Symmetries in the Schrödinger Equation

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Abstract

We have discussed the various symmetries in Quantum Mechanics, in particular the continuous transformations that leave the free particle Schrödinger equation invariant. The ideas of symmetry operators and their generators has been discussed, and the commutators of the generators have been obtained. The Schrödinger group has been studied. Some discussion has also been made on gauge symmetries, symmetries and conservation laws in classical fields, and discrete symmetries.

Contents

Chapter 1

Symmetry operators

The idea of symmetry plays a fundamental role in Physics. The symmetries followed by a physical system often help in understanding the underlying theory. In this chapter, we will introduce the concept of symmetry, transformation operators and generators. We will also study the generators of some space-time transformation. Most of the material in this chapter is based on Refs. [1, 9].

1.1 Concept of symmetry

We know that in Quantum Mechanics, the state of a system is given by a vector or ray $|\Psi\rangle$ in Hilbert space, while the observables correspond to operators on the space. It has been observed that the laws of nature remain invariant under certain space-time translations, $(t, x) \mapsto (t', x') = g(t, x)$. The transformations may include translations in space and time, rotations, Galilean boosts (in non-relativistic theory) etc. Corresponding to every such transformation in coordinate space, there would be a transformation in the Hilbert space states, $|\Psi\rangle \mapsto |\Psi'\rangle$, and of the operators, $A \mapsto A'$. For the laws of Physics to remain invariant under such transformations, some relations must be satisfied.

1. If the system was initially in an eigenstate $|\phi\rangle$ of an operator A with eigenvalue a, $A|\phi\rangle = a|\phi\rangle$; then the transformed state $|\phi'\rangle$ should also be an eigenstate of the transformed operator A' with the same eigenvalue a, $A'|\phi'\rangle = a|\phi'\rangle$. This is because the eigenvalue is obtained by measuring the observable corresponding to A, and it should not change due to the transformation.

2. If the initial state $|\Psi\rangle$ of a system was not an eigenstate of A but a linear combination of eigenstates $|\phi_n\rangle$ ($|\Psi\rangle = \sum_n c_n|\phi_n\rangle$, where $A|\phi_n\rangle = a_n|\phi_n\rangle$), then the observable corresponding to A upon measurement will give a_n with probability $|c_n|^2 = |\langle\phi_n|\Psi\rangle|^2$. Let the transformed state be $|\Psi'\rangle = \sum_n c'_n|\phi'_n\rangle$. Then the probability should remain invariant, i.e. $|c_n|^2 = |c'_n|^2$.

In general, inner products are transformed as $|\langle\Psi_1|\Psi_2\rangle|^2 = |\langle\Psi'_1|\Psi'_2\rangle|^2$.

For every transformation g, there is a corresponding operator T_g in Hilbert space which maps the initial state vector $|\Psi\rangle$ to the final state $|\Psi'\rangle = T_g|\Psi\rangle$. There is a theorem by Wigner which states that any symmetry operator must be unitary or antiunitary. We can see that unitary operators satisfy the condition of symmetry. Let us take two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, which are transformed as $|\Psi'_1\rangle = T_g|\Psi_1\rangle$ and $|\Psi'_2\rangle = T_g|\Psi_2\rangle$. Then $\langle\Psi'_1|\Psi'_2\rangle = \langle\Psi_1|T_g^\dagger T_g|\Psi_2\rangle$. Clearly, this is equal to $\langle\Psi_1|\Psi_2\rangle$ if T_g is unitary, i.e. $T_g^\dagger T_g = I$.

An antiunitary operator U is one that is antilinear (i.e. $U(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1^* U|\Psi_1\rangle + c_2^* U|\Psi_2\rangle$, c_1 and c_2 are complex scalars), and gives $\langle\Psi_1'|\Psi_2'\rangle = \langle\Psi_1|\Psi_2\rangle^*$ where $|\Psi_1'\rangle = U|\Psi_1\rangle$, $|\Psi_2'\rangle = U|\Psi_2\rangle$. All continuous transformation operators, as well as most discrete transformation operators are unitary. One example of an antiunitary operator is time reversal, and we will discuss it in the chapter on discrete symmetries.

