SSY285 - Home Assignment M1

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1 Dynamic model of DC motor with flywheel

Consider the system shown in Figure 1 which consists of an electric DC-motor (having separate magnetization) that drives a flywheel and that is influenced by external torque.

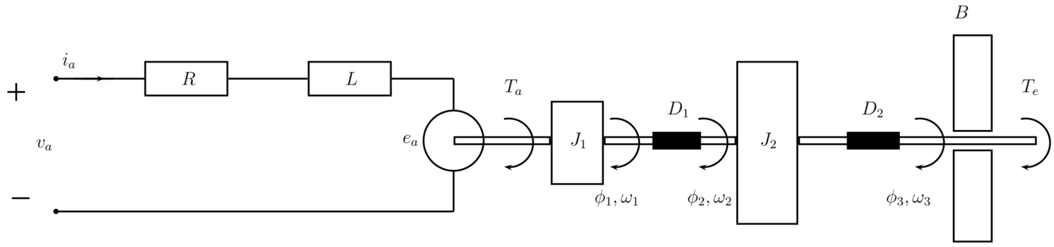

Figure 1: *DC motor with flywheel*

The system is characterized by the following variables and parameters such as external voltage applied to the rotor v_a, rotor current, i_a, induced rotor voltage, e_a, rotor produced torque T_a, external torque applied to flywheel axis T_e, angles ϕ_1, ϕ_2, ϕ_3, angular speeds ω_1, ω_2, ω_3. Moreover, L is the rotor inductance, R is the rotor resistance, K_E is the coefficient related to induced voltage e_a to rotor speed, K_T is the rotor torque constant (driving torque to rotor current i_a), J_1 is the rotor inertia, J_2 is the flywheel inertia. Both D_1 and D_2 represent flexibility of the axis (torsional springs) on each side of flywheel, B denotes dynamic (linear) friction proportional to the angular speed.

Question a)

Given the relations $e_a = K_E\omega_1$ and $T_a = K_T i_a$, formulate a mathematical model for this system, using basic laws of electricity and mechanics. How many linear differential equations and at what order are needed to describe this system completely? Which are the system inputs in this case?

We start analysing the system of the figure 1 using fundamental physics laws. The equation (1) derives from applied Kirchhoff's voltage law to the circuit representing the motor windings, where the counter EMF voltage $e_a = K_e\omega_1$ as given in the exercise.

$$\mathrm{v_a} = L\frac{di_a}{dt} + \underbrace{\mathrm{K_E}\,\omega_1}_{\mathrm{e_a}} + R\,i_a \tag{1}$$

The equation (2) is obtained applying Newton's laws of angular movement to the system composed by the rotor with inertia J_1. The rotor shaft is subjected to a motor torque T_a and to the spring torque, generated by the difference between the shaft angles ϕ_1 and ϕ_2. Here, we have replaced the torque of the motor Ta by its direct relation with the current of the motor i_a as we have other terms depending on i_a in equation (1).

$$J_1\,\dot{\omega}_1 = \underbrace{\mathrm{K_T}\,i_a}_{\mathrm{T_a}} - D_1\,\left(\phi_1 - \phi_2\right) \tag{2}$$

Applying a similar procedure then in (3) in the next shaft, we obtain the equation (4) as shown below:

$$J_2\,\dot{\omega}_2 = D_1\,\left(\phi_1 - \phi_2\right) - D_2\,\left(\phi_2 - \phi_3\right) \tag{3}$$

Finally, the last equation of the system (4) is obtaining via analyzing the torque forces occurring after the flywheel in the output shaft. Since no inertia has been defined, the angular velocity is determined directly by the following equation:

$$0 = B\,\omega_3 - \mathrm{T_e} - D_2\,\left(\phi_2 - \phi_3\right) \tag{4}$$

One may notice that if we time differentiate equation 4, we can get the acceleration of the shaft:

$$\dot{\omega_3} = \left(\dot{\mathrm{T}}_\mathrm{e} + D_2\,\left(\omega_2 - \omega_3\right)\right)/B \tag{5}$$

The equation 5 should not be included in the state-space formulation. However, it is interesting to notice that if we introduce a step input torque T_e in the system, $\dot{T_e}$ and consequently $\dot{\omega_3}$ might also assume infinite values for a short period of time.

Finally, we put equations (1), (2), (3), (4) together to create the mathematical representation of system. We can see in the equations that the **order of the system is 6**, because there are 2 order-2 equations and 2 order-1 equations. In order to obtain the state space model we reduce the order of 2-order equations, as follows:

$$\dot{\phi}_1 = \omega_1 \tag{6}$$
$$\dot{\phi}_2 = \omega_2 \tag{7}$$

Now we can define the state vector, looking into equations (1) to (7) for all variables that are time-differentiated. Hence, the state vector is define as:

$$x = \left[i_a \quad \phi_1 \quad \omega_1 \quad \phi_2 \quad \omega_2 \quad \phi_3\right]^T \tag{8}$$

and the inputs of the system as vector $u = [v_a \quad T_e]^T$.

