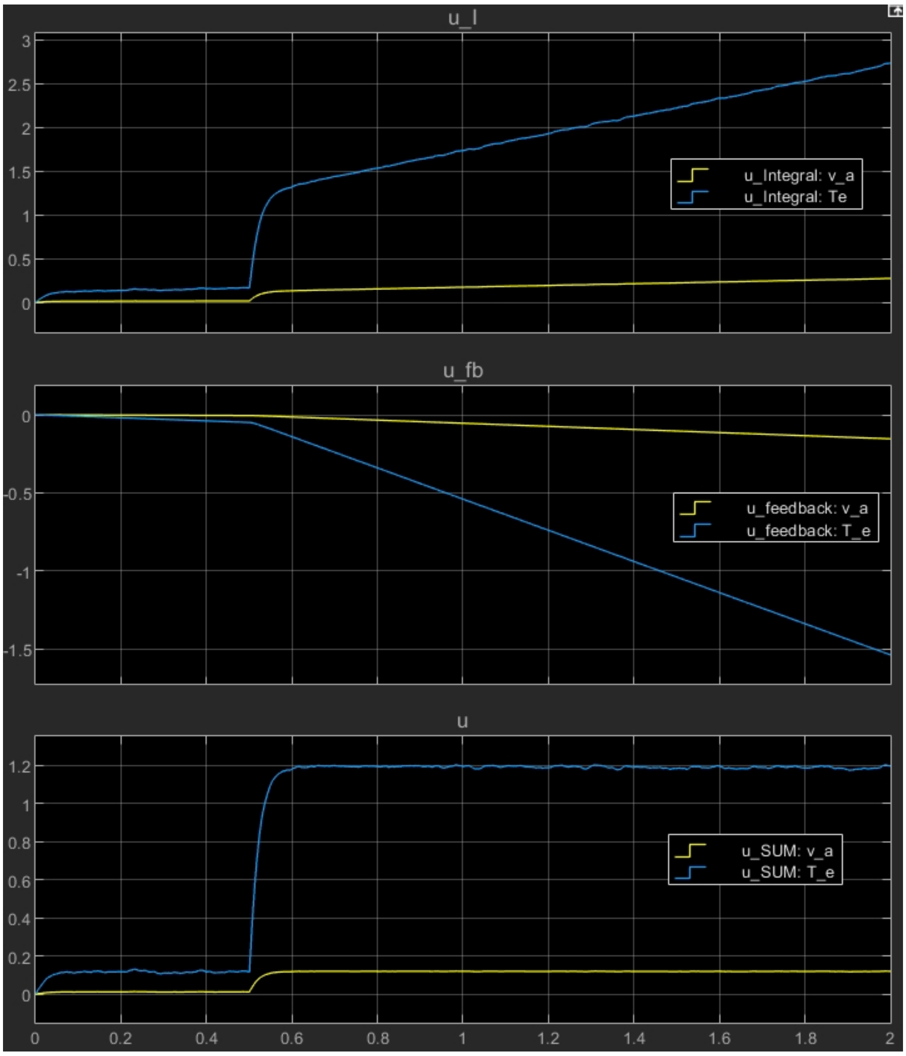

Figure 4: *Input during simulation where blue is T_e and yellow is v_a*