Let A be an operator, and $|\phi\rangle$ be an eigenstate of A with eigenvalue a ($A|\phi\rangle = a|\phi\rangle$). The transformed operator A' will have eigenstate $|\phi'\rangle = T_g|\phi\rangle$ (T_g is the transformation operator) with the same eigenvalue a. Therefore, $A'T_g|\phi\rangle = aT_g|\phi\rangle \implies T_g^{-1}A'T_g|\phi\rangle = a|\phi\rangle$. Since we can express any state vector as a linear combination of eigenstates of A, this implies that $T_g^{-1}A'T_g = A \implies A' = T_g A T_g^{-1}$. This is the transformation relation for the operators.

1.2 Generators

Let us consider a family of unitary operators $U(s)$ depending on a single continuous parameter s. Let $U(0) = I$ be the identity operator, and $U(s_1 + s_2) = U(s_1)U(s_2)$. As an example, $U(s)$ may correspond to "translation along the x-axis by s". Since $U(s)$ is continuous, $\lim_{s\to 0} U(s) = I$. For small s, we can Taylor expand $U(s)$ up to the first order to get

$$U(s) = I + \left.\frac{dU}{ds}\right|_{s=0} s + \mathcal{O}(s^2). \tag{1.1}$$

Similarly,

$$U^\dagger(s) = I + \left.\frac{dU^\dagger}{ds}\right|_{s=0} s + \mathcal{O}(s^2). \tag{1.2}$$

Therefore,

$$\begin{aligned} UU^\dagger &= I \\ \implies I + s\left(\frac{dU}{ds} + \frac{dU^\dagger}{ds}\right)_{s=0} + \mathcal{O}(s^2) &= I \\ \implies \left(\frac{dU}{ds} + \frac{dU^\dagger}{ds}\right)_{s=0} &= 0 \\ \implies \left.\frac{dU}{ds}\right|_{s=0} &= iK,\ \ K = K^\dagger. \end{aligned}$$

K is a Hermitian operator and is called the *generator* of the transformation. For any s and small ϵ,

$$\begin{aligned} U(s+\epsilon) &= U(s)U(\epsilon) \\ &= U(s)\left(I + iK\epsilon\right) \\ &= U(s) + iU(s)K\epsilon. \\ \therefore \frac{U(s+\epsilon) - U(s)}{\epsilon} &= iU(s)K \\ \implies \frac{dU}{ds} &= iKU(s). \end{aligned}$$

Integrating, we get

$$U(s) = e^{isK}. \tag{1.3}$$

Thus, the generator K determines the transformation for all values of s. It is much more convenient to work with the generators than with the transformation operators themselves, and we will discuss some of the generators in the following section.

In most cases, we take the transformation to be close to the identity (i.e. the parameter s is small), and then we can write up to the first order as

$$U(s) = I + isK. \tag{1.4}$$

Before coming to the generators of the specific transformations, let us discuss *commutators* of the generators. The transformations in general do not commute, i.e. the final result after applying a number of transformations depends on the order in which they are applied (for two transformation operators U and T, $UT \neq TU$).

