Two-Dimensional Geometric Transformations:

With the procedures for displaying output primitives and their attributes, we can create variety of pictures and graphs. In many applications, there is also a need for altering or manipulating displays. Design applications and facility layouts are created by arranging the orientations and sizes of the component parts of the scene. And animations are produced by moving the "camera" or the objects in a scene along animation paths. Changes in orientation, size, and shape are accomplished with geometric transformations that alter the coordinate descriptions of objects.

The basic geometric transformations are translation, rotation, and scaling. Other transformations that are often applied to objects include reflection and shear. We first discuss methods for performing geometric transformations and then consider how transformation functions can be incorporated into graphics packages.

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BASIC TRANSFORMATIONS:

Translation:

A translation is applied to an object by repositioning it along a straight-line path from one coordinate location to another. We translate a two-dimensional point by adding translation distances, t_x and t_y , to the original coordinate position (x, y) to move the point to a new position (x ' , y') (Fig. 5-1).

$$x' = x + t_x, \quad y' = y + t_y \qquad \text{........} \quad (5.1)$$

The translation distance pair (t_x, t_y) is called a **translation vector or shift vector.** We can express the translation equations 5.1 as a single matrix equation by using column vectors to represent coordinate positions and the translation vector:

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$$\mathbf{P} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{P}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{5-2}$$

This allows us to write the two-dimensional translation equations in the matrix form:

$$\mathbf{P}' = \mathbf{P} + \mathbf{T} \tag{5-3}$$

Sometimes matrix-transformation equations are expressed in terms of coordinate row vectors instead of column vectors. In this case, we would write the matrix representations as P = [x y] and T = [t_x t_y] Since the column-vector representation for a point is standard mathematical notation, and since many graphics packages, for example, GKS and PHIGS, also use the column-vector representation, we will follow this convention.

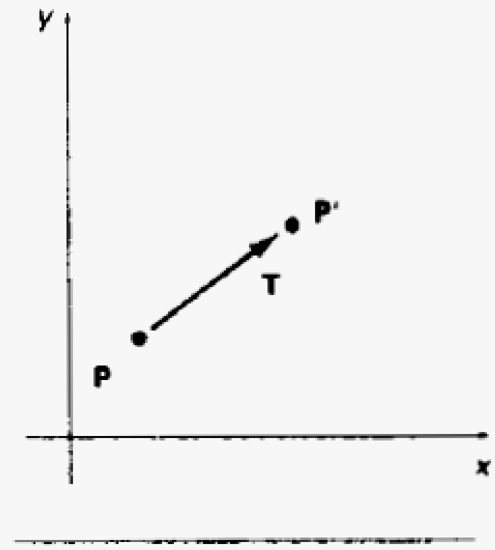

Figure 5-1
Translating a point from position **P** to position **P'** with translation vector **T**.

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Translation is a rigid-body transformation that moves objects without deformation. That is, every point on the object is translated by the same amount. A straight Line segment is translated by applying the transformation equation 5-3 to each of the line endpoints and redrawing the line between the new endpoint positions. Polygons are translated by adding the translation vector to the coordinate position of each vertex and regenerating the polygon using the new set of vertex coordinates and the current attribute settings.
Figure 5-2 illustrates the application of a specified translation vector to move an object from one position to another.

Similar methods are used to translate curved objects. To change the position of a circle or ellipse, we translate the center coordinates and redraw the figure in the new location. We translate other curves (for example, splines) by displacing the coordinate positions defining the objects, then we reconstruct the curve paths using the translated coordinate points.

