1) Q-3 pg 59

$\Theta(g(n)) = \{ f(n) : \exists\, c_1, c_2$ s.t. $f(n) \ge c_2\, g(n)$ and $f(n) \le c_1\, g(n)$ $\forall\, n \ge n_0 \}$

n_0 ← const. from $f(n)$ expression → max → n^{20}

③ $f(n)$:

ⓐ $(n^2+10)^{10} \rightarrow 1n^{20} + K_1 n^{19} + K_2 n^{18} + \dots + K_{19} n + K_{20}$

$\Rightarrow n_0 = 10$

$\Rightarrow g(n) = n^{20}$ $\quad c_1 = 2,\; c_2 = 0.5$

$\Theta(g(n)) = \Theta(n^{20})$

$i = n-2-0$

$j = n-1-i$ * $\quad i = n-2$

$j = +1, 2, \; n-1$

$\Rightarrow j \left\{ \sum_{i=0}^{n-1} (n-1-i) \right\} \cdot i$

Proof: the func. $g(n) = n^{20}$ grows at a certain rate and so does the degree of $f(n)$. However, the constants $c_1 = 2, c_2 = 0.5$ respectively allows $\Theta(g(n))$ to bound the maximum order of growth of $f(n)$

mathematically: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \left(\frac{n^2+1}{n^2}\right)^{10} = \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^{10}$

$= 1$ ($\because n \to \infty$, $\frac{1}{n^2} \to 0$)

$=$ const (>0)

Since, it yields a constant (>0) the order of growth is same for $f(n)$ & $g(n)$, $\Theta(g(n)) = \Theta(n^{20})$

ⓑ $f(n) = \sqrt{10n^2 + 7n + 3}$ → max term is $10n^2$ so $\sqrt{10n^2} = \sqrt{10}\, n$

So, $g(n) = n$ and $\Theta(g(n)) = \Theta(n)$ since if $n \ge 3$ and $c_1 = 3$ and $c_2 = 1$ then $\Theta(g(n))$ bounds $f(n)$

mathematically: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty} \left(\frac{10n^2+7n+3}{n^2}\right)^{1/2}$

$= \lim_{n\to\infty} \left(10 + \frac{7}{n} + \frac{3}{n^2}\right)^{1/2} = \sqrt{10} > 0$

$[\because n \to \infty$ so, $\frac{1}{n^2}, \frac{1}{n} \to 0]$

since, it yields a positive non zero constant, $\sqrt{10}$ so the functions $f(n)$ & $g(n)$ grow at same rate. $\Theta(g(n)) = \Theta(n)$

ⓒ $f(n) = 2n \lg(n+2)^2 + (n+2)^2 \lg \frac{n}{2}$

$= 2n \cdot 2 \lg(n+2) + (n+2)^2 \lg n - \lg 2$

$= (n+2)^2 \lg n + 2 \cdot 2n \lg(n+2) - \lg 2$

→ max term: $(n+2)^2 \lg n$

So $g(n) = n^2 \lg n$ and $\Theta(g(n)) = \Theta(n^2 \lg(n))$

since, for $n \ge \lg 2$ the constants, $c_1 = 2$ and $c_2 = 0.5$ $\Theta(g(n))$ bounds $f(n)$

mathematically: $\lim_{n\to\infty} \frac{(n+2)^2 \lg n + 4n \lg(n+2) - \lg 2}{n^2 \lg(n)}$

$= \lim_{n\to\infty} \left[\frac{(n+2)^2 \cancel{\lg n}}{n^2 \cancel{\lg n}}\right] + \lim_{n\to\infty} \left[\frac{4\cancel{n} \lg(n+2)}{\cancel{n^2}\, n \lg(n)}\right] - \lim_{n\to\infty} \left[\frac{\lg 2}{n^2 \lg(n)}\right]$

NOTE.

