NULL STRUCTURES IN GENERAL RELATIVITY

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF SCIENCE

by

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to the

School of Physical Sciences

National Institute of Science Education and Research

Bhubaneswar

24th April, 2018

Dedicated to the memory of my friend, Debashish Jena

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfilment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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The thesis work reported in the thesis entitled "Null Structures in General Relativity" was carried out under my supervision, in the School of Physical Sciences at NISER, Bhubaneswar, India.

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ACKNOWLEDGEMENTS

I would like to thank my thesis advisor, Dr Yogesh Kumar Srivastava, for his support, motivation and insightful suggestions. I would also like to thank the School of Physical Sciences, National Institute of Science Education and Research for providing me with the requisite training as well as the opportunity to work on this project. Lastly, I wish to thank my parents and grandparents for their continuous support and encouragement and my classmates for making my years of study at NISER enjoyable and fruitful.

ABSTRACT

In this thesis, I have studied and analysed various null structures in General Relativity. I have studied linearised and exact plane wave metrics, their properties, and Penrose limits. The Garfinkle-Vachaspati transformation, used in metrics admitting a null hypersurface orthogonal Killing vector fields to generate new solutions of the Einstein equations, has also been discussed, and the transformation has been applied on some known metrics such as the D3 brane metric. I have also tried to find a generalised form of the Garfinkle-Vachaspati transformation, which can act on metrics having a covariantly constant spacelike vector as well as the null hypersurface orthogonal Killing vector, and derived the equations which the transformation parameters must satisfy in this case for the Einstein-Maxwell system as well as a system with a 3-form field. I have solved the transformation equations for the plane wave metric and the self-dual string metric.

Contents

Chapter 1

Introduction

General relativity is at present the best classical theory that describes gravity, and its predictions has been repeatedly verified experimentally. In general relativity, spacetime is considered to be Lorentzian manifold with a metric $g_{\mu\nu}$. The metric gives the distance between two nearby points in spacetime, and various properties of the manifold such as curvature can be derived from it. In the following sections, the topics in general relativity that are relevant to this project have been outlined briefly, following the approach of Wald[1] and Schutz.[2] At the end of the chapter, the outline of this thesis is given.

1.1 Convention

In special and general relativity, two conventions for the metric signature are prevalent. A D-dimensional Lorentzian manifold has one timelike and $(D-1)$ spacelike orthogonal basis vectors. We have worked the mostly positive signature $(-,+,\ldots,+)$ in this thesis, according to which timelike vectors have negative norm and spacelike have positive. The opposite is true in the mostly negative signature.

The Ricci tensor $R_{\mu\nu}$ has been defined as

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{1.1}$$

where $R^{\alpha}{}_{\mu\nu\lambda}$ is the Riemann tensor. The Riemann and Ricci tensor will be discussed in section 1.3.

In this thesis, we will choose units such that $c = 1$. The usual convention for geometrised units is to define $G = 1$, but I have sometimes defined G to be some other dimensional constant such as 4π for convenience.

1.2 Covariant derivative and Christoffel symbol

Since the vectors at different points on the spacetime manifold belong to different vector spaces, we need to define a way to compare vectors at different points, which boils down to defining a derivative operator on vectors and tensors in general. The covariant derivative is defined as a linear operator ∇ mapping (p,q) tensors to $(p,q{+}1)$ tensors which satisfies certain properties. For a vector v^μ, the components of the covariant derivative $(\nabla v)_\nu{}^\mu \equiv \nabla_\nu v^\mu$ for a given coordinate system are related to the partial derivatives $\partial_\nu v^\mu$ of the components of v^μ as

$$\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu{}_{\nu\alpha} v^\alpha, \tag{1.2}$$

where $\Gamma^\mu{}_{\nu\alpha}$ are called Christoffel symbols. For a general tensor $T^{\mu_1\cdots\mu_p}{}_{\nu_1\ldots\nu_q}$, we have

$$\begin{aligned}\nabla_\lambda T^{\mu_1\cdots\mu_p}{}_{\nu_1\ldots\nu_q} = \partial_\lambda T^{\mu_1\cdots\mu_p}{}_{\nu_1\ldots\nu_q} &+ \sum_{i=1}^{p} \Gamma^{\mu_i}{}_{\alpha\lambda} T^{\mu_1\cdots\mu_{i-1}\alpha\mu_{i+1}\cdots\mu_p}{}_{\nu_1\ldots\nu_q} \\ &- \sum_{j=1}^{q} \Gamma^{\beta}{}_{\nu_j\lambda} T^{\mu_1\cdots\mu_p}{}_{\nu_1\ldots\nu_{j-1}\beta\nu_{j+1}\ldots\nu_q}.\end{aligned} \tag{1.3}$$

The covariant derivative is not uniquely defined in general, and thus provides additional structure to the manifold other than the metric. However, we usually impose the condition that the inner product of two vectors is invariant under parallel transport, which implies that

$$\nabla_\lambda g_{\mu\nu} = 0, \tag{1.4}$$

which uniquely determines the covariant derivative with the corresponding Christoffel symbol

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}). \tag{1.5}$$

A geodesic is usually defined as a curve whose tangent vector $v^\mu = \mathrm{d}x^\mu \, \mathrm{d}\tau$ satisfies

$$\begin{aligned} v^\nu \nabla_\nu v^\mu &= 0 \\ \implies \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \Gamma^\mu{}_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} \frac{\mathrm{d}x^\lambda}{\mathrm{d}t} &= 0. \end{aligned} \tag{1.6}$$

The distance between two points is minimum (or stationary in general) along a geodesic.

1.3 Riemann and Ricci tensors

The curvature of a manifold is given in terms of the Riemann tensor $R^\alpha{}_{\mu\nu\lambda}$, given in terms of the Christoffel symbol as

$$R^\alpha{}_{\mu\nu\lambda} = \Gamma^\alpha{}_{\mu\lambda,\nu} - \Gamma^\alpha{}_{\mu\nu,\lambda} + \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda} - \Gamma^\alpha{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu}. \tag{1.7}$$

The Riemann tensor is related to the commutator of covariant derivatives of a vector as

$$\nabla_\alpha \nabla_\beta v^\mu - \nabla_\beta \nabla_\alpha v^\mu = R^\mu{}_{\nu\alpha\beta} v^\nu, \tag{1.8}$$

and the deviation ξ^μ between two nearby geodesics with tangent vectors given by the vector field v^μ evolves as

$$v^\nu \nabla_\nu (v^\lambda \nabla_\lambda \xi^\mu) = R^\alpha{}_{\mu\nu\beta} v^\mu v^\nu \xi^\beta. \tag{1.9}$$

The Ricci tensor is defined by contracting the Riemann tensor as

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \tag{1.10}$$

the Ricci scalar as

$$R = R^\mu{}_\mu, \tag{1.11}$$

and the Einstein tensor as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

The dependence of the metric on the energy-momentum distribution of the matter in the universe is obtained from the Einstein tensor via the Einstein equation,

$$G_{\mu\nu} = T_{\mu\nu}, \tag{1.12}$$

where $T_{\mu\nu}$ is the stress-energy-momentum tensor.

1.4 Lie derivative and Killing vector

The Lie derivative gives another way of comparing vectors at different points, using the transformations generated by a vector field instead of the concept of parallel transport. Let us consider a smooth vector field v^μ. For any point $p = (x_0^\mu)$ on tha manifold M, we can find an integral curve $x^\mu(t)$ of the vector field passing through p such that $x^\mu(0) = x_0^\mu$ and $\mathrm{d}x^\mu \,/\, \mathrm{d}t = v^\mu(x^\nu(t))$ for some parameter t. Then for $t \in \mathbb{R}$ (or at least some neighbourhood of 0), we can define $\phi_t : M \to M$ as $\phi_t(p)^\nu = x^\nu(t)$, i.e. it physically corresponds to 'shifting every point along the vector field by an amount given by t'. ϕ_t are smooth and invertible (which are called diffeomorphisms), and they form a group as $\phi_{t+s} = \phi_t \circ \phi_s$.

Once the diffeomorphisms ϕ_t connecting different points on the manifold are defined, we can use them to define functions ϕ_t^* connecting tensors at different points. The Lie derivative of a tensor field $T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q}$ is then defined as

$$\mathcal{L}_v T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q} = \lim_{t \to 0} \frac{\phi_{-t}^* T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q} - T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q}}{t}, \tag{1.13}$$

which intuitively gives the 'flow' of the tensor along the vector field v^μ. The Lie derivative of a tensor is a tensor of the same rank. Its definition requires a vector field, but it does not need any additional structure like the covariant derivative. If $v = \partial_x$ is a coordinate basis vector, the Lie derivative is simply the partial derivative with respect to that coordinate:

$$\mathcal{L}_{\partial_x} T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} = \partial_x T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}. \tag{1.14}$$

In general, the Lie derivative of a scalar is

$$\mathcal{L}_v \Phi = v^\mu \partial_\mu \Phi, \tag{1.15}$$

and that of a general tensor is

$$\begin{aligned} \mathcal{L}_v T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} = v^\alpha \nabla_\alpha T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} &- \sum_{i=1}^{p} T^{\mu_1 \ldots \mu_{i-1} \alpha \mu_{i+1} \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \nabla_\alpha v^{\mu_i} \\ &+ \sum_{j=1}^{q} T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_{j-1} \alpha \nu_{j+1} \ldots \nu_q} \nabla_{\nu_j} v^\alpha. \end{aligned} \tag{1.16}$$

If the Lie derivative of the metric with respect to a vector field ξ^μ is zero, then it is called a Killing vector. This qualitatively means that the metric is unchanged by an transformation along that vector field, i.e. Killing vectors denote the continuous isometries (metric-preserving transformations) of the manifold. Also, for a geodesic parametrized by t, the inner product of the tangent vector with a Killing vector is conserved along the geodesic,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\xi_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right) = 0. \tag{1.17}$$

The simplest example is the Minkowski metric ($\mathrm{d}s^2 = -\,\mathrm{d}t^2 + ddx^2 + \mathrm{d}y^2 + \mathrm{d}z^2$), for which the vectors corresponding to every translation (given by ∂_μ), rotation (given by $x^i \partial_j - x^j \partial_i$) and Lorentz boost (given by $x^i \partial_t - t \partial_i$) are Killing vectors. For a static spherically symmetric such as the Schwarzschild metric, whose metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 (\mathrm{d}\theta^2 + \sin^2(\theta)\, \mathrm{d}\phi^2), \tag{1.18}$$

The Killing vectors are the timelike vector ∂_t (corresponding to time-translation) and the three spacelike vectors ∂_ϕ, $(\cos(\phi)\partial_\theta - \cot(\theta)\partial_\phi)$ and $(-\sin(\phi)\partial_\theta - \cot(\theta)\partial_\phi)$, corresponding to rotations.

Killing vectors ξ^μ obey the condition that

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{1.19}$$

which can be obtained using the fact that $\mathcal{L}_\xi g_{\mu\nu} = 0$.

1.5 Hypersurface orthgonal vector

A vector field k^μ is called hypersurface orthogonal if there exist scalar fields S and u such that

$$k_\mu = e^{-S}\partial_\mu u, \tag{1.20}$$

where the level sets of u define the hypersurfaces. The above definition turns out to be equivalent to the condition that

$$\nabla_\mu k_\nu - \nabla_\nu k_\mu = k_\mu \nabla_\nu S - k_\nu \nabla_\mu S, \tag{1.21}$$

which is useful for handling covariant derivatives of k^μ and does not require knowledge of u. To prove that the above is a necessary condition for hypersurface orthogonal vectors, let us first assume that k is a hypersurface orthogonal vector field given as

$$k_\mu = \alpha \partial_\mu u, \tag{1.22}$$

where $\alpha = e^{-S}$. Then we see that

$$\begin{aligned}
\nabla_{[\mu} k_{\nu]} &= \frac{1}{2}(\alpha_{,\mu} u_{,\nu} + \alpha u_{;\nu\mu} - \alpha_{,\nu} u_{,\mu} - \alpha u_{;\mu\nu}) \\
&= \frac{1}{2}\Big(\frac{\alpha_{,\mu}}{\alpha} k_\nu - \frac{\alpha_{,\nu}}{\alpha} k_\mu\Big) \\
&= \frac{1}{2}(k_\mu \partial_\nu(-\ln\alpha) - k_\nu \partial_\mu(-\ln\alpha)) \\
&= k_{[\mu} \nabla_{\nu]} S.
\end{aligned} \tag{1.23}$$

To prove sufficiency, we start with a vector field k^μ obeying Eq. (5.11). Then we see that

$$\begin{aligned} k_{[\alpha;\beta}k_{\gamma]} &= k_\gamma \nabla_{[\beta}k_{\alpha]} + k_\alpha \nabla_{[\gamma}k_{\beta]} + k_\beta \nabla_{[\alpha}k_{\gamma]} \\ &= k_\gamma k_{[\beta}\nabla_{\alpha]}S + k_\alpha k_{[\gamma}\nabla_{\beta]}S + k_\beta k_{[\alpha}\nabla_{\gamma]}S \\ &= k_{[\gamma}k_{\beta]}\nabla_\alpha S + k_{[\alpha}k_{\gamma]}\nabla_\beta S + k_{[\beta}k_{\alpha]}\nabla_\gamma S \\ &= 0, \end{aligned} \tag{1.24}$$

which implies that k^μ is hypersurface orthogonal, using Forbenius' theorem.[3]

If k is both Killing and hypersurface orthogonal, we have

$$\nabla_\mu k_\nu = \frac{1}{2}(k_\mu \nabla_\nu S - k_\nu \nabla_\mu S). \tag{1.25}$$

If k is null, Killing and hypersurface orthogonal, then we contract Eq. (1.21) with k^μ to obtain

$$\begin{aligned} k^\mu \nabla_{[\mu}k_{\nu]} &= k^\mu k_{[\mu}\nabla_{\nu]}S \\ \implies -2k^\mu \nabla_\nu k_\mu &= -k_\nu k^\mu \nabla_\mu S \qquad \text{(Using Eq. (1.19) and } k^\mu k_\mu = 0\text{)} \\ \implies \nabla_\nu (k^\mu k_\mu) &= -k_\nu \mathcal{L}_k S \\ \implies \mathcal{L}_k S = k^\mu \partial_\mu S &= 0. \qquad (k \text{ is null but non-zero}) \end{aligned} \tag{1.26}$$

If $S = 0$, then k is a gradient vector field. A vector is gradient as well as Killing if and only if it is covariantly constant.

1.6 Differential forms

A differential p-form $\omega_{(p)}$ is defined as a completely antisymmetric $(0, p)$ tensor, i.e.

$$\omega_{\mu_1 \ldots \mu_p} = \omega_{[\mu_1 \ldots \mu_p]}. \tag{1.27}$$

The wedge product or exterior product of two differential forms ω and μ of rank p and q respectively is a $(p+q)$-form denoted as $\omega \wedge \mu$, and given as

$$(\omega \wedge \mu)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \ldots \mu_p} \mu_{\nu_1 \ldots \nu_q]} = (-1)^{pq} (\mu \wedge \omega)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q}. \tag{1.28}$$

The exterior derivative d is an operator mapping p-forms ω to $(p+1)$-forms $\mathrm{d}\omega$, given as

$$\mathrm{d}\omega_{\nu \mu_1 \ldots \mu_p} = (p+1) \nabla_{[\nu} \omega_{\mu_1 \ldots \mu_p]}. \tag{1.29}$$

Owing to the antisymmetrisation, the exterior derivative is independent of the Christoffel symbol and thus independent of the choice of covariant derivative. It also turns out that fields in nature are usually given as differential forms, e.g. the electromagnetic field $F_{(2)} = \mathrm{d}A_{(2)}$ is a 2-form field derived from a 1-form potential. We note that

$$(\mathrm{d}^2 \omega)_{\nu_1 \nu_2 \mu_1 \ldots \mu_p} = (p+2)(p+1) \nabla_{[\nu_1} \nabla_{\nu_2} \omega_{\mu_1 \ldots \mu_p]} = 0. \tag{1.30}$$

The interior product i_k for a vector k^μ, mapping p-forms to $(p-1)$-forms, is simply given by contraction with k:

$$(i_k \omega)_{\mu_1 \ldots \mu_{p-1}} = k^\mu \omega_{\mu \mu_1 \ldots \mu_{p-1}}. \tag{1.31}$$

It satisfies the conditions that

$$i_k i_l \omega = -i_l i_k \omega, \tag{1.32}$$

$$\text{and} \quad \mathcal{L}_k \omega = \mathrm{d} i_k \omega + i_k (\mathrm{d}\omega). \tag{1.33}$$

The hodge dual or Poincaré dual of a p-form $F_{(p)}$ in a D-dimensional maifold with a metric $g_{\mu\nu}$ is a $(D-p)$-dimensional manifold, given as

$$\star F_{\mu_1 \ldots \mu_{D-p}} = \frac{1}{p!} \epsilon_{\mu_1 \ldots \mu_p} F^{\mu_{D-p+1} \ldots \mu_p}, \tag{1.34}$$

where ϵ is a D-form with $\epsilon_{012\ldots p} = \sqrt{|g|}$.

Volume integration on a D-dimensional manifold is defined in terms of D-form fields to give the volume integral of a scalar field as

$$\int f = \int \mathrm{d}^D x \, \sqrt{|g|} f. \tag{1.35}$$

The factor of $\sqrt{|g|}$ appears to make the volume element $\sqrt{|g|}\,\mathrm{d}^D x$ coordinate-independent.

1.7 Action principle of general relativity

Many theories in Physics can be written in terms of action principles, where the classical solution of a function ϕ (such as a coordinate or field) is the one at which a particular scalar functional $S[\phi]$, called the action, becomes stationary:

$$\frac{\delta S[\phi]}{\delta \phi} = 0. \tag{1.36}$$

The equation of the motion of the quantity ϕ can then be obtained from the action principle as

For general relativity, the vacuum Einstein equation satisfied by the metric can be obtained from the Hilbert action,

$$S[g_{\mu\nu}] = \int \mathrm{d}^D x \sqrt{-g} R, \tag{1.37}$$

where R is the Ricci scalar. Taking the variation, we find

$$\begin{aligned} \delta \mathcal{L} &= \delta \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \\ &= R \, \delta \sqrt{-g} + \sqrt{-g} g^{\mu\nu} \, \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \, \delta g^{\mu\nu} \, . \end{aligned} \tag{1.38}$$

We note that the variation of the determinant of the metric may be obtained as

$$
\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}(-g)^{-1/2}\,\delta\left(-\prod_i \lambda_i\right) \\
&\quad [\lambda_i \text{ are the eigenvalues of the matrix with entries } g_{\mu\nu}] \\
&= \frac{1}{2}\sqrt{-g}\sum_i \frac{\delta\lambda_i}{\lambda_i} \\
&= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\,\delta g_{\mu\nu} \quad [\lambda_i^{-1} \text{ are the eigenvalues of the matrix with entries } g^{\mu\nu}] \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\,, \quad [\because\ \delta g^{\mu\nu} = -g^{\mu\lambda}g^{\nu\sigma}\,\delta g_{\lambda\sigma}\ ^{1}] \qquad (1.39)
\end{aligned}
$$

and the variation in the Ricci tensor is given as[1]

$$g^{\mu\nu}R_{\mu\nu} = \nabla^\mu v_\mu, \tag{1.40}$$

where

$$v_\mu = \nabla^\nu\,\delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu\,\delta g_{\alpha\beta}\,. \tag{1.41}$$

This is a divergence, whose integral will contribute a boundary term, which we are ignoring. Then we have

$$
\begin{aligned}
\delta\mathcal{L} &= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} \\
&= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \sqrt{-g}G_{\mu\nu}, \qquad (1.42)
\end{aligned}
$$

which gives us the vacuum Einstein equation.

Action principles often simplify the set-up of the theory. For example, whereas it is not intuitively obvious why the left hand side of the Einstein equation (i.e. the Einstein tensor) should be of the form $R_{\mu\nu} - g_{\mu\nu}R/2$, the Lagrangian density of the Hilbert action is (up to a factor of $\sqrt{-g}$ to make the action a scalar) the Ricci scalar, which is the simplest scalar involving second derivatives of the metric. It is also helpful in setting up new theories involving other fields whose actions are known, as they can be simply added to the Hilbert action. The coupling between the metric and the field is manifested by replacing the partial derivatives in the usual expression of the action of the field with covariant derivatives. In Chapter 5, we will work with a more complicated action involving both scalar and form fields.

1.8 Plan of this project

Null structures are an important property of Lorentzian manifolds, as they do not appear in systems with Riemannian metrics. An example is the plane wave metric, which admits a null covariantly constant vector field and physically corresponds to a gravitational wave. The plane wave metric is studied as a perturbation on the Minkowski metric[4] in chapter 2, and nonperturbatively in chapter 3. The plane wave metric is seen to appear as a special limiting case of any metric,[5] and this limit, called the Penrose limit, is discussed in chapter 4.

In General Relativity, the metric is obtained as a solution of the Einstein equation, which is a highly nonlinear system of coupled partial differential equations. This makes it difficult to obtain exact solutions of the Einstein equation.[6] Some solutions can be obtained by exploiting the symmetries of the system, which are given in terms of Killing vectors. There also exist methods to generate new solutions of the Einstein equation based on known solutions. The Garfinkle-Vachaspati (GV) transformation is one such solution generating technique,[7] which can be applied on a metric admitting a null hypersurface orthogonal Killing vector. The transformation is studied in chapter 5.

In chapter 6, a generalised version of the Garfinkle-Vachaspati transformation is analysed. This transformation involves a null hypersurface orthogonal Killing vector as well as a covariantly constant spacelike vector, and the usual GV transformation as obtained as a special case of this transformation. This extended Garfinkle-Vachaspati transformation is applied on the plane wave metric and the self-dual string metric to obtain new solutions to the Einstein equations.

Chapter 2

Linearised plane gravitational waves

2.1 The linearised field equations

Gravitational waves are a prediction of general relativity, interpreted as disturbances in spacetime propagating at the speed of light. Gravitational waves had previously been detected indirectly by observing the energy lost to gravitational radiation by sources such as binary stars. In 2015 and later again in 2017, the LIGO and VIRGO detectors have made direct detections of gravitational waves from merging black holes as well as a binary neutron star merger.

In a region far away from the source of the gravitational wave, the metric may be treated as a small perturbation on the Minkowski metric, such that all analysis will be made only up to the first order in the perturbation. We begin our analysis of linearized gravitational waves (following d'Inverno[4]) by assuming a 4-dimensional spacetime whose metric differs slightly from the Minkowski metric, i.e.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}, \tag{2.1}$$

where $|\epsilon| \ll 1$. We will only consider terms up to the first order in ϵ. We also assume that the spacetime is "asymptotically flat", i.e. the metric approaches the Minkowski metric at large distances; so for a radial parameter r,

$$\lim_{r\to\infty} h_{\alpha\beta} = 0. \tag{2.2}$$

To find the inverse metric $g^{\alpha\beta}$, we first define

$$h^{\alpha\beta} \equiv \eta^{\alpha\gamma}\eta^{\beta\delta}h_{\gamma\delta}. \tag{2.3}$$

Then

$$\begin{aligned}
g_{\alpha\beta}\big(\eta^{\beta\gamma} - \epsilon h^{\beta\gamma}\big) &= (\eta_{\alpha\beta} + \epsilon h_{\alpha\beta})\big(\eta^{\beta\gamma} - \epsilon h^{\beta\gamma}\big) \\
&= \eta_{\alpha\beta}\eta^{\beta\gamma} + \epsilon\big(h_{\alpha\beta}\eta^{\beta\gamma} - \eta_{\alpha\beta}h^{\beta\gamma}\big) + \mathcal{O}\big(\epsilon^2\big) \\
&= \delta_\alpha{}^\gamma + \epsilon\big(h_{\alpha\beta}\eta^{\beta\gamma} - \eta_{\alpha\beta}\eta^{\beta\mu}\eta^{\gamma\nu}h_{\mu\nu}\big) \\
&= \delta_\alpha{}^\gamma + \epsilon\big(h_{\alpha\beta}\eta^{\beta\gamma} - \delta_\alpha{}^\mu\eta^{\gamma\nu}h_{\mu\nu}\big) \\
&= \delta_\alpha{}^\gamma + \epsilon\big(h_{\alpha\beta}\eta^{\beta\gamma} - \eta^{\gamma\nu}h_{\alpha\nu}\big) \\
&= \delta_\alpha{}^\gamma, \qquad (2.4)
\end{aligned}$$

which implies that

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \epsilon h^{\alpha\beta}. \qquad (2.5)$$

The Christoffel symbols are obtained as

$$\begin{aligned}
\Gamma^\alpha{}_{\beta\gamma} &= \frac{1}{2}g^{\alpha\mu}(g_{\beta\mu,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \\
&= \frac{1}{2}(\eta^{\alpha\mu} - \epsilon h^{\alpha\mu})(\epsilon h_{\beta\mu,\gamma} + \epsilon h_{\mu\gamma,\beta} - \epsilon h_{\beta\gamma,\mu}) \\
&= \frac{1}{2}\epsilon\eta^{\alpha\mu}(h_{\beta\mu,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu}) + \mathcal{O}\big(\epsilon^2\big) \\
&= \frac{\epsilon}{2}(h^\alpha{}_{\beta,\gamma} + h^\alpha{}_{\gamma,\beta} - h_{\beta\gamma}{}^{,\alpha}), \qquad (2.6)
\end{aligned}$$

where we use the fact that the indices terms of order ϵ can be raised using the Minkowski metric.

The Riemann tensor is

$$\begin{aligned}
R_{\alpha\beta\gamma\delta} &= g_{\alpha\mu}(\Gamma^\mu{}_{\beta\delta,\gamma} - \Gamma^\mu{}_{\beta\gamma,\delta} + \Gamma^\mu{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\mu{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}) \\
&= (\eta_{\alpha\mu} + \epsilon h_{\alpha\mu})\frac{\epsilon}{2}(h^\mu{}_{\beta,\delta\gamma} + h^\mu{}_{\delta,\beta\gamma} - h_{\beta\delta}{}^{,\mu}{}_{,\gamma} - h^\mu{}_{\beta,\gamma\delta} - h^\mu{}_{\gamma,\beta\delta} + h_{\beta\gamma}{}^{,\mu}{}_{,\delta}) + \mathcal{O}\big(\epsilon^2\big) \\
&= \frac{\epsilon}{2}\eta_{\alpha\mu}(h^\mu{}_{\delta,\beta\gamma} - h^\mu{}_{\gamma,\beta\delta} - h_{\beta\delta}{}^{,\mu}{}_{,\gamma} + h_{\beta\gamma}{}^{,\mu}{}_{,\delta}) \\
&= \frac{\epsilon}{2}(h_{\alpha\delta,\beta\gamma} - h_{\alpha\gamma,beta\delta} - h_{\beta\delta,\alpha\gamma} + h_{\beta\gamma,\alpha\delta}). \qquad (2.7)
\end{aligned}$$

The Bianchi identities

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} \equiv 0 \tag{2.8}$$

are, up to $\mathcal{O}(\epsilon)$,

$$R_{\alpha\beta\gamma\delta,\mu} + R_{\alpha\beta\mu\gamma,\delta} + R_{\alpha\beta\delta\mu,\gamma} \equiv 0, \tag{2.9}$$

which are identically satisfied by the Riemann tensor given in Eq. (2.7).

The Ricci tensor is

$$\begin{aligned} R_{\alpha\beta} &= \eta^{\gamma\delta} R_{\gamma\alpha\delta\beta} \\ &= \eta^{\gamma\delta} \frac{\epsilon}{2} (h_{\gamma\beta,\alpha\delta} + h_{\alpha\delta,\gamma\beta} - h_{\gamma\delta,\alpha\beta} - h_{\alpha\beta,\gamma\delta}) \\ &= \frac{\epsilon}{2} \big(h^{\delta}{}_{\beta,\alpha\delta} + h^{\gamma}{}_{\alpha,\gamma\beta} - h_{,\alpha\beta} - \Box h_{\alpha\beta} \big), \end{aligned} \tag{2.10}$$

where we define

$$h \equiv \eta^{\gamma\delta} h_{\gamma\delta} = h^{\gamma}{}_{\gamma} \tag{2.11}$$

and

$$\Box \equiv \eta^{\gamma\delta} \partial_\gamma \partial_\delta = \partial^\gamma \partial_\gamma = -\frac{\partial^2}{\partial t^2} + \nabla^2, \tag{2.12}$$

called the d'Alembertian operator.

The Ricci scalar is

$$\begin{aligned} R &= R^{\alpha}{}_{\alpha} \\ &= \frac{\epsilon}{2} \big(h^{\delta\alpha}{}_{,\alpha\delta} + h^{\gamma\alpha}{}_{,\gamma\alpha} - \Box h - \Box h \big) \\ &= \epsilon \big(h^{\alpha\delta}{}_{,\alpha\delta} - \Box h \big), \end{aligned} \tag{2.13}$$

and the Einstein tensor is

$$\begin{aligned} G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \\ &= \frac{\epsilon}{2} \big(h^{\delta}{}_{\beta,\alpha\delta} + h^{\gamma}{}_{\alpha,\gamma\beta} - h_{,\alpha\beta} - \Box h_{\alpha\beta} - \eta_{\alpha\beta} \big(h^{\gamma\delta}{}_{\gamma\delta} - \Box h \big) \big) + \mathcal{O}\big(\epsilon^2\big) \\ &= \frac{\epsilon}{2} \big(h^{\delta}{}_{\beta,\alpha\delta} + h^{\gamma}{}_{\alpha,\gamma\beta} - h_{,\alpha\beta} - \Box h_{\alpha\beta} - \eta_{\alpha\beta} h^{\gamma\delta}{}_{,\gamma\delta} + \eta_{\alpha\beta} \Box h \big). \end{aligned} \tag{2.14}$$

It is convenient to introduce new variables $\psi_{\alpha\beta}$, defined as

$$\psi_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h. \tag{2.15}$$

Then the Ricci tensor can be expressed in terms of $\psi_{\alpha\beta}$ as

$$\begin{aligned}
R_{\alpha\beta} &= \frac{\epsilon}{2}(h^{\gamma}{}_{\alpha,\beta\gamma} + h^{\gamma}{}_{\beta,\alpha\gamma} - \Box h_{\alpha\beta} - h_{,\alpha\beta}) \\
&= \frac{\epsilon}{2}\left(\psi^{\gamma}{}_{\alpha,\beta\gamma} + \frac{1}{2}\delta^{\gamma}{}_{\alpha}h_{,\beta\gamma} + \psi^{\gamma}{}_{\beta,\alpha\gamma} + \frac{1}{2}\delta^{\gamma}{}_{\beta}h_{,\alpha\gamma} - \Box h_{\alpha\beta} - h_{,\alpha\beta}\right) \\
&= \frac{\epsilon}{2}(\psi^{\gamma}{}_{\alpha,\beta\gamma} + \psi^{\gamma}{}_{\beta,\alpha\gamma} - \Box h_{\alpha\beta}),
\end{aligned} \tag{2.16}$$

from which the Ricci scalar is obtained as

$$\begin{aligned}
R &= \frac{\epsilon}{2}\big(\psi^{\gamma\delta}{}_{,\delta\gamma} + \psi^{\gamma\delta}{}_{,\delta\gamma} - \Box h\big) \\
&= \frac{\epsilon}{2}\big(2\psi^{\gamma\delta}{}_{,\gamma\delta} - \Box h\big),
\end{aligned} \tag{2.17}$$

and the Einstein tensor as

$$\begin{aligned}
G_{\alpha\beta} &= \frac{\epsilon}{2}\left(\psi^{\gamma}{}_{\alpha,\beta\gamma} + \psi^{\gamma}{}_{\beta,\alpha\gamma} - \Box\psi_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\Box h - \eta_{\alpha\beta}\psi^{\gamma\delta}{}_{,\gamma\delta} + \frac{1}{2}\eta_{\alpha\beta}\Box h\right) \\
&= \frac{\epsilon}{2}\big(\psi^{\gamma}{}_{\alpha,\beta\gamma} + \psi^{\gamma}{}_{\beta,\alpha\gamma} - \Box\psi_{\alpha\beta} - \eta_{\alpha\beta}\psi^{\gamma\delta}{}_{,\gamma\delta}\big)
\end{aligned} \tag{2.18}$$

This implies that the Einstein equations

$$G_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{2.19}$$

will reduce to wave equations in $\psi_{\alpha\beta}$ if we impose the condition

$$\psi^{\alpha}{}_{\beta,\alpha} = 0 \tag{2.20}$$

$$\iff h^{\alpha}{}_{\beta,\alpha} - \frac{1}{2}h_{,\beta} = 0. \tag{2.21}$$

We will be mostly considering the vacuum Einstein equation, though similar work may be done in presence of sources as long as $T_{\alpha\beta}$ infinitesimal of order ϵ. For finite matter or energy flux, Eq. (2.1) can anyway not be expected to hold.

