Chapter 04 : Fourier Series.

By adding infinite sin or cos wares we can make other functions. We use fourier series to represent a periodic signal as an infinite sum of sin (or cos) waves.

4.1) Determination of fourier coefficient

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(nx\frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \left(\sin nx\frac{\pi}{L}\right)$$

- Full series of sin and cos with a name of all coefficients, where;
- $f(x)$ is the function that we want (square ware)
- L is the half of the period of the function
- a_0, a_n, b_n are the coefficients that we need to calculate.

$n = 1, 2, \dots$

$a_1 \cos\left(\frac{n\pi}{L}\right) \quad a_2 = \cos\left(\frac{2n\pi}{L}\right) \dots$

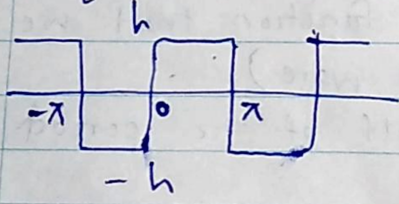
a_0, a_n, b_n

1) $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

2) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{nx\pi}{L}\right)dx$

3) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{nx\pi}{L}\right)dx$

Ex: Square Wave



$2L = 2\pi$

$L = \pi$

i) $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_0 = 0$ (Because net area is positive (0 to π) and negative ($-\pi$ to 0) same value)

2) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

($-\pi$ to 0) and (0 to π)

ii) [illegible]

$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (-h)\cos(nx)\,dx$

~~$a_1 = \frac{-h}{\pi}\int \cos x\,dx$~~ [illegible]

(iii) [illegible]

[illegible]

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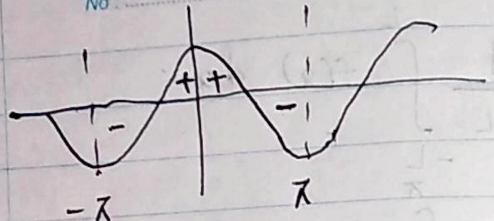
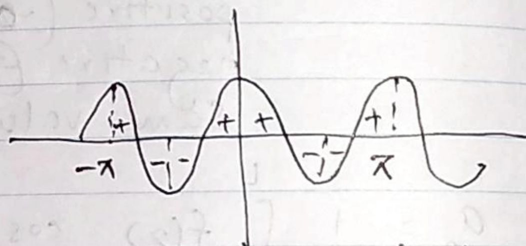


$\therefore \; a_1 = 0$

When $n = 2$

$a_2 = -\frac{h}{\pi} \cos 2x \, dx$

$a_2 = 0$



$\therefore \; a_n = 0$ (we can extend this idea to every value of n)

3) $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n x \pi}{L}\right) dx$

When $n = 1$

$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(x) \, dx$

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$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \underbrace{f(x)}_{-h} \sin(x) \, dx + \int_{0}^{\pi} \underbrace{f(x)}_{h} \sin(x) \, dx \right]$

$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -h \sin x \, dx + \int_{0}^{\pi} h \sin x \, dx \right]$

$\Rightarrow \int_{-\pi}^{0} \sin x \, dx = \left[-\cos x \right]_{-\pi}^{0}$

$= -\cos 0 - (-\cos(-\pi))$

$= -1 - (1)$

$= -2$

$\frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin x \, dx = -\frac{h}{\pi}(-2) = \frac{2h}{\pi}$

$\Rightarrow \int_{0}^{\pi} \sin x \, dx = (-\cos x)_{0}^{\pi}$

$= -\cos \pi - (-\cos 0)$

$= -(-1) + (+1) = 1 + 1$

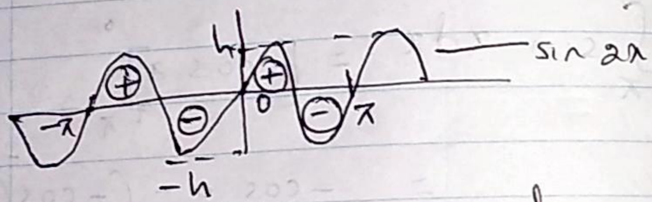
$= 2$

$\frac{1}{\pi} \int_{0}^{\pi} f(x) \sin x \, dx = \frac{h}{\pi} \times 2 = 2h/\pi$