$\frac{d}{dx} \ln[f(x)] = \frac{1}{f(x)} f'(x)$

✓ $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln a}$

✓ $\lg = \log_2$

✓ $\log_b M^P = P \log_b(M)$

✓ $\log_b(MN) = \log_b M + \log_b N$

✓ $\log_b (M/N) = \log_b M - \log_b N$

$= \lim_{n\to\infty} \left[\frac{(n+2)^2}{n^2}\right] + \lim_{n\to\infty}\left[\frac{4\lg(n+2)}{n\lg(n)}\right] - \lim_{n\to\infty}\left[\frac{1}{n^2\lg(n)}\right]$

NOTE $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$

$= \lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^2 + 4\lim_{n\to\infty}\left[\frac{\lg(n+2)}{n\lg(n)}\right] - \lim_{n\to\infty}\left[\frac{n^{-2}}{\lg(n)}\right]\right.$

L'Hopital

$= 1 + 4\lim_{n\to\infty}\frac{\frac{1}{(n+2)\ln 2}}{\lg(n) + \frac{\cancel{n}}{\cancel{n}\ln 2}} - \lim_{n\to\infty}\frac{\lg(n)(-2)n^{-3} - n^{-2}\cdot\frac{1}{n\ln 2}}{(\lg(n))^2}$

$= 1 + 4\lim_{n\to\infty}\frac{1}{(\ln(2)\lg(n)+1)(n+2)} - \lim_{n\to\infty}\frac{\frac{-2\lg(n)}{n^3} - \frac{1}{n^3\ln(2)}}{(\lg(n))^2}$

$= 1 + 4\lim_{n\to\infty}\frac{1}{\ln(2)(n+2)\lg(n)+(n+2)} - \lim_{n\to\infty}\frac{-2\ln(2)}{n^3\ln(2)(\lg(n))} - \frac{1}{n^3\ln(2)(\lg(n))^2}$

$0 \qquad 0 \qquad 0 \qquad = 1$

so, $f(n)$ & $g(n)$ has same growth rate (since, const. > 0) $= 1 > 0$.

$\theta(g(n)) = \theta(n^2 \lg n)$

(d) $f(n) = 2^{n+1} + 3^{n-1} = 2\cdot 2^n + \frac{3^n}{3}$ → max term: 3^n so, $g(n) = 3^n$

so, $\theta(g(n)) = \theta(3^n)$ for $n \geq 0$, where $c_1 = 1$, $c_2 = \frac{1}{5}$ $(< \frac{1}{3})$ to bound $f(n)$

$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{2^{n+1}+3^{n-1}}{3^n} = 2\lim_{n\to\infty}\left(\frac{2}{3}\right)^n + \frac{1}{3}\lim_{n\to\infty} 1$

$= 2\cdot\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots 0\right] + \frac{1}{3}$

binomial exp.

$= 2\cdot 3 + \frac{1}{3} = \left(6 + \frac{1}{3}\right) > 0$

so, the constant shows $g(n) = 3^n$ and $f(n)$ grow at same rate. $\theta(g(n)) = \theta(3^n)$

(e) $f(n) = \lfloor\log_2 n\rfloor$ $\quad g(n) = \log_2 n$ and $\theta(g(n)) = \theta(\log_2 n)$ since for $n \geq 2$ for $c_1 = 2$ and $c_2 = \frac{1}{2}$ the $\theta(g(n))$ bounds $f(n)$

mathematically: $\lim_{n\to\infty}\frac{f(n)}{g(n)}$

$= \lim_{n\to\infty}\frac{\lfloor\log_2 n\rfloor}{\log_2 n}$

$= \lim_{n\to\infty}\frac{K_n \cancel{\log_2 n}}{\cancel{\log_2 n}}$ where $K_n \leq 1$ [$\because \lfloor\ \rfloor$, floor func. reduces a value]

$= K_n > 0$

so, the constant proves that $\theta(g(n))$ bounds $f(n)$, $\theta(g(n)) = \theta(\log_2 n)$

2

Q-5 Pg 60

Given functions: $f_1(n) = (n-2)!$, $\longrightarrow \theta(g_1(n)) = \theta(n!)$ G

$f_2(n) = 5\lg(n+100)^{10} = 50\lg(n+100)$, $\longrightarrow \theta(g_2(n)) = \theta(\lg n)$ A