2.2 The Einstein gauge

While the condition (Eq. (2.20)) need not be satisfied in general, we shall see that it is always possible to transform to a suitable set of coordinates such that the condition will be satisfied. Of course, the starting condition of Eq. (2.1) has already restricted the coordinate freedom to some extent, since even a flat spacetime may not have the metric $\eta_{\alpha\beta}$ in non-Cartesian coordinates. However, Eq. (2.1) remains satisfied for an infinitesimal coordinate transformation,

$$x^\alpha \mapsto x'^\alpha = x^\alpha + \epsilon\xi^\alpha. \tag{2.22}$$

Then

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta + \epsilon\xi^\alpha{}_{,\beta} \tag{2.23}$$

and

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha{}_\beta - \epsilon\frac{\partial\xi^\alpha}{\partial x'^\beta} = \delta^\alpha{}_\beta - \epsilon\xi^\alpha{}_\beta + \mathcal{O}\big(\epsilon^2\big), \tag{2.24}$$

and the transformed metric tensor is thus obtained as

$$\begin{aligned}
g'_{\alpha\beta} &= g_{\gamma\delta}\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} \\
&= (\eta_{\gamma\delta} + \epsilon h_{\gamma\delta})(\delta^\gamma{}_\alpha - \epsilon\xi^\gamma{}_{,\alpha})\big(\delta^\delta{}_\beta - \epsilon\xi^\delta{}_{,\beta}\big) + \mathcal{O}\big(\epsilon^2\big) \\
&= \eta_{\alpha\beta} + \epsilon(h_{\alpha\beta} - \xi_{\beta,\alpha} - \xi_{\alpha,\beta}) + \mathcal{O}\big(\epsilon^2\big) \\
&= \eta_{\alpha\beta} + \epsilon h'_{\alpha\beta},
\end{aligned} \tag{2.25}$$

where

$$h'_{\alpha\beta} = h_{\alpha\beta} - 2\xi_{(\alpha,\beta)}. \tag{2.26}$$

An infinitesimal transformation thus preserves the condition that the metric is almost flat. The change in the Riemann tensor is given by

$$\begin{aligned}
R'_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} &= \frac{\epsilon}{2}(\xi_{\alpha,\delta\beta\gamma} + \xi_{\delta,\alpha\beta\gamma} + \xi_{\beta,\gamma\alpha\delta} + \xi_{\gamma,\beta\alpha\delta} - \xi_{\alpha,\gamma\beta\delta} - \xi_{\gamma,\alpha\beta\delta} - \xi_{\beta,\delta\alpha\gamma} - \xi_{\delta,\beta\alpha\gamma}) \\
&= 0,
\end{aligned} \tag{2.27}$$

which implies that the Ricci tensor, Ricci scalar and Einstein tensor also remain unchanged. The transformation given by Eq. (2.26) thus leaves the Physics invariant, and is called a gauge transformation by analogy with the gauge transformations of the electromagnetic potential. We will use gauge transformation to obtain the coordinates satisfying the condition given by Eq. (2.20), known as the Einstein gauge condition.

Under a gauge transformation, $\psi_{\alpha\beta}$ is transformed as

$$\begin{aligned}\psi_{\alpha\beta} \mapsto \psi'_{\alpha\beta} &= h'_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h' \\ &= h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} - \frac{1}{2}\eta_{\alpha\beta}(h - 2\xi^{\gamma}{}_{\gamma}) \\ &= \psi_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^{\gamma}{}_{\gamma}, \end{aligned} \tag{2.28}$$

which implies that

$$\begin{aligned}\psi'^{\alpha}{}_{\beta,\alpha} &= \psi^{\alpha}{}_{\beta,\alpha} - \xi^{\alpha}_{\beta,\alpha} - \xi_{\beta}{}^{,\alpha}{}_{,\alpha} + \delta^{\alpha}{}_{\beta}\xi^{\gamma}{}_{,\gamma\alpha} \\ &= \psi^{\alpha}{}_{\beta,\alpha} - \xi^{\alpha}{}_{,\alpha\beta} - \Box\xi_{\beta} + \xi^{\gamma}{}_{,\beta\gamma} \\ &= \psi^{\alpha}{}_{\beta,\alpha} - \Box\xi_{\beta}. \end{aligned} \tag{2.29}$$

Therefore, $\psi_{\alpha\beta}$ will satisfy the Einstein gauge condition given in Eq. (2.20) if ξ_{β} are chosen to satisfy

$$\Box\xi_{\beta} = \psi^{\alpha}{}_{\beta,\alpha}. \tag{2.30}$$

This does not completely fix the gauge, and any further transformation with gauge parameter ζ_{β} will still preserve the Einstein gauge condition provided that

$$\Box\zeta_{\beta} = 0. \tag{2.31}$$

In Einstein gauge, the field equations are

$$-\frac{\epsilon}{2}\Box\psi_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{2.32}$$

and the vacuum field equations will thus be

$$\Box\psi_{\alpha\beta} = 0. \tag{2.33}$$

Taking the trace,

$$\eta^{\alpha\beta}\Box\psi_{\alpha\beta} = \Box(\psi^{\alpha}{}_{\alpha}) = \Box\left(h - \frac{1}{2}\times 4h\right) = -\Box h = 0. \tag{2.34}$$

From Eqs. (2.33), (2.34) and (2.15) we get the vacuum equation for $h_{\alpha\beta}$ as

$$\Box h_{\alpha\beta} = 0. \tag{2.35}$$

2.3 Linearised plane gravitational wave propagating in the x-direction

To solve the vacuum field equations for a plane wave propagating in the x-direction, we take the coordinates

$$\left(x^0, x^1, x^2, x^3\right) = (t, x, y, z), \tag{2.36}$$

and assume the metric to be independent of y and z,

$$h_{\alpha\beta} = h_{\alpha\beta}(t, x) \tag{2.37}$$

$$\implies h_{\alpha\beta,2} = h_{\alpha\beta,3} = 0 \tag{2.38}$$

Then the independent components of the Riemann tensor are found to be

$$R_{\alpha\beta 23} = 0 \quad \forall\alpha, \beta \in \{0, 1, 2, 3\}, \tag{2.39}$$

$$\begin{aligned} R_{0101} &= \frac{\epsilon}{2}(2h_{01,01} - h_{00,11} - h_{11,00}), \\ R_{01\alpha\mu} &= \frac{\epsilon}{2}(h_{0\mu,1\alpha} - h_{1\mu,0\alpha})\forall \quad \alpha \in \{0, 1\}, \mu \in \{2, 3\}, \end{aligned} \tag{2.40}$$

$$R_{\alpha\mu\beta\nu} = \frac{\epsilon}{2}h_{\mu\nu,\alpha\beta} \quad \forall\alpha, \beta \in \{0, 1\},\ u, \nu \in \{2, 3\}. \tag{2.41}$$

The other components can be obtained using

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}. \tag{2.42}$$

We now impose the vacuum condition $G_{\alpha\beta} = 0 \implies R_{\alpha\beta} = 0$. In terms of the components, this condition may be written as

$$R_{00} = R_{0101} + R_{0202} + R_{0303} = 0, \tag{2.43}$$

$$R_{01} = R_{0212} + R_{0313} = 0, \tag{2.44}$$

$$R_{02} = -R_{0112} + R_{0323} = -R_{0112} = 0, \tag{2.45}$$

$$R_{03} = -R_{0113} - R_{0223} = -R_{0113} = 0, \tag{2.46}$$

$$R_{11} = -R_{0101} + R_{1212} + R_{1313} = 0, \tag{2.47}$$

$$R_{12} = -R_{0102} + R_{1323} = -R_{0102} = 0, \tag{2.48}$$

$$R_{13} = -R_{0103} - R_{1223} = -R_{0103} = 0, \tag{2.49}$$

$$R_{22} = -R_{0202} + R_{1212} + R_{2323} = -R_{0202} + R_{1212} = 0, \tag{2.50}$$

$$R_{23} = -R_{0203} + R_{1213} = 0, \tag{2.51}$$

$$R_{33} = -R_{0303} + R_{1313} + R_{2323} = -R_{0303} + R_{1313} = 0. \tag{2.52}$$

From Eqs. (2.43), (2.50) and (2.52), we have

$$R_{0101} + R_{1212} + R_{1313} = 0, \tag{2.53}$$

which along with Eq. (2.47) gives

$$R_{0101} = 0. \tag{2.54}$$

Thus, each of the components of the Riemann tensor in the group of equations (2.40) vanishes. The only non-zero components are thus the ones given in the Eq. (2.41), which only involve the components h_{22}, h_{23} and h_{33}. Since the remaining components of $h_{\alpha\beta}$ do not contribute to the Riemann tensor, we expect the existence of a coordinate system in which h_{22}, $h_{23} = h_{32}$ and h_{33} are the only non-zero components of $h_{\alpha\beta}$. We shall obtain this coordinate system for a plane wave using gauge transformation.

First, we impose the condition

$$h_{\alpha\beta} = h_{\alpha\beta}(t - x), \tag{2.55}$$

which is stricter that Eq. (2.37) and represents a plane wave propagating in the x-direction. The Einstein gauge condition given in Eq. (2.21) then implies (lowering indices using $\eta = \mathrm{diag}(-1, 1, 1, 1)$)

$$-h_{00,0} + h_{01,1} - \frac{1}{2}h_{,0} = 0, \tag{2.56}$$
$$-h_{01,0} + h_{11,1} - \frac{1}{2}h_{,1} = 0, \tag{2.57}$$
$$-h_{02,0} + h_{12,1} = 0, \tag{2.58}$$
$$-h_{03,0} + h_{13,1} = 0. \tag{2.59}$$

Using a dot to denote differentiation with respect to $x - t$, the equations may be rewritten as

$$\dot{h}_{00} + \dot{h}_{01} + \frac{1}{2}\dot{h} = 0, \tag{2.60}$$
$$\dot{h}_{01} + \dot{h}_{11} - \frac{1}{2}\dot{h} = 0, \tag{2.61}$$
$$\dot{h}_{02} + \dot{h}_{12} = 0, \tag{2.62}$$
$$\dot{h}_{03} + \dot{h}_{13} = 0. \tag{2.63}$$

These are integrated to give

$$h_{00} + h_{01} + \frac{h}{2} = f_1, \tag{2.64}$$
$$h_{01} + h_{11} - \frac{h}{2} = f_2, \tag{2.65}$$
$$h_{02} + h_{12} = f_3, \tag{2.66}$$
$$h_{03} + h_{13} = f_4, \tag{2.67}$$

where f_i is a function of y and z only for all i. However, since $\lim_{x\to\infty} h_{\alpha\beta} = 0 \forall y, z$, we get $f_i = 0 \forall i$. Therefore,

$$h_{12} = -h_{02}, \tag{2.68}$$

$$h_{13} = -h_{03}, \tag{2.69}$$

$$h_{01} = -\frac{1}{2}(h_{00} + h_{11}) \quad \text{(Adding Eqs. (2.64) and (2.65))}, \tag{2.70}$$

$$h_{33} = -h_{22} \quad \text{(Substituting Eq. (2.70) in Eq. (2.64) and using the definition of } h\text{)}. \tag{2.71}$$

We can now apply a further gauge transformation, where the gauge parameter ζ_α satisfies $\Box\zeta_\alpha = 0$ to preserve the Einstein gauge condition, so that the transformed metric satisfies

$$h'_{00} = h'_{02} = h'_{03} = h'_{11} = 0. \tag{2.72}$$

This, along with Eq. (2.26), implies

$$\begin{aligned} h_{00} - 2\zeta_{0,0} &= 0, \\ h_{02} - \zeta_{0,2} - \xi_{2,0} &= 0, \\ h_{03} - \zeta_{0,3} - \xi_{3,0} &= 0, \\ h_{11} - 2\zeta_{1,1} &= 0. \end{aligned} \tag{2.73}$$

If we assume that $\zeta_\alpha = \zeta_\alpha(t-x)$, then it automatically satisfies Eq. (2.31). This gauge transformation leaves h_{22}, h_{23} and h_{33} unchanged. Also, if a dot denotes differentiation with respect to $t - x$, the group of equations (2.73) implies

$$\dot{\zeta}_0 = \frac{h_{00}}{2}, \tag{2.74}$$

$$\dot{\zeta}_1 = -\frac{h_{11}}{2}, \tag{2.75}$$

$$\dot{\zeta}_2 = h_{02}, \tag{2.76}$$

$$\dot{\zeta}_3 = h_{03}. \tag{2.77}$$

Thus, a gauge transformation with the gauge parameter satisfying the above conditions transforms the metric to the form where h_{22}, h_{23}, h_{32} and h_{33} are the only non-zero components of $h_{\alpha\beta}$. Since $h_{32} = h_{23}$ for symmetric $h_{\alpha\beta}$ and $h_{33} = -h_{22}$ as obtained in Eq. (2.71), the metric depends on only the two functions $h_{22}(t-x)$ and $h_{23}(t-x)$. These two functions are responsible for the two polarization states ($+$ and $\times$) of gravitational waves.

While we have studied linearised gravitational waves as a perturbation $h_{\alpha\beta}$ on background Minkowski metric, it may also sometimes be useful to think of $h_{\alpha\beta}$ or $\psi_{\alpha\beta}$ as a $(0,2)$ tensor field in Minskowski spacetime, akin to the $(0,1)$ electromagnetic potential A_α. The Riemann tensor defined in Eq. (2.7) is then the field strength tensor (analogous to $F_{\alpha\beta}$ for electromagnetism), which satisfies Eq. (2.19) as an analogue of the Maxwell equation. The gauge transformation described in Eq. (2.26) leaves the field strength unchanged, and the Einstein gauge of Eq. (2.20) is a convenient gauge to get he form of a plane wave, as is the Lorenz gauge for electromagnetism.

Chapter 3

Exact plane gravitational waves

We have seen gravitational waves for the special case where the wave can be treated as a perturbation on a background Minkowski metric. While this treatment is sufficient for predicting gravitational waves which can be experimentally confirmed, it is of interest to study plane wave metrics which are exact solutions of the Einstein equations. Such metrics, which have been studied in this chapter, will also be seen to have some interesting properties, such as having all curvature invariants zero. To extrapolate gravitational waves to the general case, it is convenient to use a different set of coordinates. Let us define $U = x - t$ and $V = (x+t)/2$. We also replace y with y^1 and z with y^2. Then the plane wave metric given in Eq. (2.1) is given as

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + (\delta_{ij} + \epsilon h_{ij}(U))\,\mathrm{d}y^i\,\mathrm{d}y^j\,, \tag{3.1}$$

where $i, j \in \{1, 2\}$. We generalize this idea to the non-perturbative case, and define the plane wave metric as

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + g_{ij}(U)\,\mathrm{d}y^i\,\mathrm{d}y^j\,. \tag{3.2}$$

This is called the plane wave metric in Rosen coordinates.

3.1 Plane wave in Rosen coordinates with diagonal transverse metric

Before studying the general plane wave metric, let us consider the case where $g_{12} = g_{21} = 0$. For convenience, we define $g_{ii} = (\tilde{g}_{ii})^2$. Then the metric is given by

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + (\tilde{g}_{11})^2(U)(\mathrm{d}y^1)^2 + (\tilde{g}_{22})^2(U)(\mathrm{d}y^2)^2. \tag{3.3}$$

The non zero Christoffel symbols are

$$\Gamma^{V}{}_{11} = -\tilde{g}_{11}\dot{\tilde{g}}_{11}, \tag{3.4}$$

$$\Gamma^{V}{}_{22} = -\tilde{g}_{22}\dot{\tilde{g}}_{22}, \tag{3.5}$$

$$\Gamma^{1}{}_{U1} = \frac{\dot{\tilde{g}}_{11}}{\tilde{g}_{11}} = \Gamma^{1}{}_{1U}, \tag{3.6}$$

$$\Gamma^{2}{}_{U2} = \frac{\dot{\tilde{g}}_{22}}{\tilde{g}_{22}} = \Gamma^{2}{}_{2U}, \tag{3.7}$$

where $\dot{\tilde{g}}_{ii} = \partial_U \tilde{g}_{ii}$. The non-zero independent components of the Riemann tensor are

$$R_{U1U1} = -\tilde{g}_{11}\ddot{\tilde{g}}_{11}, \tag{3.8}$$

$$R_{U2U2} = -\tilde{g}_{22}\ddot{\tilde{g}}_{22}, \tag{3.9}$$

and the only non-zero component of the Ricci tensor is

$$R_{UU} = -\frac{\ddot{\tilde{g}}_{11}}{\tilde{g}_{11}} - \frac{\ddot{\tilde{g}}_{22}}{\tilde{g}_{22}} = -R. \tag{3.10}$$

Thus the vacuum field equations reduce to the single equation,

$$\frac{\ddot{\tilde{g}}_{11}}{\tilde{g}_{11}} + \frac{\ddot{\tilde{g}}_{22}}{\tilde{g}_{22}} = 0. \tag{3.11}$$

If we define

$$h = \frac{\ddot{\tilde{g}}_{11}}{\tilde{g}_{11}} = -\frac{\ddot{\tilde{g}}_{22}}{\tilde{g}_{22}}, \tag{3.12}$$

then we get a vacuum plane wave solution by determining g_{11} and g_{22} for any arbitrary choice of h.

Another set are coordinates which are often convenient to work with are the Brinkmann coordinates (u, v, x^1, x^2), in which the above plane wave metric takes the form

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v + h(u)\Big(\big(x^1\big)^2 - \big(x^2\big)^2\Big)\,\mathrm{d}u^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2. \tag{3.13}$$

To show the equivalence of the two expressions of the metric, we transform from Rosen to Brinkmann coordinates as

$$
\begin{aligned}
u &= U, && (3.14)\\
v &= V - \frac{1}{2}\big(\tilde{g}_{11}\dot{\tilde{g}}_{11}(y^1)^2 + \tilde{g}_{22}\dot{\tilde{g}}_{22}(y^2)^2\big), && (3.15)\\
x^1 &= \tilde{g}_{11}y^1, && (3.16)\\
x^2 &= \tilde{g}_{22}y^2. && (3.17)
\end{aligned}
$$

Then we see that

$$
\begin{aligned}
\mathrm{d}s^2 &= h(u)\Big(\big(x^1\big)^2 - \big(x^2\big)^2\Big)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\\
&= h\big(\tilde{g}_{11}^2(y^1)^2 - \tilde{g}_{22}^2(y^2)^2\big)\,\mathrm{d}U^2 + \mathrm{d}U\,\big(2\,\mathrm{d}V - 2y^1\tilde{g}_{11}\dot{\tilde{g}}_{11}\,\mathrm{d}y^1 - (y^1)^2\dot{\tilde{g}}_{11}^2\,\mathrm{d}U\big)\\
&\quad + \mathrm{d}U\,\big(-(y^1)^2\tilde{g}_{11}\ddot{\tilde{g}}_{11}\,\mathrm{d}U - 2y^2\tilde{g}_{22}\dot{\tilde{g}}_{22}\,\mathrm{d}y^2 - (y^2)^2\dot{\tilde{g}}_{22}^2\,\mathrm{d}U - (y^2)^2\tilde{g}_{22}\ddot{\tilde{g}}_{22}\,\mathrm{d}U\big)\\
&\quad + \tilde{g}_{11}^2(\mathrm{d}y^1)^2 + (y^1)^2\dot{\tilde{g}}_{11}^2\,\mathrm{d}U^2 + 2y^1\tilde{g}_{11}\dot{\tilde{g}}_{11}\,\mathrm{d}y^1\,\mathrm{d}U + \tilde{g}_{22}^2(\mathrm{d}y^2)^2 + (y^2)^2\dot{\tilde{g}}_{22}^2\,\mathrm{d}U^2\\
&\quad + 2(y^2)\tilde{g}_{22}\dot{\tilde{g}}_{22}\,\mathrm{d}y^2\,\mathrm{d}U\\
&= 2\,\mathrm{d}U\,\mathrm{d}V + g_{11}(\mathrm{d}y^1)^2 + g_{22}(\mathrm{d}y^2)^2. && (3.18)
\end{aligned}
$$

The general expression of a plane wave metric in Brinkmann coordinates will be discussed in the following section.

3.2 Obtaining the plane wave metric in Brinkmann coordinates

We now wish to study general plane waves in a systematic manner. Following Blau,[5] we start by noting for the metric given in Eq. (3.2), the vector ∂_V is null (since $g_{VV} = 0$), nowhere vanishing (since it is a coordinate vector field), and covariantly constant, since

$$\begin{aligned}
\nabla_\nu (\partial_V)^\mu &= \partial_\nu \delta_V{}^\mu + \Gamma^\mu{}_{\nu\alpha} \delta_V{}^\alpha \\
&= \Gamma^\mu{}_{\nu V} \\
&= \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,V} + g_{\alpha V,\nu} - g_{\nu V,\alpha}) \\
&= 0. \qquad (3.19)
\end{aligned}$$

We take the existence of a nowhere vanishing covariantly constant null vector field to be a defining property of a plane wave, and derive the general metric admitting such a field.

Let us consider a $(d+2)$ dimensional space with a metric $g_{\mu\nu}$ and a parallel null vector field Z,

$$Z^\mu{}_{;\nu} = 0. \qquad (3.20)$$

We can lower the index, recalling that $g_{\alpha\beta;\nu} = 0$. Then we can rewrite Eq. (3.20) as the following two conditions:

$$Z_{\mu;\nu} + Z_{\nu;\mu} = 0, \qquad (3.21)$$

$$Z_{\mu;\nu} - Z_{\nu;\mu} = 0. \qquad (3.22)$$

The first condition implies that Z is a Killing vector field, while the second says that Z is also a gradient vector field. If Z^μ is nowhere zero, we can choose a parameter along the integral curves of Z as a coordinate v, i.e. $Z = \partial_v$. This implies $Z^\mu = \delta^\mu{}_v$, and thus

$$Z_\mu = g_{\mu v}. \qquad (3.23)$$

Since Z is null, we have

$$Z_\mu Z^\mu = g_{\mu v} Z^\mu = Z_v = g_{vv} = 0. \qquad (3.24)$$

Also, since $Z = \partial_v$ is a Killing vector, translation along the v-axis is an isometry, and hence the components of the metric are independent of v,

$$g_{\mu\nu,v} = 0. \tag{3.25}$$

Since the terms with the Christoffel symbols cancel out, Eq. (3.22) may be written as

$$Z_{\nu,\mu} = Z_{\mu,\nu}, \tag{3.26}$$

which implies by Poincaré lemma the existence of a function $u(x^\mu)$ such that

$$Z_\mu = g_{\mu v} = u_{,\mu}. \tag{3.27}$$

Thus, the general form of a metric admitting a parallel null vector field, using coordinates $\{u, v, x^a\}$, $a = 1, \ldots, d$, is given by

$$\begin{aligned}
\mathrm{d}s^2 &= g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\
&= g_{uu}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + 2g_{au}\,\mathrm{d}x^a\,\mathrm{d}u + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \\
&[\because g_{uv} = u_{,u} = 1,\ g_{vv} = 0,\ g_{av} = u_{,a} = 0] \\
&= 2\,\mathrm{d}u\,\mathrm{d}v + K(u, x^c)\,\mathrm{d}u^2 + 2A_a(u, x^c)\,\mathrm{d}x^a\,\mathrm{d}u + g_{ab}(u, x^c)\,\mathrm{d}x^a\,\mathrm{d}x^b \qquad (3.28) \\
&[\text{Defining } K = g_{uu} \text{ and } A_a = g_{au}].
\end{aligned}$$

We now choose the class of metrics with $g_{ab} = \delta_{ab}$, so that

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v + K\,\mathrm{d}u^2 + 2A_a\,\mathrm{d}x^a\,\mathrm{d}u + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b\,. \tag{3.29}$$

Such a metric is called a plane-fronted metric with parallel rays, or pp-waves for short. There are coordinate transformations which leave the form of this metric invariant, for example the transformation $v \mapsto v + \Lambda(u, x^a)$. Then the metric is given by

$$\begin{aligned}
\mathrm{d}s^2 &= 2\,\mathrm{d}u\,\mathrm{d}v + 2\Lambda_{,u}\,\mathrm{d}u^2 + 2\Lambda_{,a}\,\mathrm{d}u\,\mathrm{d}x^a + K\,\mathrm{d}u^2 + 2A_a\,\mathrm{d}x^a\,\mathrm{d}u + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \\
&= 2\,\mathrm{d}u\,\mathrm{d}v + (K + 2\Lambda_{,u})\,\mathrm{d}u^2 + 2(A_a + \Lambda_{,a})\,\mathrm{d}x^a\,\mathrm{d}u + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b\,, \qquad (3.30)
\end{aligned}$$

which is in the same form as a pp-wave metric if the coefficients transform as

$$\begin{aligned} K &\mapsto K + 2\Lambda_{,u}, \\ A_a &\mapsto A_a + \Lambda_{,a}. \end{aligned} \tag{3.31}$$

A plane wave is a special kind of pp-wave, where $A_a = 0$ and K is quadratic in x^a, $K(u, x^c) = A_{ab}(u)x^a x^b$. The metric is then

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v + A_{ab}(u)x^a x^b\,\mathrm{d}u^2 + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b\,. \tag{3.32}$$

This is the metric of a plane wave in Brinkmann coordinates. The metric obtained in Eq. (3.13) is a special case of this in 4 dimensions when $A_{11} = h(u)$, $A_{22} = -h(u)$ and $A_{12} = A_{21} = 0$.

It is sometimes convenient to work in a noncoordinate basis, where the basis one-forms are given as

$$E^A = E^A{}_\mu\,\mathrm{d}x^\mu\,. \tag{3.33}$$

The expression of the metric tensor in such a basis will be

$$g = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = g_{AB}E^A \otimes E^B. \tag{3.34}$$

If we choose a basis $\{E^+, E^-, E^a\}$ such that $g_{++} = g_{--} = 0$, $g_{+-} = 1$ and $g_{ab} = \delta_{ab}$, then it is called a pseudo-orthonormal frame. For a plane wave metric in Brinkmann coordinates, if we choose

$$\begin{aligned} E^+ &= \mathrm{d}u\,, \\ E^- &= \mathrm{d}v + \frac{1}{2}A_{ab}x^a x^b\,\mathrm{d}u\,, \\ E^a &= \mathrm{d}x^a\,, \end{aligned} \tag{3.35}$$

then we get a pseudo-orthonormal basis, since

$$\begin{aligned} g &= \mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u + A_{ab}x^a x^b\,\mathrm{d}u \otimes \mathrm{d}u + \delta_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b \\ &= \mathrm{d}u \otimes \left(\mathrm{d}v + \frac{1}{2}A_{ab}x^a x^b\,\mathrm{d}u\right) + \left(\mathrm{d}v + \frac{1}{2}A_{ab}x^a x^b\,\mathrm{d}u\right) \otimes \mathrm{d}u + \delta_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b \\ &= E^+ \otimes E^- + E^- \otimes E^+ + \delta_{ab}E^a \otimes E^b. \end{aligned} \tag{3.36}$$

3.3 Geodesics and the light-cone gauge

We now study the geodesics of a plane wave metric in Brinkmann coordinates. The geodesics satisfy the Euler-Lagrange equations with the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\
&= \dot{u}\dot{v} + \frac{1}{2} A_{ab}(u) x^a x^b \dot{u}^2 + \frac{1}{2} \delta_{ab} \dot{x}^a \dot{x}^b
\end{aligned} \tag{3.37}
$$

(derivatives are with respect to an affine parameter τ), with the constraint

$$
2\mathcal{L} = \begin{cases} 0 & \text{for massless particles,} \\ -1 & \text{for massive particles.} \end{cases} \tag{3.38}
$$

Since $\mathcal{L}$ is independent of v, the momentum conjugate to v,

$$
p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \dot{u} \tag{3.39}
$$

is conserved. For $p_v = 0$, the Lagrangian reduces to that for a flat metric in the transverse coordinates, and the geodesics are straight lines. For $p_v \neq 0$, we choose the light-cone gauge

$$
u = p_v \tau. \tag{3.40}
$$

Then the Euler-Lagrange equations for the transverse coordinates are

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a} \right) = \partial_a \mathcal{L} \\
\implies\ &\delta_{ab} \ddot{x}^b = A_{ab} x^b p_v^2,
\end{aligned} \tag{3.41}
$$

which is the equation of motion of a non-relativistic harmonic oscillator,

$$
\ddot{\delta}_{ab} x^b = -\omega_{ab}^2 x^b, \tag{3.42}
$$

with the frequency matrix

$$
\omega_{ab}^2 = -p_v^2 A_{ab}. \tag{3.43}
$$

The u-equation of motion is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{u}}\right) &= \partial_u\mathcal{L} \\
\implies \ddot{v} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(A_{ab}x^a x^b p_v\right) &= \frac{1}{2}\dot{A}_{ab}x^a x^b p_v \\
\implies \ddot{v} + \frac{1}{2}\dot{A}_{ab}x^a x^b p_v + 2A_{ab}\dot{x}^a x^b p_v &= 0
\end{aligned} \tag{3.44}$$

This is automatically satisfied by the constraint equation

$$\mathcal{L} = p_v\dot{v} + \frac{1}{2}A_{ab}x^a x^b p_v^2 + \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b = 0 \quad \text{or} \quad 1, \tag{3.45}$$

which, upon differentiating both sides by τ, gives

$$\begin{aligned}
0 &= p_v\ddot{v} + \frac{1}{2}\dot{A}_{ab}x^a x^b p_v^2 + A_{ab}\dot{x}^a x^b p_v^2 + \delta_{ab}\dot{x}^a\ddot{x}^b \\
&= p_v\left(\ddot{v} + \frac{1}{2}\dot{A}_{ab}x^a x^b p_v + 2A_{ab}\dot{x}^a x^b p_v\right),
\end{aligned} \tag{3.46}$$

which is nothing but Eq. (3.44).

Multiplying Eq. (3.41) with x^a and substituting in Eq. (3.45), we obtain

$$\begin{aligned}
p_v\dot{v} + \frac{1}{2}x^a\ddot{x}^a + \frac{1}{2}\delta_{ab}x^a x^b x &= 0 \\
\implies p_v\dot{v} &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(x^a\dot{x}^a) \\
\implies v &= -\frac{1}{2p_v}x^a\dot{x}^a + v_0.
\end{aligned} \tag{3.47}$$

Thus, v can be obtained once x^a are known. One trivial geodesic is given by

$$u = p_v\tau, v = v_0, x^a = 0. \tag{3.48}$$

Then the geodesic is entirely parametrised by u, and we can choose $\tau = u$ to have $p_v = 1$. In general, Eq. (3.41) can be solved analytically for some particular functional forms of $A_{ab}(u)$, two of which are shown below.

One case where the equation can be solved easily is when $A_{ab} = \omega_a^2 \delta_{ab}$ ($\omega \in \mathbb{R} \cup i\mathbb{R}$), i.e. constant and diagonal. (Note that the a in ω_a is not a tensor index.) Then we have

$$x^a = A^a e^{p_v \omega_a \tau} + B^a e^{-p_v \omega_a \tau} \quad (a \text{ is not summed over}), \tag{3.49}$$

and

$$v = v_0 - \frac{1}{2} \sum_a \omega_a \big((A^a)^2 e^{2p_v \omega_a \tau} - (B^a)^2 e^{-2p_v \omega_a \tau} \big). \tag{3.50}$$

Clearly, x^a varies with τ exponentially if ω_a is real, and sinusoidally if it is purely imaginary. The τ-dependence of v can be a sum of both of these two kinds of functions, since some of the ω^a can be positive and some other negative. While we took A_{ab} to be diagonal, similar analysis may be done for any constant A_{ab} by diagonalising it.

The other case we will consider here is when $A_{ab}(u)$ has an inverse square dependence on u,

$$A_{ab} = \frac{m_a}{u^2} \delta_{ab} = \frac{m_a}{p_v^2 \tau^2} \delta_{ab}. \tag{3.51}$$

Then the coordinates are obtained as

$$x^a = C^a \tau^{n_+^a} + D^a \tau^{n_-^a}, \tag{3.52}$$

and

$$v = v_0 - \frac{1}{2p_v} \sum_a \big(C^a \tau^{n_+^a} + D^a \tau^{n_-^a} \big) \big(n_+^a C^a \tau^{n_+^a - 1} + n_-^a D^a \tau^{n_-^a - 1} \big), \tag{3.53}$$

where we have defined

$$n_\pm^a = \frac{1 \pm \sqrt{1 + 4m_a}}{2}. \tag{3.54}$$

This is in general well-defined for $\tau > 0$. At $\tau = 0$, x^a and v are both undefined for non-positive or imaginary $n_\pm^a$ (though the limit as $\tau \to 0$ may exist for imaginary $n_\pm^a$), and v is also undefined for $0 < n_\pm^a \leq 1/2$.