We consider two transformations, generated by the operators A and B respectively. Let the infinitesimal parameters be ϵ_1 and ϵ_2 respectively. Now we consider the transformation represented by $e^{-i\epsilon_2 B} e^{-i\epsilon_1 A} e^{i\epsilon_2 B} e^{i\epsilon_1 A}$. If the operators commute, this reduces to the identity. But in general, we get

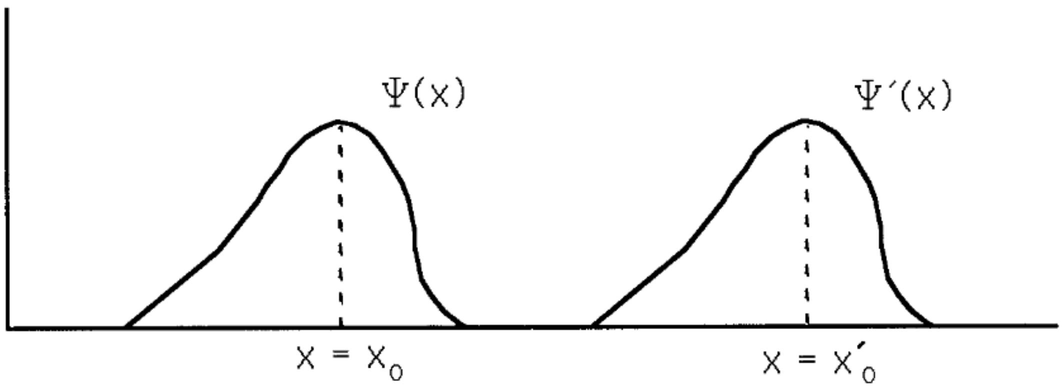
$$\begin{aligned}
&e^{-i\epsilon_2 B} e^{-i\epsilon_1 A} e^{i\epsilon_2 B} e^{i\epsilon_1 A} \\
&= \left(I - i\epsilon_2 B - \frac{1}{2}\epsilon_2^2 B^2\right)\left(I - i\epsilon_1 A - \frac{1}{2}\epsilon_1^2 A^2\right)\left(I + i\epsilon_2 B - \frac{1}{2}\epsilon_2^2 B^2\right) \\
&\quad \left(I + i\epsilon_1 A - \frac{1}{2}\epsilon_1^2 A^2\right) + \mathcal{O}(\epsilon^3) \\
&= I + i\epsilon_1 (A - A) + i\epsilon_2 (B - B) + \epsilon_1^2 \left(-\frac{1}{2}A^2 - \frac{1}{2}A^2 + A^2\right) \\
&\quad + \epsilon_2^2 \left(-\frac{1}{2}B^2 - \frac{1}{2}B^2 + B^2\right) + \epsilon_1\epsilon_2 \left(-BA + BA + AB - BA\right) + \mathcal{O}(\epsilon^3) \\
&= I + \epsilon_1\epsilon_2 [A, B] + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{1.5}$$

Here, $[A, B] = AB - BA$ is called the *commutator* of A and B. Since the composition of symmetry transformations again gives a symmetry transformation, $[A, B]$ can be expressed as a linear combination of the transformation generators. Thus the commutators between the symmetry generators play an important role in understanding the transformations.

1.3 Generators of space-time translations in position basis

Now we are going to find the generators of the space-time transformations. Of course, they will be operators in Hilbert space. We will find them in the position basis, where the states $|\Psi\rangle$ are expressed as wave functions $\Psi(t, x) = \langle x|\Psi\rangle$, and the operators will be linear combinations of constants and differential operators.

Let us first observe the effect of a transformation operator on the wave function. We



[ht]

Figure 1.1: Relation between coordinate transformation and function transformation for the translation operator. (Reproduced from Ref. [1])

consider a general transformation $x \mapsto g(x)$. Then we see that

$$\begin{aligned} T_g|x\rangle &= |g(x)\rangle \\ \implies \Psi'(x) &= \langle x|T_g|\Psi\rangle \\ &= \langle T_g^\dagger x|\Psi\rangle \\ &= \langle g^{-1}(x)|\Psi\rangle \\ &= \Psi(g^{-1}(x)). \end{aligned} \tag{1.6}$$

Thus we see that there is an inverse relation between the transformations of coordinates and the transformations in function space. This can be appreciated from Fig. 1.1. If we translate the system to the right by c, then the modified wave function is related to the original one as $\Psi'(x+c) = \Psi(x)$. For example, if the function was previously localized at $x = x_0$, it will now be localized at $x_0' = x_0 + c$. Therefore, $\Psi'(x) = \Psi(x-c)$. Similar logic holds for all transformations.

