Question 5

For the first formula: $D(n) = 2$, $M(n) = n$, $A(n)+S(n) = [(n-1)+(n-1)]+(n+1) = 3n-1$.
For the second formula: $D(n) = 2$, $M(n) = n+1$, $A(n)+S(n) = [(n-1)+(n-1)]+2 = 2n$.

Question 6

1. What does this algorithm compute?

   Answer: It sorts the numbers in a non-increasing order.

2. What is the input size?

   Answer: n

3. What is the basic operation?

   Answer: Comparison.

4. How many times is the basic operation executed?

   Answer:
   $\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2}[(n-1)-(i+1)+1] = \sum_{i=0}^{n-2}[n-i-1] = \sum_{i=0}^{n-2}(n-1) - \sum_{i=0}^{n-2} i = (n-1)^2 - (n-1)(n-2)/2 = n(n-1)/2$

5. What is the efficiency class of this algorithm?

   Answer: $\Theta(n^2)$

Question 7

1. $A(n) = 3A(n-1)$ for $n > 1$, $A(1) = 4$

$$
\begin{aligned}
A(n) &= 3A(n-1) \qquad \text{(Sub. } A(n-1) = 3A(n-2)\text{)} \\
&= 3[3A(n-2)] = 3^2A(n-2) \qquad \text{(Sub. } A(n-2) = 3A(n-3)\text{)} \\
&= 3^2[3A(n-3)] = 3^3A(n-3) \\
&= \ldots \\
&= 3^iA(n-i) \qquad \text{(Sub. } i = n-1\text{)} \\
&= 3^{n-1}A(1) = 4(3^{n-1}) \in \Theta(3^n).
\end{aligned}
$$

2. $A(n) = A(n-1) + 5$ for $n > 1$, $A(1) = 0$

$$
\begin{aligned}
A(n) &= A(n-1) + 5 \qquad \text{(Sub. } A(n-1) = A(n-2) + 5\text{)} \\
&= [A(n-2) + 5] + 5 = A(n-2) + 2 \times 5 \qquad \text{(Sub. } A(n-2) = A(n-3) + 5\text{)} \\
&= [A(n-3) + 5] + 2 \times 5 = A(n-3) + 3 \times 5 \\
&= \ldots \\
&= A(n-i) + 5i \qquad \text{(Sub. } i = n-1\text{)} \\
&= A(1) + 5 \times (n-1) \\
&= 5(n-1) \in \Theta(n)
\end{aligned}
$$

3. $A(n) = A(n-1) + n$ for $n > 0$, $A(0) = 0$

$$\begin{aligned}
A(n) &= A(n-1) + n \qquad \text{(Sub. } A(n-1) = A(n-2) + (n-1)\text{)} \\
&= [A(n-2) + (n-1)] + n \\
&= A(n-2) + (n-1) + n \qquad \text{(Sub. } A(n-2) = A(n-3) + (n-2)\text{)} \\
&= [A(n-3) + (n-2)] + (n-1) + n \\
&= A(n-3) + (n-2) + (n-1) + n \\
&= ... \\
&= A(n-i) + (n-i+1) + (n-i+2) + ... + n \qquad \text{(Sub. } i = n\text{)} \\
&= A(0) + 1 + 2 + ... + n \\
&= n(n+1)/2 \in \Theta(n^2)
\end{aligned}$$

4. $A(n) = A(n/5) + 1$ for $n > 1$, $A(1) = 1$ (solve for $n = 5^k$)

$$\begin{aligned}
A(5^k) &= A(5^{k-1}) + 1 \qquad \text{(Sub. } A(5^{k-1}) = A(5^{k-2}) + 1\text{)} \\
&= [A(5^{k-2}) + 1] + 1 \\
&= A(5^{k-2}) + 2 \qquad \text{(Sub. } A(5^{k-2}) = A(5^{k-3}) + 1\text{)} \\
&= [A(5^{k-3}) + 1] + 2 \\
&= A(5^{k-3}) + 3 \\
&= ... \\
&= A(5^{k-i}) + i \qquad \text{(Sub. } i = k\text{)} \\
&= A(5^{k-k}) + k \qquad \text{(Sub. } k = \log_5 n\text{)} \\
&= 1 + \log_5 n \in \Theta(\log n)
\end{aligned}$$

5. $A(n) = 2A(n/2) + n - 1$ for $n > 1$, $A(1) = 0$ (solve for $n = 2^k$)

$$\begin{aligned}
A(2^k) &= 2A(2^{k-1}) + 2^k - 1 \qquad \text{(Sub. } A(2^{k-1}) = 2A(2^{k-2}) + 2^{k-1} - 1\text{)} \\
&= 2[2A(2^{k-2}) + 2^{k-1} - 1] + 2^k - 1 \\
&= 2^2A(2^{k-2}) + 2(2^k) - 2 - 1 \qquad \text{(Sub. } A(2^{k-2}) = 2A(2^{k-3}) + 2^{k-2} - 1\text{)} \\
&= 2^2[2A(2^{k-3}) + 2^{k-2} - 1] + 2(2^k) - 2 - 1 \\
&= 2^3A(2^{k-3}) + 3(2^k) - 2^2 - 2 - 1 \qquad \text{(Sub. } A(2^{k-3}) = 2A(2^{k-4}) + 2^{k-3} - 1\text{)} \\
&= 2^3[2A(2^{k-4}) + 2^{k-3} - 1] + 3(2^k) - 2^2 - 2 - 1 \\
&= 2^4A(2^{k-4}) + 4(2^k) - 2^3 - 2^2 - 2 - 1 \\
&= ... \\
&= 2^iA(2^{k-i}) + i(2^k) - \Sigma_{j=0}^{i-1}2^j \qquad \text{(Sub. } \Sigma_{j=0}^{i-1}2^j = 2^i - 1\text{)} \\
&= 2^iA(2^{k-i}) + i(2^k) - (2^i - 1) \\
&= 2^iA(2^{k-i}) + i(2^k) - 2^i + 1 \qquad \text{(Sub. } i = k\text{)} \\
&= A(2^{k-k}) + k(2^k) - 2^k + 1 \qquad \text{(Sub. } 2^k = n, \quad k = \log_2 n\text{)} \\
&= n\log_2 n - n + 1 \in \Theta(n \log n)
\end{aligned}$$