$$\therefore b_1 = \frac{2h}{\pi} + \frac{2h}{\pi} = \frac{4h}{\pi}$$

<u>n = 2</u>

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin(2x)\ dx.$$



$$b_2 = \int_{-\pi}^{0} (-h) \sin 2x\ dx$$

$$\left.\begin{aligned} &\text{(i)} \quad = -h \int_{-\pi}^{0} \sin 2x\ dx \\ &\text{(ii)} \quad = h \int_{0}^{\pi} \sin 2x\ dx \end{aligned}\right\} 0$$

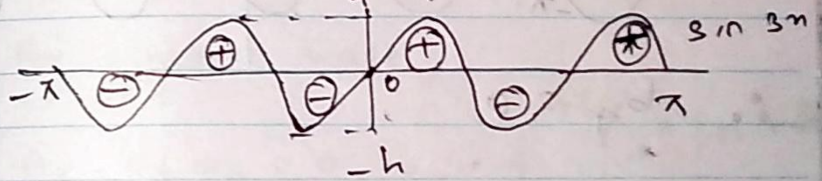
$$\therefore b_2 = 0$$

<u>n = 3</u>

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin 3x\ dx.$$

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin 3x\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin 3x\ dx$$

$$= -\frac{h}{\pi}\int_{-\pi}^{0} \sin 3x\ dx + \frac{h}{\pi}\int_{0}^{\pi} \sin 3x\ dx$$



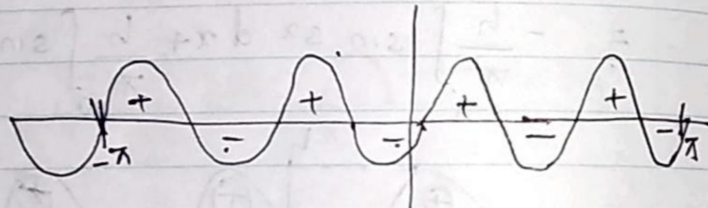
$$b_3 = \frac{b_1}{3} = \frac{4h}{3\pi}$$

Two areas cancelled and one is remaining
So it is like b_1 integral, but with only one - third of the area.

n = 4

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(4x)\,dx$$

$$= -\frac{h}{\pi}\int_{-\pi}^{0}\sin 4x\,dx + \frac{h}{\pi}\int_{0}^{\pi}\sin 4x\,dx$$

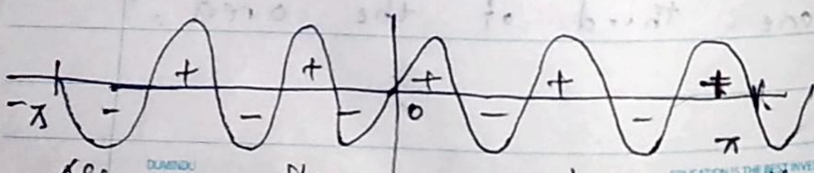


$\therefore\ b_4 = 0$

n = 5

$$b_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(5x)\,dx$$

$$= -\frac{h}{\pi}\int_{-\pi}^{0}\sin 5x\,dx + \frac{h}{\pi}\int_{0}^{\pi}\sin 5x\,dx$$



After cancelling one part remains

$\therefore\ b_5 = b_1/5$

For even

$$b_n = 0$$

For odd

$$b_n = \frac{b_1}{n} = \frac{4h}{n\pi}$$

(all except one are cancel, for a result of $1/n$)

For square wave,

$$a_0 = 0,\ a_n = 0,\ b_n = \begin{cases} 4h/n\pi; & n \text{ odd} \\ 0; & n \text{ even} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= 0 + \sum 0 \cos(\,) + \sum_{n=1}^{\infty}\left(\frac{4h}{n\pi}\right)\sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \cancel{\frac{4h}{\pi}}\ \frac{4h}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\left(\frac{\sin nx}{n}\right)$$