The light-cone Hamiltonian is defined as

$$H_{\mathrm{lc}} = -p_u, \tag{3.55}$$

where p_u is the momentum conjugate to u in the light-cone gauge. From the expression of the Lagrangian in Eq. (3.37), we have

$$\begin{aligned} p_u &= \frac{\partial \mathcal{L}}{\partial \dot{u}} \\ &= g_{u\mu}\dot{x}^\mu \\ &= \dot{v} + A_{ab}(p_v\tau)x^a x^b p_v \\ &= -p_v^{-1} H_{\mathrm{ho}}(\tau), \end{aligned} \tag{3.56}$$

where H_{ho} is the harmonic oscillator Hamiltonian,

$$H_{\mathrm{ho}} = \frac{1}{2}\big(\delta_{ab}\dot{x}^a\dot{x}^b - p_v^2 A_{ab}x^a x^b\big). \tag{3.57}$$

Thus the light-cone Hamiltonian is related to the harmonic oscillator Hamiltonian as

$$H_{\mathrm{lc}} = p_v^{-1} H_{\mathrm{ho}}. \tag{3.58}$$

We have previously showed how to construct a pseudo-orthonormal frame at any point for a plane wave metric. Since parallel propagation preserves inner products, such a frame basis may be parallel propagated along any geodesic passing through that point to obtain a pseudo-orthonormal frame E^A at every point $x^\mu(\tau)$ on the geodesic, whose basis vectors are covariantly constant along the geodesic,

$$\dot{x}^\mu E^A{}_{\nu;\mu} = 0. \tag{3.59}$$

For a null geodesic, we may take E_+ to be the tangent vector to the geodesic, since $g_{++} = 0$ implies that E_+ is null, and the tangent vector is parallel propagated by the definition of a geodesic. One frame satisfying this is given by the vectors

$$\begin{aligned} E_+ &= p_v\partial_u + \dot{v}\partial_v + \dot{x}^a\partial_a, \\ E_- &= p_v^{-1}\partial_v, \\ E_a &= \partial_a - p_v^{-1}\dot{x}^a\partial_v \end{aligned} \tag{3.60}$$

(unique up to orthogonal transformation of the transverse vectors), and the one-forms

$$\begin{aligned} E^+ &= p_v^{-1}\,\mathrm{d}u\,, \\ E^- &= -(\dot{v} - p_v^{-1}\dot{x}^2)\,\mathrm{d}u + p_v\,\mathrm{d}v + \dot{x}^a\,\mathrm{d}x^a\,, \\ E^a &= \mathrm{d}x^a - p_v^{-1}\dot{x}^a\,\mathrm{d}u\,. \end{aligned} \tag{3.61}$$

3.4 Curvature of plane waves

The only independent non-zero components of the Riemann tensor of a plane wave metric are

$$R_{uaub} = -A_{ab}, \tag{3.62}$$

and the only non-zero component of the Ricci tensor is

$$R_{uu} = -\operatorname{tr} A. \tag{3.63}$$

The Ricci scalar is zero, $R = 0$, and the only non-zero component of the Einstein tensor is

$$G_{uu} = R_{uu} = -\operatorname{tr} A. \tag{3.64}$$

Thus, the vacuum Einstein equations reduce to the condition that

$$\operatorname{tr} A = 0. \tag{3.65}$$

If $\operatorname{tr} A \neq 0$, then $G_{uu} \neq 0$, and we have plane wave solutions to Einstein equations where T_{uu} is the only non-zero component of the energy-momentum tensor. One example of such a stress-energy tensor is obtained for an electromagnetic potential vector $(A_u(x^1, x^2), A_v = 0, A_1(u), A_2(u))$. The physically realisable functional forms are to be found from the electromagnetic field action in curved spacetime.

3.5 Curvature invariants of plane waves

The curvature invariants (i.e. scalars constructed from the Riemann tensor, its covariant derivatives and the metric) of a plane wave are zero. This can be shown by the following argument.

Firstly, we note that a general curvature invariant will consist of the Riemann tensor and its covariant derivatives (in the form $\nabla_{\mu_1} \cdots \nabla_{\mu_n} R^{\mu}{}_{\nu\lambda\rho}$), with an appropriate number of factors of the inverse metric $g^{\alpha\beta}$ to construct a scalar. Now, under a constant rescaling of the metric, $g_{\alpha\beta} \mapsto \lambda g_{\alpha\beta}$, the Christoffel symbols are conserved, and thus the Riemann tensor and its covariant derivatives are also unchanged. However, at least one factor of the inverse metric is required to construct a scalar, due to which the curvature invariant will not remain unchanged under constant rescaling of the metric.

Therefore, if there exists a coordinate transformation under which the metric undergoes a constant nontrivial rescaling (called a homothety), then the curvature invariant at any fixed point of the transformation has to be zero. This is because the curvature invariant is a scalar and hence should not change under coordinate transformation. On the other hand, the particular coordinate transformation considered induces a rescaling of the metric, under which the curvature invariant should also undergo a scale change. The only way this can hold is if the quantity is zero at that point,.

Finally, we show that for every point x, there exists a homothety whose fixed point is x, i.e. which does not change the coordinates of x. This is more convenient to show in Rosen coordinates. Then the metric is invariant under translation of V and y^i, so without loss of generality, we may take x to be the point $\left(U, 0, \vec{\mathbf{0}}\right)$. Now we take the coordinate transformation

$$\left(U, V, y^i\right) \mapsto \left(U, \lambda^2 V, \lambda y^i\right) \qquad \lambda \in \mathbb{R}\backslash\{0,1\}. \tag{3.66}$$

Clearly, $\left(U, 0, \vec{\mathbf{0}}\right)$ is a fixed point of this transformation. Also, under this transformation, the metric is transformed as

$$\mathrm{d}s^2 \mapsto 2\,\mathrm{d}U\,\lambda^2\,\mathrm{d}V + g_{ij}(U)\lambda\,\mathrm{d}y^i\,\lambda\,\mathrm{d}y^j = \lambda^2\,\mathrm{d}s^2\,,$$

so this transformation induces a rescaling of the metric keeping x as a fixed point. Therefore, curvature invariants of a plane wave metric are zero everywhere.

3.6 Singularities of the plane wave metric

Even though the curvature invariants are zero for a plane wave metric, it is still possible for singularities to exist. To see this, we start with the geodesic deviation equation,

$$\frac{D^2}{D\tau^2}\,\delta x^\mu = R^\mu{}_{\alpha\beta\nu}\dot{x}^\alpha\dot{x}^\beta\,\delta x^\nu\,, \tag{3.67}$$

where τ is an affine parameter and δx^μ is the separation between two nearby geodesics. If we choose u to be the affine parameter, then the equation for the transverse coordinates becomes

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}u^2}\,\delta x^a &= R^a{}_{\alpha\beta\nu}\dot{x}^\alpha\dot{x}^\beta\,\delta x^\nu\\ &= R^a{}_{uub}\dot{u}^2\,\delta x^b\\ &= A_{ab}\,\delta x^b\,.\end{aligned} \tag{3.68}$$

From this equation, we see that there exists a singularity at a point if and only if $A_{ab}(u)$ diverges at that point.

3.7 Relation between Rosen and Brinkmann coordinates

We wish to obtain the Brinkmann coordinates (u, v, x^a) satisfying

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v + A_{ab}(u)x^a x^b\,\mathrm{d}u^2 + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b\,, \tag{3.32 revisited}$$

where the Rosen coordinates (U, V, y^i) are given with the general plane wave metric

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + g_{ij}(U)\,\mathrm{d}y^i\,\mathrm{d}y^j\,. \tag{3.2 revisited}$$

The transverse metric is flat in Brinkmann coordinates but non-flat in Rosen coordinates. Thus, the transformation of the transverse coordinates should be of the form

$$x^a = E^a{}_i y^i, \tag{3.69}$$

where $E^a{}_i$ are functions of u and

$$g_{ij} = E^a{}_i E^b{}_j \delta_{ab}. \tag{3.70}$$

Denoting the inverse transformation by $E^i{}_a$ such that $y^i = E^i{}_a x^a$, we get

$$\begin{aligned}
g_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j &= E^a{}_i\,\mathrm{d}y^i\,E^b{}_j\,\mathrm{d}y^j\,\delta_{ab}\\
&= \Big(\mathrm{d}x^a - y^i\dot{E}^a{}_i\,\mathrm{d}u\Big)\Big(\mathrm{d}x^b - y^j\dot{E}^b{}_j\,\mathrm{d}u\Big)\delta_{ab}\\
&= \Big(\mathrm{d}x^a - E^i{}_c\dot{E}^a{}_i x^c\,\mathrm{d}u\Big)\Big(\mathrm{d}x^b - E^j{}_d\dot{E}^b{}_j x^d\,\mathrm{d}u\Big)\delta_{ab}\\
&= \mathrm{d}\vec{\mathbf{x}}^2 + \dot{E}^a{}_i E^i{}_c \dot{E}_{aj} E^j{}_d x^c x^d\,\mathrm{d}u^2 - 2\dot{E}_{ai}E^i{}_c x^c\,\mathrm{d}x^a\,\mathrm{d}u
\end{aligned} \tag{3.71}$$

Thus we get the terms $\delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ as well as a $\mathrm{d}u^2$ term whose coefficient is quadratic in x^a, along with cross terms of the form $\mathrm{d}x^a\,\mathrm{d}u$. If E's satisfy the symmetry condition

$$\dot{E}_{ai}E^i{}_b = \dot{E}_{bi}E^i{}_a, \tag{3.72}$$

then we can use the transformations

$$U = u, \tag{3.73}$$

$$V = v + \frac{1}{2}\dot{E}_{ai}E^i{}_b x^a x^b \tag{3.74}$$

so that

$$\begin{aligned}
2\,\mathrm{d}U\,\mathrm{d}V &= 2\,\mathrm{d}u\,(\mathrm{d}v + \frac{1}{2}(\ddot{E}_{ai}E^i{}_b x^a x^b\,\mathrm{d}u + \dot{E}_{ai}\dot{E}^i{}_b x^a x^b\,\mathrm{d}u + \dot{E}_{ai}E^i{}_b x^b\,\mathrm{d}x^a \\
&\quad + \dot{E}_{ai}E^i{}_b x^a\,\mathrm{d}x^b)) \\
&= 2\,\mathrm{d}u\,\mathrm{d}v + \left(\ddot{E}_{ai}E^i{}_b + \dot{E}_{ai}\dot{E}^i{}_b\right)x^a x^b\,\mathrm{d}u^2 + \dot{E}_{ai}E^i{}_b x^b\,\mathrm{d}x^a\,\mathrm{d}u \\
&\quad + \dot{E}_{ai}E^i{}_b x^a\,\mathrm{d}x^b\,\mathrm{d}u \\
&= 2\,\mathrm{d}u\,\mathrm{d}v + \left(\ddot{E}_{ai}E^i{}_b + \dot{E}_{ai}\dot{E}^i{}_b\right)x^a x^b\,\mathrm{d}u^2 + 2\dot{E}_{ai}E^i{}_b x^b\,\mathrm{d}x^a\,\mathrm{d}u \\
&\quad \text{(using Eq. (3.72))}.
\end{aligned} \tag{3.75}$$

Finally, we note that

$$\begin{aligned}
\dot{E}^a{}_i E^i{}_c \dot{E}_{aj} E^j{}_d x^c x^d &= \dot{E}^c{}_i E^i{}_b \dot{E}_{cj} E^j{}_a x^b x^a \\
&= \left(0 - E^c{}_i \dot{E}^i{}_b\right)\dot{E}_{aj} E^j{}_c x^a x^b \\
&\quad \text{(using Eq. (3.72) and that } E^c{}_i E^i{}_b = \delta^c{}_b) \\
&= -\delta^j{}_i \dot{E}^i{}_b \dot{E}_{aj} \\
&= -\dot{E}_{ai}\dot{E}^i{}_b.
\end{aligned} \tag{3.76}$$

Hence, adding Eqs. (3.71) and (3.75) and using Eq. (3.76), we get the plane wave metric in Brinkmann coordinates, with A_{ab} given as

$$A_{ab} = \ddot{E}_{ai}E^i{}_b. \tag{3.77}$$

We have previously shown the transformation for the case when $d = 2$ and the transverse metric in Rosen coordinates is diagonal. Now we will show the transformations for a diagonal transverse metric g_{ij} arise as a special case of the transformation equations given above. Clearly, if we choose the transformation operators $E^a{}_i$ for transverse coordinates in this case as

$$E^a{}_i = \sqrt{g_{ii}}\delta^a{}_i, \tag{3.78}$$

then Eq. (3.70) is satisfied. Then the transverse coordinates transformation as

$$x^i = \sqrt{g_{ii}}y^i \tag{3.79}$$

which are exactly the transformations we had seen previously in Eqs. (3.16) and (3.17).

We also see that

$$\dot{E}_{aj}E^j{}_a = \frac{\dot{\sqrt{g_{ii}}}}{\sqrt{g_{ii}}}\delta_{ai}, \quad \dot{E}_{aj}E^j{}_b = \dot{E}_{bj}E^j{}_a = 0 \quad \text{for} \quad a \neq b, \tag{3.80}$$

so Eq. (3.72) is also satisfied. Then the transformations for the null coordinates are obtained as

$$U = u, \tag{3.81}$$

$$V = v + \frac{1}{2}\sum_i \frac{\dot{\sqrt{g_{ii}}}}{\sqrt{g_{ii}}}\left(x^i\right)^2,$$

$$\implies v = V - \frac{1}{2}\sum_i \sqrt{g_{ii}}\dot{\sqrt{g_{ii}}}\left(y^i\right)^2, \quad \text{(from Eq. (3.79))} \tag{3.82}$$

which are again the same transformations we had obtained previously in Eqs. (3.14) and (3.15). For the Brinkamnn metric, A_{ab} is seen to be (from Eq. (3.77))

$$A_{ab} = \frac{\ddot{\sqrt{g_{ii}}}}{\sqrt{g_{ii}}}\delta^i{}_a\delta^i{}_b, \tag{3.83}$$

which gives us the same metric as in Eq. (3.13).

Chapter 4

Penrose Limits

4.1 Construction of Penrose limit for a metric

Penrose[8] has shown that under a particular limiting case, every metric approaches the plane wave metric. This qualitatively means that all metrics appear to be gravitational waves to highly boosted observers.

The Penrose limit provides for every metric $g_{\mu\nu}$ and null geodesic γ a plane wave metric in a limiting case. Firstly, we choose U, an affine parameter for the geodesic γ, to be a coordinate, so that e_U is a parallel null vector field. Then we may find a metric of the form

$$\mathrm{d}{s_\gamma}^2 = 2\,\mathrm{d}U\,\mathrm{d}V + a\big(U,V,Y^k\big)\,\mathrm{d}V^2 + 2b_i\big(U,V,Y^k\big)\,\mathrm{d}V\,\mathrm{d}Y^i + g_{ij}\big(U,V,Y^k\big)\,\mathrm{d}Y^i\,\mathrm{d}Y^j\,, \tag{4.1}$$

in a method similar to what we did in the last chapter. Every curve parametrized by U (with V and Y^k constant) is a null geodesic, and γ is the geodesic with $V = Y^k = 0$.

Now we change transform the coordinates as

$$(U, V, Y^k) = (u, \lambda^2 v, \lambda y^k) \quad (\lambda \in \mathbb{R}\backslash\{0\}), \tag{4.2}$$

and the metric in the new set of coordinates is

$$\mathrm{d}{s_{\gamma,\lambda}}^2 = 2\lambda^2\,\mathrm{d}u\,\mathrm{d}v + \lambda^4 a\,\mathrm{d}v^2 + 2\lambda^3 b\,\mathrm{d}v\,\mathrm{d}y^i + \lambda^2 g_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j \tag{4.3}$$

The Penrose limit is then given by

$$\mathrm{d}s^2 = \lambda^{-2}\lim_{\lambda\to 0}\mathrm{d}{s_{\gamma,\lambda}}^2 = 2\,\mathrm{d}u\,\mathrm{d}v + g_{ij}(U,0,0)\,\mathrm{d}y^i\,\mathrm{d}y^j\,. \tag{4.4}$$

This is the metric of a plane wave in Rosen coordinates. Thus, the Penrose limit may be obtained by ignoring the coefficients a and b_i and restricting g_{ij} to the null geodesic γ. The coordinates can be further transformed to obtain the metric in the Brinkmann form, as shown in the previous chapter.

4.2 Obtaining the adapted coordinates

From Eq. (4.1), it can be seen that the vector ∂_U is null (since $g_{UU} = 0$), and parallel along itself (since $\nabla_U \partial_U = 0 \iff \Gamma^\mu{}_{UU} = 0 \forall \mu$, which holds as $g_{U\nu}$ is constant for all ν). Thus, every curve with constant V and Y^k parametrized by U is a null geodesic; γ is the one for which $V = Y^k = 0$. Therefore, to obtain the adapted coordinates, we first need to embed γ in a congruence of null geodesics. An affine parameter of the geodesics can then be chosen to be U, and V and Y^k serve to label different geodesics within the congruence. We take work with a manifold of $D = d + 2$ dimensions.

First of all, we note that every such geodesic will have $\dot{U} = 1$ and $\dot{V} = \dot{Y}^k = 0$. Since the conjugate momenta for a geodesic satisfy $p_\mu = g_{\mu\nu}\dot{x}^\nu$, we get $p_V = 1$, $p_U = p_a = 0$ Therefore, in a general system of coordinates,

$$p_\mu = \frac{\partial V}{\partial x^\mu} p_V = \partial_\mu V. \tag{4.5}$$

Finally, we impose the condition that the geodesic is null. Then the tangent vectors, as well as the conjugate momenta, ar null, and we thus obtain

$$g^{\mu\nu} \partial_\mu V \partial_\nu V = 0. \tag{4.6}$$

This is called the Hamilton-Jacobi equation by analogy with the Hamilton-Jacobi equation of Classical Mechanics. Equation (4.6) is first order in the D variables x^μ; thus, its solution will have D constants. One of them is an additive constant, which has no physical significance as Eq. (4.6) only involves the derivatives of V. Another is a scale factor, which also does not affect the conditions $g_{VV} = g_{Vk} = 0$; the condition that $g_{UV} = 0$ can be ensured by applying a suitable scale transformation on the affine parameter U. Thus we have $D - 2 = d$ constants which serve to distinguish the particular geodesic congruence in which γ belongs; their values are obtained from the known geodesic γ.

Once V is known, we can find

$$\dot{x}^\mu = g^{\mu\nu}\partial_\nu V, \tag{4.7}$$

where the dot represents differentiation with respect to U. This is a system of first order equations in one independent variable and D dependent variables, and thus can be solved to obtain the geodesic $x^\mu(U)$, given the initial values $x^\mu(U = 0) = x_0^\mu$. The initial points are thus responsible for identifying different geodesics within the congruence. x_0^μ can be any point in the manifold, but we only need a set of points such that every geodesic has exactly one point in that set, i.e. a surface which every geodesic intersects once. This avoids choosing different points on the same geodesic as initial points. The surface can be chosen arbitrarily, but it should be a Cauchy surface as every point on the manifold can be reached from the surface by a null geodesic. The surface is taken to be spacelike, with a timelike normal.

Now we have the congruence of geodesics, given as $x^\mu(U)$. We also have the coordinate $V(x^\mu)$. What remains is to find the transverse coordinates Y^k, which along with V distinguish the different geodesics within the congruence. V and Y^k will thus parametrize the $D - 1$ dimensional initial point surface. First of all, we check that

$$\dot{V} = \dot{x}^\mu \partial_\mu V = g^{\mu\nu}\partial_\nu V \partial_\mu V = 0, \tag{4.8}$$

i.e. V is constant along a geodesic as expected. In most situations, one can easily find a suitable set of Y^k from the starting coordinates x^μ, where we label the indices as $1 \leq \mu \leq D$. Since we need only $D-1$ coordinates to identify a geodesic, one coordinate, say x^D, may be taken to depend only on the affine parameter U. The transverse coordinates Y^k can then be defined as $Y^k = x^k - \int_0^U \partial_U x^k \, \mathrm{d}U$ for $1 \leq k \leq D-2$. This makes Y^k constant along a geodesic as we require. $\partial_U x^k$ is simply $\dot{x}^k$ on a curve of constant V and Y^k, which is of course already known from Eq. (4.7). x^{D-1} can then be found by inverting $V = V(x^\mu)$, since all other coordinates are already known. Thus we obtain $x^\mu = x^\mu(U, V, Y^k)$, which can be inverted to obtain the adapted coordinates as functions of the coordinates we had started with. Finally, we check that the required conditions on the metric are indeed satisfied by these coordinates, since we have

$$g_{UU} = g_{\mu\nu}\frac{\partial x^\mu}{\partial U}\frac{\partial x^\nu}{\partial U} = 0 \quad (U \text{ parametrises a null geodesic}) \quad , \tag{4.9}$$

$$g_{UV} = g_{\mu\nu}\frac{\partial x^\mu}{\partial U}\frac{\partial x^\nu}{\partial V} = g_{\mu\nu}g^{\mu\rho}\partial_\rho V\frac{\partial x^\nu}{\partial V} = \partial_\nu V\frac{\partial x^\nu}{\partial V} = \frac{\partial V}{\partial V} = 1, \tag{4.10}$$

$$g_{Ui} = g_{\mu\nu}\frac{\partial x^\mu}{\partial U}\frac{\partial x^\nu}{\partial Y^i} = g_{\mu\nu}g^{\mu\rho}\partial_\rho V\frac{\partial x^\nu}{\partial Y^i} = \frac{\partial V}{\partial Y^i} = 0. \tag{4.11}$$

4.3 The Penrose limit from the adapted coordinates

As explained before, the next step to obtain the Penrose limit once the adapted coordinates $\left(U, V, Y^k\right)$ are found is to do the coordinate transformation $\left(U, V, Y^k\right) = \left(u, \lambda^2 v, \lambda y^k\right)$. This coordinate transformation can be physically interpreted as the Lorentz transformation $(U, V, Y^k) \mapsto (\lambda U, \lambda^{-1} V, Y^k)$, followed by a scale change by λ^{-1}. We also redefine the metric as $g \mapsto \lambda^{-2} g$. Then the metric in the new coordinates is, as seen before,

$$\mathrm{d}{s_{\gamma,\lambda}}^2 = 2\,\mathrm{d}u\,\mathrm{d}v + \lambda^2 b(u, v, y^k)\,\mathrm{d}v^2 + 2\lambda a_i\,\mathrm{d}v\,\mathrm{d}y^i + g_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j\,. \tag{4.12}$$

The Penrose limit is then obtained by taking $\lambda \to 0$, which can be physically interpreted as increasing the boost of the Lorentz transformation to infinity while rescaling the metric correspondingly.

From the coordinate transformation, it is clear that any point at a finite distance from γ (i.e. with a finite V or Y^k) will correspond to a point at infinity. Thus, the Penrose limit gives an infinitesimal neighbourhood of the null geodesic, just as the tangent space at a point maybe treated as an infinitesimal neighbourhood of a point. However, while the tangent space is flat, the Penrose limit has a plane wave structure.

4.4 The Penrose limit of the Schwarzschild metric

As an example, let us now find the Penrose limit of the Schwarzschild metric,

$$\mathrm{d}s^2 = -f(t)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r\,2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \tag{4.13}$$

where

$$f(r) = 1 - \frac{2m}{r}. \tag{4.14}$$

To begin with, we may use the spherical symmetry of the metric to argue that any geodesic may be made to lie in a plane of constant ϕ by choosing the coordinates appropriately. We then take the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\Big(-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2\Big), \tag{4.15}$$

where the dot represents differentiation with respect to an affine parameter τ. The Euler-Lagrange equations for t and θ are integrated to obtain

$$\dot{\theta} = Lr^{-2} \tag{4.16}$$

$$\text{and} \quad \dot{t} = f(r)^{-1}E, \tag{4.17}$$

where E and L are the constant energy and angular momentum respectively. They can be calculated from the provided geodesic γ, and are the same for all geodesics in the congruence. We now use the condition that $\mathcal{L} = 0$ for null geodesics, which gives

$$\begin{aligned} -f(r)E^2 f(r)^{-2} + f(r)^{-1}\dot{r}^2 + r^2 L^2 r^{-4} = 0 \\ \implies \dot{r}^2 = E^2 - L^2 f(r) r^{-2} \equiv E^2 - 2V_{\text{eff}}(r), \end{aligned} \tag{4.18}$$

where $V_{\text{eff}}(r) \equiv L^2 f(r) r^{-2}/2$ is called the effective potential, since differentiating Eq. (4.18) once with respect to τ gives us the Newtonian equation of motion,

$$2\dot{r}\ddot{r} = -2V'_{\text{eff}}(r)\dot{r} \implies \ddot{r} = -V'_{\text{eff}}(r). \tag{4.19}$$

Now we have to find the adapted coordinates. We choose U to be the parameter τ. $V(x^\mu)$ has to satisfy Eq. (4.6). We use the ansatz

$$V = -Et + L\theta + \rho(r), \tag{4.20}$$

whereupon the Hamilton-Jacobi equation gives

$$\begin{aligned} -f(r)^{-1}(-E)^2 + f(r)\rho'(r)^2 + r^{-2}L^2 = 0 \\ \implies \rho'(r)^2 = f(r)^{-2}E^2 - L^2 f(r)^{-1} r^{-2} = f(r)^{-2}\dot{r}^2 \quad \text{(using Eq. (4.18))} \\ \implies \rho(r) = \int f(r)^{-1}\dot{r}\,\mathrm{d}r = \int f(r)^{-1}\dot{r}^2\,\mathrm{d}\tau\,. \end{aligned} \tag{4.21}$$

We need two transverse coordinates, which are constant along a particular geodesic. One of them is of course ϕ, the other we can obtain from θ by removing the U-dependence as explained before. We can choose r to depend only on U, and then t is obtained from Eq. (4.20). The coordinate differentials or basis one-forms are then given in terms of the new coordinates $\left(U, V, \tilde{\theta}, \tilde{\phi}\right)$ by

$$\mathrm{d}\phi = \mathrm{d}\tilde{\phi}\,, \tag{4.22}$$

$$\begin{aligned}\mathrm{d}\theta &= \dot{\theta}(U)\,\mathrm{d}U + \mathrm{d}\tilde{\theta} \\ &= Lr(U)^{-2}\,\mathrm{d}U + \mathrm{d}\tilde{\theta}\,,\end{aligned} \tag{4.23}$$

$$\mathrm{d}r = \dot{r}(U)\,\mathrm{d}U\,, \tag{4.24}$$

$$\begin{aligned}\mathrm{d}t &= -E^{-1}\,\mathrm{d}V + E^{-1}L\,\mathrm{d}\theta + E^{-1}\,\mathrm{d}\rho(r(U)) \\ &= -E^{-1}\,\mathrm{d}V + E^{-1}L^2 r(U)^{-2}\,\mathrm{d}U + E^{-1}L\,\mathrm{d}\tilde{\theta} + E^{-1}f(r(U))^{-1}\dot{r}^2\,\mathrm{d}U \\ &= -E^{-1}\,\mathrm{d}V + E^{-1}L^2 r^{-2}\,\mathrm{d}U + E^{-1}L\,\mathrm{d}\tilde{\theta} + E^{-1}f(r)^{-1}\big(E^2 - L^2 f(r) r^{-2}\big)\,\mathrm{d}U \\ &= -E^{-1}\,\mathrm{d}V + E^{-1}L\,\mathrm{d}\tilde{\theta} + Ef(r(U))^{-1}\,\mathrm{d}U\,.\end{aligned} \tag{4.25}$$

e note that the term appearing with $\mathrm{d}U$ in the last equation is the previously obtained expression of $\dot{t}(U)$, as expected. Now we can find the metric in adapted coordinates to get

$$\begin{aligned}\mathrm{d}s_\gamma{}^2 &= -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \\ &= -f\Big(E^{-1}\,\mathrm{d}V + E^{-1}L\,\mathrm{d}\tilde{\theta} + Ef^{-1}\,\mathrm{d}U\Big)^2 + f^{-1}(\dot{r}\,\mathrm{d}U)^2 + r^2\Big(\mathrm{d}\tilde{\theta} + Lr^{-2}\,\mathrm{d}U\Big)^2 \\ &\quad + r^2\sin^2\theta\,\mathrm{d}\tilde{\phi}^2 \\ &= -f\Big(E^{-2}\,\mathrm{d}V^2 + E^{-2}L^2\,\mathrm{d}\tilde{\theta}^2 + E^2 f^{-2}\,\mathrm{d}U^2 - 2E^{-2}L\,\mathrm{d}V\,\mathrm{d}\tilde{\theta} + 2Lf^{-1}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}U\Big) \\ &\quad - f\big(-2f^{-1}\,\mathrm{d}U\,\mathrm{d}V\big) + f^{-1}\dot{r}^2\,\mathrm{d}U^2 + r^2\,\mathrm{d}\tilde{\theta}^2 + L^2 r^{-2}\,\mathrm{d}U^2 + 2L\,\mathrm{d}\tilde{\theta}\,\mathrm{d}U \\ &\quad + r^2\sin^2\theta\,\mathrm{d}\tilde{\phi}^2 \\ &= 2\,\mathrm{d}U\,\mathrm{d}V - E^{-2}f(r(U))\,\mathrm{d}V^2 + E^{-2}r(U)^2\dot{r}(U)^2\,\mathrm{d}\tilde{\theta}^2 + 2E^{-2}Lf(r(U))\,\mathrm{d}V\,\mathrm{d}\tilde{\theta} \\ &\quad + r(U)^2\sin^2\left(\tilde{\theta} + L\int r(U)^{-2}\,\mathrm{d}U\right)\mathrm{d}\tilde{\phi}^2\,.\end{aligned} \tag{4.26}$$

This is in the required form for the adapted coordinates. The Penrose limit metric will have $g_{VV} = g_{Vi} = 0$, and g_{ij} restricted to the geodesic with $\tilde{\theta} = \tilde{\phi} = 0$. Thus, without relabelling the coordinates, we the Penrose limit is obtained as

$$\mathrm{d}\tilde{s}^2 = 2\,\mathrm{d}U\,\mathrm{d}V - E^{-2} r(U)^2 \dot{r}(U)^2\,\mathrm{d}\tilde{\theta}^2 + r(U)^2 \sin^2\left(L\int r(U)^{-2}\,\mathrm{d}U\right)\mathrm{d}\tilde{\phi}^2\,. \tag{4.27}$$

If $L = 0$, then $\tilde{\theta} = 0 \implies \theta = 0$, which is a coordinate singularity for the Schwarzschild metric. So in that case, we define replace $\tilde{\theta}$ with $\tilde{\theta} + \theta_0$, so that the metric becomes

$$\mathrm{d}\tilde{s}^2 = 2\,\mathrm{d}U\,\mathrm{d}V - E^{-2} r(U)^2 \dot{r}(U)^2\,\mathrm{d}\tilde{\theta}^2 + r(U)^2 \sin^2\theta_0\,\mathrm{d}\tilde{\phi}^2\,. \tag{4.28}$$

Let us work with $L \neq 0$ for now. The Rosen transverse metric is diagonal, and thus A_{ab} of the corresponding Brinkmann metric is (using $u = U$)

$$\begin{aligned}
A_{11}(u) &= \frac{\frac{\mathrm{d}^2}{\mathrm{d}u^2} r(u)\dot{r}(u)}{r(u)\dot{r}(u)} \\
&= \frac{3\dot{r}\ddot{r} + r\dddot{r}}{r\dot{r}} \\
&= -\frac{3V'_{\text{eff}}(r(u))}{r(u)} + V''_{\text{eff}}(r(u)) \quad \text{(by Eq. (4.19))},
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
A_{22}(u) &= \frac{\frac{\mathrm{d}^2}{\mathrm{d}u^2} r(u)\sin\big(L\int r(u)^{-2}\,\mathrm{d}u\big)}{r(u)\sin\big(L\int r(u)^{-2}\,\mathrm{d}u\big)} \\
&= \frac{\ddot{r}\sin\big(L\int r(u)^{-2}\,\mathrm{d}u\big) + 2Lr^{-2}\dot{r}\cos\big(L\int r(u)^{-2}\,\mathrm{d}u\big)}{r\sin\big(L\int r(u)^{-2}\,\mathrm{d}u\big)} \\
&\quad + \frac{-L^2 r^{-3}\sin\big(L\int r(u)^{-2}\,\mathrm{d}u\big) - 2Lr^{-2}\dot{r}\cos\big(L\int r(u)^{-2}\,\mathrm{d}u\big)}{r\sin\big(L\int r(u)^{-2}\,\mathrm{d}u\big)} \\
&= \frac{-V'_{\text{eff}}(r(u))}{r(u)} - L^2 r(u)^{-4}.
\end{aligned} \tag{4.30}$$

Using $f(r) = 1 - 2m/r$, we get $V_{\text{eff}}(r) = L^2 r^{-2}/2 - mL^2 r^{-3}$, which gives

$$A_{11}(u) = -A_{22}(u) = \frac{3mL^2}{r(u)^5}. \tag{4.31}$$

We see that $\operatorname{tr} A = 0$, i.e. the Penrose limit is Ricci-flat, which is expected as the Schwarzschild metric satisfies the vacuum Einstein equation. We can also see that for a general $f(r)$, we have $\operatorname{tr} A = 0$ if and only if $2f(r) - r^2 f''(r) - 2 = 0$. The vacuum

Einstein equation, on the other hand, gives $rf'(r)+f(r)-1=0$, which automatically satisfies the $\operatorname{tr} A = 0$ condition, as

$$\begin{aligned} rf'(r) + f(r) - 1 &= 0 \\ \implies rf''(r) + 2f'(r) &= 0 \\ \implies r^2 f''(r) + 2(1 - f(r)) &= 0 \\ \implies 2f(r) - r^2 f''(r) - 2 &= 0. \end{aligned} \tag{4.32}$$

The Einstein equation is satisfied only for $f(r) = 1 - 2m/r$, whereas the flat Penrose metric condition is satisfied for $f(r) = 1 + 1 + 2m/r + Kr^2$ for any constant K. However, since we expect the metric to be flat at infinity, the Kr^2 term is anyway not realisable physically.