There is a point to be noted here. Because the physically significant quantity is $|\Psi|^2$ and not the wave function Ψ itself, Ψ and $e^{ih}\Psi$ will give the same probability densities. For constant h, this does not matter. But this holds *even if h is a function of position and time*. In that case, whether the wave function satisfies Schrödinger equation or not depends on the function h. We will come across this problem later. Also, we may in some cases need to introduce factor to make the transformed wave function normalized. Thus, the most general expression will be

$$T_g\Psi(t,x) = A(t)e^{ih(t,x)}\Psi(g^{-1}(t,x)), \tag{1.7}$$

where A is a normalization factor and h is a function introduced to make the wave function satisfy Schrödinger equation. A can not be a function of position (since that would change the probability density of the particle being found at different positions), but it may be a function of time.

To obtain the generators, we will find the transformed wave function $\Psi'(x) = \Psi(g^{-1}x)$ for a small transformation parameter, and Taylor expand to the first order to get the generator using Eq. (1.4). The factors A and $e^{i\theta}$ are 1 in most cases, and we will simply ignore them for simplicity except where they are necessary.

1.3.1 Generator of translation along x-axis, P_x

We first consider translation along x-axis by a infinitesimal parameter c; $x \mapsto x+c$. Then we have $\Psi(x) \mapsto \Psi'(x) = \Psi(x-c)$. Now we Taylor expand:

$$\begin{aligned}
\Psi(x-c) &= \Psi(x)|_{c=0} - c\,\frac{\partial \Psi}{\partial x}\bigg|_{c=0} + \mathcal{O}(c^2) \text{ (Taylor expanding about } c=0\text{)} \\
&= \Psi(x) - c\frac{\partial \Psi}{\partial x} \\
&= \left(1 - c\frac{\partial}{\partial x}\right)\Psi(x) \\
&= \left(1 + (-ic)\,(-i)\frac{\partial}{\partial x}\right)\Psi(x) \\
&= (1 - icP_x)\,\Psi(x).
\end{aligned} \tag{1.8}$$

Hence, the generator of translation along x-axis in position basis is $P_x = -i\frac{\partial}{\partial x}$. Similarly we can find the generators for space translation along the other axes ($P_i = -i\partial_i$), and time-translation ($H = i\frac{\partial}{\partial t}$). The negative sign in the translation generators is by convention.

1.3.2 Generator of rotation about z-axis, J_z

Rotation by a small angle θ about the z-axis gives

$$\begin{aligned}
(x,y,z) &\mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) \text{ (}\theta\text{ is small rotation angle)} \\
\implies \Psi'(x,y,z) &= \Psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, z) \\
&= \Psi(x + y\theta, y - x\theta, z) + \mathcal{O}(\theta^2) \\
&= \Psi(x,y,z) + \theta\left(y\frac{\partial \Psi}{\partial x} + (-x)\frac{\partial \Psi}{\partial y}\right) + \mathcal{O}(\theta^2) \\
&= \left(1 + \theta\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right)\Psi(x,y,z) \\
&= \left(1 - i\theta\, i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right)\Psi(x,y,z) \\
&= (1 - i\theta J_z)\,\Psi(x,y,z).
\end{aligned} \tag{1.9}$$

Hence, the generator of rotation about z-axis is $J_z = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = (xP_y - yP_x)$. Similarly, J_x and J_y can be found.

1.3.3 Generator of dilation, D

The commonly encountered space-time symmetries in non-relativistic Quantum Mechanics are translations in space and time, rotations, and Galilean boosts (which we will discuss later). However, as we will see in the next chapter, there are two more symmetries for the non-relativistic free particle Schrödinger equation. One of them is the transformation

$$(t,x,y,z) \mapsto (\lambda^2 t, \lambda x, \lambda y, \lambda z) \text{ (}\lambda\text{ is a real parameter)}, \tag{1.10}$$

called *dilation*. We will now study its generator.

