UMA 011

Numerical Analysis

1) Errors
2) Approximations
3) Initial guess
4) roots Solutions of Zeroes
5) Graphs

Significant digits

1.00 — 3
1.001 → 4
3.007 → 4
3.112 → 4
7.2000 — 5

0.00007 → 1

100 → 3
1000 → 4
1×10^2 → 1
1×10^3 → 1

100.00 → 5
100.000 → 6

Floating point representation

$$425.926 = 0.425926\times10^3$$
$$= 0.0425926\times10^4$$
$$= 0.00425926\times10^5$$

↓

$$\boxed{x = \pm(0.a_1 a_2 a_3 a_4 \cdots a_n a_{n+1} \cdots) \times 10^{e}}$$

Mantissa

base

Floating pt representations

a_i is single digit 0 to 9 $e \to$ exponent

Infinite representation

Normalized representation (spl case of floating)

$m \leq e \leq M$

$$x = \pm(0.a_1 a_2 a_3 \cdots a_n a_{n+1} \cdots) \times 10^{e}$$

$M = 999$

$m = -999$

$a_1 \neq 0$

a_i is single digit 0 to 9

Floating & also Normalized rep

$$\boxed{\begin{array}{l} 0.2034 \\ \quad 0.2034 \times 10^{0} \end{array}}$$

$0.547 \times 10^{992} + 0.532 \times 10^{992}$

$-$ ✓

$\div$ ✓

$\times$ ✗

$0.547 \times 0.532 \times 10^{(992+992)}$

$0.547 \times 0.532 \times 10^{(1984)} \to$ Overflow

$0.547 \times 10^{-992} + 0.532 \times 10^{-992}$

$-$ ✓

$\div$ ✓

$\times$ ✗

$0.547 \times 0.532 \times 10^{-(992+992)}$

$0.547 \times 0.532 \times 10^{-(1984)} \to$ underflow

Chopping

dropping digit

$$x = \pm (0 \cdot a_1 a_2 \cdots \underline{a_n a_{n+1} \cdots}) \times 10^e, \; a_1 \neq 0$$

(1) ignore all digits after a_n

$$fl(x) = (0 \cdot \underline{a_1 a_2 \cdots a_n}) \times 10^e$$

retaining digit

(ii) Rounding

$$fl(x) = \begin{cases} \pm (0 \cdot a_1 a_2 \cdots a_n) \times 10^e & 0 \le a_{n+1} < 5 \\ \pm (0 \cdot a_1 a_2 \cdots a_n) \times 10^e & 5 \le a_{n+1} < 10 \end{cases}$$

→ Dropping digit

0.234785001 $\quad$ 0.234775001 → 0.23478

after 5 if Non zero no is present

then

0.23479

0.2347775 → 0.23478

0.234785 → 0.23478

if even → leave as it is

if odd →

Errors

Absolute → Relative

|excat value − Approx value|

$$\frac{|\text{excat value} - \text{Approx value}|}{|\text{excat value}|}$$

relative error %

$$\left| \frac{E.V - A.V}{E.V} \right| \times 100\%$$

$\left.\begin{array}{l}\text{Rounding error} \\ \text{chopping error}\end{array}\right\}$

less error = best

$\frac{8}{3} = 2.66666\text{------}$

2.666

suppose this is excat

Rounding 2.67

Chopping 2.66

R.E < C.E

* Relative Rounding error ??

Evaluate $x^3 + x^2 + 1$ by using ③ point digit Arithmetic

$x = 1.721$

If not mentioned the we use rounding by default

$x = 1.72$

$x^2 = (1.72)^2 = 2.95(84$

$x^3 = x^2 \times x$ $= 2.96$

$x^3 = 5.091 = 5.09$

$x^3 + x^2$ $\quad 5.09 + 2.96 = 8.05$

$x^3 + x^2 + 1 = 9.05$ Ans

Q) $\dfrac{\sin x + x^3}{x} + 5^x$ using 3 point Arithmetic digit

at $x = 0.723$

$\sin x = 0.0126$

$x^2 = 0.522)72$

$= 0.523$

$x^3 = x^2 \times x$

$= 0.378)1$

$0.012 + 0.378$

$= 0.390$

$\dfrac{0.390}{0.723}$

$= 0.539)41$

$= 0.539$

$5^{0.723} = 3.20151$

$= 3.20)2$

$3.20 + 0.539$

3.739

$= 3.74$

9) $f(x) = \dfrac{1}{\sqrt{x+1} - \sqrt{x}}$ $\qquad f(x) = \dfrac{1}{1 - \sin x}$