$f_3(n) = 2^{2n}$, $\longrightarrow \theta(g_3(n)) = \theta(4^n)$ F

$f_4(n) = 0.001n^4 + 3n^3 + 1$, $\longrightarrow \theta(g_4(n)) = \theta(n^4)$ D

$f_5(n) = \ln^2 n = (\ln n)^2$, $\longrightarrow \theta(g_5(n)) = \theta(\ln^2 n)$ B

$f_6(n) = \sqrt[3]{n}$, $\longrightarrow \theta(g_6(n)) = \theta(\sqrt[3]{n})$ C

$f_7(n) = 3^n$ $\longrightarrow \theta(g_7(n)) = \theta(3^n)$ E

NOTE:
$\theta(\log n)$
$< \theta(n^\alpha)$
$< \theta(a^n)$
$< \theta(n!)$
$< \theta(n^n)$

Order of growth sorted in ascending order: $\boxed{Ans}$

$5\lg(n+100)^{10}$
$\ln^2 n$
$\sqrt[3]{n}$
$0.001n^4 + 3n^3 + 1$
3^n
2^{2n}
$(n-2)!$

(Use Appendix A)

Q-1 Pg 67 (3)

(a) $1 + 3 + 5 + 7 + \dots + 999$

$a = 1$ $\quad$ $sum = \frac{n}{2}[2a + (n-1)d]$

$d = 2$ $\quad$ $= \frac{500}{2}[2 + (500-1)2]$

$n = 500$

$(t_n = a + (n-1)d$

$\frac{t_n - a}{d} + 1 = n)$

$= \boxed{250000}$

(b) $\underbrace{2+4}_{\times 2} + 8 + 16 + \dots + 1024$

$a = 2$

$r = 2$ $\quad$ $Sum = a\frac{1-r^n}{1-r}$

$a_n = a_1 \cdot r^{n-1}$

$1024 = 2 \cdot 2^{n-1}$ $\quad$ $= 2 \times \frac{1 - 2^{10}}{1-2}$

$1024 = 2^n$ $\quad$ $= \boxed{2046}$

$n = \log_2 1024 = 10$

$\sum_{i=1}^{5} i$

(c) $\sum_{i=3}^{n+1} 1 = \sum_{i=1}^{n+1} 1 - \sum_{i=1}^{2} 1$

$= n + 1 - 2$

$= \boxed{n-1}$

(d) $\sum_{i=3}^{n+1} i = \sum_{i=1}^{n+1} i - \sum_{i=1}^{2} i$ $\quad$ (sum formula)

$= (1 + 2 + \dots + n+1) - (1+2)$

$= \frac{(n+1)(n+2)}{2} - 3$

$= \frac{n^2 + 3n + 2 - 6}{2} = \boxed{\frac{n^2 + 3n - 4}{2}}$

$n(n-1)\left[\frac{2n+2}{6}\right]$

$= \frac{n(n^2-1)}{3}$

(e) $\sum_{i=0}^{n-1} i(i+1)$

$= \sum_{i=1}^{n-1} i(i+1) = \sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i$ $\quad$ (sum of i^2)

$= \frac{(n-1)(n-1+1)(2n-2+1)}{6} + \frac{(n-1)(n)}{2}$

$= \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2}$

$= \boxed{\frac{n(n^2-1)}{3}}$

(f) $\sum_{j=1}^{n} 3^{j+1} = 3\sum_{j=1}^{n} 3^j = 3\left(\sum_{j=0}^{n} 3^j - 3^0\right)$

$= 3 \cdot \sum_{j=0}^{n} 3^j - 3 = \frac{3^{n+1} - 1}{3-1} \times 3 - 3$

$= \frac{3^{n+2} - 3 - 6}{2}$

$= \boxed{\frac{3^{n+2}}{2} - 9}$

(h) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{i+1-i}{i(i+1)}$

$= \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i+1}$ $\quad$ [Partial fraction]

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 + \frac{1}{n+1}$ (since a diff. of two sum series so only first & last terms exist middle terms are cancelled out)

