DATE

1. Given k_1 and k_2 are 2 Kernel functions. i.e they are symmetric and are $\boxed{\text{Positive semi-definite}}$ in nature

Since they are PSD, they solve the property of

$u^T C u \geq 0$ where $u \in \mathbb{R}^n$ & C is a matrix with

$\boxed{\text{We prove that } u^T E u \geq 0}$ $c_{ij} = K_1(x_i, x_j)\ \forall$ ~~$x_1, \ldots x_n \in S$~~

If $x_1, x_2 \ldots x_n \in S$ (Sample space) $\forall x_1, \ldots x_n \in S$

a) ~~a)~~ To Prove whether

$$K(x,z) = K_1(x,z) + K_2(x,z) \text{ is a valid Kernel}$$

or not.

Proof: wkt $u \in \mathbb{R}^n$

$K_1(x,z)$ = kernel fn $\Rightarrow u^T C u \geq 0$ for $c_{ij} = K_1(x_i, x_j)$

$K_2(x,z) \Rightarrow u^T D u \geq 0$ for $d_{ij} = K_2(x_i, x_j)$

To Prove

$K(x,z) \Rightarrow u^T E u \geq 0$ for $e_{ij} = K(x_i, x_j)$

Proof:

$K(x,z) \Rightarrow u^T E u$

$\Rightarrow u^T (C+D) u$ $[\because$ as $E = D + C$ $(K = K_1 + K_2)]$

$\Rightarrow u^T C u + u^T D u$

we already know that $u^T C u \geq 0$ and $u^T D u \geq 0$ from definition

$\therefore$ $\boxed{K(x,z) = K_1(x,z) + K_2(x,z) \text{ is a valid kernel}}$

b) ~~b)~~ $K(x,z) = K_1(x,z) \cdot K_2(x,z)$ is valid or not

From before

$K(x,z) = u^T E u$

$$= \sum_i \sum_j u_i u_j e_{ij} = \sum_{ij} u_i u_j [c_{ij} \cdot d_{ij}]$$

One property of PSD is that the matrix can be written as

$C = A^T A$ ~~for any matrix~~ [~~as $C = \lambda U$~~ and

$D = B^T B$ [$C = U \lambda U^T$ (Diagonalized) simplified to $A^T A$] similarly to D also]

$c_{ij} = a_i^T a_j = \sum_k a_{ik} a_{jk}$

$d_{ij} = b_i^T a_j = \sum_l b_{il} b_{jl}$

$$= \sum_{kl} \sum_{ij} u_i u_j a_{ik} b_{il} a_{jk} b_{jl}$$

$$= \sum_{kl} \underbrace{\left(\sum_i u_i a_{ik} b_{il} \right) \cdot \left(\sum_j u_j a_{jk} b_{jl} \right)}_{\text{same except for variable } i, j}$$

$$= \sum_{kl} \left(\sum_i u_i a_{ik} b_{il} \right)^2$$

Sum of squared elements is always ≥ 0

$\Rightarrow$

$$\sum_{kl} \left(\sum_i u_i a_{ik} b_{il} \right)^2 \geq 0 \Rightarrow \boxed{K(x,z) = K_1(x,z) \cdot K_2(x,z) \text{ is a valid kernel.}}$$

c) $K(x,z) = h(K_1(x,z))$ where h is a poly function with positive coeff

To Prove this we take the case of Proving $K(x,z) = \alpha . K_1(x,z)$ is a kernel where $\alpha > 0$

$K_1(x,z) = u^T C u \geq 0$

$K(x,z) = u^T E u \Rightarrow u^T (\alpha . C) u$

$= \alpha . u^T C u \geq 0$ as $\alpha > 0$ & $u^T C u \geq 0$

$\therefore$ $\boxed{K(x,z) = \alpha . K_1(x,z) \text{ is a valid kernel}}$

As

$K(x,z) = \alpha K_1(x,z)$, $K_1(x,z) + K_2(x,z)$ & $K_1(x,z) . K(x,z)$ are valid kernels, any combinations of these are also valid kernels (polynomial fn)

$\Rightarrow$ $\boxed{K(x,z) \text{ is a valid kernel}}$

d) $K(x,z) = \exp(K_1(x,z))$

From Taylor series expansion, we know that

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\exp(K_1(x,z)) = 1 + K_1(x,z) + \frac{K_1(x,z) . K_1(x,z)}{2} + \dots$$

This is just infinite series of multiplications & additions

DATE

As we have already proved that

$K_1(x,z) + K_2(x,z)$; $K_1(x,z) \cdot K_2(x,z)$; $\alpha K_1(x,z)$ $[\alpha > 0]$

are all valid kernels

$K(x,z) = \exp(K_1(x,z))$ is also a valid kernel fn.

e) To check whether $k(x,z) = \exp\left(\frac{-\|x-z\|_2^2}{\sigma^2}\right)$ is valid or not

we can expand $\exp\left(\frac{-\|x-z\|_2^2}{\sigma^2}\right)$ as

$$\exp\left(\frac{-\|x-z\|_2^2}{\sigma^2}\right) = \exp\left(\frac{-\|x\|_2^2}{\sigma^2}\right) \cdot \exp\left(\frac{-\|z\|_2^2}{\sigma^2}\right) \cdot \boxed{\exp\left(\frac{x \cdot z}{\sigma^2}\right)}$$

Kernel fn

This is just a kernel fn multiplied by a constant funcn ~~of~~ on its individual variables

$\Rightarrow K(x,z) = \exp\left(\frac{-\|x-z\|_2^2}{\sigma^2}\right)$ is a kernel fn