The Penrose limit of the Scwarzschild metric in Brinkmann coordinates is thus

$$\mathrm{d}\tilde{s}^2 = 2\,\mathrm{d}u\,\mathrm{d}v + \frac{3mL^2}{r(u)^5}\left((x^1)^2 - (x^2)^2\right)\mathrm{d}u^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2. \tag{4.33}$$

The exact form will of course depend on the function $r(u)$, which can be found from the geodesic γ.

4.5 The covariant description of Penrose limits

We have seen how to find the Penrose limit for a general metric, given a null geodesic. However, the procedure employed was a complicated one involving a number of coordinate transformations, as well as a metric rescaling taking a limit at one point. To recap, we first had to find the coordinates adapted to the null geodesic to get the metric in a convenient form, then we had to do a further coordinate transformation (equivalent to a Lorentz boost along with a scale change), rescale the metric, take a limiting case for the coordinate transformation and rescaling to get the plane wave metric in Rosen coordinates, and then finally transform to Brinkmann coordinates which are more convenient to handle plane waves and do not have spurious singularities. Not only is this procedure tedious to work out, it is expected intuitively that the matrix A_{ab} of the Brinkmann metric depends only on the initial metric $g_{\mu\nu}$ and should be directly obtainable from it. We will study the physical meaning of A_{ab} and thus discuss how to obtain it, using a covariant description of the Penrose limit.

4.5.1 Penrose limit and curvature

We have seen before that the Riemann tensor component $\tilde{R}_{uaub}$ of a plane wave metric is equal to $-A_{ab}$. Let us see if this can be related to the Riemann tensor of the metric $g_{\mu\nu}$.

We start with the adapted coordinates $\left(U, V, Y^k\right)$. This may seem cumbersome, as the purpose of this chapter is to get the Penrose limit directly without having to do all the intermediate coordinate transformations, but it will be helpful for now, and we will eventually get a coordinate independent description. Then we have $g_{UV} = 1,\ g_{UU} = g_{Ui} = 0$. The inverse metric will have $g^{UV} = 1,\ g^{VV} = g^{Vi} = 0$. In what follows, Greek indices represent all coordinates, while Roman indices denote transverse coordinates only. Then we have

$$\Gamma^{\alpha}{}_{\beta U} = \frac{1}{2} g^{\alpha\gamma} g_{\beta\gamma,U}. \tag{4.34}$$

which means $\Gamma^{\alpha}{}_{UU} = 0$ and $\Gamma^{i}{}_{\alpha U} = \frac{1}{2} g^{ij} g_{\alpha j,U}$. Also, $\Gamma^{\alpha}{}_{jU} = \frac{1}{2} g^{\alpha k} g_{jk,U}$, which is non-zero only if α is a transverse coordinate. This implies that

$$\begin{aligned} R^{i}{}_{UjU} &= \Gamma^{i}_{UU,j} - \Gamma^{i}{}_{Uj,U} + \Gamma^{i}{}_{\alpha j} \Gamma^{\alpha}{}_{UU} - \Gamma^{i}{}_{\alpha U} \Gamma^{\alpha}{}_{jU} \\ &= -\Gamma^{i}{}_{Uj,U} - \Gamma^{i}{}_{kU} \Gamma^{k}{}_{jU}, \end{aligned} \tag{4.35}$$

which depends only on the transverse metric g_{ij} of the adapted coordinates, and not on the other metric components a and b_i. This is fortunate, since the information of

a and b_i ar lost when we take the limit, and so any quantity dependent on them can not be expected to remain unchanged in the Penrose limit metric. g_{ij}, however, only gets restricted to the geodesic γ (i.e. $g_{ij}\big|_{\text{Penrose}} = g_{ij}\big|_{\gamma}$) when the limit is taken, and so the Riemann tensor component of the Penrose limit can be found from that of the original metric as $\tilde{R}^i{}_{UjU} = R^i{}_{UjU}\big|_{\gamma}$. We also note that $\Gamma^V{}_{\alpha\beta} = -\frac{1}{2}g_{\alpha\beta,U}$ is non-zero only if none of α or β is U, and thus $R^V{}_{UjU} = 0$.

Now we introduce a pseudo-orthonormal basis (E_+, E_-, E_a) for the metric, which is parallel propagated along γ. Since the tangent vector is parallel propagated by definition of a geodesic, we can choose one of the basis vectors, say E_+, to be the tangent vector ∂_U of the geodesic. Then, since $g(E_+, E_a) = 0$, we have $E_a{}^V = 0$, i.e. $E_a = E_a{}^i\partial_i + E_a{}^U\partial_U$. Also, $g(E_+, E_-) = 1$ implies $E_-{}^V = 1$. We can find $E_a{}^i$ from $g(E_a, E_b) = E_a{}^i E_b{}^j g_{ij} = \delta_{ab}$, i.e. $E_a{}^i$ depends only on g_{ij} and not on a or b_i. Thus $E_a{}^i$ for the Penrose limit can also be found from the metric directly just as we had found $E^i{}_{UjU}$ before; $\tilde{E}_a{}^i = E_a{}^i\big|_{\gamma}$. Finally, we use the fact that E_a is parallel propagated along the geodesic to obtain

$$
\begin{aligned}
\nabla_u E_a{}^i &= 0 \\
\implies \dot{E}_a{}^i &= -\Gamma^i{}_{U\nu} E_a{}^\nu \\
\implies \dot{E}_{ai} &= \partial_U \big(g_{ij} E_a{}^j \big) \\
&= g_{ij} \dot{E}_a{}^j + g_{ij,U} E_a{}^j \\
&= g_{ij,U} E_a{}^j - g_{ij} \Gamma^j{}_{u\nu} E_a{}^\nu \\
\implies \dot{E}_{ai} E_b{}^i &= g_{ij,U} E_a{}^j E_b{}^i - g_{ij} \Gamma^j{}_{U\nu} E_a{}^\nu E_b{}^i \\
&= g_{ij,U} E_a{}^j E_b{}^i - \frac{1}{2} g_{ij} g^{j\alpha} g_{\alpha\nu,U} E_a{}^\nu E_b{}^i \\
&= g_{ij,U} E_a{}^j E_b{}^i - \frac{1}{2} g_{i\nu,U} E_a{}^\nu E_b{}^i \\
&= g_{ij,U} E_a{}^j E_b{}^i - \frac{1}{2} g_{ij,U} E_a{}^j E_b{}^i \quad (\because g_{Ui} = g_{Vi} = 0) \\
&= g_{ij,U} E_b{}^j E_a{}^i - \frac{1}{2} g_{ij,U} E_b{}^j E_a{}^i \quad (\because g_{ij} = g_{ji}) \\
&= \dot{E}_{bi} E_a{}^i.
\end{aligned}
\tag{4.36}
$$

This happens to be the same condition that we had imposed without reasoning when studying the transformation from Rose to Brinkmann coordinates. Of course, the index a here refers to a pseudo-orthonormal frame and not a Brinkmann coordinate, but that need not bother us as we have seen before that for the geodesic γ (with $U = u$ the only varying coordinate), we can always find a pseudo-orthonormal frame such that $E_a = \partial_a$ and $E^a = dx^a$.

Finally, we are in a position to find the matrix A_{ab} of the Penrose limit in Brinkmann coordinates. For the plane wave metric, we know that $A_{ab} = -\tilde{R}_{uaub} =$

$\tilde{R}_{+a+b}$, since the frame E_A is chosen such that $E_+ = \partial_u$. But R_{+a+b} can be written as

$$\begin{aligned} R_{+a+b} &= R_{a+b+} \\ &= E_a{}^i E_b{}^j R_{iUjU} \quad (\because E_a{}^V = 0, R_{UUjU} = R_{iUUU} = 0) \\ &= E_a{}^i E_b{}^j g_{ik} R^k{}_{UjU}. \quad (\because g_{iU} = 0, R^V{}_{UjU} = 0) \end{aligned} \tag{4.37}$$

But we have seen that all the quantities on the right hand side can be obtained for the Penrose limit by simply restricting the corresponding quantities of the original metric to the null geodesic γ.

Thus we get a prescription for finding the Penrose limit without going through all the intermediate transformations, rescaling and limit-taking. Given a null geodesic γ in a manifold specified by the metric $g_{\mu\nu}$, we choose a pseudo-orthonormal parallel propagated frame (E_+, E_-, E_a) where E_+ is a tangent vector along the geodesic, and take u to be the affine parameter along the geodesic such that $E_+ = \partial_u$. E_- is chosen such that $g(E_-, E_-) = 0$ and $g(E_+, E_-) = 1$, and the other basis vectors satisfy $g(E_a, E_+) = g(E_a, E_-) = 0$ and $g(E_a, E_b) = \delta_{ab}$, and are unique up to orthogonal transformations. We shall use Greek indices for components in the coordinate basis, upper case Roman indices for components in the pseudo-orthonormal basis, and lower case Roman indices to indicate transverse components only in the pseudo-orthonormal basis. Then the wave profile $A_{ab}(u)$ of the Penrose limit of the given metric is obtained as

$$A_{ab}(u) = - \left. R_{a+b+} \right|_{\gamma} . \tag{4.38}$$

4.5.2 Geodesic deviation and Penrose limits

During our discussion on plane waves, we had seen that A_{ab}, being a component of the Riemann tensor, appears in the geodesic equation for the congruence of geodesics which includes γ,

$$\frac{\partial^2}{\partial u^2} Z^a = A_{ab}(u) Z^b, \tag{4.39}$$

where Z is the vector connecting nearby geodesics. Let us now study the null geodesic deviation equation, and see how it is related to the Penrose limit.

We once again start with a manifold with a metric $g_{\mu\nu}$, and a null geodesic γ. As before, we construct a parallel pseudo-orthonormal frame E^A along the null geodesic congruence, where E_+ is chosen to be the tangent vector $\dot{x}^\mu \partial \mu = \partial_u$ of the geodesic. u is the affine parameter. Let Z be the connecting vector. Then Z and E_+ commute,

$$\nabla_{E_+} Z = \nabla_Z E_+. \tag{4.40}$$

Since the frame is parallel along the geodesics, the left hand side of Eq. (4.40) can be written as

$$\nabla_{E_+} Z = \nabla_u (Z^A E_A) = (\partial_u Z^A) E_A. \tag{4.41}$$

To simplify the right hand side of Eq. (4.40), we first note that

$$g(E_+, E_+) = 0 \implies \nabla_A g(E_+, E_+) = 2g(\nabla_A E_+, E_+) = 0 \implies (\nabla_A E_+)^- = 0. \tag{4.42}$$

Thus, $(\nabla_Z E_+)^- = (\nabla_{E_+} Z)^- = \partial_u Z^- = 0$, and we can assume without loss of generality that $Z^- = 0$. Then we see that

$$\begin{aligned} \nabla_Z E_+ &= Z^B (\nabla_B E_+)^A E_A \\ &= Z^b (\nabla_b E_+)^A E_A \quad (\because Z^- = 0, \nabla_{E_+} E_+ = 0) \\ &= Z^b (\nabla_b E_+)^a E_a + Z^b (\nabla_b E_+)^+ E_+. \quad (\because (\nabla_b E_+)^- = 0) \end{aligned} \tag{4.43}$$

Therefore, from Eq. (4.40) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} Z^a = (\nabla_b E_+)^a Z^b = B^a{}_b Z^b, \tag{4.44}$$

where

$$B^a{}_b = (\nabla_b E_+)^a = E^a{}_\nu E_b{}^\mu \nabla_{E_+}\mu E_+{}^\nu. \tag{4.45}$$

Equation (4.44) is a system of equations from which the transverse components Z^a can be obtained. Once they are known, Z^+ can be found from

$$\frac{\mathrm{d}}{\mathrm{d}u} Z^+ = Z^b (\nabla_b E_+)^+ \tag{4.46}$$

It is useful to note that the trace of B is

$$\begin{aligned}
\operatorname{tr} B &= B^a{}_a \\
&= (\nabla_a E_+)^a \\
&= (\nabla_A E_+)^A \quad (\because \nabla_+ E_+ = 0, (\nabla_- E_+)^- = 0 \text{ by Eq. (4.42)}) \\
&= E^A{}_\nu E_A{}^\mu \nabla_\mu E_+^\nu \\
&= \nabla_\mu E_+^\mu \\
&= \nabla_\mu \dot{x}^\mu \\
&= \partial_\mu \dot{x}^\mu + \Gamma^\mu{}_{\mu\nu} \dot{x}^\nu \\
&= \partial_\mu \dot{x}^\mu + \frac{1}{2g}\partial_\mu g \quad (g = \det g_{\mu\nu}) \quad (\text{from Wald}^1) \\
&= \partial_\mu \dot{x}^\mu + \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} \\
&= \frac{1}{\sqrt{-g}}\partial_u\big(\sqrt{-g}\,\dot{x}^\mu\big).
\end{aligned} \tag{4.47}$$

To get the second order geodesic deviation equation for the transverse components of Z, we differentiate Eq. (4.44) to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} Z^a = \dot{B}^a{}_b Z^b + B^a{}_c B^c{}_c Z^b = A^a{}_b Z^b, \tag{4.48}$$

where $A^a{}_b = \dot{B}^a{}_b + B^a{}_c B^c{}_b$. Comparing this with the geodesic deviation equation $\frac{\mathrm{d}^2}{\mathrm{d}u^2} Z^a = R^a{}_{cdb} E_+{}^c E_+{}^d Z^b = -R^a{}_{+b+} Z^b$, we again get $A_{ab} = R_{a+b+}\big|_\gamma$ as expected.

Chapter 5

The Garfinkle-Vachaspati transformation

Since the Einstein equations are nonlinear, it is not possible to generate new solutions of the metric from known solutions by taking linear combinations. However, Garfinkle and Vachaspati[7] showed if the known metric admits a null hypersurface orthogonal Killing vector, then there exists a transformation which can be used to generate new metrics satisfying the Einstein equations. The Garfinkle-Vachaspati (GV) transformation can be thought of as the addition of wave degrees of freedom, and is thus used to study oscillating sources. In this chapter, we will mainly follow the analysis given by Kaloper et al.[9] to study this transformation and its properties.

5.1 The action and equations of motion

Let us consider a D-dimensional spacetime with a metric and some fields. The action is taken to be of the following form:

$$S = \int \mathrm{d}^D x \sqrt{-g}\left(R(g_{\mu\nu}) - \frac{1}{2}\sum_a h_a(\phi_c)(\nabla\phi_a)^2 - \frac{1}{2}\sum_p f_p(\phi_c)F^2_{(p+1)} \right) = \int \mathrm{d}^D x\, \mathcal{L}, \tag{5.1}$$

where the corresponding Lagrangian density is

$$\begin{aligned} \mathcal{L} = \sqrt{-g}\bigg(& R(g_{\mu\nu}) - \frac{1}{2}\sum_a h_a(\phi_c)\nabla_\mu\phi_a\nabla^\mu\phi_a \\ & - \frac{1}{2}\sum_p f_p(\phi_c)[F_{(p+1)}]_{\mu_1\ldots\mu_{p+1}}[F_{(p+1)}]^{\mu_1\ldots\mu_{p+1}} \bigg). \end{aligned} \tag{5.2}$$

The metric is $g_{\mu\nu}$ with determinant g and corresponding Ricci scalar R. There are also a number of scalar fields ϕ_a (a labels the different fields and is not a vector index), and a number of $(p+1)$-form fields $F_{(p+1)}$ which are obtained as exterior derivatives of p-form fields $A_{(p)}$; $F_{(p+1)} = \mathrm{d}A_{(p)}$.

To obtain the equations of motion, we will work with the different terms in the action separately for convenience. Let us call $\mathcal{L}_1$ the term in the Lagrangian with the Ricci scalar, $\mathcal{L}_2$ the term with $(\nabla\phi_a)^2$, and $\mathcal{L}_3$ the term with F^2_{p+1}; the corresponding terms in the action are respectively S_1, S_2 and S_3. We note that the S_1 is simply the usual Hilbert action[1] for vacuum Einstein equation, and we can write as shown in section 1.7,

$$\frac{\delta S_1}{\delta g^{\mu\nu}} = \sqrt{-g}\bigg(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\bigg) = \sqrt{-g}G_{\mu\nu}, \tag{5.3}$$

where $G_{\mu\nu}$ is the Einstein tensor. Taking variation with respect to the inverse metric $g^{\mu\nu}$, we obtain for the other terms in the action,

$$\begin{aligned}
\delta\mathcal{L}_2 &= -\frac{1}{2}\,\delta\big(\sqrt{-g}\big)\sum_a h_a(\phi_c)(\nabla\phi_a)^2 - \frac{1}{2}\sqrt{-g}\sum_a h_a(\phi_c)\nabla_\mu\phi_a\nabla_\nu\phi_a\,\delta g^{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g}\sum_a h_a(\phi_c)\bigg(\nabla_\mu\phi_a\nabla_\nu\phi_a - \frac{1}{2}g_{\mu\nu}(\nabla\phi_a)^2\bigg)\,\delta g^{\mu\nu} \\
\implies \frac{\delta S_2}{\delta g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}\sum_a h_a(\phi_c)\bigg(\nabla_\mu\phi_a\nabla_\nu\phi_a - \frac{1}{2}g_{\mu\nu}(\nabla\phi_a)^2\bigg), \qquad (5.4)\\
\delta\mathcal{L}_3 &= -\frac{1}{2}\sum_p f_p(\phi_c)\bigg(\,\delta\big(\sqrt{-g}\big)\,F^2_{(p+1)} \\
&\qquad + \sqrt{-g}\sum_{i=1}^{p+1}[F_{(p+1)}]^{\nu_1\ldots}{}_{\mu_i}{}^{\ldots\nu_{p+1}}[F_{(p+1)}]_{\nu_1\ldots\nu_i\ldots\nu_{p+1}}\,\delta g^{\mu_i\nu_i}\bigg) \\
&= -\frac{1}{2}\sqrt{-g}\sum_p f_p(\phi_c)\bigg((p+1)[F_{(p+1)}]_\mu{}^{\nu_2\ldots\nu_{p+1}}[F_{(p+1)}]_{\nu\nu_2\ldots\nu_{p+1}} \\
&\qquad - \frac{1}{2}g_{\mu\nu}F^2_{(p+1)}\bigg)\,\delta g^{\mu\nu} \\
\implies \frac{\delta S_3}{\delta g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}\sum_p f_p(\phi_c)\bigg((p+1)[F_{(p+1)}]_\mu{}^{\nu_2\ldots\nu_{p+1}}[F_{(p+1)}]_{\nu\nu_2\ldots\nu_{p+1}} - \frac{1}{2}g_{\mu\nu}F^2_{(p+1)}\bigg).
\end{aligned}$$

$$(5.5)$$

Combining, we can write the equation of motion for the metric (Einstein's equation) in mixed-index form as

$$R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R = \frac{1}{2}T^{\mu}{}_{\nu}, \tag{5.6}$$

where

$$\begin{aligned} T^{\mu}{}_{\nu} = &\sum_a h_a(\phi_c)\Big(\nabla^{\mu}\phi_a\nabla_{\nu}\phi_a - \frac{1}{2}\delta^{\mu}{}_{\nu}(\nabla\phi_a)^2\Big) \\ &+ \sum_p f_p(\phi_c)\Big((p+1)[F_{(p+1)}]^{\mu\nu_1\ldots\nu_p}[F_{(p+1)}]_{\nu\nu_1\ldots\nu_p} - \frac{1}{2}\delta^{\mu}{}_{\nu}F^2_{(p+1)}\Big). \end{aligned} \tag{5.7}$$

To find the other field equations, we now take variation with respect to ϕ_a to obtain

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{2}\Big(\sqrt{-g}\sum_c \frac{\partial h_c}{\partial\phi_a}(\nabla\phi_c)^2\,\delta\phi_a + 2\sqrt{-g}h_a\partial^{\mu}\phi_a\,\delta(\partial_{\mu}\phi_a) \\ &\quad + \sqrt{-g}\sum_p \frac{\partial f_p}{\partial\phi_a}F^2_{(p+1)}\,\delta\phi_a\Big) \\ &= -\frac{1}{2}\Big(\sqrt{-g}\sum_c \frac{\partial h_c}{\partial\phi_a}(\nabla\phi_c)^2 - 2\partial_{\mu}(\sqrt{-g}h_a\partial^{\mu}\phi_a) \\ &\quad + \sqrt{-g}\sum_p \frac{\partial f_p}{\partial\phi_a}F^2_{(p+1)}\Big)\,\delta\phi_a \\ \implies -2\frac{\delta S}{\delta\phi_a} &= \sqrt{-g}\sum_c \frac{\partial h_c}{\partial\phi_a}(\nabla\phi_c)^2 - 2\partial_{\mu}(\sqrt{-g}h_a\partial^{\mu}\phi_a) + \sqrt{-g}\sum_p \frac{\partial f_p}{\partial\phi_a}F^2_{(p+1)} = 0, \end{aligned} \tag{5.8}$$

and similarly variation of a p-form field $A_{(p)}$ (where $F_{(p+1)} = \mathrm{d}A_{(p)}$) gives

$$\begin{aligned} \delta\mathcal{L} &= -\sqrt{-g}f_p(\phi)\partial_{[\mu_1}[A_{(p)}]_{\mu_2\ldots\mu_{(p+1)}]}\,\delta\big(\partial^{[\mu_1}[A_{(p)}]^{\mu_2\ldots\mu_{(p+1)}]}\big) \\ &= \partial^{[\mu_1}(\sqrt{-g}f_p(\phi)[F_{(p+1)}]_{\mu_1\ldots\mu_{(p+1)}})\,\delta\big([A_{(p)}]^{\mu_2\ldots\mu_{(p+1)}]}\big) \\ \implies \frac{\delta S}{\delta[A_{(p)}]^{\mu_1\ldots\mu_p}} &= \partial^{\mu}(\sqrt{-g}f_p(\phi)[F_{(p+1)}]_{\mu\mu_1\ldots\mu_p}) = 0. \end{aligned} \tag{5.9}$$

5.2 Properties of the null hypersurface orthogonal Killing vector field

Now, let us assume the existence of a solution $(g, \phi_a, A_{(p)})$ of the field equations which admits a vector field k^μ satisfying the following:

$$k^\mu k_\mu = 0, \qquad \text{(Null)} \tag{5.10}$$

$$\nabla_\mu k_\nu - \nabla_\nu k_\mu = k_\mu \nabla_\nu S - k_\nu \nabla_\mu S, \qquad \text{(Hypersurface orthogonal)} \tag{5.11}$$

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0. \qquad \text{(Killing)} \tag{5.12}$$

Equations (5.11) and (5.12) together also give

$$\nabla_\mu k_\nu = \frac{1}{2}(k_\mu \nabla_\nu S - k_\nu \nabla_\mu S), \tag{5.13}$$

which we will often find useful. Also, the Riemann tensor contracts with k as

$$\begin{aligned} k_\alpha R^\alpha{}_{\beta\mu\nu} &= \nabla_\nu \nabla_\mu k_\beta - \nabla_\mu \nabla_\nu k_\beta \\ &= \frac{1}{2}\nabla_\nu (k_\mu \nabla_\beta S - k_\beta \nabla_\mu S) - \frac{1}{2}\nabla_\mu (k_\nu \nabla_\beta S - k_\beta \nabla_\nu S) \\ &= k_{[\mu} \nabla_{\nu]} \nabla_\beta S - \frac{1}{2} k_{[\mu} \nabla_{\nu]} S \nabla_\beta S. \end{aligned} \tag{5.14}$$

We have seen in section 1.5 that k^μ will satisfy

$$\mathcal{L}_k S = k^\mu \partial_\mu S = 0. \tag{5.15}$$

Thus we see that S has a vanishing Lie derivative with respect to k. We assume that this holds for the matter fields ϕ_a and $F_{(p+1)}$ as well, i.e.

$$\mathcal{L}_k \phi_a = k^\mu \partial_\mu \phi_a = 0, \tag{5.16}$$

$$\mathcal{L}_k F_{(p+1)} = (\mathrm{d} i_k + i_k \mathrm{d}) F_{(p+1)} = \mathrm{d} i_k F_{(p+1)} = 0. \tag{5.17}$$

The last equation uses $\mathcal{L}_k = \mathrm{d} i_k + i_k \mathrm{d}$ (where i_k denotes the interior product with k and d the exterior derivative), and the Bianchi identity $\mathrm{d}F_{(p+1)} = \mathrm{d}^2 A_{(p)} = 0$. An additional assumption is that the $(p+1)$-form fields satisfy a transversality constraint with respect to k,

$$i_k F_{(p+1)} = k \wedge \theta_{(p-1)}, \tag{5.18}$$

where the k on the right hand side denotes the one-form field $k_\mu \, dx^\mu$ corresponding to the vector field $k = k^\mu \partial_\mu$. We also note that $(i_k)^2 F_{(p+1)} = -k \wedge i_k \theta_{(p-1)} = 0$, which implies that $i_k \theta_{(p-1)} = 0$.

A plane wave metric admits a null covariantly constant (i.e. gradient and Killing) vector field.[5] The vector k in this case is null, Killing and hypersurface orthogonal (though not necessarily a gradient field), and thus any solution having such a vector field may be interpreted as gravitational waves. Since k is Killing, we can take $k = \partial_v$ (i.e. v parametrizes the integral curves of k), where v is a cyclic coordinate: $g_{\mu\nu,v} = 0$. The vanishing Lie derivatives of the matter fields imply that they are independent of v as well.

5.3 Properties of the Garfinkle-Vachaspati transformation

The Garfinkle-Vachaspati technique involves defining a new metric as

$$g'_{\mu\nu} = g_{\mu\nu} + e^S \Psi k_\mu k_\nu, \tag{5.19}$$

Where the indices of k have been lowered using the original metric. It will be seen that the configuration $(g', \phi_a, A_{(p)})$, with the new metric and the original matter fields, will be a solution of the equations of motion if Ψ satisfies certain conditions as will be shown later. Thus, the transformation given in Eq. (5.19), with appropriately chosen Ψ, can be used to obtain new solutions to the field equations from a known solution.

The metric transformation clearly does not affect the condition that k is null. The hypersurface orthogonality is also preserved, as Eq (5.11) can be written in terms of partial derivatives instead of covariant derivatives (the terms with Christoffel symbols in the left hand side cancel out), and thus does not depend on the metric. We also impose the condition that the scalar field Ψ, like all the other fields, has a vanishing Lie derivative with respect to k; $\mathcal{L}_k\Psi = k^\mu\partial_\mu\Psi = 0$. This gives $\mathcal{L}_k g'_{\mu\nu} = 0$, i.e. the vector field is Killing with respect to the new metric as well. In the following, we will sometimes write $e^S\Psi$ as κ for convenience, where $\mathcal{L}_k\kappa = 0$.

5.3.1 Matter field equations are unchanged

Let us first check that the transformed field configuration satisfies the matter field equations. First of all, we note that

$$g'^{\mu\nu} = g^{\mu\nu} - e^S\Psi k^\mu k^\nu, \tag{5.20}$$

which can be verified as follows:

$$\begin{aligned} g'_{\mu\nu}g'^{\nu\lambda} &= (g_{\mu\nu} + \kappa k_\mu k_\nu)(g^{\nu\lambda} - \kappa k^\nu k^\lambda) \\ &= \delta_\mu{}^\lambda + \kappa k_\mu k^\lambda - \kappa k_\mu k^\lambda - \kappa^2(k^\nu k_\nu)k_\mu k^\lambda \\ &= \delta_\mu{}^\lambda. \end{aligned} \tag{5.21}$$

To check that the determinant of the metric remains unchanged by the transformation, we write Eq. (5.19) as $g'_{\mu\nu} = g_{\mu\lambda}(\delta^\lambda{}_\nu + \kappa k^\lambda k_\nu)$. Thus, if we write $\mathbf{g}$ to be the matrix whose elements are $g_{\mu\nu}$, then the transformation may be written in matrix form as $\mathbf{g}' = \mathbf{g}(\mathbf{I}+\mathbf{K})$, where $K^\lambda{}_\nu = \kappa k^\lambda k_\nu$. Therefore, $g' = g\det(\mathbf{I}+\mathbf{K})$. Let us now define $\det(\mathbf{I}+\mathbf{K}) = x$. Now we see that $[\mathbf{K}^2]^\mu{}_\nu = \kappa^2 k^\mu k_\lambda k^\lambda k_\nu = 0$, i.e. $\mathbf{K}$ is nilpotent. Therefore, $(\mathbf{I}+\mathbf{K})^2 = \mathbf{I}+2\mathbf{K}$, and if we suppose that $(\mathbf{I}+\mathbf{K})^m = \mathbf{I}+m\mathbf{K}$ for some m, then

$$(\mathbf{I}+\mathbf{K})^{m+1} = (\mathbf{I}+\mathbf{K})^m(\mathbf{I}+\mathbf{K}) = (\mathbf{I}+m\mathbf{K})(\mathbf{I}+\mathbf{K}) = \mathbf{I}+(m+1)\mathbf{K}. \tag{5.22}$$

So we can say that by the principle of mathematical induction, $(\mathbf{I}+\mathbf{K})^n = \mathbf{I}+n\mathbf{K}$ for any natural number n. Thus, $\det(\mathbf{I}+n\mathbf{K}) = \det(\mathbf{I}+\mathbf{K})^n = x^n$. But for matrices of size $D \times D$, $\det(\mathbf{I}+n\mathbf{K})$ should be a polynomial in n having degree at most D. The only values of x for which x^n is a polynomial function in n of finite degree are 0 and 1. But if $\det(\mathbf{I}+\mathbf{K}) = 0$, then there exists some non-zero vector $\mathbf{v}$ such that $(\mathbf{I}+\mathbf{K})\mathbf{v} = 0$, i.e. $\mathbf{K}\mathbf{v} = -\mathbf{v}$. Then $\mathbf{K}^2\mathbf{v} = \mathbf{v} = \mathbf{0}$ since $\mathbf{K}$ is nilpotent, which is a contradiction. Thus we can say that $\det(\mathbf{I}+\mathbf{K}) = 1$, and thus $g' = g$.

To show that the matter field equations are not affected by the transformation, it remains to be proven that raising all indices of a $(p+1)$-form field gives the same expression with respect to the new metric as before. To show this, we first verify that this is true for a 2-form, using the transversality condition stated before. Then we see that

$$\begin{aligned}
[F'_{(2)}]^{\mu_1\mu_2} &= [F_{(2)}]^{\mu_1\mu_2} - \kappa k^{\mu_1}k^{\nu_1}[F_{(2)}]_{\nu_1}{}^{\mu_2} - \kappa k^{\mu_2}k^{\nu_2}[F_{(2)}]^{\mu_1}{}_{\nu_2} + \kappa^2 k^{\mu_1}k^{\nu_1}k^{\mu_2}k^{\nu_2}[F_{(2)}]_{\nu_1\nu_2} \\
&= [F_{(2)}]^{\mu_1\mu_2} - \kappa\theta_{(0)}k^{\mu_1}k^{\mu_2} + \kappa\theta_{(0)}k^{\mu_2}k^{\mu_1} + \kappa^2\theta_{(0)}k^{\mu_1}k^{\mu_2}k^{\nu_2}k_{\nu_2} \\
&\quad [k^{\nu_1}[F_{(2)}]_{\nu_1\nu_2} = \theta_{(0)}k_{\nu_2}, \quad k^{\nu_2}[F_{(2)}]_{\nu_1\nu_2} = -\theta_{(0)}k_{\nu_1} \quad \text{by transversality condition}] \\
&= [F_{(2)}]^{\mu_1\mu_2}.
\end{aligned} \tag{5.23}$$

This gives us an idea of how the raised-indices components appear in general. The terms of second and higher order in κ vanish because of the transversality and null conditions, and the first order terms cancel out. For a general $(p+1)$-form, this can be seen as

$$
\begin{aligned}
[F'_{(p+1)}]^{\mu_1\cdots\mu_{p+1}} - [F_{(p+1)}]^{\mu_1\cdots\mu_{p+1}} &= -\kappa \sum_{i=1}^{p+1} k^{\mu_i} k^{\nu_i} [F_{(p+1)}]^{\mu_1\cdots}{}_{\nu_i}{}^{\cdots\mu_{p+1}} \\
&= -\kappa k^{\mu_1} k^{[\mu_2} [\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} - \kappa \sum_{i=2}^{p+1} (-1)^{i-1} k^{\mu_i} k^{[\mu_1} [\theta_{(p-1)}]^{\mu_2\cdots\underline{\mu_i}\cdots\mu_{p+1}]} \\
&\quad \text{[Underlined index is omitted]} \\
&= -\kappa (p+1) k^{[\mu_1} k^{\mu_2} [\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} \\
&= 0,
\end{aligned}
\tag{5.24}
$$

where the penultimate step is obtained by noting that

$$
\begin{aligned}
&(p+1) k^{[\mu_1} k^{\mu_2} [\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} \\
&\quad = k^{\mu_1} k^{[\mu_2} [\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} + \sum_{i=2}^{p+1} (-1)^{-p(i-1)} k^{\mu_i} k^{[\mu_{i+1}} [\theta_{(p-1)}]^{\mu_{i+2}\cdots\mu_{p+1}\mu_1\cdots\mu_{i-1}]} \\
&\quad = k^{\mu_1} k^{[\mu_2} [\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} + \sum_{i=2}^{p+1} (-1)^{-p(i-1)+(i-1)(p+1)} k^{\mu_i} k^{[\mu_1} [\theta_{(p-1)}]^{\mu_2\cdots\underline{\mu_i}\cdots\mu_{p+1}]} \\
&\quad = k^{\mu_1} k^{[\mu_2} [\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} + \sum_{i=2}^{p+1} (-1)^{i-1} k^{\mu_i} k^{[\mu_1} [\theta_{(p-1)}]^{\mu_2\cdots\underline{\mu_i}\cdots\mu_{p+1}]}.
\end{aligned}
\tag{5.25}
$$

Thus we see that the field equation (5.9) is satisfied by the new field configuration. Similarly, the fact that $\mathcal{L}_k \phi_a = 0$ implies that $g'^{\mu\nu}\partial_\nu\phi_a = g^{\mu\nu}\partial_\nu\phi_a$, and thus Eq. (5.8) is also satisfied by the transformed metric with the original matter fields.