$f(x) = \sqrt{x+1} - \sqrt{x}$ $\qquad f(x) = \dfrac{1}{1 - \cos x}$

$$f(x) = \frac{1}{x-1}$$

$$f(x) = \frac{1}{\sqrt{x+1} - \sqrt{x}} \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$\frac{\sqrt{x+1} + \sqrt{x}}{x+1-x} = \sqrt{x+1} + \sqrt{x}$$

9) $f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)$ using 6-digit calc

Propose a modification for large x

$f(x) = 100\left(\sqrt{101} - \sqrt{100}\right)$ $\quad x = 100.000$

$\sqrt{101} = 10.0498125$

$\simeq 10.0499$

$\sqrt{100} = 10.0000$

$\sqrt{101} - \sqrt{100} = 10.0499 - 10.0000 = 0.0499000$

$= 0.499 \times 10^{-1}$

$f(x) = 100\,(0.0499000) = 4.99000 \Rightarrow 0.499 \times 10^1$

Loss of Significance

$$f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right) \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$f(x) = \frac{x}{\sqrt{x+1} + \sqrt{x}} = \frac{100}{\sqrt{101} + \sqrt{100}}$$

$$= \frac{100}{10.0499 + 10.0000} = \frac{100}{20.0499}$$

$f(100) = 4.98756$

$f(100) = 4.99 \times 10^1$

$f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)$

$x = 10^{-3} = 0.001$

$ax^2 + bx + c = 0$

$a, b, c \in R, a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$1.002x^2 + 11.01x + 0.01265 = 0$

$a = 1.002$, $b = 11.01$, $c = 0.01265$

$a = 0.1002 \times 10^1$, $b = 0.1101 \times 10^2$, $c = 0.1265 \times 10^{-1}$

$$\boxed{\begin{aligned} x_1 &= -0.001149075659911 \\ x_2 &= -10.98687487643590 \end{aligned}}$$

Exact Solution upto 14 decimal Places

$1.31x^2 + 2.2773x + 7.234 = 0$

use four digit rounding arithmetic and find solution of given quadratic eqn

$0.23457891 - 0.1111111$

$|x - y|$ loss of significance if $x \approx y$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1.002$
$b = 11.01$
$c = 0.01265$

$b^2 = (11.01)^2 = 121.2$ $\quad 4a = 4(1.002)$
$= 4.008$

$4ac = 4.008 \times 0.01265$

$= 0.05070$

$p^2 - 4ac =$

$= 121.2 - 0.05070$

$= 121.1$

$2a = 2 \times 1.002$

$= 2.004$

$\sqrt{p^2 - 4ac} = 11.00$

$$x = \frac{-11.01 \pm 11.00}{2 \times 1.002} = \frac{-11.01 \pm 11.00}{2.004}$$

loss of Significance

$$\frac{-0.01}{2.004}, \frac{-22.01}{2.004}$$

$x_1 = -0.004990, x_2 = -10.98$

$x_1^* = -0.001149, x_2^* = -10.99$

Relative error in first root $\dfrac{|x_1^* - x_1|}{|x_1^*|}$

$$= \frac{|-0.001149 + 0.004990|}{|-0.001149|}$$

$= 3.34265968$

Relative errOo = 334 %

Relative error in 2nd root $\left|\dfrac{x_2^* - x_2}{x_2^*}\right| = \left|\dfrac{-10.99 + 10.98}{-10.99}\right|$

$= 0.00092\,6127$

% error 0.092% (error)

"aisa kyun ho rha hai"???

loss of Significance ki vajah se hi

ab remedy dekho

In first root $b \simeq \sqrt{b^2 - 4ac}$

$\therefore -b + \sqrt{b^2 - 4ac}$ will lead to loss of Significance.