$= \boxed{\frac{n}{n+1}}$

(g) $\sum_{i=1}^{n} \sum_{j=1}^{n} ij = \sum_{i=1}^{n} i \cdot \sum_{j=1}^{n} j$

$= \sum_{j=1}^{n} j + 2\sum_{j=1}^{n} j + \dots + n\sum_{j=1}^{n} j$ $\quad$ (sum formula)

$\sum_{j=1}^{n} j\,(1 + 2 + \dots + n) = (1+2+\dots+n)(1+2+\dots+n)$

$= \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2} = \boxed{\frac{n^2(n+1)^2}{4}}$

4

A-2 pg 67

a) $\sum_{i=0}^{n-1}(i^2+1)^2 = \sum_{i=0}^{n-1}(i^4+2i^2+1) = \sum_{i=0}^{n-1} i^4 + 2\sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} 1$

$= \frac{1}{5}(n-1)^5 + 2\cdot\frac{1}{3}(n-1)^3 + n \in \underbrace{\Theta(n^5)}_{\text{max term}} + \Theta(n^3) + \Theta(n) = \boxed{\Theta(n^5)}$

b) $\sum_{i=2}^{n-1} \lg i^2 = 2\sum_{i=2}^{n-1} \lg i = 2\sum_{i=1}^{n-1} \lg i - 2 \underbrace{\lg 1}_{0}$

$= 2 \times (n-1)\lg(n-1) = 2[n\lg(n-1) - \lg(n-1)]$

$\in \Theta(n\lg n) - \Theta(\lg n)$

$\therefore \Theta(g(n)) = \boxed{\Theta(n\lg(n))}$

c) $\sum_{i=1}^{n}(i+1)2^{i-1} = \frac{1}{2}\sum_{i=1}^{n} i\cdot 2^i + \frac{1}{2}\sum_{i=0}^{n} 2^i - \frac{1}{2}\cdot 2^0$

$= \frac{1}{2}[(n-1)2^{n+1}+2] + \frac{1}{2}[2^{n+1} - 1 - 1]$

$= (n-1)\cdot 2^n + 1 + 2^n - 1 \quad \in \Theta(n2^n) + \Theta(2^n)$

$\Theta(g(n)) = \boxed{\Theta(n2^n)}$

d) $\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}(i+j) = \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}(i+j)$

$= \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i-1} i + \sum_{j=0}^{i-1} j\right]$

$= \sum_{i=1}^{n-1}\left[i^2 + \frac{i(i-1)}{2}\right] \quad \left[\because \sum_{j=0}^{i-1} i = i\sum_{j=0}^{i-1} 1 = i\cdot i = i^2;\ \sum_{j=0}^{i-1} j = \frac{(i-1)(i)}{2}\right]$

$= \frac{3}{2}\sum_{i=1}^{n-1} i^2 - \frac{1}{2}\sum_{i=1}^{n-1} i$

sum of sq. sum formula

$= \frac{\cancel{3}}{2}\,\frac{(n)(n+1)(2n+1)}{\cancel{6}\,2} - \frac{1}{2}\,\frac{(n)(n+1)}{2}$

$= \frac{2n^3 + \cancel{n^2} + 2n^2 + \cancel{n} - \cancel{n^2} - \cancel{n}}{4} = \frac{n^3+n^2}{2}$

$\in \Theta(n^3) + \Theta(n^2)$

$\Theta(g(n)) = \boxed{\Theta(n^3)}$

Note: $\sum_{i=1}^{n} x_i^2 = n-1$ adds, n multiplies

5
Q-3 Pg 67

Avg $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ → $n-1$ additions, 1 division

Formula 1: $\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$ → n subtr., $n-1$ additions, n multiplications, ~~1 subtr~~, 1 division

Here,
Additions: $(n-1)+(n-1)$
Subtr. : $n+1$
Add & sub : $3n-1$
multiply : n
Division : 2

add/sub: $3n-1$
multiply: n
divide: 2

Formula 2: $\frac{\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 / n}{n-1}$

$n-1$ additions, n mult. ; $n-1$ additions, 1 mult, 1 div. ; 1 sub2, 1 div, 1 sub