5.3.2 Effect on Einstein equations; transformation of Ricci tensor

We now need to find the suitable condition on Ψ to ensure that Eq. (5.6) is also satisfied. Clearly, the previously obtained identities show that the expression (5.7) of the stress-energy tensor is not affected by the transformation, and hence the right hand side of Eq. (5.6) is unchanged. Let us now find how the Ricci tensor is transformed by the metric transformation (5.19). Let ∇ and ∇' represent the covariant derivatives with respect to the metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ respectively, and $\Gamma^{\mu}{}_{\alpha\beta}$ and $\Gamma'^{\mu}{}_{\alpha\beta}$ be the corresponding Christoffel symbols, where $\Omega^{\mu}{}_{\alpha\beta} = \Gamma'^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\alpha\beta}$. Then we see that (following the derivation by Chandrasekhar[10])

$$
\begin{aligned}
\nabla'_{\alpha} g'_{\mu\nu} &= 0 \\
\implies \nabla_{\alpha}(g_{\mu\nu} + \kappa k_{\mu} k_{\nu}) - \Omega^{\beta}{}_{\alpha\mu} g'_{\beta\nu} - \Omega^{\beta}{}_{\alpha\nu} g'_{\mu\beta} &= 0 \\
\implies \Omega^{\beta}{}_{\alpha\mu} g'_{\beta\nu} + \Omega^{\beta}{}_{\alpha\nu} g'_{\mu\beta} &= \nabla_{\alpha}(\kappa k_{\mu} k_{\nu}). \quad [\nabla_{\alpha} g_{\mu\nu} = 0] \\
\text{Similarly,} \quad \Omega^{\beta}{}_{\nu\alpha} g'_{\beta\mu} + \Omega^{\beta}{}_{\nu\mu} g'_{\alpha\beta} &= \nabla_{\nu}(\kappa k_{\alpha} k_{\mu}), \\
\Omega^{\beta}{}_{\mu\nu} g'_{\beta\alpha} + \Omega^{\beta}{}_{\mu\alpha} g'_{\nu\beta} &= \nabla_{\mu}(\kappa k_{\nu} k_{\alpha}). \\
\therefore \Omega^{\beta}{}_{\mu\nu} g'_{\alpha\beta} &= \frac{1}{2}(\nabla_{\mu}(\kappa k_{\nu} k_{\alpha}) + \nabla_{\nu}(\kappa k_{\alpha} k_{\mu}) \\
&\quad - \nabla_{\alpha}(\kappa k_{\mu} k_{\nu})) \\
\implies \Omega^{\alpha}{}_{\mu\nu} &= \frac{1}{2}(\nabla_{\mu}(\kappa k_{\nu} k^{\alpha}) + \nabla_{\nu}(\kappa k^{\alpha} k_{\mu}) \\
&\quad - \nabla^{\alpha}(\kappa k_{\mu} k_{\nu})). \qquad (5.26)
\end{aligned}
$$

The above expression holds in general and does not depend on the properties of k^{μ}. If we now impose the null, Killing and hypersurface orthogonality conditions of the vector field and the fact that the Lie derivative of κ with respect to k^{μ} is zero, then $\Omega^{\alpha}{}_{\mu\nu}$ is seen to have the following properties:

$$
\begin{aligned}
\Omega^{\mu}{}_{\mu\nu} &= \frac{1}{2}(\nabla_{\mu}(\kappa k_{\nu} k^{\mu}) + \nabla_{\nu}(\kappa k^{\mu} k_{\mu}) - \nabla^{\mu}(\kappa k_{\mu} k_{\nu})) = 0, \qquad (5.27) \\
k^{\mu} \Omega^{\alpha}{}_{\mu\nu} &= k^{\mu} \frac{1}{2}(\nabla_{\mu}(\kappa k_{\nu} k^{\alpha}) + \nabla_{\nu}(\kappa k^{\alpha} k_{\mu}) - \nabla^{\alpha}(\kappa k_{\mu} k_{\nu})) = 0, \\
k_{\alpha} \Omega^{\alpha}{}_{\mu\nu} &= k_{\alpha} \frac{1}{2}(\nabla_{\mu}(\kappa k_{\nu} k^{\alpha}) + \nabla_{\nu}(\kappa k^{\alpha} k_{\mu}) - \nabla^{\alpha}(\kappa k_{\mu} k_{\nu})) = 0 \\
\implies k^{\mu} \nabla_{\alpha} \Omega^{\alpha}{}_{\mu\nu} &= -\Omega^{\alpha}{}_{\mu\nu} \nabla_{\alpha} k^{\mu} = 0, \qquad (5.28) \\
\Omega^{\alpha}{}_{\mu\beta} \Omega^{\beta}{}_{\alpha\nu} &= \frac{1}{4}(\nabla_{\mu}(\kappa k_{\beta} k^{\alpha}) + \nabla_{\beta}(\kappa k^{\alpha} k_{\mu}) - \nabla^{\alpha}(\kappa k_{\mu} k_{\beta}))(\nabla_{\alpha}(\kappa k_{\nu} k^{\beta}) \\
&\quad + \nabla_{\nu}(\kappa k^{\beta} k_{\alpha}) - \nabla^{\beta}(\kappa k_{\alpha} k_{\nu})) \\
&= 0. \qquad (5.29)
\end{aligned}
$$

Equation 5.27 is a consequence of the fact that $g = \det(\mathbf{g})$ is unchanged, since

$$
\begin{aligned}
\Gamma^{\mu}{}_{\mu\nu} &= \frac{1}{2} g^{\mu\lambda}(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\nu\mu,\lambda}) \\
&= \frac{1}{2} g^{\mu\lambda} g_{\lambda\mu,\nu} - g^{(\mu\lambda)} g_{\nu[\mu,\lambda]} \\
&= \frac{1}{2}\operatorname{tr}\bigl(\mathbf{g}^{-1}\partial_\nu \mathbf{g}\bigr) \\
&= \frac{1}{2}\sum_i \lambda_i^{-1}\partial_\nu \lambda_i \\
&= \frac{1}{2}\sum_i \partial_\nu \ln|\lambda_i| && [\lambda_i \text{ are the eigenvalues of } \mathbf{g}] \\
&= \frac{1}{2}\partial_\nu \ln\prod_i |\lambda_i| \\
&= \partial_\nu \ln\sqrt{|g|}. && (5.30)
\end{aligned}
$$

Thus it can be seen that

$$
\begin{aligned}
R'_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\
&= \Gamma'^{\alpha}{}_{\mu\nu,\alpha} - \Gamma'^{\alpha}{}_{\mu\alpha,\nu} + \Gamma'^{\alpha}{}_{\beta\alpha}\Gamma'^{\beta}{}_{\mu\nu} - \Gamma'^{\alpha}{}_{\beta\nu}\Gamma'^{\beta}{}_{\mu\alpha} \\
&= R_{\mu\nu} + \Omega^{\alpha}{}_{\mu\nu,\alpha} - \Omega^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} + \Omega^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} + \Omega^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} \\
&\quad - \Omega^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} - \Omega^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} - \Gamma^{\beta}{}_{\nu\alpha}\Omega^{\alpha}{}_{\mu\beta} + \Gamma^{\beta}{}_{\alpha\nu}\Omega^{\alpha}{}_{\mu\beta} \\
&= R_{\mu\nu} + \nabla_\alpha \Omega^{\alpha}{}_{\mu\nu} \\
\implies R'^{\mu}{}_{\nu} &= R^{\mu}{}_{\nu} - e^S \Psi k^\mu k^\lambda R_{\lambda\nu} + \nabla_\alpha\bigl(g^{\mu\lambda}\Omega^{\alpha}{}_{\lambda\nu}\bigr). && (5.31)
\end{aligned}
$$

Now,

$$
\begin{aligned}
k^\lambda R_{\lambda\nu} &= k^\lambda R^{\alpha}{}_{\lambda\alpha\nu} = -k_\lambda R^{\lambda\alpha}{}_{\alpha\nu} \\
&= -\frac{1}{2}k_\alpha \nabla_\nu \nabla^\alpha S + \frac{1}{2}k_\nu \nabla^2 S + \frac{1}{4}k_\alpha \nabla_\nu S \nabla^\alpha S - \frac{1}{4}k_\nu \nabla_\alpha S \nabla^\alpha S && [\text{Using Eq. (5.14)}] \\
&= \frac{1}{2}\nabla_\nu k_\alpha \nabla^\alpha S + \frac{1}{2}k_\nu \nabla^2 S - \frac{1}{4}k_\nu \nabla_\alpha S \nabla^\alpha S && [\because \mathcal{L}_k S = 0] \\
&= \frac{1}{2}k_\nu \nabla^2 S, && (5.32)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_\alpha(g^{\mu\lambda}\Omega^\alpha{}_{\lambda\nu}) &= \frac{1}{2}\nabla_\alpha(\nabla^\mu(\kappa k^\alpha k_\nu) + \nabla_\nu(\kappa k^\alpha k^\mu) - \nabla^\alpha(\kappa k^\mu k_\nu)) \\
&= \frac{1}{2}\nabla_\alpha(k^\alpha k_\nu \nabla^\mu \kappa + \frac{1}{2}\kappa k_\nu(k^\mu\nabla^\alpha S - k^\alpha \nabla^\mu S) + \frac{1}{2}\kappa k^\alpha(k^\mu \nabla_\nu S \\
&\quad - k_\nu \nabla^\mu S)) + \frac{1}{2}\nabla_\alpha(k^\alpha k^\mu \nabla_\nu \kappa + \frac{1}{2}\kappa k^\mu(k_\nu \nabla^\alpha S - k^\alpha \nabla_\nu S) \\
&\quad + \frac{1}{2}\kappa k^\alpha(k_\nu \nabla^\mu S - k^\mu \nabla_\nu S)) - \frac{1}{2}\nabla_\alpha(k^\mu k_\nu \nabla^\alpha \kappa + \frac{1}{2}\kappa k_\nu(k^\alpha \nabla^\mu S \\
&\quad - k^\mu \nabla^\alpha S) + \frac{1}{2}\kappa k^\mu(k^\alpha \nabla_\nu S - k_\nu \nabla^\alpha S)) \\
&= \frac{1}{2}\nabla_\alpha(k^\alpha k_\nu \nabla^\mu \kappa + k^\alpha k^\mu \nabla_\nu \kappa - k^\mu k_\nu \nabla^\alpha \kappa + \kappa(2k^\mu k_\nu \nabla^\alpha S - k^\alpha k_\nu \nabla^\mu S \\
&\quad - k^\alpha k^\mu \nabla_\nu S)) \\
&= \frac{1}{2}(k^\alpha k_\nu \nabla_\alpha \nabla^\mu \kappa + k^\alpha k^\mu \nabla_\alpha \nabla_\nu \kappa - k^\mu k_\nu \nabla^2 \kappa) - \frac{1}{4}k^\mu(k_\alpha \nabla_\nu S \\
&\quad - k_\nu \nabla_\alpha S)\nabla^\alpha \kappa - \frac{1}{4}k_\nu(k_\alpha \nabla^\mu S - k^\mu \nabla_\alpha S)\nabla^\alpha \kappa + k^\mu k_\nu \nabla_\alpha \kappa \nabla^\alpha S \\
&\quad + \frac{1}{2}\kappa k^\mu(k_\alpha \nabla_\nu S - k_\nu \nabla_\alpha S)\nabla^\alpha S + \frac{1}{2}\kappa k_\nu(k_\alpha \nabla^\mu S \\
&\quad - k^\mu \nabla_\alpha S)\nabla^\alpha S + \kappa k^\mu k_\nu \nabla^2 S - \frac{1}{2}k^\alpha k_\nu \nabla_\alpha \kappa \nabla^\mu S - \frac{1}{2}k^\alpha k_\nu \kappa \nabla_\alpha \nabla^\mu S \\
&\quad - \frac{1}{2}k^\alpha k^\mu \nabla_\alpha \kappa \nabla_\nu S - \frac{1}{2}\kappa k^\alpha k^\mu \nabla_\alpha \nabla_\nu S \\
&= -\frac{1}{2}e^S \Psi k^\mu k_\nu \nabla_\alpha S \nabla^\alpha S + k^\mu k_\nu \nabla_\alpha(e^S\Psi)\nabla^\alpha S + e^S \Psi k^\mu k_\nu \nabla^2 S \\
&\quad - \frac{1}{2}k^\mu k_\nu \nabla^2(e^S \Psi) \\
&= \frac{1}{2}k^\mu k_\nu(-e^S\Psi\nabla_\alpha S\nabla^\alpha S + 2e^S\Psi\nabla_\alpha S \nabla^\alpha S + 2e^S \nabla_\alpha \Psi \nabla^\alpha S + 2e^S \Psi \nabla^2 S \\
&\quad - \nabla_\alpha(e^S \Psi \nabla^\alpha S + e^S \nabla^\alpha \Psi)) \\
&= \frac{1}{2}k^\mu k_\nu(e^S\Psi\nabla_\alpha S \nabla^\alpha S + 2e^S \nabla_\alpha \Psi \nabla^\alpha S + 2e^S\Psi\nabla^2 S - e^S \Psi \nabla_\alpha S \nabla^\alpha S \\
&\quad - e^S \nabla_\alpha \Psi \nabla^\alpha S - e^S \Psi \nabla^2 S - e^S \nabla_\alpha S \nabla^\alpha \Psi - e^S \nabla^2 \Psi) \\
&= \frac{1}{2}k^\mu k_\nu(e^S \Psi \nabla^2 S - e^S \nabla^2 \Psi). \qquad (5.33)
\end{aligned}
$$

Therefore, the change in the Ricci tensor is obtained as

$$R'^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}e^S k^\mu k_\nu \nabla^2 \Psi. \qquad (5.34)$$

Thus, Eq. (5.20) can be used to generate solutions for the given Lagrangian, provided that Ψ satisfies $k^\mu \partial_\mu \Psi = 0$, and the Laplace equation $\nabla^2 \Psi = 0$ in the background of the original metric $g_{\mu\nu}$.

5.4 Curvature invariants are unchanged under the Garfinkle-Vachaspati transformation

It was seen before that the Ricci tensor can be made to remain unchanged under a Garfinkle-Vachaspati transformation if the associated function Ψ satisfies the Laplace equation. The Ricci scalar, however, can be seen to be invariant under the transformation even without imposing that condition. In fact, it will be seen in this section that *any* scalar constructed from the metric, the Riemann tensor, the matter fields, and their covariant derivatives will not change upon the application of a GV transformation. This means that the existence of plane wave terms in the metric can not be detected from any curvature scalar, and any curvature singularity introduced in the transformed metric can be found only from the Riemann tensor components in a parallelly propagated frame and not from any scalar.[1,11]

Let us now formally describe the above statement. We have a spacetime with a metric g and some scalar or $(p+1)$-form matter fields, and there exists a vector k which is null, hypersurface orthogonal and Killing with respect to the original metric. The Lie derivative of any matter field is zero with respect to k. We transform the metric as

$$g'_{\mu\nu} = g_{\mu\nu} + \kappa k_\mu k_\nu, \tag{5.35}$$

where the scalar κ satisfies $\mathcal{L}_k \kappa = 0$.

We have already seen some of the identities satisfied by the vector field k in the previous section. We will need some more identities to prove the invariance of the curvature scalars. First of all, the commutator of covariant derivatives of k is given by the Riemann tensor as

$$\nabla_\alpha \nabla_\beta k_\mu - \nabla_\beta \nabla_\alpha k_\mu = -R_{\nu\mu\alpha\beta} k^\nu .$$

$$\text{Permuting indices,} \quad \nabla_\mu \nabla_\alpha k_\beta - \nabla_\alpha \nabla_\mu k_\beta = -R_{\nu\beta\mu\alpha} k^\nu .$$

Adding, using Eq. (5.12) and the Bianchi identity,

$$\begin{aligned} 2\nabla_\alpha \nabla_\beta k_\mu + \nabla_\beta \nabla_\mu k_\alpha - \nabla_\mu \nabla_\beta k_\alpha &= R_{\nu\alpha\beta\mu} k^\nu \\ \implies 2\nabla_\alpha \nabla_\beta k_\mu + R_{\alpha\nu\beta\mu} k^\nu &= R_{\nu\alpha\beta\mu} k^\nu \\ \implies \nabla_\alpha \nabla_\beta k_\mu &= R_{\nu\alpha\beta\mu} k^\nu . \end{aligned} \tag{5.36}$$

The Lie derivative of a $(0, q)$ tensor field T with respect to a vector field v is defined as[9]

$$\mathcal{L}_v T_{\nu_1 \dots \nu_q} = v^\mu \nabla_\mu T_{\nu_1 \dots \nu_q} + T_{\mu \nu_2 \dots \nu_q} \nabla_{\nu_1} v^\mu + \cdots + T_{\nu_1 \dots \nu_{q-1} \mu} \nabla_{\nu_q} v^\mu . \tag{5.37}$$

We can see that the Lie derivative operator with respect to the vector field k commutes with the covariant derivative as follows:

$$
\begin{aligned}
\nabla_\lambda \mathcal{L}_k T_{\nu_1\dots\nu_q} &= \nabla_\lambda(k^\mu \nabla_\mu T_{\nu_1\dots\nu_q}) + \nabla_\lambda(T_{\mu\nu_2\dots\nu_q}\nabla_{\nu_1}k^\mu) + \dots + \nabla_\lambda(T_{\nu_1\dots\nu_{q-1}\mu}\nabla_{\nu_q}k^\mu) \\
&= \nabla_\lambda k^\mu \nabla_\mu T_{\nu_1\dots\nu_q} + k^\mu \nabla_\lambda\nabla_\mu T_{\nu_1\dots\nu_q} + \nabla_{\nu_1}k^\mu\nabla_\lambda T_{\mu\nu_2\dots\nu_q} + \cdots \\
&\quad + \nabla_{\nu_q}k^\mu\nabla_\lambda T_{\nu_1\dots\nu_{q-1}\mu} + T_{\mu\nu_2\dots\nu_q}\nabla_\lambda\nabla_{\nu_1}k^\mu + \dots + T_{\nu_1\dots\nu_{q-1}\mu}\nabla_\lambda\nabla_{\nu_q}k^\mu \\
&= \left(k^\mu\nabla_\lambda\nabla_\mu T_{\nu_1\dots\nu_q} + R_{\rho\lambda\nu_1}{}^\mu k^\rho T_{\mu\nu_2\dots\nu_q} + \dots + R_{\rho\lambda\nu_q}{}^\mu k^\rho T_{\nu_1\dots\nu_{q-1}\mu}\right) \\
&\quad + \nabla_\lambda k^\mu\nabla_\mu T_{\nu_1\dots\nu_q} + \nabla_{\nu_1}k^\mu\nabla_\lambda T_{\mu\nu_2\dots\nu_q} + \dots + \nabla_{\nu_q}k^\mu\nabla_\lambda T_{\nu_1\dots\nu_{q-1}\mu} \\
&= \left(k^\mu\nabla_\lambda\nabla_\mu T_{\nu_1\dots\nu_q} + k^\mu R_{\nu_1}{}^\rho{}_{\mu\lambda} T_{\rho\nu_2\dots\nu_q} + \dots + k^\mu R_{\nu_q}{}^\rho{}_{\mu\lambda} T_{\nu_1\dots\nu_{q-1}\rho}\right) \\
&\quad [\text{Swapping } \mu \leftrightarrow \rho] \\
&\quad + \nabla_\lambda k^\mu\nabla_\mu T_{\nu_1\dots\nu_q} + \nabla_{\nu_1}k^\mu\nabla_\lambda T_{\mu\nu_2\dots\nu_q} + \dots + \nabla_{\nu_q}k^\mu\nabla_\lambda T_{\nu_1\dots\nu_{q-1}\mu} \\
&= k^\mu\nabla_\mu\nabla_\lambda T_{\nu_1\dots\nu_q} + \nabla_\lambda k^\mu\nabla_\mu T_{\nu_1\dots\nu_q} + \nabla_{\nu_1}k^\mu\nabla_\lambda T_{\mu\nu_2\dots\nu_q} + \cdots \\
&\quad + \nabla_{\nu_q}k^\mu\nabla_\lambda T_{\nu_1\dots\nu_{q-1}\mu} \\
&= \mathcal{L}_k\nabla_\lambda T_{\nu_1\dots\nu_q}. \qquad (5.38)
\end{aligned}
$$

While we showed this for a $(0, q)$ tensor (all lower indices), it can be generalised to a tensor with some upper indices, since $\nabla_\mu g^{\alpha\beta} = \mathcal{L}_k g^{\alpha\beta} = 0$. It ca be also be proven that the Lie derivative of the Riemann tensor with respect to k is zero. Since $\mathcal{L}_k$ and the covariant derivative commute, we can say that any tensor obtained by taking covariant derivatives of S, κ or the Riemann tensor also has a vanishing Lie derivative with respect to k.

5.4.1 Contraction of k with a tensor

To show the invariance of the curvature scalar under k, we first show that the contraction of any tensor constructed from the Riemann tensor, any scalar whose lie derivative with respect to k vanishes (such as S and κ) and their covariant derivatives can be written as a linear combination of terms, each of which can be written as the product of k (or its corresponding one-form) and a tensor of lower rank. Thus, if T is a $(0, p+q+1)$ tensor constructed in the above way, then we have

$$k^{\mu}T_{\nu_1\dots\nu_p\mu\lambda_1\dots\lambda_q} = \sum_{i=1}^{p} k_{\nu_i}\theta^{(i)}_{\nu_1\dots\underline{\nu_i}\dots\nu_p\lambda_1\dots\lambda_q} + \sum_{i=1}^{q} k_{\lambda_i}\theta^{(p+i)}_{\nu_1\dots\nu_p\lambda_1\dots\underline{\lambda_i}\dots\lambda_q}, \tag{5.39}$$

where the underlined index is omitted. $\theta(1), \dots, \theta^{(p+q)}$ are $p+q$ tensors of rank $p+q-1$.

To prove this, let us start from the simplest cases. The only tensors of rank 1 that can be constructed in the stated way are of the form $\nabla_{\mu}B$, where the scalar B can be S or κ. Its contraction with k^{μ} is simply the lie derivative of B, which is zero. The contraction of k with the Riemann tensor is known from Eq. (5.14). One way to construct tensors of other rank is by taking covariant derivatives of the above tensors. We can show that the contraction property holds for such cases using the principle of mathematical induction on the index of the tensor. Let us suppose that Eq. 5.39 holds for all tensors of rank q or less. A tensor of rank $q+1$ may be constructed by taking the covariant derivative of T. Now we see that

$$\begin{aligned}
k^{\mu}\nabla_{\nu_1}T_{\mu\nu_2\dots\nu_q} &= \nabla_{\nu_1}(k^{\mu}T_{\mu\nu_2\dots\nu_q}) - T_{\mu\nu_2\dots\nu_q}\nabla_{\nu_1}k^{\mu} \\
&= \sum_{i=2}^{q}\nabla_{\nu_1}(k_{\nu_i}\theta^{(i)}_{\nu_2\dots\underline{\nu_i}\dots\nu_q}) - \frac{1}{2}T_{\mu\nu_2\dots\nu_q}(k_{\nu_1}\nabla^{\mu}S - k^{\mu}\nabla_{\nu_1}S) \\
&= \sum_{i=2}^{q}k_{\nu_i}\nabla_{\nu_1}\theta^{(i)}_{\nu_2\dots\underline{\nu_i}\dots\nu_q} + \frac{1}{2}\sum_{i=2}^{q}(k_{\nu_1}\nabla_{\nu_i}S - k_{\nu_i}\nabla_{\nu_1}S)\theta^{(i)}_{\nu_2\dots\underline{\nu_i}\dots\nu_q} \\
&\quad - \frac{1}{2}k_{\nu_1}T_{\mu\nu_2\dots\nu_q}\nabla^{\mu}S + \frac{1}{2}\nabla_{\nu_1}S\sum_{i=2}^{q}k_{\nu_i}\theta^{(i)}_{\nu_2\dots\underline{\nu_i}\dots\nu_q} \\
&= \frac{1}{2}k_{\nu_1}\left(\sum_{i=2}^{q}\nabla_{\nu_i}S\theta^{(i)}_{\nu_2\dots\underline{\nu_i}\dots\nu_q} - T_{\mu\nu_2\dots\nu_q}\nabla^{\mu}S\right) + \sum_{i=2}^{q}k_{\nu_i}\left(\nabla_{\nu_1}\theta^{(i)}_{\nu_2\dots\underline{\nu_i}\dots\nu_q}\right) \\
&= \sum_{i=1}^{q}k_{\nu_i}\bar{\theta}^{(i)}_{\nu_1\dots\underline{\nu_i}\dots\nu_q},
\end{aligned} \tag{5.40}$$

which is in the desired form. The other situation that may arise is $k^{\mu}\nabla_{\mu}T_{\nu_1\dots\nu_q}$. In this case, we use the fact that $\mathcal{L}_k T = 0$ to obtain

$$
\begin{aligned}
k^\mu \nabla_\mu T_{\nu_1\ldots\nu_q} &= -T_{\mu\nu_2\ldots\nu_q}\nabla_{\nu_1}k^\mu - \cdots - T_{\nu_1\ldots\nu_{q-1}\mu}\nabla_{\nu_q}k^\mu \\
&= -\frac{1}{2}T_{\mu\nu_2\ldots\nu_q}k_{\nu_1}\nabla^\mu S - \cdots - \frac{1}{2}T_{\nu_1\ldots\nu_{q-1}\mu}k_{\nu_q}\nabla^\mu S \\
&\quad + \frac{1}{2}T_{\mu\nu_2\ldots\nu_q}k^\mu\nabla_{\nu_1}S + \cdots + \frac{1}{2}T_{\nu_1\ldots\nu_{q-1}\mu}k^\mu\nabla_{\nu_q}S \\
&= \frac{1}{2}\sum_{i=1}^{q} k_{\nu_i}\left(\sum_{j\neq i}\theta^{(i,j)}_{\nu_1\ldots\underline{\nu_i}\ldots\underline{\nu_j}\ldots\nu_q}\nabla_{\nu_j}S - T_{\nu_1\ldots\nu_{i-1}\mu\nu_{i+1}\ldots\nu_q}\nabla^\mu S\right) \\
&= \sum_{i=1}^{q} k_{\nu_i}\bar{\theta}^{(i)}_{\nu_1\ldots\underline{\nu_i}\ldots\nu_q}, \qquad (5.41)
\end{aligned}
$$

$$
\text{where} \quad k^\mu T_{\nu_1\ldots\nu_{j-1}\mu\nu_{j+1}\ldots\nu_q} = \sum_{\substack{i=1\\ i\neq j}}^{q} k_{\nu_i}\theta^{(i,j)}_{\nu_1\ldots\underline{\nu_i}\ldots\underline{\nu_j}\ldots\nu_q}. \qquad (5.42)
$$

Thus we see that any tensor constructed from the Riemann tensor, scalars with vanishing Lie derivative with respect to k, and their covariant derivatives (called a 'primary tensor') satisfies the contraction property. To prove that contractions of two or more such primary tensors also satisfies this, we need to check that the 'secondary tensors' (i.e. the θ tensors obtained after the contraction) satisfy this property as well. It can be seen that the secondary tensors obtained from contracting ∇B or the Riemann tensor with k satisfy the contraction property, and they also have zero Lie derivative with respect to k. Let us assume that the θ tensors coming from the contraction of T with k satisfy these two properties. Then, from the above equations, we see that the secondary tensors $\bar{\theta}$ appearing from the contraction of ∇T with k consist of terms such as the covariant derivative of θ, product of θ with ∇S, and contraction of T with ∇S. It can be checked that each of these terms satisfy the contraction and vanishing Lie derivative properties as well, since $\mathcal{L}_k S = \mathcal{L}_k T = \mathcal{L}_k \theta = 0$ and $[\mathcal{L}_k, \nabla] = 0$.

Now we are in a position to check that the contraction property holds for contractions of primary tensors. Let $T^{(1)}$ and $T^{(2)}$ be two primary tensors. Then we have

$$
\begin{aligned}
k^{\mu}T^{(1)}{}_{\mu\nu_1\ldots\nu_p\lambda_1\ldots\lambda_q}T^{(2)}{}^{\lambda^1\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} &= \sum_{i=1}^{p} k_{\nu_i}\theta^{(i)}_{\nu_1\ldots\underline{\nu_i}\ldots\nu_p\lambda_1\ldots\lambda_q}T^{(2)}{}^{\lambda_1\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \\
&\quad+\sum_{i=1}^{q} k_{\lambda_i}\theta^{(p+i.1)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q}T^{(2)}{}^{\lambda_1\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \quad [\text{if } q>0] \\
&= (\ldots)+\sum_{i=1}^{q}\theta^{(p+i.1)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q}\sum_{j=1}^{r}k_{\omega_j}\theta^{(i,q+j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q}{}_{\omega_1\ldots\underline{\omega_j}\ldots\omega_r} \\
&\quad+\sum_{i=1}^{q}\theta^{(p+i.1)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q}\sum_{\substack{j=1\\ j\neq i}}^{q}k^{\lambda_j}\theta^{(i,j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \quad [\text{if } q>1] \\
&= (\ldots)+\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q}\theta^{(i,j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q}{}_{\omega_1\ldots\omega_r}\sum_{l=1}^{p}k_{\nu_l}\theta^{(p+i,1,l)}_{\nu_1\ldots\underline{\nu_l}\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q} \\
&\quad+\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q}\theta^{(i,j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q}{}_{\omega_1\ldots\omega_r}\sum_{\substack{l=1\\ l\neq i,j}}^{q}k_{\lambda_l}\theta^{(p+i,1,p+l)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\underline{\lambda_l}\ldots\lambda_q} \quad [\text{if } q>2] \\
&= \ldots,
\end{aligned} \tag{5.43}
$$

where the terms already in the desired form have not been written in subsequent steps for the sake of simplicity. Since $T^{(1)}$, $T^{(2)}$ as well as the θ tensors satisfy the contraction property and q is finite, the above steps can be carried out till no contraction with k remains in any term. Thus the contraction identity is seen to be valid for contractions of two primary tensors, which can be generalised to contractions of any number of primary tensors.

We also check that the Levi-Civita tensor ϵ also satisfies the contraction property. To do this, we first use the Killing condition to say there exists a u such that $k^{\mu}=\delta^{\mu}{}_{u}$ and $g_{\mu\nu,u}=0$, and the hypersurface orthogonal condition to state the existence of v such that $k_{\mu}=e^{-S}\partial_{\mu}v$. We choose a coordinate system with u and v as two coordinates. Then we have

$$\begin{aligned} k^\mu \epsilon_{\mu\alpha\ldots\beta} &= \delta^\mu{}_u \epsilon_{\mu\alpha\ldots\beta} \\ &= \epsilon_{u\alpha\ldots\beta}. \end{aligned} \tag{5.44}$$

Since ϵ is completely antisymmetric, every non-zero term in the above must have v as a free index, we can thus write $i_k \epsilon = k \wedge \theta$, where θ is a $(D-2)$-form. Hence, the Levi-Civita tensor also satisfies the contraction property.