We are then ready to solve the system, which can be developed symbolically and solved using MATLAB. Part of the the code for this is shown below. For practical reasons the variables are indexed with two numbers i and j as such

$$\omega_{i,j} \tag{9}$$

where i distinguishes the variables index and j is the time derivative order of the same variable. E.g.:

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_{i,1} = \omega_{i,2} \quad \text{where } i = 1, 2, 3 \tag{10}$$

To describe the entire dynamical system, one must include **6 order-1 differential equations** and is shown in the listing (1). Here the equations (1) to (7) are rearranged to match the listing order of the variables of the space vector defined in (8).

Listing 1: *Dynamic equations for the DC motor*

```
eqs = [        v_a == ia(1)*R + L*ia(2) + K_e*w1(1) ;
        J_1*w1(2) == K_t*ia(1) - D_1*(phi1(1)-phi2(1)) ;
        J_2*w2(2) == D_1*(phi1(1)-phi2(1)) - D_2*(phi2(1)-phi3(1)) ;
                0 == D_2*(phi2(1)-phi3(1)) - B*phi3(2) + T_e ;
         phi1(2) == w1(1) ;
         phi2(2) == w2(1) ];

x = [ia; phi1; w1; phi2; w2; phi3]  % State space variables
u = [v_a; T_e]                      % Input variables

% eqs -> x(:,2) = x(:,1) * A +  B * u
[A,B] = get_state_space(eqs, x(:,2), x(:,1), u)
```

The standard state space representation then follows:

$$\begin{bmatrix} i_{a,2} \\ \phi_{1,2} \\ \omega_{1,2} \\ \phi_{2,2} \\ \omega_{2,2} \\ \phi_{3,2} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_e}{L} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{K_t}{J_1} & -\frac{D_1}{J_1} & 0 & \frac{D_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{D_1}{J_2} & 0 & -\frac{D_1+D_2}{J_2} & 0 & \frac{D_2}{J_2} \\ 0 & 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{A} \begin{bmatrix} i_{a,1} \\ \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{B} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{11}$$

Question b)

Assume that the inductance is very small, $L \approx 0$, then choose state variables to compose a state vector $x(t)$, and inputs for vector $u(t)$ of the underlying system. Pay attention to select the state and control inputs such that you formulate a continuous time state equation under the form of $\dot{x}(t) = Ax(t) + Bu(t)$. What are the matrices A and B?

By assuming a very low inductance, $v_L \approx 0$, we can get rid of the state-space variable i_a (equation 1 becomes algebraic) because it will not be a result anymore from a differential equation, but from a direct correlation between the inputs and the other state-space variables.

Therefore, in order to eliminate the variable i_a from our model, we solve (1) for i_a and then replace its value in the other equations. The whole procedure can be outlined according to the following code 2.

Listing 2: *MATLAB code finding A and B*

```
eqs_s = subs(eqs, ia(2), 0);         % replace i_a(2) [i_a_t] in the first equation by 0
ia_v = solve(eqs_s(1), ia(1));       % solve the first equation for i_a
eqs_s = subs(eqs_s, ia(1), ia_v);    % Replace i_a in the equation system
eqs_s = eqs_s(2:end);                % Eliminate the first equation because is 0.

% eqs_s -> x(:,2) = x(:,1) * Am +  Bm * u
[A,B] = get_state_space(eqs_s, x(:,2), x(:,1), u)
```

And finally, we get the following state-space matrices A and B:

$$\begin{bmatrix} \phi_{1,2} \\ \omega_{1,2} \\ \phi_{2,2} \\ \omega_{2,2} \\ \phi_{3,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{D_1}{J_1} & -\frac{K_e\,K_t}{J_1\,R} & \frac{D_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{D_1}{J_2} & 0 & -\frac{D_1+D_2}{J_2} & 0 & \frac{D_2}{J_2} \\ 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{A} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{K_t}{J_1\,R} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{B} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{12}$$

Question c)

The output $y(t)$ of the system in a state space model is related to state and input, as $y(t) = Cx(t) + Du(t)$. Give the matrices C and D for the following two cases; (1) $y_1(t) = \phi_2$, $y_2(t) = \omega_2$ and (2) $y_1(t) = i_a$, $y_2 = \omega_3$!