Here,
Addition: $(n-1)+(n-1)$
Subtraction: $1+1=2$
Add & sub: $2n$
multiplication: $n+1$
Division: 2

add/sub: $2n$
multiply: $n+1$
divide: 2

6 Algorithm X(A[0..n-1])

```
// Input : A contains n real numbers
for i ← 0 to n-2 do
    for j ← i+1 to n-1 do
        if A[j] > A[i]
            swap A[i] and A[j]
```

6.① The algorithm sorts an array in descending order using Brute Force Method (Bubble sort)

6.② Input size is n

6.③ Basic Operation in inner loop is Comparison since it happens in every iteration

~~Ascending order of test~~

6.④ Worst case: $\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1)-(i+1)+1 = \sum_{i=0}^{n-2} n-i-1 = (n-1)\sum_{i=0}^{n-2} 1 - \sum_{i=0}^{n-2} i$

$= (n-1)(n-1) - \frac{(n-2)(n-1)}{2}$

$= \frac{(n-1)(2n-2-n+2)}{2}$

$= \frac{n(n-1)}{2}$

$= \frac{1}{2}n^2 - \frac{n}{2} \in \Theta(n^2)$
$-\Theta(n)$

6.⑤ Efficiency class $\Theta(n^2)$ [max term in: $\Theta(n^2) - \Theta(n)$ is $\Theta(n^2)$]

7

(1) $A(n) = 3A(n-1)$

$A(1) = 4$

method 1: $x(n) = a\,x(n-1) + f(n)$

$x(n) = a\,x(n/b) + f(n)$

$A(n) = 3A(n-1)$; $A(n-1) = 3A(n-2) = 3^2 A(n-3)$

$= 3^3 A(n-3)$ $\quad A(n-2) = 3A(n-3)$

$A(n) = 3^i A(n-i)$

if $i = n-1$ $\quad A(n) = 3^{n-1} A(1)$

$\boxed{= 4 \cdot 3^{n-1}} \in \Theta(3^n)$, Efficiency class: $\boxed{\Theta(3^n)}$

(2) $A(n) = A(n-1) + 5$

$A(1) = 0$

$A(n) = A(n-1) + 5$; $A(n-1) = A(n-2) + 5 = A(n-3) + 5 + 5$

$= A(n-3) + 5$ $\quad A(n-2) = A(n-3) + 5$

$+5$

$+5$

$= A(n-3) + 3 \cdot 5$

$A(n) = A(n-i) + 5i$

$i = n-1$

$A(n) = \overset{0}{A(1)} + 5(n-1)$

$= \boxed{5n - 5}$; Efficiency class: $\boxed{\Theta(n)}$

$\in \Theta(n)$

(3) $A(n) = A(n-1) + n$

$A(0) = 0$

$A(n) = A(n-1) + n$; $A(n-1) = A(n-2) + (n-1) = A(n-3) + (n-2) + (n-1)$

$= A(n-3) + n$ $\quad A(n-2) = A(n-3) + (n-2)$

$+(n-1)$

$+(n-2)$

$\therefore$ Efficiency class: $\Theta(n^2)$

$A(n) = A(n-3) + (n-1) + (n-2)$

$+n$

$A(n) = A(n-i) + n + (n-1) + (n-2)$

$+ \dots + (n-i+1)$

$i = n$ $\quad = A(0) + n + (n-1) + (n-2)$

$+ \dots + 2 + 1$

$= 0 + \dfrac{n(n+1)}{2} = \boxed{\dfrac{n^2+n}{2}}$

$\in \Theta(n^2)$

⑦ (5) $A(n) = 2A(n/2) + n - 1$

$A(1) = 0$

$n = 2^k$ $\quad A(2^k) = 2A(2^{k-1}) + 2^k - 1$

$A(2^{k-1}) = 2A(2^{k-2}) + 2^{k-1} - 1 = 2^2 A(2^{k-3}) + 2 \cdot 2^{k-2} - 2 \cdot 1 + 2^{k-1} - 1$

$\quad = 2^2 A(2^{k-3}) + 2 \cdot 2^{k-2} + 2^{k-1} - 2 \cdot 1 - 1$

$A(2^{k-2}) = 2A(2^{k-3}) + 2^{k-2} - 1$ $\quad = 2^2 A(2^{k-3}) + 2 \cdot 2^{k-1} - 3$