To avoid it let us rewrite $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

$$x = \frac{b^2 - b^2 + 4ac}{-2a(b + \sqrt{b^2 - 4ac})} = \frac{-2c}{b + \sqrt{b^2 - 4ac}}$$

$$x = \frac{-2c}{b + \sqrt{b^2 - 4ac}} \rightarrow$$

$$\therefore x = \frac{-2 \times 0.01265}{11.01 + 11.00} = -0.001149$$

exact

Consider the equation $x^2 + 62.10x + 1 = 0$

$a = 1.000 \quad b = 62.10 \quad c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-b) \pm \sqrt{b^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - 4a}}{2a}$$

$$x = \frac{b + \sqrt{b^2 - 4ac}}{2a}, \frac{b - \sqrt{b^2 - 4ac}}{2a}$$

$$1.002x^2 - 11.01x + 0.01265 = 0$$

Condition and Stability

$f(x) = \sqrt{x+1} - \sqrt{x}$

$\downarrow$

unstable or

ill-conditioned

$f(x) = \dfrac{1}{\sqrt{x+1} - \sqrt{x}}$

well conditioned

or stable

Condition number :- (IC)

IC ≤ 1 stable

IC > 1 unstable

$f(x) = x^2 + x$ $\quad x = 10.01$

$f(10.01) = a_1$

$f(10.005) = a_2$

$$IC = \frac{\text{relative change in output}}{\text{relative change in Input}}$$

$x \rightarrow$ input $\quad f(x)$ output

$x^* \rightarrow$ input $\quad f(x^*)$ output

$$\text{relative change in output} = \frac{f(x^*) - f(x)}{f(x)}$$

where x is exact output

x^* is change output

relative change in input $= \dfrac{x^* - x}{x}$

$$IC = \frac{\dfrac{f(x^*) - f(x)}{f(x)}}{\dfrac{x^* - x}{x}} = \left|\frac{f(x^*) - f(x)}{x^* - x}\right| \left|\frac{x}{f(x)}\right|$$

Since $x^* \to x$

$$IC = \left|\frac{f'(x) \cdot x}{f(x)}\right| \begin{array}{l} \leq 1 \rightarrow \text{stable} \\ > 1 \text{ unstable or ill condition} \end{array}$$

Q) $\boxed{f(x) = \dfrac{10}{1-x^2}}$ verify wether this fn is stable or unstable

$$f'(x) = -(1-x^2)^{-2}(-2x)10 = \frac{20x}{|1-x^2|^2}$$

$$IC = \left|\frac{20x}{(1-x^2)^2} \cdot \frac{x}{10} (1-x^2)\right| = \frac{2x^2}{(1-x^2)}$$

$$\kappa = \left|\frac{2x^2}{1-x^2}\right| = \frac{2x^2}{|1-x^2|}$$

$\kappa < 1 \quad x \in (-1, 1)$

$\kappa >> 1$

$f(x) = \sqrt{x+1} - \sqrt{x}$ at $x = 12345$ $\quad \left|\frac{f'(x)\, x}{f(x)}\right|$

$x_0 : x = 12345$

$x_1 = x_0 + 1 \,; \quad \kappa(x_0) = \left|\frac{x_0 \cdot 1}{x_0+1}\right|$

$\sqrt{x+1} = x_2 : \sqrt{x_1}$

$$\kappa(x_1) = \left|\frac{x_1 \cdot \frac{1}{2\sqrt{x_1}}}{\sqrt{x_1}}\right| = \frac{1}{2}$$