5.4.2 Transformation of the Riemann tensor

We have seen before that the change in the Christoffel symbol is given by

$$\begin{aligned} \Omega^\mu{}_{\alpha\beta} &= \frac{1}{2}(\nabla_\alpha(\kappa k^\mu k_\beta) + \nabla_\beta(\kappa k^\mu k_\alpha) - \nabla^\mu(\kappa k_\alpha k_\beta)) \\ &= \frac{1}{2}(k^\mu k_\beta \nabla_\alpha \kappa + k^\mu k_\alpha \nabla_\beta \kappa - k_\alpha k_\beta \nabla^\mu \kappa) \\ &\quad + \frac{\kappa}{4}(k_\alpha k_\beta \nabla^\mu S - k^\mu k_\beta \nabla_\alpha S + k^\mu k_\alpha \nabla_\beta S - k^\mu k_\beta \nabla_\alpha S + k_\beta k_\alpha \nabla^\mu S - k^\mu k_\alpha \nabla_\beta S \\ &\quad + k^\mu k_\beta \nabla_\alpha S) + \frac{\kappa}{4}(-k^\mu k_\alpha \nabla_\beta S - k^\mu k_\beta \nabla_\alpha S + k_\alpha k_\beta \nabla^\mu S - k_\alpha k^\mu \nabla_\beta S + k_\alpha k_\beta \nabla^\mu S) \\ &= \frac{1}{2}(k^\mu k_\beta \nabla_\alpha \kappa + k^\mu k_\alpha \nabla_\beta \kappa - k_\alpha k_\beta \nabla^\mu \kappa) - \frac{\kappa}{2}(k^\mu k_\beta \nabla_\alpha S + k^\mu k_\alpha \nabla_\beta S - 2 k_\alpha k_\beta \nabla^\mu S). \end{aligned} \tag{5.45}$$

The shift in the Riemann tensor is thus given by

$$\begin{aligned} R'^\alpha{}_{\beta\gamma\delta} &= \Gamma'^\alpha{}_{\beta\delta,\gamma} - \Gamma'^\alpha{}_{\beta\gamma,\delta} + \Gamma'^\alpha{}_{\mu\gamma}\Gamma'^\mu{}_{\beta\delta} - \Gamma'^\alpha{}_{\mu\delta}\Gamma'^\mu{}_{\beta\gamma} \\ &= R^\alpha{}_{\beta\gamma\delta} + \nabla_\gamma \Omega^\alpha{}_{\beta\delta} - \nabla_\delta \Omega^\alpha{}_{\beta\gamma} + \Omega^\alpha{}_{\mu\gamma}\Omega^\mu{}_{\beta\delta} - \Omega^\alpha{}_{\mu\delta}\Omega^\mu{}_{\beta\gamma}. \end{aligned} \tag{5.46}$$

But we see that

$$\begin{aligned} \Omega^\alpha{}_{\mu\gamma}\Omega^\mu{}_{\beta\delta} &= [\frac{1}{2}(k^\alpha k_\gamma \nabla_\mu \kappa + k^\alpha k_\mu \nabla_\gamma \kappa - k_\mu k_\gamma \nabla^\alpha \kappa) - \frac{\kappa}{2}(k^\alpha k_\gamma \nabla_\mu S + k^\alpha k_\mu \nabla_\gamma S \\ &\quad - 2 k_\mu k_\gamma \nabla^\alpha S)] \times [\frac{1}{2}(k^\mu k_\delta \nabla_\beta \kappa + k^\mu k_\beta \nabla_\delta \kappa - k_\beta k_\delta \nabla^\mu \kappa) - \frac{\kappa}{2}(k^\mu k_\delta \nabla_\beta S \\ &\quad + k^\mu k_\beta \nabla_\delta S - 2 k_\beta k_\delta \nabla^\mu S)] \\ &= -\frac{1}{4} k^\alpha k_\beta k_\gamma k_\delta \nabla_\mu \kappa \nabla^\mu \kappa + \frac{3\kappa}{4} k^\alpha k_\beta k_\gamma k_\delta \nabla_\mu \kappa \nabla^\mu S \quad - \frac{\kappa^2}{2} k^\alpha k_\beta k_\gamma k_\delta \nabla_\mu S \nabla^\mu S, \end{aligned} \tag{5.47}$$

which is symmetric under the interchange of γ and δ. Thus, the Riemann tensor is

$$R'^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} + \nabla_\gamma \Omega^{\alpha}{}_{\beta\delta} - \nabla_\delta \Omega^{\alpha}{}_{\beta\gamma}, \tag{5.48}$$

and the expression with all covariant indices will be

$$\begin{aligned} R'_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + g_{\alpha\mu}\nabla_\gamma \Omega^{\alpha}{}_{\beta\delta} - g_{\alpha\mu}\nabla_\delta \Omega^{\alpha}{}_{\beta\gamma} + \kappa k_\alpha \mu R^{\mu}{}_{\beta\gamma\delta} \\ &= R_{\alpha\beta\gamma\delta} + \nabla_\gamma (\frac{1}{2}(k_\alpha k_\delta \nabla_\beta \kappa + k_\alpha k_\beta \nabla_\delta \kappa - k_\beta k_\delta \nabla_\alpha \kappa) - \frac{\kappa}{2}(k_\alpha k_\delta \nabla_\beta S + k_\alpha k_\beta \nabla_\delta S \\ &\quad - 2k_\beta k_\delta \nabla_\alpha S)) \\ &\quad - \nabla_\delta \Omega^{\alpha}{}_{\beta\gamma}. \end{aligned} \tag{5.49}$$

Since every term in $\Omega^{\alpha}{}_{\beta\delta}$ as well as $\Omega^{\alpha}{}_{\beta\gamma}$ is bilinear in k, we use Eq. (5.13) to show that the shift in the Riemann tensor is also bilinear in k. It can also be shown that the shift any tensor obtained by taking any number of covariant derivatives of the Riemann tensor is also at least bilinear in k.

Since the determinant of the metric is unchanged under the transformation, the Levi-Civita tensor also remains the same.

5.4.3 Invariance of curvature scalars

An arbitrary curvature scalar will be a product of some number of tensors obtained from the Riemann tensor and its covariant derivatives and Levi-Civita tensors, contracted with the sufficient number of inverse metrics:

$$\mathcal{I}' = \prod_{j=1}^{N} T'_{\nu_1^j \ldots \nu_{q_j}^j} \prod_{k=1}^{K} \epsilon'_{\lambda_1^k \ldots \lambda_D^k} \prod_{l=1}^{M} g'^{\alpha_l \beta_l}, \quad \text{where} \quad \sum_{j=1}^{N} q_j + KD = 2M. \tag{5.50}$$

Now we know that $T' = T + \chi$, where χ is at least bilinear in k, $\epsilon' = \epsilon$, and $g^{\alpha\beta} = g^{\alpha\beta} - \kappa k^\alpha k^\beta$. So we see that

$$\begin{aligned} \mathcal{I}' &= \prod_{j=1}^{N} \left(T_{\nu_1^j \ldots \nu_{q_j}^j} + \chi_{\nu_1^j \ldots \nu_{q_j}^j} \right) \prod_{k=1}^{K} \epsilon_{\lambda_1^k \ldots \lambda_D^k} \prod_{l=1}^{M} \left(g^{\alpha_l \beta_l} - \kappa k^{\alpha_l} k^{\beta_l} \right) \\ &= \mathcal{I} + \mathcal{J}, \end{aligned} \tag{5.51}$$

where $\mathcal{I}$ is the same curvature scalar for the original metric, and $\mathcal{J}$ consists of all the other terms. Since T satisfies the contraction property and χ is at least bilinear in k, we cause the fact that k is null to prove that $\mathcal{J} = 0$, i.e. the curvature scalar is unchanged.

5.5 Neutral oscillating string

Garfinkle-Vachaspati transformation is widely used in General Relativity, for instance, to generate hair modes of a black hole metric.[12] As a simpler example[13] of the GV transformation, let us consider the following metric g and fields ϕ and $B_{\mu\nu}$ in $D = 10$ dimensions,

$$\mathrm{d}s^2 = -e^{2\phi}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\,, \tag{5.52}$$

$$e^{-2\phi} = 1 + \frac{Q}{r^{D-4}}, \tag{5.53}$$

$$B_{uv} = \frac{1}{2}\big(e^{2\phi} - 1\big). \tag{5.54}$$

The above can be thought of as the metric corresponding to a neutral straight string, which couples with the 2-form field $B_{\mu\nu}$ the same way a charged point particle interacts with the electromagnetic 1-form potential. We can see that both the vectors ∂_u and ∂_v are null (since $g_{uu} = g_{vv}$), hypersurface orthogonal (since $g_{u\mu} = -e^{2\phi}\partial_\mu v$ and $g_{v\mu} = -e^{2\phi}\partial_\nu u$, so $S = -2\phi$ in both cases) and Killing (since $g_{\mu\nu,u} = g_{\mu\nu,v} = 0$). Applying the GV technique with the vector $k = \partial_u$, we get

$$g' = g + e^S \Psi k \otimes k = g + e^{-S}\Psi(v,\vec{x})\,\mathrm{d}v \otimes \mathrm{d}v\,, \tag{5.55}$$

which gives the new metric

$$\mathrm{d}s^2 = -e^{2\phi}(\mathrm{d}u\,\mathrm{d}v - \Psi(v,\vec{x})\,\mathrm{d}v^2) + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\,, \tag{5.56}$$

leaving the fields ϕ and $B_{\mu\nu}$ unchanged. It can be seen that Ψ is independent of u by noting that $\mathcal{L}_k\Psi = 0$. Also, since $\partial_u\Psi = 0$ and $g_{\mu v}$ is non-zero only for $\mu = u$, the condition that $\nabla^2\Psi = 0$ reduces to $\partial^2\Psi = 0$, i.e. Ψ satisfies the Laplace equation in the flat transverse space given by x^i. The solution of this may be written in terms of the $(D-2)$-dimensional spherical harmonics Y_ℓ as

$$\Psi(v,\vec{x}) = \sum_{\ell\geq 0}\left(a_\ell(v)r^\ell + b_\ell(v)r^{-D+4-\ell}\right)Y_\ell. \tag{5.57}$$

It can be seen[13] that only the terms of order r^1 correspond to string sources, and only such terms are kept in the expression of $\Psi(v,\mathbf{x})$ to obtain

$$\Psi(v,\vec{x}) = \vec{f}(v)\cdot\vec{x}. \tag{5.58}$$

The metric now does not appear to be asymptotically flat. However, this can be remedied by changing the coordinates as

$$\begin{aligned} v &= v', \\ u &= u' - 2\dot{\vec{F}}\cdot\vec{x}' + 2\dot{\vec{F}}\cdot\vec{F} - \int^{v'}\dot{F}^2\,\mathrm{d}v'', \\ \vec{x} &= \vec{x}' - \vec{F}, \end{aligned} \tag{5.59}$$

where $\vec{f}(v) = -2\ddot{\vec{F}}(v)$ and $\dot{F}^2 = \dot{\vec{F}}\cdot\dot{\vec{F}}$, where the dot denotes derivative with respect to v. The metric in the new coordinates becomes

$$\begin{aligned} \mathrm{d}s^2 &= -e^{2\phi}(\mathrm{d}u\,\mathrm{d}v - \vec{f}\cdot\vec{x}\,\mathrm{d}v^2) + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} \\ &= -e^{2\phi}((\mathrm{d}u' - 2\ddot{\vec{F}}\cdot\vec{x}\,\mathrm{d}v' - 2\dot{\vec{F}}\cdot\mathrm{d}\vec{x} + 2\ddot{\vec{F}}\cdot\vec{F}\,\mathrm{d}v' + 2\dot{\vec{F}}\cdot\dot{\vec{F}}\,\mathrm{d}v' - \dot{F}^2\,\mathrm{d}v')\,\mathrm{d}v' \\ &\quad + 2\ddot{\vec{F}}\cdot(\vec{x}' - \vec{F})\,\mathrm{d}v'^2) + (\mathrm{d}\vec{x}' - \dot{\vec{F}}\,\mathrm{d}v')\cdot(\mathrm{d}\vec{x}' - \dot{\vec{F}}\,\mathrm{d}v') \\ &= -e^{2\phi}(\mathrm{d}u' - 2\dot{\vec{F}}\cdot\mathrm{d}\vec{x} + \dot{F}^2\,\mathrm{d}v')\,\mathrm{d}v' + \mathrm{d}\vec{x}'\cdot\mathrm{d}\vec{x}' - 2\dot{\vec{F}}\cdot\mathrm{d}x'\,\mathrm{d}v' + \dot{F}^2\,\mathrm{d}v'^2 \\ &= -e^{2\phi}\,\mathrm{d}u\,\mathrm{d}v + 2(e^{2\phi}-1)\dot{\vec{F}}\cdot\mathrm{d}\vec{x}\,\mathrm{d}v - \left(2e^{2\phi}\ddot{\vec{F}}\cdot\vec{F} + (e^{2\phi}-1)\dot{F}^2\right)\mathrm{d}v^2\,, \end{aligned} \tag{5.60}$$

which is asymptotically flat.

The above analysis can be extended for multi-string solutions as well, giving

$$\Psi(v,\vec{x}) = \vec{f}(v)\cdot\vec{x}, \tag{5.61}$$

$$e^{-2\phi(\vec{x})} = 1 + \sum_i \frac{Q}{|\vec{x}-\vec{x}_i|^{D-4}}, \tag{5.62}$$

which can be then made asymptotically flat by making the appropriate coordinate transformation as before. It is interesting to note that even though the equations of motion are in general are nonlinear, the multi-string solutions can be simply written as a linear combination of single-string solutions in this case, as both $e^{-2\phi}$ and Ψ follow the Laplace equation, which is linear.

5.6 The D3 brane

As another example, we will now apply the transformation on the D3 brane, which has a Killing horizon but no singularity. We will first study the features of the D3 brane and then study how it is transformed.

5.6.1 Properties of a D3 brane metric

We now turn our attention to the D3 brane, which is a $(3+1)$-dimensional object embedded in a 10-dimensional spacetime. Its metric is given as[14]

$$\mathrm{d}s^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + \left(1 + \frac{L^4}{r^4}\right)^{1/2} (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2). \tag{5.63}$$

The D3 brane metric is a solution for the action

$$S = \int \mathrm{d}^{10}x\,\sqrt{-g}\left(R(g) - \frac{1}{240}F_{(5)}^2\right), \tag{5.64}$$

which is a special case of the action given in Eq. (5.1) for 10 dimensions with no scalar field and one 5-form field $F_{(5)} = \mathrm{d}C_{(4)}$, where the 4-form potential C is given as $C_{0123} = \Big((1 + L^4/r^4)^{-1} - 1\Big)$.

We first make the transformation

$$r = \frac{L\rho}{(1-\rho^4)^{1/4}}, \qquad \rho = \left(1 + \frac{L^4}{r^4}\right)^{-1/4}, \tag{5.65}$$

so that the metric in the new coordinates becomes

$$\mathrm{d}s^2 = \rho^2\eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + L^2\rho^{-2}\left(\frac{\mathrm{d}\rho^2}{(1-\rho^4)^{5/2}} + \frac{\rho^2}{(1-\rho^4)^{1/2}}\,\mathrm{d}\Omega_5{}^2\right). \tag{5.66}$$

The metric components diverge at $\rho = 0$, but that is not a true singularity as the metric can be made regular by choosing a different set of coordinates (as will be shown later). It is, however, a Killing horizon, as all the Killing vectors ∂_i (corresponding to translations along x^i) become null at $\rho = 0$. We see that the metric is symmetric under the transformation $\rho \mapsto -\rho$, and thus the inside region of the horizon is a copy of the outside region. D3 brane metric thus has no singularity.

At ρ close to 0 ($\rho \ll 1$), the metric approaches that of $AdS_5 \times S^5$. While the $L^2\,\mathrm{d}\Omega_5{}^2$ term in the metric obviously corresponds to a sphere of radius L, it can be seen that the other terms correspond to a 5-dimensional anti-de Sitter space by going over to the embedding coordinates

$$\begin{gathered} X^i = \rho x^i \quad (i \in \{1,2,3\}), \quad X^5 = \rho x^0, \\ X^0 = \frac{L}{2\rho}\left(1 + \rho^2 + \frac{\rho^2}{L^2}(\vec{x}^2 - (x^0)^2)\right), \quad X^4 = \frac{L}{2\rho}\left(1 - \rho^2 + \frac{\rho^2}{L^2}(\vec{x}^2 - (x^0)^2)\right). \end{gathered} \tag{5.67}$$

These coordinates satisfy the constraint

$$(X^0)^2 + (X^5)^2 - \sum_{i=1}^{4}(X^i)^2 = \rho^2\left((x^0)^2 - \sum_{i=1}^{3}(x^i)^2\right) + \frac{L^2}{4\rho^2}4\rho^2\left(1 + \frac{\rho^2}{L^2}(\vec{x}^2 - (x^0)^2)\right) = L^2, \tag{5.68}$$

with the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}\big(X^0\big)^2 - \mathrm{d}\big(X^5\big)^2 + \sum_{i=1}^{4}\mathrm{d}\big(X^i\big)^2 . \tag{5.69}$$

5.6.2 Nonsingular coordinates of the D3 brane metric

The D3 metric is devoid of any singularity, even though it does have a horizon at $\rho = 0$. However, in the coordinates chosen, the metric components diverge at $\rho \to 0$. This can be avoided by choosing a different set of coordinates. Let us take the coordinates

$$\begin{gathered} v = x^1 + x^0 = -\frac{1}{V}, \qquad u = x^1 - x^0 = U + \frac{1}{VW^2} + \frac{(X^2)^2 + (X^3)^2}{V}, \\ \rho = LVW, \qquad x^{2,3} = \frac{X^{2,3}}{V}. \end{gathered} \tag{5.70}$$

For any timelike or null geodesic going towards $\rho \to 0$, v approaches infinity. This is equivalent to the Schwarzschild metric, where any timelike or null geodesic going as $r \to 2GM$ has $t \to \infty$.[1] In the new coordinates, this will correspond to $V \to 0$. ρ appears only in the definition $\rho = LVW$, where W remains finite as $\rho \to 0$ and $V \to 0$. Then the metric becomes

$$\begin{aligned}
\mathrm{d}s^2 &= \rho^2\Big(\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2\Big) + \frac{L^2}{\rho^2}\left(\frac{\mathrm{d}\rho^2}{(1-\rho^4)^{5/2}} + \frac{\rho^2}{(1-\rho^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\right) \\
&= L^2V^2W^2\left(\mathrm{d}U - \frac{\mathrm{d}V}{V^2W^2} - \frac{2\,\mathrm{d}W}{VW^3} + \frac{2X^2\,\mathrm{d}X^2 + 2X^3\,\mathrm{d}X^3}{V} - \frac{(X^2)^2+(X^3)^2}{V^2}\,\mathrm{d}V\right) \\
&\quad\times \frac{\mathrm{d}V}{V^2} + L^2V^2W^2\left(\frac{\mathrm{d}\big(X^2\big)^2 + \mathrm{d}\big(X^3\big)^2}{V^2} + \frac{(X^2)^2+(X^3)^2}{V^4}\,\mathrm{d}V^2\right. \\
&\quad\left. - \frac{2X^2\,\mathrm{d}X^2 + 2X^3\,\mathrm{d}X^3}{V^3}\,\mathrm{d}V\right) \\
&\quad + \frac{1}{V^2W^2}\left(\frac{L^2(V^2\,\mathrm{d}W^2 + W^2\,\mathrm{d}V^2 + 2VW\,\mathrm{d}V\,\mathrm{d}W)}{(1-L^4V^4W^4)^{5/2}} + \frac{L^2V^2W^2}{(1-L^4V^4W^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\right) \\
&= L^2W^2\left(\mathrm{d}U - \frac{\mathrm{d}V}{V^2W^2} - \frac{2\,\mathrm{d}W}{VW^3}\right)\mathrm{d}V + L^2W^2\Big(\mathrm{d}\big(X^2\big)^2 + \mathrm{d}\big(X^3\big)^2\Big) \\
&\quad + \frac{1}{V^2W^2}\left(\frac{L^2(V^2\,\mathrm{d}W^2 + W^2\,\mathrm{d}V^2 + 2VW\,\mathrm{d}V\,\mathrm{d}W)}{(1-L^4V^4W^4)^{5/2}} + \frac{L^2V^2W^2}{(1-L^4V^4W^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\right) \\
&= L^2W^2\Big(\mathrm{d}U\,\mathrm{d}V + \mathrm{d}\big(X^2\big)^2 + \mathrm{d}\big(X^3\big)^2\Big) + \frac{L^2\,\mathrm{d}W^2}{W^2(1-L^4V^4W^4)^{5/2}} \\
&\quad + \frac{L^2}{(1-L^4V^4W^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2 + \left(\sum_{r=1}^{\infty}\frac{L^{4r+2}V^{4r-2}W^{4r-1}}{2^r r!}\prod_{k=0}^{r-1}(5+2k)\right) \\
&\quad\times (W\,\mathrm{d}V^2 + 2V\,\mathrm{d}V\,\mathrm{d}W). \qquad [\rho < 1]
\end{aligned} \tag{5.71}$$

This is seen to be finite when $V \to 0$. Thus, the new coordinates show that $\rho = 0$ is not a singularity of the metric; in fact, no singularity exists for this metric.

5.6.3 Garfinkle-Vachaspati transformation of the D3 brane metric

For the D3 brane, it can be easily seen that the vector ∂_u is null (since $g_{uu} = 0$), hypersurface orthogonal (since $g_{u\mu} = \rho^2\partial_\mu v$) and Killing (since $g_{\mu\nu,u} = 0$). Therefore, we can apply a GV transformation on the metric with this vector. The associated function Ψ will be independent of u (as $\mathcal{L}_{\partial_u}\Psi = 0$) and will satisfy the Laplace equation $\nabla^2\Psi = 0$. For simplicity, we assume for now that Ψ is also independent of the coordinates x^2 and x^3 and is thus a function of only v and the coordinates of the scaled $\mathbb{R}^6$ submanifold. The metric can be written as $\mathbf{g} = H^{-1/2}\boldsymbol{\eta}_{\mathbb{R}^{3+1}} \otimes H^{1/2}\mathbf{g}_{\mathbb{R}^6}$, where $H = 1 + L^4/r^4$. Therefore, its determinant will be $g = H^{-2}(-1) \times H^3 g_{\mathbb{R}^6} = -Hg_{\mathbb{R}^6}$, where $g_{\mathbb{R}^6}$ is the determinant of the 6-dimensional Euclidean space in spherical coordinates, and the determinant of the Minkowski metric in cartesian coordinates is -1. Then we see that

$$
\begin{aligned}
\nabla^2\Psi &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) \\
&= \frac{1}{H^{1/2}\sqrt{g_{\mathbb{R}^6}}}\partial_i(H^{1/2}\sqrt{g_{\mathbb{R}^6}}H^{-1/2}(g_{\mathbb{R}^6})^{ij}\partial_j\Psi) \qquad [i{,}j \text{ are coordinates of } \mathbb{R}^6] \\
&= H^{-1/2}\frac{1}{\sqrt{g_{\mathbb{R}^6}}}\partial_i(\sqrt{g_{\mathbb{R}^6}}(g_{\mathbb{R}^6})^{ij}\partial_j\Psi) \\
&= H^{-1/2}\nabla^2_{\mathbb{R}^6}\Psi = 0,
\end{aligned}
\tag{5.72}
$$

i.e. Ψ satisfies the Laplace equation in the flat transverse space. This can be solved similar to the example of the neutral oscillating string,[13] and we thus obtain $\Psi(v,\vec{r}) = \vec{f}(v)\cdot\vec{r}$, where $\vec{r}$ is the position vector in $\mathbb{R}^6$. Therefore, the GV transformed D3 brane metric is

$$
\mathrm{d}s^2 = H^{-1/2}(\mathrm{d}u\,\mathrm{d}v + \vec{f}(v)\cdot\vec{r}\,\mathrm{d}v^2 + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2) + H^{1/2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2). \tag{5.73}
$$

To make the metric asymptotically flat, we apply the coordinate transformation

$$
u \to u - \vec{r}\cdot\int^v \mathrm{d}v'\,\vec{f}(v'). \tag{5.74}
$$

Then the metric becomes

$$
\mathrm{d}s^2 = H^{-1/2}\left(\mathrm{d}u\,\mathrm{d}v - \mathrm{d}v\,\mathrm{d}\vec{r}\cdot\int^v \mathrm{d}v'\,\vec{f}(v') + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2\right) + H^{1/2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2), \tag{5.75}
$$

which is asymptotically flat.

Writing $\vec{r} = r\hat{n}$, such that $\mathrm{d}r^i = r\,\mathrm{d}n^i + n^i\,\mathrm{d}r$ we can rewrite the metric as

$$\mathrm{d}s^2 = H^{-1/2}\Big(\mathrm{d}u\,\mathrm{d}v - r\,\mathrm{d}v\,\mathrm{d}n^i \int^v \mathrm{d}v'\, f_i(v') - n_i\,\mathrm{d}v\,\mathrm{d}r \int^v \mathrm{d}v'\, f_i(v') + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2\Big) + H^{1/2}(\mathrm{d}r^2 + r^2\,\mathrm{d}{\Omega_5}^2). \tag{5.76}$$

It is to be noted that the 6 variables n^i are not independent coordinates as $\sum_i (n^i)^2 = 1$. They can be related to the 5 angular coordinates of $\mathbb{R}^6$.

Let us define $f_i(v) = \ddot{F}_i(v)$, where the dots denote differentiation with respect to v. We see that the GV transformation has introduced two additional terms in the metric. Since we have already seen that the original metric is regular at the horizon $r = 0$, we only need to check if the new terms diverge at $r = 0$ or not. The first term can be written in terms of the $\{U, V, W, \ldots\}$ coordinates as

$$- L^2V^2W^2 \frac{L^2VW}{(1 - L^4V^4W^4)^{1/4}} \frac{\mathrm{d}V}{V^2}\,\mathrm{d}n^i\,\dot{F}_i(v) = -\frac{L^4VW^3}{(1 - L^4V^4W^4)^{1/4}} \dot{F}_i(v)\,\mathrm{d}V\,\mathrm{d}n^i\,, \tag{5.77}$$

which remains finite as $V \to 0$. The second term is

$$-L^2V^2W^2 n_i \frac{\mathrm{d}V}{V^2} \frac{L^2(V\,\mathrm{d}W + W\,\mathrm{d}V)}{(1 - L^4V^4W^4)^{5/4}} \dot{F}_i(v) = -L^4W^2n^i\dot{F}_i(v) \frac{V\,\mathrm{d}V\,\mathrm{d}W + W\,\mathrm{d}V^2}{(1 - L^4V^4W^4)^{5/4}}, \tag{5.78}$$

which is also regular as $v \to 0$. Thus we have a coordinates in which the metric components are regular at the horizon. However, the derivatives of the metric components may diverge as $V \to 0$, so this is only a C^0 extension. We will try to find a C^1 extension of this metric and then use the method described by Kimura et al[11] to see if a C^2 solution exists.

Chapter 6

Extended Garfinkle-Vachaspati transformation

6.1 Introduction

The Garfinkle-Vachaspati transformation discussed before involves a null hypersurface orthogonal Killing vector. The metric transformation itself somewhat resembles the Kerr-Schild form in which some black hole metrics can be written. However, it has been found[15] that there are certain metrics which do not have a Kerr-Schild form but can be instead be written as

$$g'_{\mu\nu} = g_{\mu\nu} + Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu), \tag{6.1}$$

where k is null, l is spacelike, and the two vectors are orthogonal with respect to the original metric. A similar transformation has also been used[16] to add travelling-wave disturbances to supergravity. This suggests that the above transformation may be used to devise a solution generating technique, of which the Garfinkle-Vachaspati transformation is a special case.. To implement this extended Garfinkle-Vachaspati transformation, We also impose the conditions that k is hypersurface orthogonal and Killing (i.e. $\nabla_\mu k_\nu = k_{[\mu}\nabla_{\nu]}S$) and l is covariantly constant. Also, we stipulate that every field has Lie derivative zero with respect to k as well as l. We can now do the analysis in a manner similar to that followed by Kaloper at al.[9] to check how such a transformation would affect the Physics of the problem.

6.2 Transformation of the inverse metric

First of all, let us find the inverse metric $g'^{\mu\nu}$. We expect to be of the form

$$g'^{\mu\nu} = g^{\mu\nu} + \tilde{H}k^\mu k^\nu + \tilde{K}(k^\mu l^\nu + l^\mu k^\nu) + \tilde{G}l^\mu l^\nu. \tag{6.2}$$

Then we see that

$$\begin{aligned}
g'_{\mu\nu}g'^{\nu\lambda} &= (g_{\mu\nu} + Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))\Big(g^{\nu\lambda} + \tilde{H}k^\nu k^\lambda + \tilde{K}(k^\nu l^\lambda + l^\nu k^\lambda) + \tilde{G}l^\nu l^\lambda\Big) \\
&= \delta_\mu{}^\lambda + \tilde{H}k_\mu k^\lambda + \tilde{K}(k_\mu l^\lambda + l_\mu k^\lambda) + \tilde{G}l_\mu l^\lambda + Hk_\mu k^\lambda + K(k_\mu l^\lambda + l_\mu k^\lambda) \\
&\quad + K\tilde{K}(l_\nu l^\nu)k_\mu k^\lambda + K\tilde{G}(l_\nu l^\nu)k_\mu l^\lambda \\
&= \delta_\mu{}^\lambda + (\tilde{H} + H + K\tilde{K}l^2)k_\mu k^\lambda + (\tilde{K} + K + K\tilde{G}l^2)k_\mu l^\lambda + (\tilde{K} + K)l_\mu k^\lambda \\
&\quad + \tilde{G}l_\mu l^\lambda = \delta_\mu{}^\lambda \\
\implies 0 &= (\tilde{H} + H + K\tilde{K}l^2)k_\mu k^\lambda + (\tilde{K} + K + K\tilde{G}l^2)k_\mu l^\lambda + (\tilde{K} + K)l_\mu k^\lambda + \tilde{G}l_\mu l^\lambda.
\end{aligned} \tag{6.3}$$

Contracting both sides with l^μ, we get

$$(\tilde{K} + K)l^2 k^\lambda + \tilde{G}l^2 l^\lambda = 0. \tag{6.4}$$

This upon contraction with l_λ gives $\tilde{G} = 0$, which in turn implies $\tilde{K} = -K$. Then we obtain $(\tilde{H} + H - K^2 l^2)k_\mu k^\lambda = 0 \implies \tilde{H} = -H + l^2K^2$. Thus, the transformed inverse metric is given by

$$g'^{\mu\lambda} = g^{\mu\lambda} - (H - l^2K^2)k^\mu k^\nu - K(k^\mu l^\nu + l^\mu k^\nu). \tag{6.5}$$

The inverse metric is thus seen to have a term of second order in the coefficient K. The change in the metric is written as $h_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu} = Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu)$. Then the inverse metric transforms as $g'^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu$.

We can also see that

$$k'_\nu = g'_{\mu\nu}k^\mu = g_{\mu\nu}k^\mu = k_\nu, \tag{6.6}$$

$$\text{and} \quad l'_\nu = g'_{\mu\nu}l^\mu = g_{\mu\nu}l^\mu + Kl^2k_\nu = l_\nu + Kl^2k_\nu. \tag{6.7}$$

So we see that the lowering of the index of l gives different results for the original and transformed metrics. However, the orthogonality of k and l in the original metric imply that the norms of both the vector fields, as well as their orthogonality, are preserved under the transformation. Whereas k^μ retains all of its properties, l^μ is Killing but not gradient (or even hypersurface orthogonal) in the new metric. In the present analysis, we will raise and lower indices using the original metric; use of the transformed metric will be denoted by primes such as $l'_\nu = {g'}_{\mu\nu} l^\mu$.

6.3 Effect on the determinant of the metric

Let us check if the above transformation changes the determinant g of metric. The following analysis closely resembles that done for the usual GV transformation, with only minor modifications. We start by writing

$$g'_{\mu\nu} = g_{\mu\lambda}(\delta^\lambda{}_\nu + h^\lambda{}_\nu), \tag{6.8}$$

which can be written as a matrix equation $\mathbf{g}' = \mathbf{g}(\mathbf{I} + \mathbf{H})$, where $\mathbf{g}$ is the matrix with entries $g_{\mu\nu}$, $\mathbf{I}$ the identity matrix and $\mathbf{H}$ the matrix with entries $h^\mu{}_\nu = Hk^\mu k_\nu + K(k^\mu l_\nu + l^\mu k_\nu)$. Then we see that $[\mathbf{H}^2]^\mu{}_\nu = K^2 l^2 k^\mu k_\nu$ and $[\mathbf{H}^3]^\mu{}_\nu = 0$, i.e. $\mathbf{H}$ is nilpotent of index 3. So we can say that

$$(\mathbf{I} + \mathbf{H})^n = \mathbf{I} + n\mathbf{H} + \frac{n(n-1)}{2}\mathbf{H}^2 \qquad [n \in \mathbb{N}]. \tag{6.9}$$

Let us now take $\det(\mathbf{I} + \mathbf{H}) = x$. Then, taking the determinant of both sides od the above equation gives that

$$x^n = \det\left(\mathbf{I} + n\mathbf{H} + \frac{n(n-1)}{2}\mathbf{H}^2\right), \tag{6.10}$$

where the left hand side is an exponential function in n, and the right hand side is a polynomial in n of degree at most $2D$ for a D-dimensional manifold. This can hold for every natural number n only if x is 0 or 1. But $x = 0$ implies the existence of a column vector $\mathbf{v} \neq \mathbf{0}$ such that $(\mathbf{I}+\mathbf{H})\mathbf{v} = \mathbf{0} \implies \mathbf{H}\mathbf{v} = -\mathbf{v}$. Then $\mathbf{H}^2\mathbf{v} = -\mathbf{H}\mathbf{v} = \mathbf{v}$, and $\mathbf{H}^3\mathbf{v} = -\mathbf{v} = \mathbf{0}$, which contradicts the condition that $\mathbf{v} \neq \mathbf{0}$. Thus we have $\det(\mathbf{I}+\mathbf{H}) = 1$, i.e. the determinant of the metric is unchanged under the generalised GV transformation.

