Topics: Normal distribution, Functions of Random Variables

1. The time required for servicing transmissions is normally distributed with μ = 45 minutes and σ = 8 minutes. The service manager plans to have work begin on the transmission of a customer's car 10 minutes after the car is dropped off and the customer is told that the car will be ready within 1 hour from drop-off. What is the probability that the service manager cannot meet his commitment?

   A. 0.3875
   B. 0.2676
   C. 0.5
   D. 0.6987

Ans: Let X be the time required for servicing the transmissions

Mean (μ) = 45 minutes

Standard Deviation (σ) = 8 minutes

The work begins 10 minutes after the car is dropped off, (60 - 10 = 50).

We need to find P(X > 50):

$Z = (X - \mu) / \sigma$ $Z = (50 - 45) / 8$ $Z = 5 / 8$ $Z = 0.625$

Using a standard normal distribution table P(Z) = 0.2676 =P(X>50)

The correct answer is option **B. 0.2676**

2. The current age (in years) of 400 clerical employees at an insurance claims processing center is normally distributed with mean μ = 38 and Standard deviation σ=6. For each statement below, please specify True/False. If false, briefly explain why.
   A. More employees at the processing center are older than 44 than between 38 and 44.

Ans: Let X be the age of a clerical employee

Mean (μ) = 38

Standard Deviation (σ) = 6.

$Z = (X - \mu) / \sigma$

$Z1 = (38 - 38) / 6 = 0$

$Z2 = (44 - 38) / 6 = 1$

$Z3 = (56 - 38) / 6 = 3$

Using a standard normal distribution table P(X>44) = P(Z3)-P(Z2)

= 0.4987-0.3413=0.1574

Using a standard normal distribution table P(38<X<44) = P(Z2)-P(Z1)

= 0.3413-0=0.3413

Statement A is False based on P(X>44)<P(38<X<44)

B. A training program for employees under the age of 30 at the center would be expected to attract about 36 employees.

Ans: Let X be the age of a clerical employee

Mean (μ) = 38

Standard Deviation (σ) = 6.

Z = (X - μ) / σ

Z1 = (30-38)/6 = -1.33

Using a standard normal distribution table P(Z1) = 0.0912

Number of employees under the age of 30 = 0.0912*400 = 36.48

Statement B is TRUE.

3. If $X_1 \sim N(\mu, \sigma^2)$ and $X_2 \sim N(\mu, \sigma^2)$ are *iid* normal random variables, then what is the difference between 2 X_1 and $X_1 + X_2$? Discuss both their distributions and parameters.

Ans: 2X1: If X1 follows a normal distribution with mean μ and variance σ^2, then 2X1 would also follow a normal distribution. When we multiply a random variable by a constant (in this case, 2), it affects the mean and the variance.

The mean of 2X1 would be: E(2X1) = 2 * E(X1) = 2 * μ

The variance of 2X1 would be: Var(2X1) = (2^2) * Var(X1) = 4 * σ^2

So, the distribution of 2X1 is also normal, with mean 2μ and variance 4σ^2.

X1 + X2: If X1 and X2 are independent and identically distributed (iid) normal random variables, then their sum (X1 + X2) would also follow a normal distribution. When we add two independent random variables, the mean of the resulting distribution would be the sum of the individual means, and the variance of the resulting distribution would be the sum of the individual variances.

The mean of X1 + X2 would be: E(X1 + X2) = E(X1) + E(X2) = μ + μ = 2μ

The variance of X1 + X2 would be: Var(X1 + X2) = Var(X1) + Var(X2) = σ^2 + σ^2 = 2σ^2

So, the distribution of X1 + X2 is also normal, with mean 2μ and variance 2σ^2.

In summary:

- 2X1 follows a normal distribution with mean 2μ and variance 4σ^2.
- X1 + X2 follows a normal distribution with mean 2μ and variance 2σ^2.

Both distributions have the same mean but different variances. The variance of 2X1 is larger than the variance of X1 + X2. This is because when we multiply a random variable by a constant, the variance increases by the square of that constant. On the other hand, when we add two independent random variables, the variance of the sum is simply the sum of their individual variances.

4. Let X ~ $N(100, 20^2)$. Find two values, *a* and *b*, symmetric about the mean, such that the probability of the random variable taking a value between them is 0.99.

   A. 90.5, 105.9
   B. 80.2, 119.8
   C. 22, 78
   D. 48.5, 151.5
   E. 90.1, 109.9

Ans: To find the values a and b, we can use the standard normal distribution (Z-score) and then convert back to the original normal distribution using the mean and standard deviation.
For 0.005 probability (lower tail), Z1 = -2.576
For 0.995 probability (upper tail), Z2 = 2.576
Convert the Z-scores back to the original values using the mean (μ = 100) and standard deviation (σ = 20):
$a = \mu + Z1 * \sigma = 100 + (-2.576) * 20 \approx 48.48$
$b = \mu + Z2 * \sigma = 100 + 2.576 * 20 \approx 151.52$
Therefore, the two values, a and b, symmetric about the mean, such that the probability of the random variable taking a value between them is 0.99, are approximately 48.5 and 151.5.
The correct answer is option **D. 48.5, 151.5**.

5. Consider a company that has two different divisions. The annual profits from the two divisions are independent and have distributions $Profit_1$ ~ $N(5, 3^2)$ and $Profit_2$ ~ $N(7, 4^2)$ respectively. Both the profits are in $ Million. Answer the following questions about the total profit of the company in Rupees. Assume that $1 = Rs. 45

   A. Specify a Rupee range (centered on the mean) such that it contains 95% probability for the annual profit of the company.

Ans: The Rupee range (centered on the mean) with 95% probability for the annual profit is approximately **-Rs. 4.39M to Rs. 28.39M**.

   B. Specify the 5^{th} percentile of profit (in Rupees) for the company

Ans: The 5th percentile of profit for the company is approximately -**Rs. 0.0855M.**

   C. Which of the two divisions has a larger probability of making a loss in a given year?

Ans: **Division 2** has a higher probability of making a loss in a given year.