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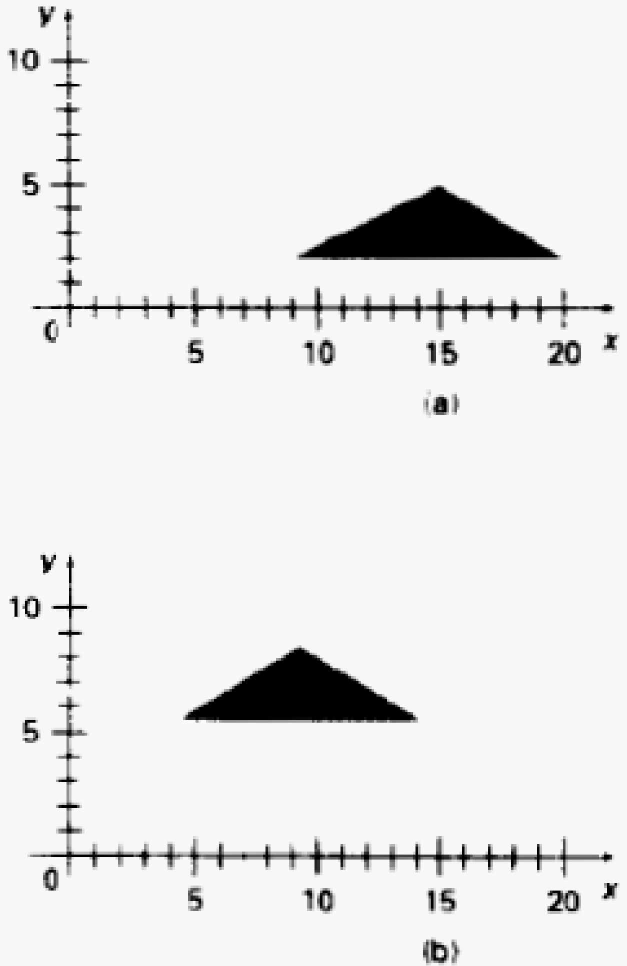

Figure 5-2
Moving a polygon from position (a) to position (b) with the translation vector (-5.50, 3.75).

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Rotation:

A two-dimensional rotation is applied to an object by repositioning it along a circular path in the xy plane. To generate a rotation, we specify a rotation angle θ and the position (x_r, y_r) of the **rotation point (or pivot point)** about which the object is to be rotated (Fig. 5-3). Positive values for the rotation angle define counterclockwise rotations about the pivot point, as in Fig. 5-3, and negative values rotate objects in the clockwise direction. This transformation can also be described as a rotation about a rotation axis that is perpendicular to the xy plane and passes through the pivot point.

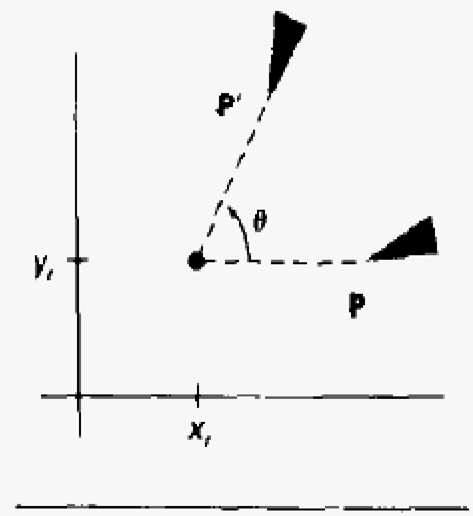

Figure 5-3
Rotation of an object through angle θ about the pivot point (x_r, y_r).

Two-Dimensional Geometric Transformations:

We first determine the transformation equations for rotation of a point position P when the pivot point is at the coordinate origin. The angular and coordinate relationships of the original and transformed point positions are shown in Fig. 5-4. In this figure, r is the constant distance of the point from the origin, angle Φ is the original angular position of the point from the horizontal, and θ is the rotation angle. Using standard trigonometric identities, we can express the transformed coordinates in terms of angles θ and Φ as