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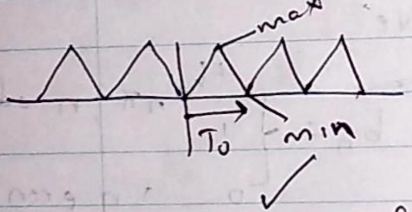
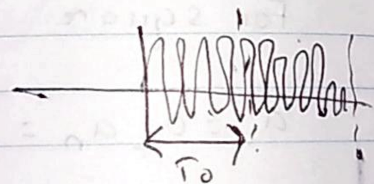
$$= \frac{4h}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin 5x}{5} + \dots\right]$$

Conditions for existance of fourier series (Dirichlet conditions)

(1) function should have finitie number of maxima and minima over the range of time period.



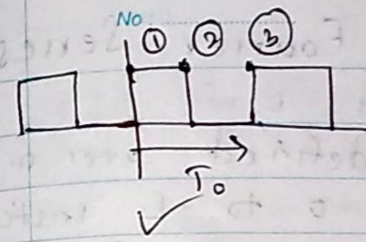
Finite max and min.



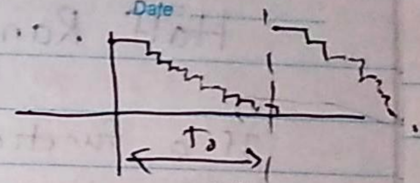
X

Infinite max and min

(2) Function should have finite number of discontinuities over the range of the time period.



Finite number of discontinuities.



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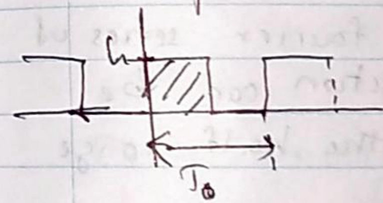
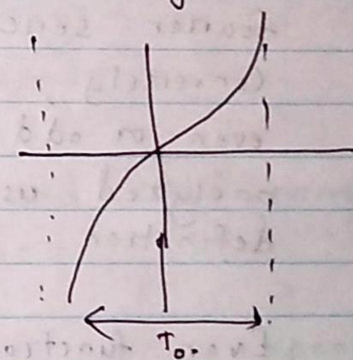
Infinite number of discontinuities.

(3) Function should be absolutely integrable over the range of the time period.



$$\int_0^{T_0} f(x)\,dx = h \times T_0/2$$



Area – ∞

Not absolutely integrable over the time period.

Half Range Fourier Series.

If a function defined over a half range say 0 to L instead of the full range $-L$ to $+L$, it may be expanded in a series of sine terms only or cosine terms only. The series produce is then called a half range fourier series.

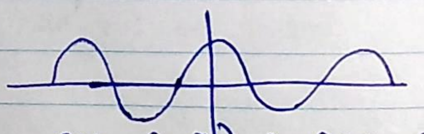
Conversely, the fourier series of even or odd function can be analyzed using the half range definition.

Even functions

Symmetric about the y-axis.

$$f(-x) = f(x)$$

Eg :- cos function.



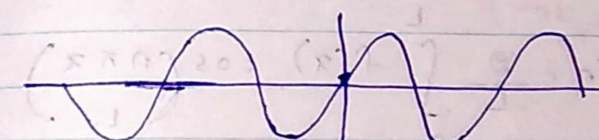
$\cos(-x) = \cos x$

Odd functions

Symmetric about the origin

$$f(-x) = -f(x)$$

Eg :- sin function.



$\sin(-x) = -\sin x$

Even functions and half range cosine Series.

An even function can be expanded using half it's range from 0 to L or $-L$ to 0 or L to 2L. That is the range of integration is L. The fourier series of the half range even function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

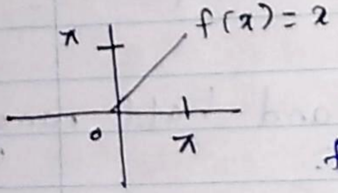


figure 1

- In figure 1 & $f(x) = x$ is sketched from $x = 0$ to $x = \pi$

- An even function means that it must be symmetrical about the y-axis / f(x) axis; and this is shown in the following figure 2. by dotted lines.