$\sqrt{x} = x_3 := \sqrt{x_0}$

$$\kappa(x_0) = \frac{1}{2}$$

$x_4 := x_2 - x_3$

$$\kappa(x_3) = \frac{(-1) \cdot x_3}{|x_2 - x_3|}$$

Negative vala for κ

$$= \left|\frac{x_3}{x_2 - x_3}\right|$$

$\simeq 24690.5 >> 1$

unstable

$f(x) = \sqrt{x+1} - \sqrt{x} \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$

$x_0: x = 12345$ stable

$x_1: = x_0 + 1$ stable

$x_3: \sqrt{x_0}$ stable

$x_4 = x_2 + x_3$ $\quad k(x_3) = \left| \frac{x_3 \cdot 1}{x_2 + x_3} \right| < 1$

$x_5: = \frac{1}{x_4}$ $\quad k(x_4) = \left| \frac{x_4 \cdot \left(\frac{-1}{x_4^2}\right)}{\frac{1}{x_4}} \right| = 1$

Algebraic & Trancedental

$x^3 + x + 1 = 0$

$x^5 - x + 7 = 0$

$x^{15} - x^{14} + x + 1 = 0$

$x^3 - x^2 - x = 0$

$\rightarrow$

$\sin x + e^x = 0$

$x^3 + \sin x - 7 = 0$

$\tan^{-1} x + e^x = 0$

ILATE

1. bisection Method
2. Fixed point iteration
3. Newton's method $\rightarrow$ 3A ~~=~~ secent
4. Modified Newton's Method.

Initial Guess

Intermediate value property

if $f(x)$ is a continuous function on $[a,b]$ &

$f(a).f(b) < 0$ then $\exists$ at least one real number 'k'

$k \in (a,b)$ s.t $f(k)=0$

$f(x) = x^3 + x - 1$ Initial guess

$f(0) = -ve$

$f(1) = +ve$

$f(2) = +ve$

$f(3) = +ve$

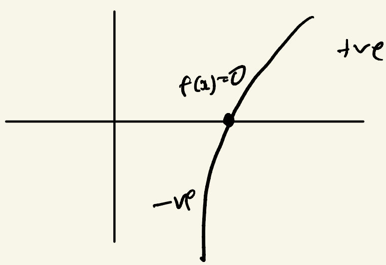
$f(-1) = -ve$

$f(-2) = -ve$

$f(-3) = -ve$

$\boxed{[0,1]}$

$f(x) = x^3+x-1$ $[0,1]$



Bisection Method

$x^3 - x^2 - x - 3 = 0$ correct to 10^{-3} (3 decimals)

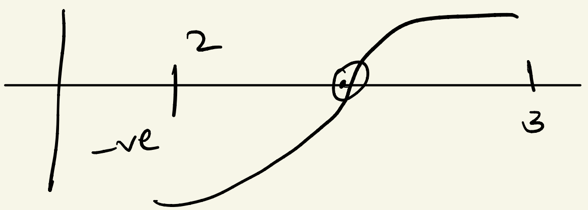
$f(0) = -ve$

$f(1) = -ve$

$f(2) = -ve$

$f(3) = +ve$

{ we take one extra to round off }



	a	f(a)	b	f(b)	$c = \frac{a+b}{2}$	f(c)
	2	-ve	3	+ve	2.5	+ve
	2	-ve	2.5	+ve	2.25	+ve
	2	-ve	2.25	+ve	2.125	-ve
$\Rightarrow$	2.125	-ve	2.25	+ve	2.187	+ve
	2.125	-ve	2.187	+ve	2.158	+ve
	2.125	-ve	2.156	+ve	2.140	+ve
	2.125	-ve	2.140	+ve	2.1325	+ve
	2.125	-ve	2.1325	+ve	2.1287	-ve
	2.1287	-ve	2.1325	+ve	2.1306	+ve
	2.1287	-ve	2.1306	+ve	2.1296	-ve
	2.1296	-ve	2.1306	+ve	2.1301	-ve
	2.1301	-ve	2.1306	+ve	2.13035	-ve
	2.1304	-ve	2.1306	+ve	2.1305	+ve
	2.1304	-ve	2.1305	+ve	2.13045	+ve
	2.1304	-ve	2.1304	+ve		

$\boxed{2.130}$ root

Fixed - Point iteration

22-8-2023 (1)

'α' $\quad x = g(x)$.

