SSY285 - Home Assignment M2

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1 Analysis of linear state-space model of a DC-motor with flywheel

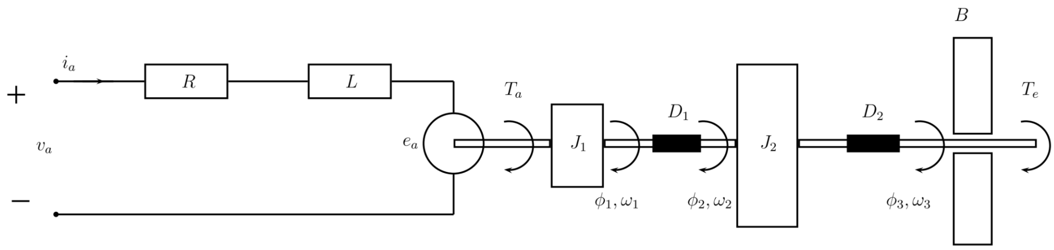

Figure 1: *DC motor with flywheel*

$$\begin{bmatrix} \phi_{1,2} \\ \omega_{1,2} \\ \phi_{2,2} \\ \omega_{2,2} \\ \phi_{3,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{D_1}{J_1} & -\frac{K_e\,K_t}{J_1\,R} & \frac{D_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{D_1}{J_2} & 0 & -\frac{D_1+D_2}{J_2} & 0 & \frac{D_2}{J_2} \\ 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{A} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{K_t}{J_1\,R} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{B} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{1}$$

$$y = \begin{bmatrix} \phi_{1,1} \\ \omega_{2,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{C1} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{2}$$

$$y = \begin{bmatrix} i_a \\ \omega_{3,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{K_e}{R} & 0 & 0 & 0 \\ 0 & 0 & \frac{D_2}{B} & 0 & -\frac{D_2}{B} \end{bmatrix}}_{C2} \begin{bmatrix} \phi_{1,1} \\ \omega_{1,1} \\ \phi_{2,1} \\ \omega_{2,1} \\ \phi_{3,1} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{D} \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{3}$$

a) Investigate if the system is controllable and observable for nonzero, positive and finite parameter values of the motor resistance R. Use the same parameter notation as it was introduced in assignment M1.
(Hint: Use D_1 as a symbolic variable while calculating controllability observibilty matrix and then later substitute $D_1 = 20$ to avoid numerical errors in Matlab.)

$$W_r = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \tag{4}$$

After doing Gauss-Jordan elimination on the controlability matrix obtained by 4 we can easily see the linear dependency between the columns of W_r.

$$W_r = \begin{pmatrix} 1.0 & 0 & 0 & 0 & -2.0\,10^6 & 0 & \frac{2.0\,10^9}{R} & 5.0\,10^9\,R & \frac{1.0\,10^{12}\left(5.0\,R^2-2.0\right)}{R^2} & -5.0\,10^{12}\,R \\ 0 & 1.0 & 0 & 0 & \frac{2.0\,10^5}{R} & 0 & -\frac{2.0\,10^8}{R^2} & -5.0\,10^8 & -\frac{1.0\,10^{11}\left(5.0\,R^2-2.0\right)}{R^3} & 5.0\,10^{11} \\ 0 & 0 & 1.0 & 0 & -\frac{1.0\,10^3}{R} & 0 & -\frac{1.0\,10^6\left(2.0\,R^2-1.0\right)}{R^2} & 0 & \frac{1.0\,10^9\left(4.0\,R^2-1.0\right)}{R^3} & 5.0\,10^9\,R \\ 0 & 0 & 0 & 1.0 & \frac{200.0}{R} & 0 & \frac{2.0\,10^5\,(R-1.0)}{R^2} & -5.5\,10^5 & -\frac{1.0\,10^7\left(51.0\,R^2+20.0\,R-20.0\right)}{R^3} & 5.0\,10^7 \\ 0 & 0 & 0 & 0 & 0 & 1.0 & \frac{200.0}{R} & -1.0\,10^3 & -\frac{2.0\,10^5}{R^2} & 4.5\,10^5 \end{pmatrix} \tag{5}$$

Clearly, rank(W_r)=5 because the 1st-4th and 6th columns are linear independent.