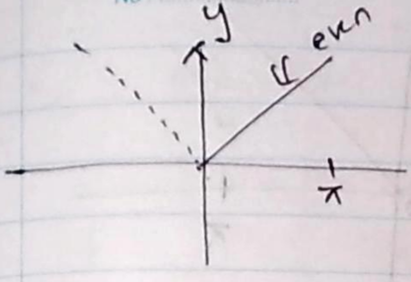
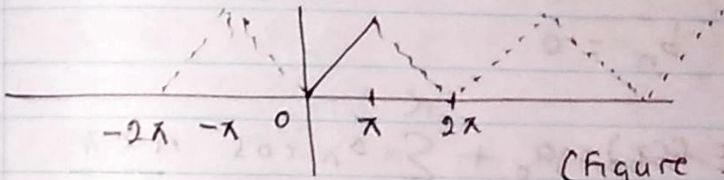


figure 2.

- Assume that the triangular wave-formed produce is periodic with period 2π. (figure 3)



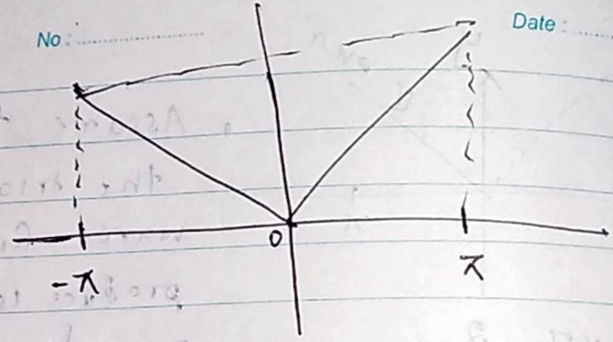
(figure 3)

Ex:- $f(x) = \begin{cases} -x \;;\; \text{if } -\pi \leq x < 0 \\ x \;;\; \text{if } 0 \leq x < \pi \end{cases}$

& $f(x)$ is periodic with period 2π

a) Sketch the function.

b) Find the fourier series using half range series.

a) 

(b) Since the function is even

$b_n = 0$

$$f(x) = a_0 + \sum_{n-1}^{\alpha} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$\boxed{\begin{matrix} 2L = 2\pi \\ L = \pi \end{matrix}}$$

$$= \frac{1}{\pi}\int_0^\pi x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^\pi$$

$$= \frac{1}{\cancel{\pi}}\frac{\cancel{\pi}^2}{2}$$

$$= \pi/2$$

$u = x$ $\quad dv = \cos nx$

$\frac{du}{dv} = 1$ $\quad v = \frac{\sin nx}{n}$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx$$

$$\int_0^\pi x\cos(nx)\,dx = x\frac{\sin nx}{n} - \int_0^\pi \frac{\sin nx}{n}\,dx$$

$$= x\frac{\sin nx}{n} - \frac{(-\cos nx)}{n^2}$$

$$= \frac{nx\sin(nx) + \cos(nx)}{n^2}$$

$$= \frac{1}{n^2}\left[\cos(nx) + nx\sin(nx)\right]$$

$$a_n = \frac{2}{\pi}\frac{1}{n^2}\left[\cos(nx) + nx\sin(nx)\right]_0^\pi$$

$$= \frac{2}{\pi n^2}\left[(\cos n\pi + 0) - (\cos 0 + 0)\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

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n even,

$$= \frac{2}{n^2\pi}\left[1-1\right] = 0$$

n odd

$$= \frac{2}{n^2\pi}\left[-1-1\right] = \frac{-4}{n^2\pi}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)$$

$$= \frac{\pi}{2} + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} -\frac{4}{n^2\pi} \cos\left(\frac{n\pi x}{L}\right)$$

$$\cancel{= \pi/2 - \left(\frac{4}{\pi}\right) \sum_{n=1}^{\infty} \cos\left(\frac{(2n-1)\pi x}{L}\right)}$$

$$= \pi/2 - (4/\pi) \sum_{n=1}^{\infty} \left[\cos\left(\frac{(2n-1)\pi x}{L}\right) \frac{1}{(2n-1)^2}\right]$$

$$\boxed{\cos n\pi = (-1)^n.}$$

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(02) Expand $f(x) = x$, $0 < x < 2$ in a half-range fourier series

(a) sine series
(b) cosine series.