$\therefore A(2^k) = 2^3 A(2^{k-3}) + 2^2 \cdot 2^{k-2} + 2 \cdot 2^{k-1} - 3 \cdot 2 + 2^k - 1$

$\quad = 2^3 A(2^{k-3}) + 3 \cdot 2^k - 3 \cdot 2 - 1$

$A(2^k) = 2^i A(2^{k-i}) + 2^k \cdot i - 2i - 1$

$i = k;\ A(2^k) = 2^k \overset{0}{A(0)} + k \cdot 2^k - 2k - 1$

$\quad = k(2^k - 2) - 1 = (n-2)\log_2 n - 1 \quad [\because n = 2^k]$

$\quad = \boxed{n \log_2 n - 2\log_2 n - 1}$

Efficiency class: $\boxed{\Theta(n \log_2 n)}$

---

④ $A(n) = A(n/5) + 1$ for $n > 1$; $A(1) = 1$ (for $n = 5^k$)

$A(5^k) = A(5^{k-1}) + 1$; $A(5^{k-1}) = A(5^{k-2}) + 1 = A(5^{k-3}) + 1 + 1$

$\quad = A(5^{k-3}) + 1$; $A(5^{k-2}) = A(5^{k-3}) + 1$

$\quad + 1$

$\quad + 1$

$\because n = 5^k$

$k = \log_5 n$

$\therefore A(5^k) = A(5^{k-i}) + i$

$i = k,\ A(5^k) = \overbrace{A(1)}^{1} + k$

$A(5^k) = 1 + k$

$\boxed{A(n) = 1 + \log_5 n}$

$\in \Theta(\log_5 n)$

$\therefore$ Efficiency class: $\Theta(\log_5 n)$

(8) ALGORITHM Y (n):

if n = 1 return 1

else return Y(n-1) + n* n

(a) $n=4 \rightarrow Y(4) = Y(3) + 4^2 = Y(2) + 3^2 + 4^2 = \underset{1}{Y(1)} + 2^2 + 3^2 + 4^2 = 1 + 2^2 + 3^2 + 4^2$

∴ calculates sum of squares

(b) Input size is n

(c) Basic operation is multiplication as it is most expensive

initial condition

(2) $M(1) = 0$ (no multiplication for $n=1$)

$M(n) = M(n-1) + 1$

$M(n-1) = M(n-2) + 1 = M(n-3) + 1 + 1$

$M(n-2) = M(n-3) + 1$

$M(n) = M(n-3) + 1 + 1 + 1$

$= M(n-3) + 3 \cdot 1$

∴ $M(n) = M(n-i) + i$

$i = n-1$, $M(n) = \overset{\to 0}{M(1)} + n-1$

$= \boxed{n-1} \in \Theta(n)$

(e) Efficiency class, $\Theta(n)$

(9) Algorithm Q(n)

if n = 1 return 1

else Q(n-1) + (2 * n) - 1 [odd]

(1) Computes n^2 (square of a number)

(or sum of first n odd numbers)

$Q(1) = 1$

$Q(n) = Q(n-1) + 2n - 1$

$1+3+5+7+9+11+13$ = sum of first n odd numbers = square of n^{th} num

$Q(1) = 1 = 1^2$

$Q(2) = \overset{1}{Q(1)} + 2 \cdot 2 - 1 = 4 = 2^2$

$Q(3) = Q(2) + 2 \cdot 3 - 1 = 9 = 3^2$

$Q(4) = Q(3) + 2 \cdot 4 - 1 = 16 = 4^2$

$Q(5) = Q(4) + 2 \cdot 5 - 1 = 25 = 5^2$

$Q(6) = Q(5) + 2 \cdot 6 - 1 = 36 = 6^2$

$Q(7) = Q(6) + 2 \cdot 7 - 1 = 49 = 7^2$

$Q(8) = Q(7) + 2 \cdot 16 - 1 = 64 = 8^2$

say $Q(n) = n^2$, in that case

$Q(n) = Q(n-1) + 2n - 1$

$= (n-1)^2 + 2n - 1$

$= n^2$ [which holds]