$\boxed{\alpha = g(\alpha)}$. $\alpha \to$ fixed Point of function 'g'.

$f(x) = x - g(x)$.

$f(\alpha) = \alpha - g(\alpha) = 0$

α is fixed point of function 'g'. & at the same time 'α' is the zero of function 'f'.

$f(x) = x^3 + 4x^2 - 10$ ✓

$\forall\, x \in [1,2]$

(i) $x = g_1(x) = x - (x^3 + 4x^2 - 10)$ ✗✗

(ii) ✓ $x^2(x+4) - 10 \Rightarrow x^2 = \dfrac{10}{x+4} \Rightarrow x = \pm\left(\dfrac{10}{x+4}\right)^{1/2}$

$x = g_2(x) = \pm\left(\dfrac{10}{x+4}\right)^{1/2}$ ✓✓

(iii) $x.x^2 + 4x^2 - 10 \Rightarrow x.x^2 = 10 - 4x^2 \Rightarrow x = \dfrac{10}{x^2} - 4$

$x = g_3(x) = \dfrac{10}{x^2} - 4$

(iv) $4x^2 = 10 - x^3 \Rightarrow x^2 = \dfrac{10 - x^3}{4} \Rightarrow x = \pm\left(\dfrac{10 - x^3}{4}\right)^{1/2}$

✓✓ $x = g_4(x) = \pm\left(\dfrac{10 - x^3}{4}\right)^{1/2}$.

(v) ✓✓ $x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$ $\Rightarrow \boxed{x = g_5(x) = x - \dfrac{f(x)}{f'(x)}}$ (*)

$$f(x) = \cos x - e^{x} = 0$$

$$x = g(x) = x - \frac{\cos x - e^{x}}{-\sin x - e^{x}} \Rightarrow x = g(x) = x + \frac{\cos x - e^{x}}{\sin x + e^{x}}.$$

$$x^{7} + \sin x + 2 = 0. \Rightarrow \boxed{x = g(x) = x - \frac{x^{7} + \sin x + 2}{7x^{6} + \cos x}.}$$

$$x^{3}.x^{4} = -\sin x - 2$$

$$\Rightarrow x = \left(\frac{-\sin x - 2}{x^{4}}\right)^{1/3}$$

$g \in [a,b]$ ∀ then $x \in [a,b]$ (2)

$g(x) = x^2 - 1$ $\qquad x \in [0,1]$

$g(x) \in [0,1]$

$g(0) = -1$ ✗

$|g(x)| = \left|\frac{x^3-1}{3}\right|$ $\qquad [3,4]$

$$g_2(x) = \left(\frac{10}{x+4}\right)^{1/2}$$

$$g_2'(x) = -\frac{1}{2}\left(\frac{10}{x+4}\right)^{\frac{1}{2}-1} \cdot \left(\frac{1}{x+4}\right)^2$$

$$= -\frac{10}{2}\left(\frac{x+4}{10}\right)^{1/2}\left(\frac{1}{x+4}\right)^2$$

$$|g_2'(x)| = \left|\frac{10}{2}\left(\frac{x+4}{10}\right)^{1/2}\left(\frac{1}{x+4}\right)^2\right| \qquad x \in [1,2] \quad 1.5$$

$$|g_2'(1.5)| \le 1$$

$x_{n+1} = g(x_n)$ $\quad n \geq 0$

$x_1 = g(x_0)$ $\quad x_0 \to$ initial guess.