The observability matrix can be found using the following equation:

$$W_o = [C \quad CA \quad CA^2 \quad \cdots \quad CA^{n-1}]^T \tag{6}$$

which rank, again, can be easily determined after doing Gauss-Jordan elimination on the observability matrix obtained by 6. We then get the following matrices for C_1 and C_2:

$$W_{o,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad W_{o,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Clearly, rank($W_{o,1}$)=5 and rank($W_{o,2}$)=4 and this does not depend on the value of R. We then conclude that the system is observable for C_1, but not observable for C_2, i.e. we can not observe all the states if we decide to measure only the current i_a and the angular velocity $w_{3,1}$.

b) If the system is not controllable or observable, is it still stabilizable or detectable for nonzero, positive and finite parameter values of the motor resistance R?

Since the system is controllable, we already know it is stabilizable. We also know that the system is observable for C_1 and therefore it is also detectable for this case.

However, the system using matrix C_2 is not observable and the detectability must be checked. We can first check if all the eigenvalues are in the LHP to check the stability of the system.

$$eig(Av) = \begin{cases} 0 \\ f_1(R) \\ f_2(R) \\ f_3(R) \\ f_4(R) \end{cases} \tag{8}$$

As noticed, one eigenvalue is zero, which means that one of the modes is not asymptotically stable. Further, we can apply the PBH test, to check if the correspondent mode is observable. The test looks like follows:

$$\begin{aligned} &\text{if} \quad fullrank\left(\begin{bmatrix} (I\lambda_i - A) \\ C \end{bmatrix}\right) \\ &\text{then} \quad \text{mode related to } \lambda_i \text{ is observable} \end{aligned} \tag{9}$$

Replacing $\lambda = 0$ in equation 9 results in a rank deficient matrix, since one of the singular value is equal zero. Therefore we found a mode that is not asymptotically stable ($\lambda = 0$) and not observable (fails PBH test), so we conclude system is not detectable for (A, C_2).

c) Given the numerical parameter values from assignment M1, repeat the above tasks by using of standard Matlab routines. Check the condition number of the controllability and observability matrices, before you conclude the analytic properties.

Given the numerical values for all parameters, we are able to calculate the controlability and observability matrices for the system using Matlab built-in functions.

```
W_r  = ctrb(Av,Bv);
W_o1 = obsv(Av,C1v);
W_o2 = obsv(Av,C2v);
```

Further, we are able to calculate the rank of the matrices to determine if the system is controllable and observable for the given matrices. However, the rank function requires a tolerance, which often can be very hard to specify, since it should be different for each system. The easiest way to check if the system is controllable or observable is to analyze the condition number. First we calculate the singular values of each matrix:

```
svd(W_r)
svd(W_o1)
svd(W_o2)
```

$$\text{svd}(W_r) = \begin{bmatrix} 100.5028e+015 \\ 7.7114e+015 \\ 284.9232e+009 \\ 1.0391e+009 \\ 98.5416e+000 \end{bmatrix} \quad \text{svd}(W_o1) = \begin{bmatrix} 1.4356e+015 \\ 5.1743e+009 \\ 52.8173e+006 \\ 187.2966e+000 \\ 468.4528e-003 \end{bmatrix} \quad \text{svd}(W_o2) = \begin{bmatrix} 2.2114e+015 \\ 468.9661e+012 \\ 946.7997e+006 \\ 7.6206e+006 \\ 165.9027e-003 \end{bmatrix} \tag{10}$$

The condition number is then defined as follows:

$$\kappa(W) = \frac{max(svd(W))}{min(svd(W))} \qquad \begin{cases} \kappa(W_r) = \dfrac{100.502 \cdot 10^{15}}{98.5416} = 1.0199 \cdot 10^{15} \\ \kappa(W_o1) = \dfrac{1.4356 \cdot 10^{15}}{0.468} = 3.0646 \cdot 10^{15} \\ \kappa(W_o2) = \dfrac{2.2114 \cdot 10^{15}}{0.165902} = 13.3295 \cdot 10^{15} \end{cases} \tag{11}$$

Notably, we got pretty high conditional numbers for the controllability and for both observability matrices. That means that there is one mode that is very hard to control and one mode that is hard to observe, for both cases using C_1 and C_2 matrices.