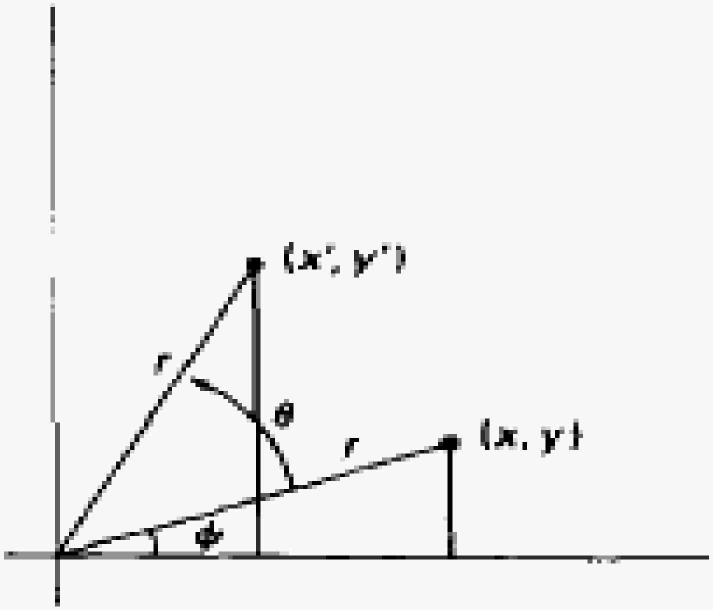

Figure 5-4
Rotation of a point from position (x, y) to position (x', y') through an angle θ relative to the coordinate origin. The original angular displacement of the point from the x axis is ϕ.

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$$x' = r\cos(\phi + \theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$y' = r\sin(\phi + \theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \tag{5-4}$$

The original coordinates of the point in polar coordinates are

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{5-5}$$

Substituting expressions 5-5 into 5-4, we obtain the transformation equations for rotating a point at position (x, y) through an angle θ about the origin:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta \tag{5-6}$$

With the column-vector representations 5-2 for coordinate positions, we can write the rotation equations in the matrix form:

$$\mathbf{P}' = \mathbf{R} \cdot \mathbf{P} \tag{5-7}$$

where the rotation matrix is

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5-8}$$

When coordinate positions are represented as row vectors instead of column vectors, the matrix product in rotation equation 5-7 is transposed so that the transformed row coordinate vector $[x' \; y']$ is calculated as

$$\mathbf{P}'^T = (\mathbf{R} \cdot \mathbf{P})^T$$
$$= \mathbf{P}^T \cdot \mathbf{R}^T$$

where $\mathbf{P}^T = [x \; y]$, and the transpose $\mathbf{R}^T$ of matrix $\mathbf{R}$ is obtained by interchanging rows and columns. For a rotation matrix, the transpose is obtained by simply changing the sign of the sine terms.

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Rotation of a point about an arbitrary pivot position is illustrated in Fig. 5-5. Using the trigonometric relationships in this figure, we can generalize Eqs. 5-6 to obtain the transformation equations for rotation of a point about any specified rotation position (x_r, y_r):

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta \qquad (5\text{-}9)$$

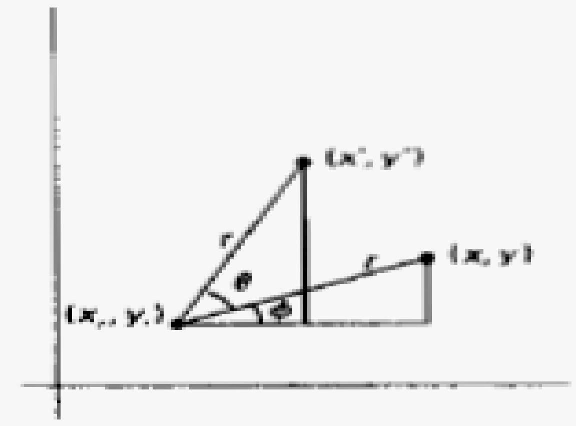

Figure 5-5
Rotating a point from position (x, y) to position (x', y') through an angle θ about rotation point (x_r, y_r).

These general rotation equations differ from Eqs. 5-6 by the inclusion of additive terms, as well as the multiplicative factors on the coordinate values. Thus, the matrix expression 5-7 could be modified to include pivot coordinates by matrix addition of a column vector whose elements contain the additive (translational) terms in Eqs. 5-9. There are better ways, however, to formulate such matrix equations, and we discuss in Section 5-2 a more consistent scheme for representing the transformation equations.