For the first case, the desired outputs are ϕ_2 and ω_2, which can be directly picked from the state-space vector with no direct feed through, i.e the D-matrix will only have zero entries

$$y = \begin{bmatrix} \phi_{1,1} \\ \omega_{2,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{13}$$

For the second case, we want to select two variables that are not part of the state-space variables. Fortunately, those variables can be expressed as functions of the state-space variables and the inputs, according to the following equations:

$$i_a = \frac{1}{R}(v_a - K_e\omega_1)) \tag{14}$$

$$\omega_3 = \frac{1}{B}(\mathrm{T_e} + D_2\ (\phi_2 - \phi_3) \tag{15}$$

We can then structure the equations above as $[i_a \quad \omega_3]^T = C\ x + D\ u$, through the following matrices and therefore obtain the desired variables as outputs of the system.

$$y = \begin{bmatrix} i_a \\ \omega_{3,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{K_e}{R} & 0 & 0 & 0 \\ 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{C} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{D} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{16}$$

Question d)

Assume the following parameter values:
$R = 1\Omega$, $K_E = 10^{-1}Vs/rad$, $K_T = 10^{-1}Nm/A$, $J_1 = 10^{-5}kgm^2$, $J_2 = 4{\cdot}10^{-5}kgm^2$, $B = 2{\cdot}10^{-3}Nms$, $D_1 = 20Nm/rad$, $D_2 = 2Nm/rad$.

Calculate (with Matlab) the eigenvalues of the A-matrix. Is the system input-output stable for both of the subcases (1) and (2) of c)?

The listing 3 shows the values of the parameters that where replaced in the matrix A that is show in the equation 12.

Listing 3: *Assigning numeric values to parameters*

```
% Giving values to the symbolic parameters
vars_sym   = [R, Ke, Kt, J1,    J2,    B,     D1, D2];
vars_value = [1, .1, .1, 1e-5, 4e-5, 2e-3, 20,  2];
```

After doing the numeric calculations, this A-matrix is obtained:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -2000000 & -1000 & 2000000 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 500000 & 0 & -550000 & 0 & 50000 \\ 0 & 0 & 1000 & 0 & -1000 \end{bmatrix} \tag{17}$$

In order to calculate the eigenvalues (λ), the following matlab code is executed:

Listing 4: *Calculating the eigenvalues*

```
double(eig(A))
ans =
   1.0e+03 *
   0.0000 + 0.0000i
  -0.9464 + 0.0000i
  -0.2708 + 0.0000i
  -0.3914 - 1.4791i
  -0.3914 + 1.4791i
```

We notice that all the eigenvalues (λ) of A-matrix are less or equal to zero, hence we can say that this system is stable. Additionally, since it is a LTI system, it is input-output stable.

Question e)

Suppose that the initial state, the applied rotor voltage and the external torque are all zero. Then the applied rotor voltage is step wise changed from 0 to 10 Volt. Investigate by simulation, how the state vector components evolve in time. Suppose that all angular velocities have reached the steady state (angles are of course increasing unboundedly). Suddenly an external (speed reducing) torque of $0.1Nm$ is applied step wise. Simulate the angular velocities as function of time.

After calculating the A, B in equations 12, we mix the C and D matrices from equations 18 and 16 to then be able to get all the angular velocities as outputs

$$y = \begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{C} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{D} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{18}$$

Further, we can get the system matrix G, that correlates the inputs $U(s)$ to the desired outputs $Y(s)$

$$\frac{Y(s)}{U(s)} = G(s) = C(sI - A)^{-1} B + D \tag{19}$$

After that we simulated the system using the command lsim from Matlab, with the desired inputs.

A plot of the input scenario is shown in figure 2, where the step in voltage v_a occurs at $t = 0$ s and the step in external torque is applied at $t = 0.04$ s. The direct feed through from T_e to ω_3 results in an instantaneous jump in the output at $t = 0.04$ s.