However, the system can still be stabilizable and detectable. We then check the eigenvalues of the A matrix:

```
eig(Av)
ans =
  -391.3884e+000 +   1.4791e+003i
  -391.3884e+000 -   1.4791e+003i
  -610.9748e-015 +   0.0000e+000i
  -270.8347e+000 +   0.0000e+000i
  -946.3886e+000 +   0.0000e+000i
```

As seen, one of the eigenvalues can be considered zero considering the computer machine precision. So we have to apply again the PBH test to check if it is stabilizable and detectable. We then do the same procedure as done in item b) and replace $\lambda = 0$ in equation 9 for (A, C_1) and (A, C_2). It follows that we get full rank for the first case and a rank deficient matrix for C_2, so (A, C_2) leads to an undetectable system.

Similarly, we apply the PHB test for controllability:

$$\begin{aligned} \text{if} \quad & fullrank\left([(I\lambda_i - A) \;\; B]\right) \\ \text{then} \quad & \text{mode related to } \lambda_i \text{ is controllable} \end{aligned} \tag{12}$$

And we get a full rank matrix, meaning that the only mode that is not stable is controllable. In this way, we conclude that there is no uncontrollable mode that is unstable, so the system is stabilizable for (A, B).

d) Choose the value $T_s = 1ms$ as the sampling interval and calculate the discrete time system matrix $A_d = e^{A_c T_s}$. (Here A_c denotes the corresponding continuous time parameter matrix.)

```
Ts=0.001;
Ad=expm(Av*Ts));
```

$$A_d = \begin{pmatrix} 0.4 & 4.5\,10^{-4} & 0.6 & 2.3\,10^{-4} & 2.6\,10^{-3} \\ -799.0 & -0.049 & 788.0 & 0.6 & 9.1 \\ 0.21 & 5.9\,10^{-5} & 0.77 & 9.2\,10^{-4} & 0.018 \\ 344.0 & 0.15 & -388.0 & 0.77 & 28.0 \\ 0.059 & 1.3\,10^{-5} & 0.57 & 3.5\,10^{-4} & 0.37 \end{pmatrix} \tag{13}$$

e) Using the same sampling interval as above, calculate the discrete time input matrix with ZOH principle, given by the expression $B_d = \int_0^{T_s} e^{A_c t} B_c \, dt$. B_d is found by first finding the matrix exponential in the following way

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} \tag{14}$$

where $\mathcal{L}$ is the laplace operator. Then simply integrating over the time step T_s and multiplying by the input matrix B

```
syms s t
matrix = inv(eye(size(Av))*s-Av)
exp_At = vpa(ilaplace(matrix))
Bd = int(exp_At,t,0,1e-3)*Bv
```

$$B_d = \begin{pmatrix} 3.2\,10^{-3} & 2.9\,10^{-4} \\ 4.5 & 1.3 \\ 1.6\,10^{-4} & 3.2\,10^{-3} \\ 0.59 & 8.8 \\ 2.9\,10^{-5} & 0.32 \end{pmatrix} \tag{15}$$

f) Check (numerically) if the resulting discrete time state-space model is minimal order and if its eigenvalues are in the region of stability!

If a system is minimal order, it is both controllable and observable. By calculating the controllability matrix and observability matrix we can inspect the ranks of these matrices and determine if the discrete system is minimal order.

```
% SVD of controllability matrix
svd(ctrb(Ad,Bd))

ans =
   35.8544
    9.1418
    0.1928
    0.0532
    0.0038

% SVD of observability matrix case 1
svd(obsv(Ad,C1v))

ans =
  598.7408
   34.5908
    1.2908
    1.0153
    0.0970

% SVD of observability matrix case 2
svd(obsv(Ad,C2v))

ans =
 1.0e+03 *
```

```
        1.4726
        0.4030
        0.0004
        0.0002
        0.0000
```

By inspection, case (A_d, B_d, C_1) is minimal order from an analytical point of view since both controllability and observability matrices are full rank (svd's greater than zero) and the condition numbers are not so large.

On the other hand, case (A_d, B_d, C_2) is not observable since one singular value is zero and therefore the system is not minimal order.

Analysis of stability:

```
>> abs(eig(Ad))

ans =
    0.6761
    0.6761
    1.0000
    0.7627
    0.3881
```

Since all eigenvalues are within the unit circle, the system is stable