6.4 Transformation of Einstein equations

6.4.1 Shift in the Christoffel symbols

We now have to find how the generalised transformation affects the Einstein equations. Firstly, we find the shift $\Omega^{\alpha}{}_{\mu\nu}$ in the Christoffel symbols using the same technique as the one used for the usual GV transformation in Eq. (5.26) to obtain

$$\begin{aligned} g'_{\beta\gamma}\Omega^{\beta}{}_{\mu\nu} &= \frac{1}{2}(\nabla_\mu(Hk_\nu k_\gamma + K(k_\nu l_\gamma + l_\nu k_\gamma)) + \nabla_\nu(Hk_\gamma k_\mu + K(k_\gamma l_\mu + l_\gamma k_\mu)) \\ &\quad - \nabla_\gamma(Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))) \\ \implies \Omega^{\alpha}{}_{\mu\nu} &= g'^{\alpha\gamma} g'{}_{\beta\gamma}\Omega^{\beta}{}_{\mu\nu} \\ &= \frac{1}{2}(\nabla_\mu(Hk_\nu k^\alpha + K(k_\nu l^\alpha + l_\nu k^\alpha)) + \nabla_\nu(Hk_\mu k^\alpha + K(k_\mu l^\alpha + l_\mu k^\alpha)) \\ &\quad - \nabla^\alpha(Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))) - \frac{1}{2}Kl^2k^\alpha(k_\mu\nabla_\nu K + k_\nu\nabla_\mu K). \end{aligned} \tag{6.11}$$

Then we see that

$$\begin{aligned} \Omega^{\mu}{}_{\mu\nu} &= \frac{1}{2}(\nabla_\mu(Hk_\nu k^\mu + K(k_\nu l^\mu + l_\nu k^\mu)) + \nabla_\nu(Hk_\mu k^\mu + K(k_\mu l^\mu + l_\mu k^\mu)) \\ &\quad - \nabla^\mu(Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))) - \frac{1}{2}Kl^2k^\mu(k_\mu\nabla_\nu K + k_\nu\nabla_\mu K) \\ &= 0, \end{aligned} \tag{6.12}$$

$$\begin{aligned} k^\mu\Omega^{\alpha}{}_{\mu\nu} &= \frac{1}{2}k^\mu(\nabla_\mu(Hk_\nu k^\alpha + K(k_\nu l^\alpha + l_\nu k^\alpha)) + \nabla_\nu(Hk_\mu k^\alpha + K(k_\mu l^\alpha + l_\mu k^\alpha)) \\ &\quad - \nabla^\alpha(Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))) - \frac{1}{2}Kl^2k^\mu k^\alpha(k_\mu\nabla_\nu K + k_\nu\nabla_\mu K) \\ &= 0, \end{aligned} \tag{6.13}$$

$$\begin{aligned}
k_\alpha \Omega^\alpha{}_{\mu\nu} &= \frac{1}{2} k_\alpha (\nabla_\mu (H k_\nu k^\alpha + K(k_\nu l^\alpha + l_\nu k^\alpha)) + \nabla_\nu (H k_\mu k^\alpha + K(k_\mu l^\alpha + l_\mu k^\alpha)) \\
&\quad - \nabla^\alpha (H k_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))) - \frac{1}{2} K l^2 k_\alpha k^\alpha (k_\mu \nabla_\nu K + k_\nu \nabla_\mu K) \\
&= 0, \qquad (6.14) \\
\Omega^\alpha{}_{\mu\beta} \Omega^\beta{}_{\alpha\nu} &= \frac{1}{4} [\nabla_\mu (H k_\beta k^\alpha + K(k_\beta l^\alpha + l_\beta k^\alpha)) + \nabla_\beta (H k_\mu k^\alpha + K(k_\mu l^\alpha + l_\mu k^\alpha)) \\
&\quad - \nabla^\alpha (H k_\mu k_\beta + K(k_\mu l_\beta + l_\mu k_\beta)) - K l^2 k^\alpha (k_\mu \nabla_\beta K + k_\beta \nabla_\mu K)] \\
&\quad \times [\nabla_\alpha (H k_\nu k^\beta + K(k_\nu l^\beta + l_\nu k^\beta)) + \nabla_\nu (H k_\alpha k^\beta + K(k_\alpha l^\beta + l_\alpha k^\beta)) \\
&\quad - \nabla^\beta (H k_\alpha k_\nu + K(k_\alpha l_\nu + l_\alpha k_\nu)) - K l^2 k^\beta (k_\alpha \nabla_\nu K + k_\nu \nabla_\alpha K)] \\
&= \frac{l^2}{4} [\nabla_\mu (K k_\beta) \nabla_\nu (K k^\beta) - \nabla_\mu (K k_\beta) \nabla^\beta (K k_\nu) + \nabla_\mu (K k^\alpha) \nabla_\alpha (K k_\nu) \\
&\quad + \nabla_\mu (K k^\alpha) \nabla_\nu (K k_\alpha) + \nabla_\beta (K k_\mu) \nabla_\nu (K k^\beta) - \nabla_\beta (K k_\mu) \nabla^\beta (K k_\nu) \\
&\quad - \nabla^\alpha (K k_\mu) \nabla_\alpha (K k_\nu) - \nabla^\alpha (K k_\mu) \nabla_\nu (K k_\alpha)] \\
&= \frac{l^2}{2} (\nabla_\mu (K k^\alpha) \nabla_\nu (K k_\alpha) - \nabla^\alpha (K k_\mu) \nabla_\alpha (K k_\nu)) \\
&= \frac{l^2}{2} (\frac{1}{4} K^2 k_\mu k_\nu \nabla^\alpha S \nabla_\alpha S - k_\mu k_\nu \nabla^\alpha K \nabla_\alpha K + K k_\mu k_\nu \nabla^\alpha K \nabla_\alpha S \\
&\quad - \frac{1}{4} K^2 k_\mu k_\nu \nabla^\alpha S \nabla_\alpha S) \\
&= \frac{l^2}{2} k_\mu k_\nu (K \nabla^\alpha K \nabla_\alpha S - \nabla^\alpha K \nabla_\alpha K). \qquad (6.15)
\end{aligned}$$

6.4.2 Shift in the Ricci tensor

Now we can find the shift in the Ricci tensor as

$$
\begin{aligned}
R'_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\
&= \Gamma'^{\alpha}{}_{\mu\nu,\alpha} - \Gamma'^{\alpha}{}_{\mu\alpha,\nu} + \Gamma'^{\alpha}{}_{\beta\alpha}\Gamma'^{\beta}{}_{\mu\nu} - \Gamma'^{\alpha}{}_{\beta\nu}\Gamma'^{\beta}{}_{\mu\alpha} \\
&= R_{\mu\nu} + \Omega^{\alpha}{}_{\mu\nu,\alpha} - \Omega^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} + \Omega^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} + \Omega^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} \\
&\quad - \Omega^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} - \Omega^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} \\
&= R_{\mu\nu} + \Omega^{\alpha}{}_{\mu\nu,\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} - \Gamma^{\beta}{}_{\alpha\nu}\Omega^{\alpha}{}_{\mu\beta} - \Gamma^{\beta}{}_{\mu\alpha}\Omega^{\alpha}{}_{\beta\nu} - \Omega^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} \\
&\quad \text{[Using Eq. (6.12) and swapping dummy indices } \alpha \text{ and } \beta \text{ in the third term]} \\
&= R_{\mu\nu} + \nabla_{\alpha}\Omega^{\alpha}{}_{\mu\nu} - \frac{l^2}{2}k_{\mu}k_{\nu}(K\nabla^{\alpha}K\nabla_{\alpha}S - \nabla^{\alpha}K\nabla_{\alpha}K) \qquad (6.16) \\
\implies R'^{\mu}{}_{\nu} &= g'^{\mu\lambda}R'_{\lambda\nu} \\
&= R^{\mu}{}_{\mu} + g^{\mu\lambda}\nabla_{\alpha}\Omega^{\alpha}{}_{\lambda\nu} + \frac{l^2}{2}k^{\mu}k_{\nu}(\nabla^{\alpha}K\nabla_{\alpha}K - K\nabla^{\alpha}K\nabla_{\alpha}S) \\
&\quad - \frac{1}{2}(H - Kl^2)k^{\mu}k_{\nu}\nabla^2 S - Kl^{\mu}k_{\nu}\nabla^2 S - Kk^{\mu}l^{\lambda}\nabla_{\alpha}\Omega^{\alpha}{}_{\lambda\nu}. \qquad (6.17)
\end{aligned}
$$

For the last step, we have used Eq. (5.32) and the fact that $l^{\lambda}R_{\lambda\nu} = 0$, which can be obtained using similar reasoning. All that remains to be found is $\nabla_{\alpha}\Omega^{\alpha}{}_{\lambda\nu}$, which is obtained as follows:

$$
\begin{aligned}
\nabla_\alpha \Omega^\alpha{}_{\lambda\nu} &= \frac{1}{2}\nabla_\alpha[\nabla_\mu(Hk_\nu k^\alpha + K(k_\nu l^\alpha + l_\nu k^\alpha)) + \nabla_\nu(Hk_\mu k^\alpha + K(k_\mu l^\alpha + l_\mu k^\alpha)) \\
&\quad - \nabla^\alpha(Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu)) - Kl^2 k^\alpha(k_\mu \nabla_\nu K + k_\nu \nabla_\mu K)] \\
&= \frac{1}{2}\nabla_\alpha[k_\nu k^\alpha \nabla_\mu H + (k_\nu l^\alpha + l_\nu k^\alpha)\nabla_\mu K + \frac{1}{2}(Hk^\alpha + Kl^\alpha)(k_\mu \nabla_\nu S - k_\nu \nabla_\mu S) \\
&\quad + \frac{1}{2}(Hk_\nu + Kl_\nu)(k_\mu \nabla^\alpha S - k^\alpha \nabla_\mu S) + k_\mu k^\alpha \nabla_\nu H + (k_\mu l^\alpha + l_\mu k^\alpha)\nabla_\nu K \\
&\quad + \frac{1}{2}(Hk^\alpha + Kl^\alpha)(k_\nu \nabla_\mu S - k_\mu \nabla_\nu S) + \frac{1}{2}(Hk_\mu + Kl_\mu)(k_\nu \nabla^\alpha S - k^\alpha \nabla_\nu S) \\
&\quad - k_\mu k_\nu \nabla^\alpha H - (k_\mu l_\nu + l_\mu k_\nu)\nabla^\alpha K - \frac{1}{2}(Hk_\nu + Kl_\nu)(k^\alpha \nabla_\mu S - k_\mu \nabla^\alpha S) \\
&\quad - \frac{1}{2}(Hk_\mu + Kl_\mu)(k^\alpha \nabla_\nu S - k_\nu \nabla^\alpha S) - Kl^2 k^\alpha(k_\mu \nabla_\nu K + k_\nu \nabla_\mu K)] \\
&= \frac{1}{2}\nabla_\alpha[k_\nu k^\alpha(\nabla_\mu H - Kl^2 \nabla_\mu K) + k_\mu k^\alpha(\nabla_\nu H - Kl^2 \nabla_\nu K) - k_\mu k_\nu \nabla^\alpha H \\
&\quad + (k_\nu l^\alpha + l_\nu k^\alpha)\nabla_\mu K + (k_\mu l^\alpha + l_\mu k^\alpha)\nabla_\nu K - (k_\mu l_\nu + l_\mu k_\nu)\nabla^\alpha K \\
&\quad - (Hk^\alpha k_\nu + Kk^\alpha l_\nu)\nabla_\mu S - (Hk^\alpha k_\mu + Kk^\alpha l_\mu)\nabla_\nu S + (2Hk_\mu k_\nu \\
&\quad + K(k_\mu l_\nu + l_\mu k_\nu))\nabla^\alpha S] \\
&= \frac{1}{2}[-k_\nu \nabla_\alpha H \frac{1}{2} k_\mu \nabla^\alpha S + Kl^2 k_\nu \nabla_\alpha K \frac{1}{2} k_\mu \nabla^\alpha S - k_\mu \nabla_\alpha H \frac{1}{2} k_\nu \nabla^\alpha S \\
&\quad + Kl^2 k_\mu \nabla_\alpha K \frac{1}{2} k_\nu \nabla^\alpha S + \frac{1}{2} k_\mu \nabla_\alpha S k_\nu \nabla^\alpha H + k_\mu \frac{1}{2} k_\nu \nabla_\alpha S \nabla^\alpha H - k_\mu k_\nu \nabla^2 H \\
&\quad - l_\nu \nabla_\alpha K \frac{1}{2} k_\mu \nabla^\alpha S - l_\mu \nabla_\alpha K \frac{1}{2} k_\nu \nabla^\alpha S + \frac{1}{2} k_\mu \nabla_\alpha S l_\nu \nabla^\alpha K + l_\mu \frac{1}{2} k_\nu \nabla_\alpha S \nabla^\alpha K \\
&\quad - (k_\mu l_\nu + l_\mu k_\nu)\nabla^2 K + Hk_\nu \nabla_\alpha S \frac{1}{2} k_\mu \nabla^\alpha S + Kl_\nu \nabla_\alpha S \frac{1}{2} k_\mu \nabla^\alpha S \\
&\quad + Hk_\mu \nabla_\alpha S \frac{1}{2} k_\nu \nabla^\alpha S + Kl_\mu \nabla_\alpha S \frac{1}{2} k_\nu \nabla^\alpha S + 2k_\mu k_\nu \nabla_\alpha H \nabla^\alpha S \\
&\quad - Hk_\mu \nabla_\alpha S k_\nu \nabla^\alpha S - Hk_\mu k_\nu \nabla_\alpha S \nabla^\alpha S \\
&\quad + (k_\mu l_\nu + l_\mu k_\nu)\nabla_\alpha K \nabla^\alpha S - K\frac{1}{2} k_\mu \nabla_\alpha S l_\nu \nabla^\alpha S - \frac{1}{2} Kl_\mu k_\nu \nabla_\alpha S \nabla^\alpha S \\
&\quad + (2Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))\nabla^2 S] \\
&= \frac{1}{2} k_\mu k_\nu(2\nabla_\alpha H \nabla^\alpha S + Kl^2 \nabla_\alpha K \nabla^\alpha S - \nabla^2 H - H\nabla^\alpha S \nabla_\alpha S + 2H\nabla^2 S) \\
&\quad + \frac{1}{2}(k_\mu l_\nu + l_\mu k_\nu)(\nabla^\alpha S \nabla_\alpha K - \nabla^2 K + K\nabla^2 S).
\end{aligned} \tag{6.18}
$$

Therefore, we obtain that

$$l^\lambda \nabla_\alpha \Omega^\alpha{}_{\lambda\nu} = \frac{l^2}{2} k_\nu (\nabla^\alpha S \nabla_\alpha K - \nabla^2 K + K \nabla^2 S). \tag{6.19}$$

Thus,

$$\begin{aligned} R'_{\mu\nu} &= R_{\mu\nu} + \frac{1}{2} k_\mu k_\nu (2\nabla_\alpha H \nabla^\alpha S + l^2 \nabla_\alpha K \nabla^\alpha K - \nabla^2 H - H \nabla^\alpha S \nabla_\alpha S + 2H\nabla^2 S) \\ &\quad + \frac{1}{2}(k_\mu l_\nu + l_\mu k_\nu)(\nabla^\alpha S \nabla_\alpha K - \nabla^2 K + K \nabla^2 S), \qquad (6.20) \\ R'^\mu{}_\nu &= R^\mu{}_\nu + \frac{1}{2} k^\mu k_\nu (2\nabla_\alpha H \nabla^\alpha S - \nabla^2 H - H \nabla_\alpha S \nabla^\alpha S + H \nabla^2 S + l^2 \nabla_\alpha K \nabla^\alpha K \\ &\quad - K l^2 \nabla_\alpha S \nabla^\alpha K + K l^2 \nabla^2 K) + \frac{1}{2}(k^\mu l_\nu + l^\mu k_\nu)(\nabla^\alpha S \nabla_\alpha K - \nabla^2 K + K \nabla^2 S) \\ &\quad - \frac{1}{2} K l^\mu k_\nu \nabla^2 S. \qquad (6.21) \end{aligned}$$

6.5 Transformation of the fields

6.5.1 2-form field

Since the shift in the Ricci tensor can not be made zero in this case without imposing very strict conditions on H and K, we can not let the matter fields be left unchanged as in the usual Garfinkle-Vachaspati transformation. Let us start with the case of a 2-form field F derived from a 1-form potential A, with the action

$$S = \int \mathrm{d}^D x \sqrt{-g} \Big(R(g_{\mu\nu}) - \frac{1}{2} F^{\alpha\beta} F_{\alpha\beta} \Big). \tag{6.22}$$

The Einstein equation is then

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = F^{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} \delta^\mu{}_\nu F^{\sigma\lambda} F_{\sigma\lambda}. \tag{6.23}$$

Let us define $P_\mu = k^\nu F_{\nu\mu}$. Then we have, from Einstein equation and using the properties of k, that

$$\begin{aligned} R^\mu{}_\nu k_\mu k^\nu &= \frac{1}{2}\nabla^2 S k_\nu k^\nu = 0 \\ \implies T^\mu{}_\nu k_\mu k^\nu &= F^{\mu\lambda}F_{\nu\lambda}k_\mu k^\nu = 0 \\ \implies k_\mu F^{\mu\lambda} k^\nu F_{\nu\lambda} &= P_\lambda P^\lambda = 0, \end{aligned} \tag{6.24}$$

and

$$k^\mu P_\mu = k^\mu k^\lambda F_{\lambda\mu} = k^{(\lambda}k^{\mu)}F_{[\lambda\mu]} = 0. \tag{6.25}$$

For any manifold with Lorentzian metric, two orthogonal null vectors have to be linearly dependent. Therefore, $k^\mu F_{\mu\nu} = ck_\nu$ for some scalar c. We also assume that $l^\mu F_{\mu\nu} = 0$.

Let us take the transformed potential to be $A'_\mu = A_\mu + ak_\mu + bl_\mu$, where a and b are scalars satisfying $\mathcal{L}_k a = \mathcal{L}_l a = \mathcal{L}_k b = \mathcal{L}_l b = 0$. This is taken to make the problem simpler, and is not a physical requirement for the potential in general since even imposing symmetry constraints will only imply that $\mathcal{L}_k F_{\mu\nu} = \mathcal{L}_l F_{\mu\nu} = 0$, which may hold even if the potential A_μ has a non-vanishing Lie derivative. Then the transformed field is

$$\begin{aligned} F'_{\mu\nu} &= \nabla_\mu A'_\nu - \nabla_\nu A'_\mu \\ &= F_{\mu\nu} + \nabla_\mu(ak_\nu + bl_\nu) - \nabla_\nu(ak_\mu + bl_\mu) \\ &= F_{\mu\nu} + k_\nu\nabla_\mu a - k_\mu\nabla_\nu a + a(k_\mu\nabla_\nu S - k_\nu\nabla_\mu S) + l_\nu\nabla_\mu b - l_\mu\nabla_\nu b. \end{aligned} \tag{6.26}$$

Raising the indices, we obtain

$$
\begin{aligned}
F'^{\alpha}{}_{\nu} &= g'^{\alpha\mu} F'_{\mu\nu} \\
&= (g^{\alpha\mu} - (H - k^2 l^2) k^{\alpha} k^{\mu} - K(k^{\alpha} l^{\mu} + l^{\alpha} k^{\mu}))(F_{\mu\nu} + k_{\nu} \nabla_{\mu} a - k_{\mu} \nabla_{\nu} a \\
&\quad + a(k_{\mu} \nabla_{\nu} S - k_{\nu} \nabla_{\mu} S) + l_{\nu} \nabla_{\mu} b - l_{\mu} \nabla_{\nu} b) \\
&= F^{\alpha}{}_{\nu} + k_{\nu} \nabla^{\alpha} a - k^{\alpha} \nabla_{\nu} a + a(k^{\alpha} \nabla_{\nu} S - k_{\nu} \nabla^{\alpha} S) + l_{\nu} \nabla^{\alpha} b - l^{\alpha} \nabla_{\nu} b \\
&\quad - (H - K^2 l^2) k^{\alpha} c k_{\nu} + K k^{\alpha} l^2 \nabla_{\nu} b - K l^{\alpha} c k_{\nu} \\
\implies F'^{\alpha\beta} &= g'^{\beta\nu} F'^{\alpha}{}_{\nu} \\
&= (g^{\beta\nu} - (H - K^2 l^2) k^{\beta} k^{\nu} - K(k^{\beta} l^{\nu} + l^{\beta} k^{\nu}))(F^{\alpha}{}_{\nu} + k_{\nu} \nabla^{\alpha} a - k^{\alpha} \nabla_{\nu} a \\
&\quad + a(k^{\alpha} \nabla_{\nu} S - k_{\nu} \nabla^{\alpha} S) + l_{\nu} \nabla^{\alpha} b - l^{\alpha} \nabla_{\nu} b - (H - K^2 l^2) k^{\alpha} c k_{\nu} \\
&\quad + K k^{\alpha} l^2 \nabla_{\nu} b - K l^{\alpha} c k_{\nu}) \\
&= F^{\alpha\beta} + k^{\beta} \nabla^{\alpha} a - k^{\alpha} \nabla^{\beta} a + a(k^{\alpha} \nabla^{\beta} S - k^{\beta} \nabla^{\alpha} S) + l^{\beta} \nabla^{\alpha} b - l^{\alpha} \nabla^{\beta} b \\
&\quad - (H - K^2 l^2) k^{\alpha} c k^{\beta} + K k^{\alpha} l^2 \nabla^{\beta} b - K l^{\alpha} c k^{\beta} - (H - K^2 l^2) k^{\beta} (-c) k^{\alpha} \\
&\quad - K k^{\beta} l^2 \nabla^{\alpha} b - K l^{\beta} (-c) k^{\alpha} \\
&= F^{\alpha\beta} + k^{\beta} \nabla^{\alpha} a - k^{\alpha} \nabla^{\beta} a + a(k^{\alpha} \nabla^{\beta} S - k^{\beta} \nabla^{\alpha} S) + l^{\beta} \nabla^{\alpha} b - l^{\alpha} \nabla^{\beta} b \\
&\quad + K l^2 (k^{\alpha} \nabla^{\beta} b - k^{\beta} \nabla^{\alpha} b) + K c (k^{\alpha} l^{\beta} - l^{\alpha} k^{\beta}). \qquad (6.27)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\nabla'_\alpha F'^{\alpha\beta} &= \nabla_\alpha F'^{\alpha\beta} + \Omega^\alpha{}_{\alpha\mu} F'^{\mu\beta} + \Omega^\beta{}_{\alpha\mu} F'^{\alpha\mu} \\
&= \nabla_\alpha F'^{\alpha\beta} \quad [\because \Omega^\alpha{}_{\alpha\mu} = 0, \quad \Omega^\beta{}_{\alpha\mu} = \Omega^\beta{}_{\mu\alpha}, \quad F'^{\alpha\mu} = -F'^{\mu\alpha}] \\
&= \nabla_\alpha (F^{\alpha\beta} + k^\beta \nabla^\alpha a - k^\alpha \nabla^\beta a + a(k^\alpha \nabla^\beta S - k^\beta \nabla^\alpha S) + l^\beta \nabla^\alpha b \\
&\quad - l^\alpha \nabla^\beta b + K l^2 (k^\alpha \nabla^\beta b - k^\beta \nabla^\alpha b) + Kc(k^\alpha l^\beta - l^\alpha k^\beta)) \\
&= \nabla_\alpha F^{\alpha\beta} - \frac{1}{2} k^\beta \nabla_\alpha S \nabla^\alpha a + k^\beta \nabla^2 a + \nabla_\alpha a \frac{1}{2} k^\beta \nabla^\alpha S - k^\beta \nabla_\alpha a \nabla^\alpha S \\
&\quad - a \nabla_\alpha S \frac{1}{2} k^\beta \nabla^\alpha S + a \frac{1}{2} k^\beta \nabla_\alpha S \nabla^\alpha S - a k^\beta \nabla^2 S + l^\beta \nabla^2 b - l^2 k^\beta \nabla_\alpha K \nabla^\alpha b \\
&\quad - K l^2 \nabla_\alpha b \frac{1}{2} k^\beta \nabla^\alpha S + K l^2 \frac{1}{2} k^\beta \nabla_\alpha S \nabla^\alpha b - K l^2 k^\beta \nabla^2 b \\
&= k^\beta (\nabla^2 a - \nabla_\alpha a \nabla^\alpha S - a \nabla^2 S - l^2 \nabla_\alpha K \nabla^\alpha b - K l^2 \nabla^2 b) + l^\beta \nabla^2 b. \qquad (6.28)
\end{aligned}
$$

Thus we see that for the shifted form field F' to satisfy the field equation, we must have

$$\nabla^2 a - \nabla_\alpha a \nabla^\alpha S - a \nabla^2 S - l^2 \nabla_\alpha K \nabla^\alpha b = 0, \qquad (6.29)$$

$$\nabla^2 b = 0. \qquad (6.30)$$

To obtain the transformed stress-energy tensor, we find

$$
\begin{aligned}
F'^{\mu\lambda}F'_{\nu\lambda} &= [F^{\mu\lambda} + k^{\lambda}\nabla^{\mu}a - k^{\mu}\nabla^{\lambda}a + a(k^{\mu}\nabla^{\lambda}S - k^{\lambda}\nabla^{\mu}S) + l^{\lambda}\nabla^{\mu}b - l^{\mu}\nabla^{\lambda}b \\
&\quad + Kl^2(k^{\mu}\nabla^{\lambda}b - k^{\lambda}\nabla^{\mu}b) + Kc(k^{\mu}l^{\lambda} - l^{\mu}k^{\lambda})][F_{\nu\lambda} + k_{\lambda}\nabla_{\nu}a - k_{\nu}\nabla_{\lambda}a \\
&\quad + a(k_{\nu}\nabla_{\lambda}S - k_{\lambda}\nabla_{\nu}S) + l_{\lambda}\nabla_{\nu}b - l_{\nu}\nabla_{\lambda}b] \\
&= F^{\mu\lambda}F_{\nu\lambda} - ck^{\mu}\nabla_{\nu}a - k_{\nu}F^{\mu\lambda}\nabla_{\lambda}a + ak_{\nu}F^{\mu\lambda}\nabla_{\lambda}S + ack^{\mu}\nabla_{\nu}S - l_{\nu}F^{\mu\lambda}\nabla_{\lambda}b \\
&\quad - ck_{\nu}\nabla^{\mu}a - k^{\mu}F_{\nu\lambda}\nabla^{\lambda}a + k^{\mu}k_{\nu}\nabla^{\lambda}a\nabla_{\lambda}a - ak^{\mu}k_{\nu}\nabla^{\lambda}a\nabla_{\lambda}S + k^{\mu}l_{\nu}\nabla^{\lambda}a\nabla_{\lambda}b \\
&\quad + ak^{\mu}F_{\nu\lambda}\nabla^{\lambda}S - ak^{\mu}k_{\nu}\nabla^{\lambda}S\nabla_{\lambda}a + a^2k^{\mu}k_{\nu}\nabla^{\lambda}S\nabla_{\lambda}S - ak^{\mu}l_{\nu}\nabla^{\lambda}S\nabla_{\lambda}b \\
&\quad + ack_{\nu}\nabla^{\mu}S + l^2\nabla^{\mu}b\nabla_{\nu}b - l^{\mu}F_{\nu\lambda}\nabla^{\lambda}b + l^{\mu}k_{\nu}\nabla^{\lambda}b\nabla_{\lambda}a - al^{\mu}k_{\nu}\nabla^{\lambda}b\nabla_{\lambda}S \\
&\quad + l^{\mu}l_{\nu}\nabla^{\lambda}b\nabla_{\lambda}b + Kl^2k^{\mu}F_{\nu\lambda}\nabla^{\lambda}b - Kl^2k^{\mu}k_{\nu}\nabla^{\lambda}b\nabla_{\lambda}a + Kl^2ak^{\mu}k_{\nu}\nabla^{\lambda}b\nabla_{\lambda}S \\
&\quad - Kl^2k^{\mu}l_{\nu}\nabla^{\lambda}b\nabla_{\lambda}b + Kcl^2k_{\nu}\nabla^{\mu}b + Kcl^2k^{\mu}\nabla_{\nu}b + Kcl^{\mu}ck_{\nu} \\
&= F^{\mu\lambda}F_{\nu\lambda} + k^{\mu}k_{\nu}(\nabla^{\lambda}a\nabla_{\lambda}a - 2a\nabla^{\lambda}a\nabla_{\lambda}S + a^2\nabla^{\lambda}S\nabla_{\lambda}S - Kl^2\nabla^{\lambda}b\nabla_{\lambda}a \\
&\quad + Kl^2a\nabla^{\lambda}b\nabla_{\lambda}S) + k^{\mu}l_{\nu}(\nabla^{\lambda}a\nabla_{\lambda}b - a\nabla^{\lambda}S\nabla_{\lambda}b - Kl^2\nabla^{\lambda}b\nabla_{\lambda}b) \\
&\quad + l^{\mu}k_{\nu}(\nabla^{\lambda}b\nabla_{\lambda}a - a\nabla^{\lambda}b\nabla_{\lambda}S + Kc^2) + l^{\mu}l_{\nu}\nabla^{\lambda}b\nabla_{\lambda}b + k^{\mu}(-c\nabla_{\nu}a \\
&\quad + ac\nabla_{\nu}S - F_{\nu\lambda}\nabla^{\lambda}a + aF_{\nu\lambda}\nabla^{\lambda}S + Kl^2F_{\nu\lambda}\nabla^{\lambda}b + Kcl^2\nabla_{\nu}b) \\
&\quad + k_{\nu}(-F^{\mu\lambda}\nabla_{\lambda}a + aF^{\mu\lambda}\nabla_{\lambda}S - c\nabla^{\mu}a + ac\nabla^{\mu}S + Kcl^2\nabla^{\mu}b) \\
&\quad + l^{\mu}(-F_{\nu\lambda}\nabla^{\lambda}b) + l_{\nu}(-F^{\mu\lambda}\nabla_{\lambda}b) + l^2\nabla^{\mu}b\nabla_{\nu}b.
\end{aligned}
\tag{6.31}
$$

Now, the Einstein equation for the transformed metric and field is

$$
R'^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R' = F'^{\mu\lambda}F'_{\nu\lambda}F'_{\nu\lambda} - \frac{1}{4}\delta^{\mu}{}_{\nu}F'^{\sigma\lambda}F'_{\sigma\lambda}, \tag{6.32}
$$

which implies

$$
\begin{aligned}
&R^\mu{}_\nu + \frac{1}{2}k^\mu k_\nu(2\nabla_\alpha H\nabla^\alpha S - \nabla^2 H - H\nabla_\alpha S\nabla^\alpha S + H\nabla^2 S + l^2\nabla_\alpha K\nabla^\alpha K \\
&\quad - Kl^2\nabla_\alpha S\nabla^\alpha K + Kl^2\nabla^2 K) + \frac{1}{2}(k^\mu l_\nu + l^\mu k_\nu)(\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S) \\
&\quad - \frac{1}{2}Kl^\mu k_\nu\nabla^2 S - \frac{1}{2}\delta^\mu{}_\nu R \\
&\quad = F^{\mu\lambda}F_{\nu\lambda} + k^\mu k_\nu(\nabla^\lambda a\nabla_\lambda a - 2a\nabla^\lambda a\nabla_\lambda S + a^2\nabla^\lambda S\nabla_\lambda S - Kl^2\nabla^\lambda b\nabla_\lambda a \\
&\qquad + Kl^2 a\nabla^\lambda b\nabla_\lambda S) + k^\mu l_\nu(\nabla^\lambda a\nabla_\lambda b - a\nabla^\lambda S\nabla_\lambda b - Kl^2\nabla^\lambda b\nabla_\lambda b) + l^\mu k_\nu(\nabla^\lambda b\nabla_\lambda a \\
&\qquad - a\nabla^\lambda b\nabla_\lambda S + Kc^2) + l^\mu l_\nu\nabla^\lambda b\nabla_\lambda b + k^\mu(-c\nabla_\nu a + ac\nabla_\nu S - F_{\nu\lambda}\nabla^\lambda a \\
&\qquad + aF_{\nu\lambda}\nabla^\lambda S + Kl^2 F_{\nu\lambda}\nabla^\lambda b + Kcl^2\nabla_\nu b) + k_\nu(-F^{\mu\lambda}\nabla_\lambda a + aF^{\mu\lambda}\nabla_\lambda S - c\nabla^\mu a \\
&\qquad + ac\nabla^\mu S + Kcl^2\nabla^\mu b) + l^\mu(-F_{\nu\lambda}\nabla^\lambda b) + l_\nu(-F^{\mu\lambda}\nabla_\lambda b) + l^2\nabla^\mu b\nabla_\nu b \\
&\qquad - \frac{1}{4}\delta^\mu{}_\nu(F^{\sigma\lambda}F_{\sigma\lambda} + 2l^2\nabla^\lambda b\nabla_\lambda b) \\
\Longrightarrow\ &\frac{1}{2}k^\mu k_\nu(2\nabla_\alpha H\nabla^\alpha S - \nabla^2 H - H\nabla_\alpha S\nabla^\alpha S + H\nabla^2 S + l^2\nabla_\alpha K\nabla^\alpha K \\
&\quad - Kl^2\nabla_\alpha S\nabla^\alpha K + Kl^2\nabla^2 K) + \frac{1}{2}(k^\mu l_\nu + l^\mu k_\nu)(\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S) \\
&\quad - \frac{1}{2}Kl^\mu k_\nu\nabla^2 S \\
&\quad = k^\mu k_\nu(\nabla^\lambda a\nabla_\lambda a - 2a\nabla^\lambda a\nabla_\lambda S + a^2\nabla^\lambda S\nabla_\lambda S - Kl^2\nabla^\lambda b\nabla_\lambda a + Kl^2 a\nabla^\lambda b\nabla_\lambda S) \\
&\qquad + k^\mu l_\nu(\nabla^\lambda a\nabla_\lambda b - a\nabla^\lambda S\nabla_\lambda b - Kl^2\nabla^\lambda b\nabla_\lambda b) + l^\mu k_\nu(\nabla^\lambda b\nabla_\lambda a - a\nabla^\lambda b\nabla_\lambda S \\
&\qquad + Kc^2) + l^\mu l_\nu\nabla^\lambda b\nabla_\lambda b + k^\mu(-c\nabla_\nu a + ac\nabla_\nu S - F_{\nu\lambda}\nabla^\lambda a + aF_{\nu\lambda}\nabla^\lambda S \\
&\qquad + Kl^2 F_{\nu\lambda}\nabla^\lambda b + Kcl^2\nabla_\nu b) + k_\nu(-F^{\mu\lambda}\nabla_\lambda a + aF^{\mu\lambda}\nabla_\lambda S - c\nabla^\mu a + ac\nabla^\mu S \\
&\qquad + Kcl^2\nabla^\mu b) + l^\mu(-F_{\nu\lambda}\nabla^\lambda b) + l_\nu(-F^{\mu\lambda}\nabla_\lambda b) + l^2\nabla^\mu b\nabla_\nu b - \frac{1}{2}\delta^\mu{}_\nu l^2\nabla^\lambda b\nabla_\lambda b.
\end{aligned}
\tag{6.33}
$$