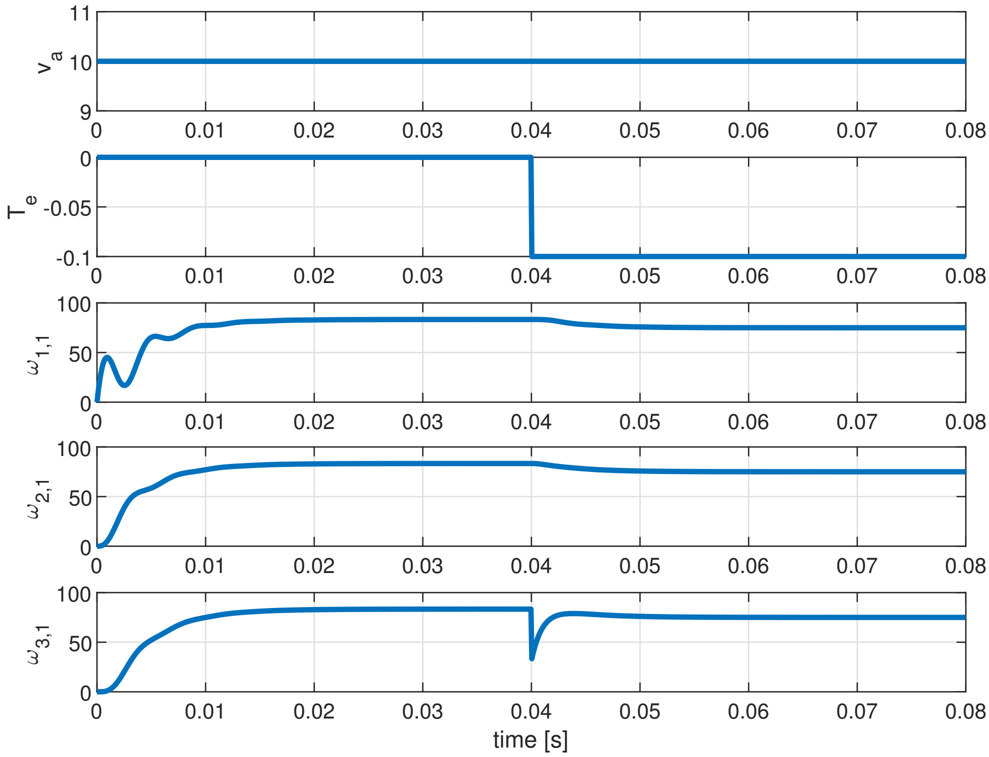

Figure 2: *MIMO response for the DC motor, with inputs v_a and T_e and outputs the angular velocities ω*

Question f)

Assume that the external torque is zero. Give the transfer function from the input, applied rotor voltage, to the output, corresponding to the case (2) in subproblem c). Calculate the poles and transmission zeros in this case. Is the system of minimum phase?

The transfer function from the input v_a to the current i_a and angular velocity ω_3 are found with MATLAB in Listing 5

Listing 5: *Calculating transfer functions*

```
% Defining laplace variable
s = tf('s')
I = eye(size(Av));

% Output matrices
Cv=[0 -0.1 0 0 0;
    0 0 1 0 -1]
```

```
    Dv=[1 0;
        0 500]
    % Calculating the transfer function matrix
    G = Cv*inv(s*I-Av)*Bv + Dv
```

The transfer functions are

$$\begin{bmatrix} i_a \\ \omega_3 \end{bmatrix} = G(s) \cdot v_a \tag{20}$$

where G(s) is obtained as follows:

Listing 6: *Computing of transfer functions*

```
>> zpk(G(:,1))

ans =
  From input to output...
       s (s+955.3) (s+41.75) (s + 1.5 + 1583.4i) (s + 1.5 - 1583.4i)
   1:  -----------------------------------------------
       s (s+946.4) (s+270.8) (s + 391.4 + 1479.1i) (s + 391.4 - 1479.1i)

       5e12 s^2 (s+946.4) (s+270.8) (s + 391.4 + 1479.1i) (s + 391.4 - 1479.1i)
   2:  -------------------------------------------------------
       s^2 (s+946.4)^2 (s+270.8)^2 ((s + 391.4 + 1479.1i) (s + 391.4 - 1479.1i))^2

Continuous-time zero/pole/gain model.
```

For MIMO system, the pole polynomial is the least common denominator of all minors of G(s). In this case the common denominator is:

$$p(s) = s \cdot (s + 946.4) \cdot (s + 270.8) \cdot (s + 391.4 + 1479.1i) \cdot (s + 391.4 - 1479.1i) = 0 \tag{21}$$

which results in the following system poles:

$$\begin{aligned} s_{1,2} &= (-0.3914 \pm j1.4791) \cdot 10^3 \\ s_3 &= -0.9464 \cdot 10^3 \\ s_4 &= -0.2708 \cdot 10^3 \\ s_5 &= 0 \end{aligned} \tag{22}$$

On the other hand, the transmission zeros of the system are obtained by the greatest common factor for the numerators of the maximal minors of G(s), when we normalize all transfer functions to the same denominator. Doing this procedure, it is clear that the common factors are:

$$\begin{aligned} z_{1,2} &= (-0.3914 \pm j1.4791) \cdot 10^3 \\ z_3 &= -0.9464 \cdot 10^3 \\ z_4 &= -0.2708 \cdot 10^3 \\ z_5 &= 0 \end{aligned} \tag{23}$$

which are exactly the same as the system poles. Since the zeros are all in the stability region, i.e. in the real part of the LHP plane (for continuous systems), the system is minimum-phase.