(a) Sine Series

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^2 x\sin\left(\frac{n x\pi}{2}\right)dx.$$

$2 = L$
$L = 2-0$
$= 2.$

$$= \cancel{\left[\frac{\sin nx - xn\cos(nx)}{n^2}\right]_0^2}$$

$$= \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2.$$

$$= -\frac{4}{n\pi}(-1)^n.$$

$$f(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n \sin\left(\frac{n\pi x}{2}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{4}{\pi} \sum_{n=1}^{n} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right)$$

(b) Cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$a_0 = \frac{2}{2} \int_0^2 x \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 = \left[\frac{4}{2} - 0\right] = 2.$$

$$a_n = \frac{2}{2} \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$$

$$= \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \cancel{\frac{4}{n\pi}}\left(0 + \frac{4}{(n\pi)^2}(-1)^n\right)$$

$$= \frac{4}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cancel{dx}$$

$$= \frac{2}{2} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$= 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right)$$

Odd functions and half range sine series.

Odd function can be expanded using half it's range from 0 to l. That is the range of integration has value l.

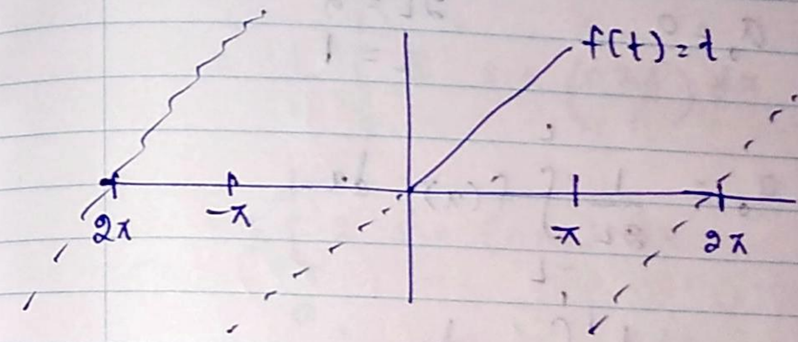
The fourier series of an odd function.

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$



It is then assumed that the ware form produced is periodic of period 2π, outside of this range shown by the dotted lines.

Ex:- Compute the fourier series of

$$f(x) = \begin{cases} 3, & 0 < x < 1 \\ -3, & -1 < x < 0 \end{cases} \quad \& \quad f(x) = f(x+2)$$

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$f(x) = f(x+2)$

$a_0 = 0$ $2L = 2$
                 $L = 1$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} 3\, dx$$

$$= \frac{1}{2}\left[3x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[3-3\right]$$

$$= 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right) dx$$

$$= 0$$

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$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{1}\int_{-1}^{1} 3\sin\left(\frac{n\pi x}{1}\right) dx$$

$$= 2\int_{0}^{1} 3\sin(n\pi x)\, dx$$

$$= 6\int_{0}^{1} \sin(n\pi x)\, dx \text{ [half range]}$$

$$= 6\left(\frac{-\cos n\pi x}{n\pi}\right)_0^1$$

$$= \frac{-6}{n\pi}\left(\cos n\pi - \cos 0\right.$$

$$= \frac{-6}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{-6}{n\pi}\left[(-1)^n - 1\right]$$

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$$b_n = -\frac{6}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & ; \; n \text{ even} \\ \frac{12}{n\pi} & ; \; n \text{ odd} \end{cases}$$

$$f(x) = \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{12}{n\pi} \sin(n\pi x) \; ; \; n = \text{odd}$$