Let us now take the trace of the above equation. We get

$$
\begin{aligned}
0 &= \frac{3}{2}l^2\nabla^\lambda b\nabla_\lambda b \\
\Longrightarrow\ \nabla^\lambda b\nabla_\lambda b &= 0.
\end{aligned}
\tag{6.34}
$$

The conditions given by Eqs. 6.30 and 6.34 imply that $\nabla_\mu b$ is a null divergenceless vertor field. Also, $\mathcal{L}_k b = 0$ gives that $\nabla_\mu k$ is also orthogonal to k_μ. Since two orthogonal null vectors have to be proportional, we get $\nabla_\mu b = dk_\mu$, where $\nabla^2 b = 0 \implies \mathcal{L}_k d = 0$. Since we k_μ is hypersurface orthogonal, $k_\mu = e^{-S}\nabla_\mu u$ for some scalar field u, we have $d = \eta e^S$, $b = \eta u$ for some constant η, which we can take to be 1. Substituting this in Eq. 6.33 gives

$$\begin{aligned}
&\frac{1}{2}k^\mu k_\nu(2\nabla_\alpha H\nabla^\alpha S - \nabla^2 H - H\nabla_\alpha S\nabla^\alpha S + H\nabla^2 S + l^2\nabla_\alpha K\nabla^\alpha K - Kl^2\nabla_\alpha S\nabla^\alpha K \\
&\quad + Kl^2\nabla^2 K) + \frac{1}{2}(k^\mu l_\nu + l^\mu k_\nu)(\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S) - \frac{1}{2}Kl^\mu k_\nu\nabla^2 S \\
&\quad = k^\mu k_\nu(\nabla^\lambda a\nabla_\lambda a - 2a\nabla^\lambda a\nabla_\lambda S + a^2\nabla^\lambda S\nabla_\lambda S) + l^\mu k_\nu(Kc^2) + k^\mu(-c\nabla_\nu a + ac\nabla_\nu S \\
&\qquad - F_{\nu\lambda}\nabla^\lambda a + aF_{\nu\lambda}\nabla^\lambda S) + k_\nu(-F^{\mu\lambda}\nabla_\lambda a + aF^{\mu\lambda}\nabla_\lambda S - c\nabla^\mu a + ac\nabla^\mu S + Kcdl^2k^\mu) \\
&\qquad + l^\mu(cdk_\nu) + l_\nu(cdk^\mu) + l^2d^2k^\mu k_\nu.
\end{aligned} \tag{6.35}$$

Contracting with l_μ and l^ν, we obtain

$$\nabla^\alpha S\nabla_\alpha K - \nabla^2 K = 2(Kc^2 + cd), \tag{6.36}$$

$$\begin{aligned}
&\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S = 2cd \\
&\qquad \implies \nabla^2 S = -2c^2 \quad (K \neq 0)
\end{aligned} \tag{6.37}$$

$$\begin{aligned}
\implies \; &\frac{1}{2}k^\mu k_\nu(-e^S\nabla^2(He^{-S}) + l^2\nabla_\alpha K\nabla^\alpha K - Kl^2\nabla_\alpha S\nabla^\alpha K + Kl^2\nabla^2 K) \\
&\qquad = k^\mu k_\nu(\nabla^\lambda a\nabla_\lambda a - 2a\nabla^\lambda a\nabla_\lambda S + a^2\nabla^\lambda S\nabla_\lambda S + l^2d^2 \\
&\qquad\quad + Kcdl^2) + k^\mu(-c\nabla_\nu a + ac\nabla_\nu S - F_{\nu\lambda}\nabla^\lambda a + aF_{\nu\lambda}\nabla^\lambda S) \\
&\qquad\quad + k_\nu(-F^{\mu\lambda}\nabla_\lambda a + aF^{\mu\lambda}\nabla_\lambda S - c\nabla^\mu a + ac\nabla^\mu S).
\end{aligned} \tag{6.38}$$

Also, Eq. 6.29 gives

$$\nabla^2 a - \nabla_\alpha a\nabla^\alpha S - a\nabla^2 S = 0. \tag{6.39}$$

We get a rather complicated system of coupled partial differential equations, which may be solvable if information about the starting metric $g_{\mu\nu}$ and field $F_{\mu\nu}$ is given.

Equation (6.37) implies that this transformation seems to be applicable only for cases where the starting metric has zero Ricci scalar R, and the field F is also null: $F^{\mu\nu}F_{\mu\nu} = 0$. To see this, we contract both sides of the Einstein equation for the original metric and field with k^ν:

$$\begin{aligned}
\left(R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R\right)k^\nu &= \left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\right) \\
\implies \left(\frac{1}{2}\nabla^2 S - \frac{D}{2}R\right)k^\mu &= \left(-c^2 - \frac{D}{4}F^{\alpha\beta}F_{\alpha\beta}\right)k^\mu \qquad \text{[Using Eq. (5.32)]} \\
\implies 2R &= F^{\alpha\beta}F_{\alpha\beta} \qquad (6.40)
\end{aligned}$$

But taking the traces of both sides of Einstein's equation gives

$$\left(1 - \frac{D}{2}\right)R = \left(1 - \frac{D}{4}\right)F^{\alpha\beta}F_{\alpha\beta}. \qquad (6.41)$$

The above two equations can hold simultaneously only if $R = F^{\alpha\beta}F_{\alpha\beta} = 0$. Hence, we can only consider metrics with zero Ricci scalar when applying this transformation. Equation (6.41) implies that for dimension other than 4, $R = 0 \implies F^{\alpha\beta}F_{\alpha\beta} = 0$. For $D = 4$, $R = 0$ must hold for an Einstein-Maxwell system, and Kramer[17] has shown that the 2-form field has to be null.

6.5.2 Example 1: Plane wave metric

As an example of this transformation, let us consider the plane wave metric,[5]

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v - B_{ab}(u)x^a x^b\,\mathrm{d}u^2 + \mathrm{d}\vec{x}^2\,, \qquad (6.42)$$

with a null 2-form (Maxwell) field derived from a 1-form potential,

$$A = A_i(u)\,\mathrm{d}x^i \implies F = A_i'(u)\,\mathrm{d}u \wedge \mathrm{d}x^i\,. \qquad (6.43)$$

In this case, the null vector $k = \partial_v$ is covariantly constant, so $S = 0$. Let us take l to be the spacelike vector ∂_{x^1}. Then we see that

$$\begin{aligned}
\nabla_\mu k^\nu &= \nabla_\mu \delta_{x^1}{}^\nu \\
&= \Gamma^\nu{}_{x^1\mu} \\
&= \frac{1}{2} g^{\nu\sigma}(g_{\sigma x^1,\mu} + g_{\sigma\mu,x^1} - g_{x^1\mu,\sigma}), \qquad (6.44)
\end{aligned}$$

which means that the only possible non-zero component of the covariant derivative of l is

$$\begin{aligned}
\nabla_u \delta_{x^1}{}^v &= \frac{1}{2} g^{uv}(g_{uu,x^1}) \\
&= -B_{1b} x^b. \qquad (6.45)
\end{aligned}$$

We take the case where $B_{1b} = 0$, so that l is covariantly constant. We note that

$$k^\mu F_{\mu\nu} = F_{v\nu} = 0, \qquad (6.46)$$

$$l^\mu F_{\mu\nu} = F_{x^1\nu} = 0, \qquad (6.47)$$

where the second is satisfied by taking $A_1 =$ constant. The system satisfies Einstein equation if $-\operatorname{tr}(B) = \sum_i A_i'(u)^2$, for example, if $B_{ij} = A_i'(u)^2\delta_{ij}$.

Then the conditions to be satisfied by a, d, H and K can be written as (substituting $c = 0$ and $S = 0$)

$$\nabla^2 K = 0, \qquad (6.48)$$

$$\nabla^2 a = 0, \qquad (6.49)$$

$$\begin{aligned}
&\frac{1}{2} k^\mu k_\nu(\nabla^2 H + \nabla_\alpha K \nabla^\alpha K) = k^\mu k_\nu(\nabla^\alpha a \nabla_\alpha a + d^2) - k^\mu F_{\nu\lambda}\nabla^\lambda a - k_\nu F^{\mu\lambda}\nabla_\lambda a \\
&\implies \nabla^2 H + \nabla_\alpha K \nabla^\alpha K = 2\nabla^\alpha a \nabla_\alpha a + 2d^2 - 4A_i'(u)\partial^i a. \qquad (6.50)
\end{aligned}$$

One solution to this is

$$\nabla^2 H = 0, \tag{6.51}$$

$$d = 0, \tag{6.52}$$

$$a = a(u), \tag{6.53}$$

$$K = \sqrt{2}a. \tag{6.54}$$

6.5.3 3-form field

Let us now consider a 3-form field $F_{(3)} = \mathrm{d}A_{(2)}$, with the action

$$S = \int \mathrm{d}^D x \sqrt{-g}\Big(R(g_{\mu\nu}) - \frac{1}{12}F^{\alpha\beta\gamma}F_{\alpha\beta\gamma}\Big). \tag{6.55}$$

The corresponding 2-form potential $A_{\mu\nu}$ is transformed as

$$A'_{\mu\nu} = A_{\mu\nu} + b(k_\mu l_\nu - k_\nu l_\mu), \tag{6.56}$$

for some scalar field b satisfying $\mathcal{L}_k b = \mathcal{L}_l b = 0$. We also assume that $k^\mu F_{\mu\nu} = k_\mu c_\nu - c_\mu k_\nu$ for some vector c_μ satisfying $k^\mu c_\mu = 0$ and $\mathcal{L}_k c^\mu = 0$. Then the transformed field strength is found to be

$$\begin{aligned}
F'_{\mu\nu\lambda} &= F_{\mu\nu\lambda} + \nabla_\mu(bk_\nu l_\lambda) + \nabla_\nu(bk_\lambda l_\mu) + \nabla_\lambda(bk_\mu l_\nu) - \nabla_\nu(bk_\mu l_\lambda) - \nabla_\lambda(bk_\nu l_\mu) \\
&\quad - \nabla_\mu(bk_\lambda l_\nu) \\
&= F_{\mu\nu\lambda} + k_\mu l_\nu \nabla_\lambda b + k_\nu l_\lambda \nabla_\mu b + k_\lambda l_\mu \nabla_\nu b - k_\nu l_\mu \nabla_\lambda b - k_\lambda l_\nu \nabla_\mu b - k_\mu l_\lambda \nabla_\nu b \\
&\quad - b(k_\mu l_\nu \nabla_\lambda S + k_\nu l_\lambda \nabla_\mu S + k_\lambda l_\mu \nabla_\nu S - k_\nu l_\mu \nabla_\lambda S - k_\lambda l_\nu \nabla_\mu S \\
&\quad - k_\mu l_\lambda \nabla_\nu S) \\
&= F_{\mu\nu\lambda} + e^S(k_\mu l_\nu \nabla_\lambda \tilde{b} + k_\nu l_\lambda \nabla_\mu \tilde{b} + k_\lambda l_\mu \nabla_\nu \tilde{b} - k_\nu l_\mu \nabla_\lambda \tilde{b} - k_\lambda l_\nu \nabla_\mu \tilde{b} \\
&\quad - k_\mu l_\lambda \nabla_\nu \tilde{b}), \quad [\tilde{b} = be^{-S}]
\end{aligned} \tag{6.57}$$

whose indices are raised to obtain

$$
\begin{aligned}
\implies F'^{\alpha}{}_{\nu\lambda} &= (g^{\alpha\mu} - (H - K^2l^2)(k^\alpha l^\mu + l^\alpha k^\mu))(F_{\mu\nu\lambda} + e^S(k_\mu l_\nu \nabla_\lambda \tilde{b} + k_\nu l_\lambda \nabla_\mu \tilde{b} \\
&\quad + k_\lambda l_\mu \nabla_\nu \tilde{b} - k_\nu l_\mu \nabla_\lambda \tilde{b} - k_\lambda l_\nu \nabla_\mu \tilde{b} - k_\mu l_\lambda \nabla_\nu \tilde{b})) \\
&= F^{\alpha}{}_{\nu\lambda} + e^S(k^\alpha l_\nu \nabla_\lambda \tilde{b} + k_\nu l_\lambda \nabla^\alpha \tilde{b} + k_\lambda l^\alpha \nabla_\nu \tilde{b} - k_\nu l^\alpha \nabla_\lambda \tilde{b} - k_\lambda l_\nu \nabla^\alpha \tilde{b} \\
&\quad - k^\alpha l_\lambda \nabla_\nu \tilde{b}) - (H - K^2l^2)k^\alpha(k_\nu c_\lambda - c_\nu k_\lambda) - Kl^\alpha(k_\nu c_\lambda - c_\nu k_\lambda) \\
&\quad - e^S K k^\alpha l^2 k_\lambda \nabla_\nu \tilde{b} + e^S K k^\alpha l^2 k_\nu \nabla_\lambda \tilde{b}, \qquad (6.58)
\end{aligned}
$$

$$
\begin{aligned}
F'^{\alpha\beta}{}_{\lambda} &= (g^{\beta\nu} - (H - K^2l^2)k^\beta k^\nu - K(k^\beta l^\nu + l^\beta k^\nu))F'^{\alpha}{}_{\nu\lambda} \\
&= F^{\alpha\beta}{}_{\lambda} + e^S(k^\alpha l^\beta \nabla_\lambda \tilde{b} + k^\beta l_\lambda \nabla^\alpha \tilde{b} + k_\lambda l^\alpha \nabla^\beta \tilde{b} - k^\beta l^\alpha \nabla_\lambda \tilde{b} - k_\lambda l^\beta \nabla^\alpha \tilde{b} \\
&\quad - k^\alpha l_\lambda \nabla^\beta \tilde{b}) - ((H - K^2l^2)k^\alpha + Kl^\alpha)(k^\beta c_\lambda - c^\beta k_\lambda) + e^S K l^2 k^\alpha (k^\beta \nabla_\lambda \tilde{b} \\
&\quad - k_\lambda \nabla^\beta \tilde{b}) - ((H - K^2l^2)k^\beta + Kl^\beta)(k_\lambda c^\alpha - c_\lambda k^\alpha) - K k^\beta l^2 e^S k^\alpha \nabla_\lambda \tilde{b} \\
&\quad + K k^\beta l^2 e^S k_\lambda \nabla^\alpha \tilde{b}, \qquad (6.59) \\
F^{\alpha\beta\gamma} &= (g^{\gamma\lambda} - (HK^2l^2)k^\gamma k^\lambda - K(k^\gamma l^\lambda + l^\gamma k^\lambda))F'^{\alpha\beta}{}_{\lambda} \\
&= F^{\alpha\beta\gamma} + e^S(k^\alpha l^\beta \nabla^\gamma \tilde{b} + k^\beta l^\gamma \nabla^\alpha \tilde{b} + k^\gamma l^\alpha \nabla^\beta \tilde{b} - k^\beta l^\alpha \nabla^\gamma \tilde{b} - k^\gamma l^\beta \nabla^\alpha \tilde{b} \\
&\quad - k^\alpha l^\gamma \nabla^\beta \tilde{b}) + K(k^\alpha l^\beta c^\gamma + l^\alpha c^\beta k^\gamma + c^\alpha k^\beta l^\gamma - l^\alpha k^\beta c^\gamma - c^\alpha l^\beta k^\gamma - k^\alpha c^\beta l^\gamma). \qquad (6.60)
\end{aligned}
$$

The field equation of motion thus gives

$$
\begin{aligned}
\nabla'_\alpha F'^{\alpha\beta\gamma} &= \nabla_\alpha F'^{\alpha\beta\gamma} \\
&= \nabla_\alpha F^{\alpha\beta\gamma} + (k^\beta l^\gamma - k^\gamma l^\beta)(e^S \nabla_\alpha S)\nabla^\alpha \tilde{b} + e^S(k^\beta l^\gamma - k^\gamma l^\beta)\nabla^2 \tilde{b} \\
&\quad + \nabla_\alpha K(k^\beta l^\gamma c^\alpha - k^\gamma l^\beta c^\alpha) + K(k^\beta l^\gamma - k^\gamma l^\beta)\nabla_\alpha c^\alpha \\
&= (k^\beta l^\gamma - k^\gamma l^\beta)(e^S \nabla_\alpha S \nabla^\alpha \tilde{b} + e^S \nabla^2 \tilde{b} + c^\alpha \nabla_\alpha K + K \nabla_\alpha c^\alpha) = 0, \qquad (6.61)
\end{aligned}
$$

which gives

$$
\nabla_\alpha(e^S \nabla^\alpha(be^{-S}) + Kc^\alpha) = 0. \qquad (6.62)
$$

To analyse the transformed Einstein's equation, we find

$$
\begin{aligned}
F'^{\mu\alpha\beta}F'_{\nu\alpha\beta} = {} & F^{\mu\alpha\beta}F_{\nu\alpha\beta} + k^\mu k_\nu(2e^{2S}l^2\nabla^\beta\tilde{b}\nabla_\beta\tilde{b} + 2Ke^S l^2 c^\beta\nabla_\beta\tilde{b}) + k^\mu l_\nu(2e^S c^\alpha\nabla_\alpha\tilde{b}) \\
& + l^\mu k_\nu(2e^S c_\alpha\nabla^\alpha\tilde{b} + 2Kc_\alpha c^\alpha),
\end{aligned} \tag{6.63}
$$

from which one can see that the $F^{\alpha\beta\gamma}F_{\alpha\beta\gamma}$ remains unchanged by the transformation. Therefore, the Einstein's equation for a 3-form field,

$$
R'^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = \frac{1}{4}F'^{\mu\alpha\beta}F'_{\nu\alpha\beta} - \frac{1}{24}F'^{\alpha\beta\gamma}F'_{\alpha\beta\gamma}, \tag{6.64}
$$

gives

$$
\begin{aligned}
& R^\mu{}_\nu + \frac{1}{2}k^\mu k_\nu(-e^S\nabla^2(He^{-S}) + l^2\nabla_\alpha K\nabla^\alpha K - Kl^2\nabla_\alpha S\nabla^\alpha K + Kl^2\nabla^2 K) \\
& \quad + \frac{1}{2}(k^\mu l_\nu + l^\mu k_\nu)(\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S) - \frac{1}{2}Kl^\mu k_\nu\nabla^2 S - \frac{1}{2}\delta^\mu{}_\nu R \\
& \quad = \frac{1}{4}F^{\mu\alpha\beta}F_{\nu\alpha\beta} + \frac{1}{4}k^\mu k_\nu(2e^{2S}l^2\nabla^\beta\tilde{b}\nabla_\beta\tilde{b} + 2Ke^S l^2 c^\beta\nabla_\beta\tilde{b}) + \frac{1}{4}k^\mu l_\nu(2e^S c^\alpha\nabla_\alpha\tilde{b}) \\
& \quad\quad + \frac{1}{4}l^\mu k_\nu(2e^S c_\alpha\nabla^\alpha\tilde{b} + 2Kc_\alpha c^\alpha) - \frac{1}{24}\delta^\mu{}_\nu F^{\alpha\beta\gamma}F_{\alpha\beta\gamma},
\end{aligned} \tag{6.65}
$$

from which we obtain

$$
\begin{aligned}
-e^S\nabla^2(He^{-S}) + l^2\nabla_\alpha K\nabla^\alpha K - Kl^2\nabla_\alpha S\nabla^\alpha K + Kl^2\nabla^2 K = {} & e^{2S}l^2\nabla^\alpha\tilde{b}\nabla_\alpha\tilde{b} \\
& + Ke^S l^2 c^\alpha\nabla_\alpha\tilde{b},
\end{aligned} \tag{6.66}
$$

$$
\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S = e^S c^\alpha\nabla_\alpha\tilde{b}, \tag{6.67}
$$

$$
\nabla^2 S = -c_\alpha c^\alpha. \tag{6.68}
$$

Using similar arguments as for the 2-form field, it can be shown from Eq. (6.68) that the transformation can be applied only if $R = F^{\alpha\beta\gamma}F_{\alpha\beta\gamma} = 0$.

6.5.4 Example 2: Self-dual string metric

Let us now apply the transformation to the following metric for $D = 10$:

$$
\begin{aligned}
\mathrm{d}s^2 = {} & \frac{1}{H_1}\,\mathrm{d}u\,\mathrm{d}v + H_1\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\phi^2 + r^2\cos^2(\theta)\,\mathrm{d}\psi^2\big) + \mathrm{d}\big(z^1\big)^2 \\
& + \mathrm{d}\big(z^2\big)^2 + \mathrm{d}\big(z^3\big)^2 + \mathrm{d}\big(z^4\big)^2,
\end{aligned} \tag{6.69}
$$

where

$$H_1 = 1 + \frac{h_1}{r^2}, \tag{6.70}$$

h_1 being a constant. The above metric is called a self-dual string metric.[18] We note that H is a harmonic function. The 3-form field $F_{(3)}$ is obtained from the 2-form potential

$$A = \frac{-h_1}{h_1 + r^2}\, \mathrm{d}u \wedge \mathrm{d}v + h_1 \sin^2(\theta)\, \mathrm{d}\phi \wedge \mathrm{d}\psi \tag{6.71}$$

as

$$F = \frac{2rh_1}{h_1 + r^2}\, \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}r + h_1 \sin(2\theta)\, \mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}\psi\,. \tag{6.72}$$

It can be seen that the above system has $R = 0$ and $F^{\alpha\beta\gamma}F_{\alpha\beta\gamma} = 0$. The above metric and field satisfy Einstein's equation.

We choose the null vector k to be ∂_v. It is clearly also Killing, since none of the metric components depend on v, and hypersurface orthogonal, as

$$k_\mu = g_{v\mu} = \frac{1}{H_1}\partial_\mu u. \tag{6.73}$$

Comparing with the usual definition $k_\mu = e^{-S}\partial_\mu u$ of a hypersurface orthogonal vector, we see that here, $e^S = H_1$. We take ∂_{z^1} to be the spacelike covariantly constant vector l. It can be seen that $\mathcal{L}_k S = \mathcal{L}_l S = 0$, and the fields satisfy

$$k^\mu F_{\mu\nu\lambda} = F_{v\nu\lambda} = k_\nu c_\lambda - c_\nu k_\lambda, \tag{6.74}$$

$$\text{where } c_\nu = \frac{-2h}{r(h + r^2)}\partial_\nu r = \partial_\nu S = \frac{1}{H}\partial_\nu H, \tag{6.75}$$

$$\text{and } l^\mu F_{\mu\nu\lambda} = F_{z^1\nu\lambda} = 0. \tag{6.76}$$

Now we can try to find the parameters K and b of the extended Garfinkle-Vachaspati transformation of the 3-form field which satisfy the necessary conditions, using the obtained expressions of S and c_α. Equation (6.68) gives

$$\nabla^2 S + \nabla_\alpha S \nabla^\alpha S = e^{-S}\nabla^2(e^S) = 0, \tag{6.77}$$

which is clearly satisfied as $e^S = H_1$ is harmonic. The other equations for this case become

$$\nabla_\alpha(e^S\nabla^\alpha(be^{-S}) + K\nabla^\alpha S) = 0, \tag{6.78}$$

$$\begin{aligned} -e^S\nabla^2(He^{-S}) + e^S\nabla_\alpha(Ke^{-S})\nabla^\alpha K + K\nabla^2 K = e^{-S}(e^S\nabla^\alpha b\nabla_\alpha(be^{-S}) \\ -be^S\nabla_\alpha S\nabla^\alpha(be^{-S})) + Ke^S\nabla^\alpha S\nabla_\alpha(be^{-S}), \end{aligned} \tag{6.79}$$

$$\nabla_\alpha(K\nabla^\alpha S) - \nabla^2 K = e^{-S}\nabla^\alpha S(\nabla_\alpha b - b\nabla_\alpha S). \tag{6.80}$$

One solution of this is

$$\nabla^2(He^{-S}) = 0 \qquad \text{[Condition imposed for usual GV transformation]}, \tag{6.81}$$

$$K = b = e^S = H_1 = 1 + \frac{h_1}{r^2}. \tag{6.82}$$

The transformed metric and field are then

$$\begin{aligned} \mathrm{d}s'^2 = \frac{1}{H_1}\,\mathrm{d}u\,\mathrm{d}v + \Psi H_1^{-1}\,\mathrm{d}u^2 + H_1\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\phi^2 + r^2\cos^2(\theta)\,\mathrm{d}\psi^2\big) \\ + 2\,\mathrm{d}u\,\mathrm{d}z^1 + \mathrm{d}\big(z^1\big)^2 + \mathrm{d}\big(z^2\big)^2 + \mathrm{d}\big(z^3\big)^2 + \mathrm{d}\big(z^4\big)^2, \end{aligned} \tag{6.83}$$

$$\text{where } \nabla^2\Psi(u, r, \theta, \phi, \psi) = 0, \tag{6.84}$$

$$\text{and } A' = \frac{-h_1}{h_1 + r^2}\,\mathrm{d}u \wedge \mathrm{d}v + h_1\sin^2(\theta)\,\mathrm{d}\phi \wedge \mathrm{d}\psi + \mathrm{d}u \wedge \mathrm{d}z^5\,. \tag{6.85}$$

The above system satisfies the field equation of motion as well as Einstein equation.

Chapter 7

Conclusion and future directions

In this project, I studied various null structures in General Relativity, including plane wave metrics and their properties, null geodesics and Penrose limits. I studied the properties of the Garfinkle-Vachaspati transformation, and derived how the transformation affected the D3 brane metric.

I have also tried to devise an extended version of the Garfinkle-Vachaspati transformation as a solution-generating technique. This is a non-trivial generalisation of the GV transformation, involving both a null hypersurface orthogonal Killing vector and a covariantly constant spacelike vector. I have derived the equations that must be satisfied by the transformation parameters for the Einstein-Maxwell system as well as for a metric with a 3-form field, and have solved them for the plane wave metric as well as for the self-dual string metric.

I have analysed the extended GV transformation for 2-form and 3-form matter fields; this may be generated to higher rank form fields. The transformation properties may be studied for a system having scalar as well as multiple form fields. The transformation in its present form can be applied to systems with form fields only when th Ricci scalar is zero; it may be possible to modify the transformation to admit more general metrics. It has been shown that the Ricci scalar remains unchanged, but it may be useful to study how the other curvature scalars are affected by the extended GV transformation. It may also be interesting to apply the transformation to metrics with horizons such as the D3 brane to see if any new singuarities are generated.

Appendix A

Some useful identities of general relativity

A.1 Kinds of vectors and their properties

$$k^\mu k_\mu = 0, \quad \text{[Null]} \tag{A.1}$$

$$l^\mu l_\mu > 0, \quad \text{[Spacelike]} \tag{A.2}$$

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0, \quad \text{[Killing]} \tag{A.3}$$

$$\nabla_\mu k_\nu - \nabla_\nu k_\mu = k_\mu \nabla_\nu S - k_\nu \nabla_\mu S, \quad \text{[Hypersurface orthogonal]} \tag{A.4}$$

$$\nabla_\mu k_\nu = \frac{1}{2}(k_\mu \nabla_\nu S - k_\nu \nabla_\mu S), \quad \text{[Killing and Hypersurface orthogonal]} \tag{A.5}$$

$$k^\mu \nabla_\mu S = 0, \quad \text{[Null, Killing and Hypersurface orthogonal]} \tag{A.6}$$

$$\nabla_\mu l_\nu = 0, \quad \text{[Covariantly constant]} \tag{A.7}$$

$$\nabla_\beta \nabla_\mu k_\nu = k_\alpha R^\alpha{}_{\beta\mu\nu}, \quad \text{[Killing]} \tag{A.8}$$

$$\nabla^\mu \nabla_\mu k_\nu = -k^\alpha R_{\alpha\nu}, \quad \text{[Killing]} \tag{A.9}$$

$$R_{\mu\nu} k^\nu = \frac{1}{2} k_\mu \nabla^2 S, \quad \text{[Null, Killing and Hypersurface orthogonal]} \tag{A.10}$$

$$k_\alpha R^\alpha{}_{\beta\mu\nu} = \frac{1}{2}(k_\mu \nabla_\nu \nabla_\beta S - k_\nu \nabla_\mu \nabla_\beta S) - \frac{1}{4}(k_\mu \nabla_\nu S \nabla_\beta S - k_\nu \nabla_\mu S \nabla_\beta S), \quad \text{[Killing and Hypersurface orthogonal]} \tag{A.11}$$

$$i_k F_{(p+1)} = k \wedge \theta_{(p-1)} \quad \text{[Transversal]}, \tag{A.12}$$

$$i_l F_{(p+1)} = 0. \tag{A.13}$$

A.2 Transformation properties

We take the metric transformation

$$g'_{\mu\nu} = g_{\mu\nu} + Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu), \tag{A.14}$$

where k^μ is null, hypersurface orthogonal and Killing, and l^μ is spacelike and covariantly constant. For $K = 0$, $H = e^S\Psi$, where $\nabla^2\Psi = 0$, this becomes the Garfinkle-Vachaspati transformation. Then we have

$$g'^{\mu\lambda} = g^{\mu\lambda} - (H - K^2)k^\mu k^\lambda - K(k^\mu l^\lambda + l^\mu k^\lambda) \tag{A.15}$$

$$\det(\mathbf{g}') = \det(\mathbf{g}), \tag{A.16}$$

$$\begin{aligned}\Gamma'^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\nu} = &\frac{1}{2}(\nabla_\mu(Hk_\nu k^\alpha + K(k_\nu l^\alpha + l_\nu k^\alpha)) + \nabla_\nu(Hk_\mu k^\alpha + K(k_\mu l^\alpha + l_\mu k^\alpha)) \\ &- \nabla^\alpha(Hk_\mu k_\nu + K(k_\mu l_\nu + l_\nu k_\mu)) - \frac{1}{2}Kl^2 k^\alpha(k_\mu\nabla_\nu K + k_\nu\nabla_\mu K)),\end{aligned} \tag{A.17}$$

$$\Omega^\mu{}_{\mu\nu} = 0, \tag{A.18}$$

$$k^\mu\Omega^\alpha{}_{\mu\nu} = 0, \tag{A.19}$$

$$k_\alpha\Omega^\alpha{}_{\mu\nu} = 0, \tag{A.20}$$

$$\Omega^\alpha{}_{\mu\beta}\Omega^\beta{}_{\alpha\nu} = \frac{l^2}{2}k_\mu k_\nu(K\nabla^\alpha K\nabla_\alpha S - \nabla^\alpha K\nabla_\alpha K), \tag{A.21}$$

$$\begin{aligned}R'_{\mu\nu} = R_{\mu\nu} &+ \frac{1}{2}k_\mu k_\nu(2\nabla_\alpha H\nabla^\alpha S + l^2\nabla_\alpha K\nabla^\alpha K - \nabla^2 H - H\nabla^\alpha S\nabla_\alpha S \\ &+ 2H\nabla^2 S) + \frac{1}{2}(k_\mu l_\nu + l_\mu k_\nu)(\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S),\end{aligned} \tag{A.22}$$

$$\begin{aligned}R'^\mu{}_\nu = R^\mu{}_\nu &+ \frac{1}{2}k^\mu k_\nu(2\nabla_\alpha H\nabla^\alpha S - \nabla^2 H - H\nabla_\alpha S\nabla^\alpha S + H\nabla^2 S \\ &+ l^2\nabla_\alpha K\nabla^\alpha K - Kl^2\nabla_\alpha S\nabla^\alpha K + Kl^2\nabla^2 K) \\ &+ \frac{1}{2}(k^\mu l_\nu + l^\mu k_\nu)(\nabla^\alpha S\nabla_\alpha K - \nabla^2 K + K\nabla^2 S) - \frac{1}{2}Kl^\mu k_\nu\nabla^2 S.\end{aligned} \tag{A.23}$$

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