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Scaling

A **scaling** transformation alters the size of an object. This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by **scaling factors** s_x and s_y to produce the transformed coordinates (x', y'):

$$x' = x \cdot s_x, \qquad y' = y \cdot s_y \tag{5-10}$$

Scaling factor s_x scales objects in the x direction, while s_y scales in the y direction. The transformation equations 5-10 can also be written in the matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{5-11}$$

or

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P} \tag{5-12}$$

where **S** is the 2 by 2 scaling matrix in Eq. 5-11.

Any positive numeric values can be assigned to the scaling factors s_x and s_y. Values less than 1 reduce the size of objects; values greater than 1 produce an enlargement. Specifying a value of 1 for both s_x and s_y leaves the size of objects unchanged. When s_x and s_y are assigned the same value, a **uniform scaling** is pro-

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produced that maintains relative object proportions. Unequal values for s_x and s_y result in a differential scaling that is often used in design applications, where pictures are constructed from a few basic shapes that can be adjusted by scaling and positioning transformations (Fig. 5-6).

Objects transformed with Eq. 5-11 are both scaled and repositioned. Scaling factors with values less than 1 move objects closer to the coordinate origin, while values greater than 1 move coordinate positions farther from the origin. Figure 5-7 illustrates scaling a line by assigning the value 0.5 to both s_x and s_y in Eq. 5-11. Both the line length and the distance from the origin are reduced by a factor of 1 /2.

Figure 5-6
Turning a square (a) into a rectangle (b) with scaling factors $s_x = 2$ and $s_y = 1$.

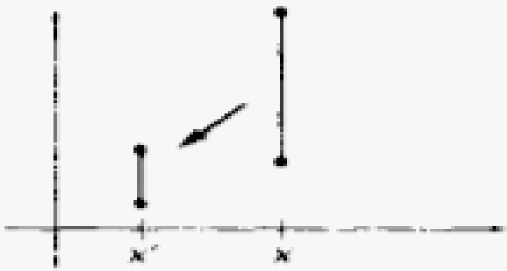

Figure 5-7
A line scaled with Eq 5-12 using $s_x = s_y = 0.5$ is reduced in size and moved closer to the coordinate origin.

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We can control the location of a scaled object by choosing a position, called the fixed point, that is to remain unchanged after the scaling transformation. Coordinates for the fixed point (x_f, y_f) can be chosen as one of the vertices, the object centroid, or any other position (Fig. 5-8). A polygon is then scaled relative to the fixed point by scaling the distance from each vertex to the fixed point. For a vertex with coordinates (x, y), the scaled coordinates (x ' , y ') are calculated as

$$x' = x_f + (x - x_f)s_x, \qquad y' = y_f + (y - y_f)s_y \tag{5-13}$$

We can rewrite these scaling transformations to separate the multiplicative and additive terms:

$$\begin{aligned} x' &= x \cdot s_x + x_f(1 - s_x) \\ y' &= y \cdot s_y + y_f(1 - s_y) \end{aligned} \tag{5-14}$$

where the additive terms $x_f(1 - s_x)$ and $y_f(1 - s_y)$ are constant for all points in the object.

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Including coordinates for a fixed point in the scaling equations is similar to including coordinates for a pivot point in the rotation equations. We can set up a column vector whose elements are the constant terms in Eqs. 5-14, then we add this column vector to the product S * P in Eq. 5-12.

Polygons are scaled by applying transformations 5.14 to each vertex and then regenerating the polygon using the transformed vertices. Other objects are scaled by applying the scaling transformation equations to the parameters defining the objects. An ellipse in standard position is resized by scaling the semimajor and semiminor axes and redrawing the ellipse about the designated center coordinates.

Uniform scaling of a circle is done by simply adjusting the radius. Then we redisplay the circle about the center coordinates using the transformed radius.