Basic operation: multiplication

(2) $M(1) = 0$ [∵ no multiply for $n = 1$]

$M(n) = M(n-1) + 1$

$M(n-1) = M(n-2) + 1 = M(n-3) + 1 + 1$

$M(n-2) = M(n-3) + 1$

$M(n) = M(n-3) + 1 + 1 + 1 = M(n-3) + 3$

$M(n) = M(n-i) + i$

$i = n-1$: $M = M(1) + n - 1 = \boxed{n-1} \in \Theta(n)$

Basic operation: add/subtraction

(3) $S(1) = 0$ [No add/subtr $n=1$] → $n-1$ → $2n-1$ → $Q(n-1) + 2n - 1$

$S(n) = S(n-1) + 3$

$S(n-1) = S(n-2) + 3 = S(n-2) + 3 + 3$

$S(n-2) = S(n-3) + 3$

$S(n) = S(n-3) + 3 + 3 + 3 = S(n-3) + 3 \cdot 3$

$S(n) = S(n-i) + 3i$

$i = n-1 \Rightarrow S(n) = S(1) + 3(n-1)$

$= \boxed{3(n-1)} \in \Theta(n)$

(10) Algorithm $W(A, l, r, k)$:

if $l > r$ return -1

else

$m \leftarrow \lfloor (l+r)/2 \rfloor$

if $k = A[m]$ return m

else if $k < A[m]$ return $W(A, l, m-1, k)$

else return $W(A, m+1, r, k)$

(1) Binary searches k in Array A recursively

(2) n is the size of sorted array A and $n = r - l$ in initial call to W

(3) Basic operation: division (most exp.) [$\because$ comparisons can be determined cheaply]

$D(1) = 0 \Rightarrow D(2^0) = 0$ [$\because$ no division]

$D(n) = D(n/2) + 1$ [1 most expensive division happens in every step]

$\Rightarrow n = 2^k$

$D(2^k) = D(2^{k-1}) + 1 = D(2^{k-2}) + 2 = D(2^{k-3}) + 3$

$D(2^{k-1}) = D(2^{k-2}) + 1 = D(2^{k-3}) + 2$

$D(2^{k-2}) = D(2^{k-3}) + 1$

$\therefore D(2^k) = D(2^{k-i}) + i \quad [1 \le i \le k]$

for $i = k \rightarrow D(2^k) = D(2^0) + k = 0 + k = \boxed{\log_2 n}$ [$\because n = 2^k$] $\rightarrow$ $\log_2 n$ eventually Non-Decreasing func

$\in \Theta(\log_2 n)$

Efficiency class, $\Theta(\log_2 n)$

so, $f(n)$ is smooth

(4) Smoothness Rule

Proof

$n = a^k$ in that case $(a \neq 2, a > 0)$

$D(a^k) = D\left(\frac{a^k}{2}\right) + 1$

$D(2^{k \log_2 a}) = D(2^{k \log_2 a - 1}) + 1$

$\therefore D(2^{k \log_2 a}) = D(2^{k \log_2 a - i}) + i$ [from prev eq.]

$i = k \log_2 a$

$D(2^{k \log_2 a}) = D(2^0) + k \log_2 a$

$D(2^{k \log_2 a}) = k \log_2 a = \boxed{\log_2 n}$

despite $n \neq 2^k$ the Binary search will still yield results in $\Theta(\log_2 n)$ asymptotic time. Since $\Theta(\log_2 n)$ is optimal

$\because a^k = n$

$\log_a n = k$

$\frac{\log_2 n}{\log_2 a} = k$

$\therefore \log_2 n = k \log_2 a$

$n = 2^{k \log_2 a}$

Answer

As per Smoothness Rule, $D(n) \in \Theta(\log_2 n)$ where $\log_2 n$ is a smooth func.

Since, $D(n) \in \Theta(\log_2 n)$ is true for $n = 2^k$ so, $D(n) \in \Theta(\log_2 n)$ is true for any n values where $k > 0$