$x_2 = g(x_1)$

$x_3 = g(x_2)$

$x_4 = g(x_3)$

$x_5 = g(x_4)$

$\boxed{|x_{n+1} - x_n| < \varepsilon}$ $\quad \varepsilon = 10^{-3}$ or 10^{-4}, or 10^{-4}

$\boxed{|\text{current approx} - \text{prev}| < \varepsilon}$

$f(x) = x^3 + 4x^2 - 10 = 0$

$f(1) =$ $\longrightarrow$ $-ve$

$f(1.1) =$

$f(1.2) =$

$f(1.3) =$ $\longrightarrow$ $-ve \to |-0.043|$

$f(1.4) =$ $\longrightarrow$ $+ve. \to |0.584|$

$f(1.5) =$ $\longrightarrow$ $+ve.$

$f(1.6) =$

$f(1.8) =$

$f(1.9) =$

$f(2) =$ $\quad +ve$

fixed-point- iteration

Newton's method

Secant method

$x_{n+1} = g(x_n)$ $\qquad x_0 = ?$

$x_1 = g(x_0)$

$x_2 = g(x_2)$

$x_3 = g(x_2)$

$|x_{n+1} - x_n| \leq \varepsilon \rightarrow$ stop error

$x^3 + 4x^2 - 10 = 0$

$g_5(n) \rightarrow$ Best case

Newton method

$f(x) = x - \cos x$

(i) By fixed-point iteration

(ii) By Newton's Method

$\Bigg\}\ 10^{-6}$

How to find initial guess

$f(0) = 0 - \cos 0 = -ve$

$f(1) = 1 - \cos 1 = +ve$

$[0, 1]$

$f(0) \Rightarrow -ve$

0.1

0.2

0.3

0.4

$0.5 = 0.5 - \cos(0.5) = -ve$

0.6

$0.7 = 0.7 - \cos(0.7) = -0.06$

$0.8 = 0.8 - \cos(0.8) = +.1033$

0.9

$[0.7, 0.8]$

$x_0 = 0.7$

$f(1) \Rightarrow +ve$

0.71

$0.72 \to 0.72 - \cos(0.72) = -ve$

$0.73 \to 0.73 - \cos(0.73) = -0.015$

$0.74 \to 0.74 - \cos(0.74) = +ve = 0.0015$

0.75

0.76

0.77

0.78

0.79

0.80

$x_0 = 0.74$

More precise initial guess

Fixed point iteration

$$f(x) = x - \cos x = 0$$

$$x = \cos x \qquad x \in [0,1]$$

$$x = g(x) \qquad g(x) = \cos x$$

① $g(x) \in [0,1] \ \forall x \in [0,1]$

② $|g'(x)| = |-\sin x| < 1 \ \forall x \in (0,1)$

$$x_0 = 0.74$$

$$x_1 = g(x_0) = g(0.74) = \cos(0.74) = 0.73846855$$

$$x_2 = g(x_1) = \cos(0.73846855) = .73833333$$

assume

$$x_3 = g(0.7383333) = \cos(0.738\,3333)$$

$$|x_{n+1} - x_n| \leq 10^{-6} \qquad x_7 = 0.736128$$

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f(x) = x - \cos x$$

$$x_{n+1} = x_n - \left(\frac{x_n - \cos x_n}{1 + \sin x_n} \right) \quad n = 0$$

$$x_1 = x_0 - \frac{(x_0 - \cos x_0)}{1 + \sin x_0} \qquad x_0 = 0.74$$

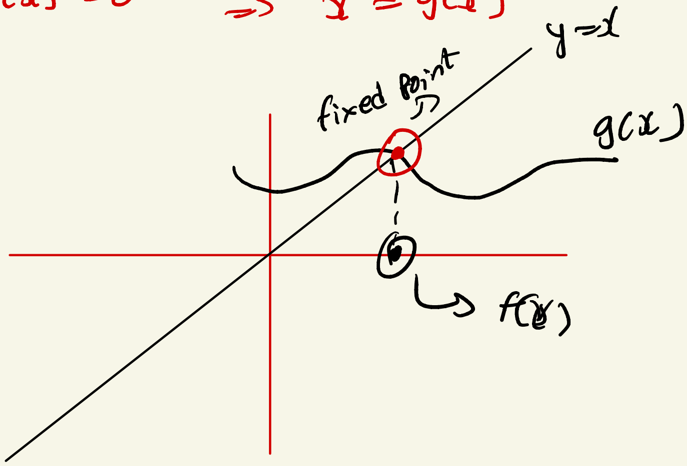
$$x_1 = 0.74 - \left(\frac{0.74 - \cos(0.74)}{1 + \sin(0.74)}\right)$$