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Reflection :

A reflection is a transformation that produces a mirror image of an object. The mirror image for a two-dimensional reflection is generated relative to an axis of reflection by rotating the object 180º about the reflection axis. We can choose an axis of reflection in the xy plane or perpendicular to the xy plane. When the reflection axis is a line in the xy plane, the rotation path about this axis is in a plane perpendicular to the xy plane. For reflection axes that are perpendicular to the xy plane, the rotation path is in the xy plane. Following are examples of some common reflections.

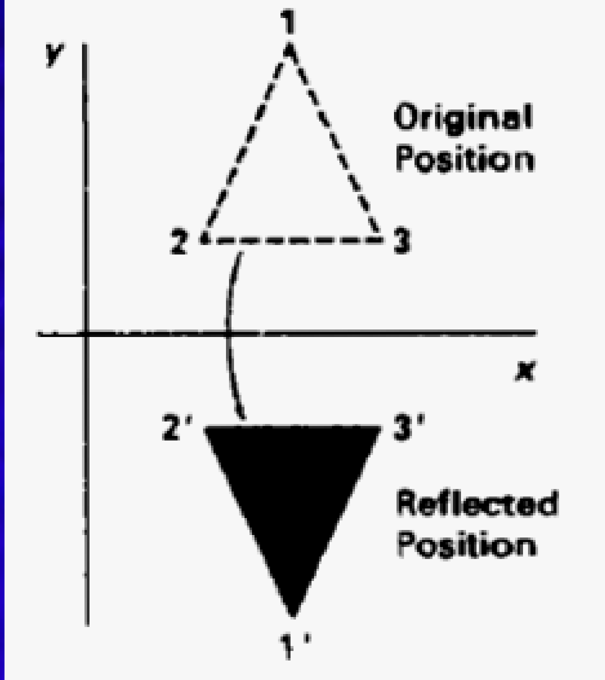

Figure 5-16
Reflection of an object about the *x* axis.

Two-Dimensional Geometric Transformations:

Reflection about the line $y = 0$, the x axis, is accomplished with the transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-48}$$

This transformation keeps x values the same, but "flips" the y values of coordinate positions. The resulting orientation of an object after it has been reflected about the x axis is shown in Fig. 5-16. To envision the rotation transformation path for this reflection, we can think of the flat object moving out of the xy plane and rotating 180° through three-dimensional space about the x axis and back into the xy plane on the other side of the x axis.

A reflection about the y axis flips x coordinates while keeping y coordinates the same. The matrix for this transformation is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-49}$$

Two-Dimensional Geometric Transformations:

Figure 5-17 illustrates the change in position of an object that has been reflected about the line $x = 0$. The equivalent rotation in this case is 180° through three-dimensional space about the y axis.

We flip both the x and y coordinates of a point by reflecting relative to an axis that is perpendicular to the xy plane and that passes through the coordinate origin. This transformation, referred to as a reflection relative to the coordinate origin, has the matrix representation:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5\text{-}50)$$

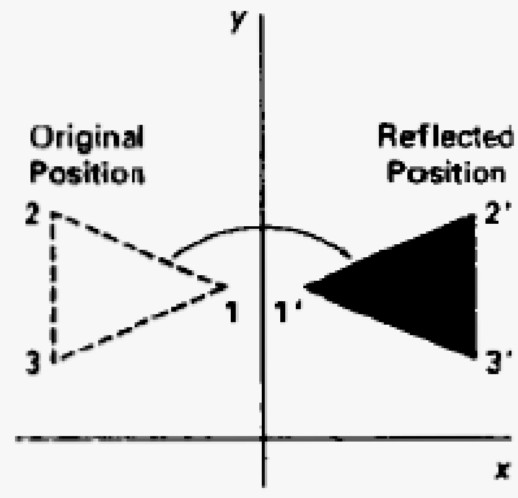

Figure 5-17
Reflection of an object about the y axis.