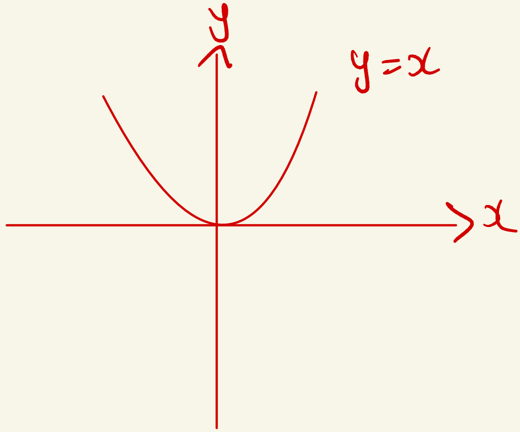
$$= 0.785981635$$

$$x_2 = x_1 - \frac{(x_1 - \cos x_1)}{1 + \sin x_1}$$

Graphical view of F.P.F

$f(x) = x - g(x) = 0 \quad \Rightarrow \quad x = g(x)$





$f(x) = (x-a)^4$

$f'(x) = 4(x-a)^3$

$f''(x) = 12(x-a)^2$

$f'''(x) = 24(x-a)$

$f^{iv}(x) = 24 \neq 0$ at $x = a$

$f(a) = f'(x) = f''(a) = f'''(a) = 0$

$f^{iv}(a) \neq 0$

$f(x) = x-a)^m$ if $x = a$ is multiple root with multiplicity (m) then

$f(a) = f'(a) = f''(a) - f^{m-1}(a) = 0$

$f^{(m)}(a) \neq 0$

Extended form?

$e^x - x - 1 = 0$ root at $x = 0$,

Let $f(x) = e^x - x - 1$

$f'(x) = e^x - 1$ $\quad f'(0) = e^0 - 1 = 1 - 1 = 0$

$f''(x) = e^x$ $\quad f''(0) = 1 \neq 0$

$e^x - x - 1 = 0$ will have multiple root with multiplicity (2) at $x = 0$.

Newton's metod for simple roots

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n \geq 0$$

Modified newton's for multiple roots

$$\boxed{x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}}$$

→ when multiplicity is known in advance..

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{\{f'(x_n)\}^2 - f(x_n) f''(x_n)}}$$

when multiplicity is not known in advance

$\boxed{n \geq 0}$

$e^x - x - 1 = 0$

Modified Newton's Method

$$x_{n+1} = x_n - \frac{m\, f(x_n)}{f'(x_n)}$$

$\boxed{m=2}$ (because second derivative $\neq 0$)

$$x_{n+1} = x_n - \frac{2\, f(x_n)}{f'(x_n)}$$

$$f(x) = e^x - x - 1$$

$$f'(x) = e^x - 1$$

$$\boxed{x_{n+1} = x_n - \frac{2\left[e^{x_n} - x_n - 1\right]}{e^{x_n} - 1}}$$

$x_0 = 1$

$f(x) = e^x - x - 1$

$f(-1) =$

$$x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{[f'(x_n)]^2 - f(x_n)\, f''(x_n)}$$

works for both simple roots & multiple root

When multiplicity is not known in advance

$$x_{n+1} = x_n - \frac{[e^{x_n} - x_n - 1]\,[e^{x_n} - 1]}{[e^{x_n} - 1]^2 - [e^{x_n} - x_n - 1]\,(e^{x_n})}$$

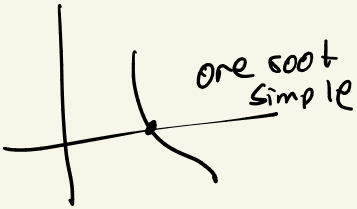
$x_0 = 1$

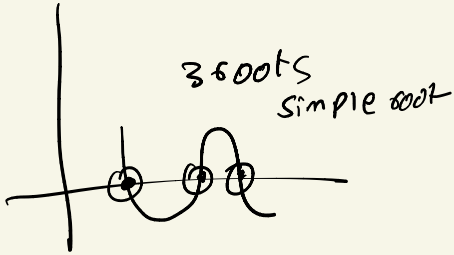
$$|x_{n+1} - x_n| \le \varepsilon$$

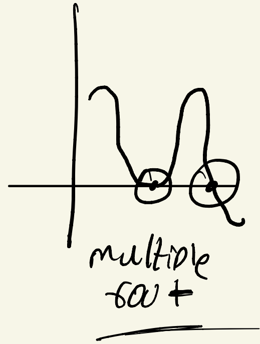
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow$$ works only for simple roots

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \rightarrow$$ works only for multiple roots

$$x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{\{f'(x_n)\}^2 - f(x_n)\, f''(x_n)}$$ $\rightarrow$ works for both







Secant Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n \geq 0$$

$$f'(x_n) \simeq \lim_{x \to x_n} \frac{f(x) - f(x_n)}{x - x_n} \qquad — (1)$$

$x \to x_{n-1}$

$$f'(x_n) \simeq \lim_{x_{n-1} \to x_n} \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} \qquad — (2)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}}$$

Secant Method

Q) Solve $x - \cos x = 0$ By Secant Method

$$x_{n+1} = x_n - \frac{f(x_n)(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}$$

$$x_2 = x_1 - \frac{f(x_1)(x_0 - x_1)}{f(x_0) - f(x_1)}$$

$f(x) = x - \cos x$

$x_0 = 0.5$, $x_1 = \pi/4 = \frac{3.14}{4} = 0.785 = 0.78$

$$x_2 = 0.78 - \frac{f(0.78)(0.5 - 0.78)}{f(0.5) - f(0.78)} = 0.785398$$

$$x_3 = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)} =$$

Order of convergence

Let (α) be the root of equation $f(x) = 0$ and $x_1, x_2, x_3, \ldots x_n \ldots$ are the Successive approximations,

The errors e_i and e_{i+1} are at i^{th} and $(i+1)^{th}$ steps are $e_i = |x_i - \alpha|$, $e_{i+1} = |x_{i+1} - \alpha|$

then the order of convergence of an iterative method is $\{\beta\}$ if β is a largest integer as

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i^{\beta}} \leq 1 \angle \quad [e_{i+1} \propto e_i^{\beta}]$$

where K is finite

Error at current step is directly pro to the β^{th} power of error at previous stem

for the Bijection method $\beta=1$ linear method

for the netwon method $\beta=2$ quadrical converger

if $\beta=3$ that method is cubically converger

for the secant method $\beta=1.618$

$\therefore$ secant method is super linear

rate of convergence

1. Newton
2. Seamt
3. Bisection

System of Linear equations

$$\left.\begin{aligned} 3x + 2y &= 7 \\ 8x - 9y &= 2 \end{aligned}\right\}$$ Solutions $\Rightarrow$ $\left.\begin{array}{l} \text{unique soln} \\ \text{No-soln} \\ \text{Infinite soln} \end{array}\right\}$

Gauss Elimination and Substitution method

$$\left.\begin{aligned} 3x_1 + 3x_2 &= 3 \\ 8x_1 + x_2 &= 1 \end{aligned}\right\}$$

$$\begin{aligned} 3x_1 + 8x_2 &= 3 \\ - \quad 3x_1 + 3x_2 &= 3 \\ \hline x_2 &= 0 \end{aligned}$$ $\rightarrow$ Elimination

$3x_1 + 8 \times 0 = 3 \Rightarrow x_1 = 1 \rightarrow$ Substitution