If the coordinate transformation is $(t,\vec{x}) \mapsto (\lambda^2 t, \lambda\vec{x})$, the wave function should be transformed as $\Psi(t,\vec{x}) \mapsto \Psi\left(\frac{t}{\lambda^2}, \frac{\vec{x}}{\lambda}\right)$. In the case of dilation, however, it turns out that the wave function $\Psi\left(\frac{t}{\lambda^2}, \frac{\vec{x}}{\lambda}\right)$ is not normalized. We will require the normalization factor A that had been mentioned before in Eq. (1.7). Let us take $\Psi'(t,x,y,z) = A\Psi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right)$.

One point to be noted is that, while the the other transformation operators reduced to the identity when the parameter was 0, here it happens for $\lambda = 1$. Also, λ can only take positive values, since otherwise we get a superposition of dilation and parity (a discrete transformation). Thus we take $\delta = \ln\lambda$ to be the parameter. δ can be any real number, and the transformation reduces to the identity for $\delta = 0$.

Firstly, we find the normalization constant A. We see that

$$\begin{aligned}
\int |\Psi'|^2\, dx\, dy\, dz &= |A|^2 \int \left|\Psi\left(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right)\right|^2\, dx\, dy\, dz \\
&= |A|^2\lambda^3 \int |\Psi(x'', y'', z'')|^2\, dx''\, dy''\, dz'' \;\; (x'' = \tfrac{x}{\lambda} \text{ etc.}) \\
&= |A|^2\lambda^3 = 1 \\
\implies A &= \lambda^{-\frac{3}{2}} \text{ (taking } A \text{ to be positive real)} \\
&= e^{-\frac{3\delta}{2}} \\
&= 1 - \frac{3\delta}{2} + \mathcal{O}(\delta^2).
\end{aligned} \tag{1.11}$$

Now,

$$\begin{aligned}
\Psi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right) =& \Psi(e^{-2\delta}t, e^{-\delta}x, e^{-\delta}y, e^{-\delta}z) \\
=& \Psi(t,x,y,z) + \delta\frac{\partial\Psi}{\partial t}(-2t) + \delta\sum_{i=1}^{3}\frac{\partial\Psi}{\partial x_i}(-x_i) + \mathcal{O}(\delta^2) \\
& \text{(Taylor expanding about } \delta = 0;\ (x_1, x_2, x_3) = (x,y,z)) \\
=& \left(1 + \delta\left(-2t\partial_t - \sum_i x_i\partial_i\right)\right)\Psi \;\; \left[\partial_t = \frac{\partial}{\partial t},\ \partial_i = \frac{\partial}{\partial x_i}\right] \\
\implies \Psi'(t,x,y,z) =& \left(1 - \delta\left(2t\partial_t + \sum_i x_i\partial_i + \frac{3}{2}\right)\right)\Psi(t,x,y,z) + \mathcal{O}(\delta^2) \\
=& \left(1 + (i\delta)\, i\left(2t\partial_t + \sum_i x_i\partial_i + \frac{3}{2}\right)\right)\Psi(t,x,y,z) \\
=& (1 + i\delta D)\Psi(t,x,y,z).
\end{aligned} \tag{1.12}$$

Therefore, $D = i\left(2t\partial_t + \sum_i x_i\partial_i + \frac{3}{2}\right) = i\left(2t\partial_0 + \vec{x}\cdot\vec{\nabla} + \frac{3}{2}\right)$.

Eigenvalues and eigenvectors of D

Many of the symmetry generators are seen to correspond to physical observables, and their eigenvalues give the possible values of the observable that can be obtained by measurement. The eigenvectors are states where the value of that observable is fixed. For example, P_x, the translation generator along x-axis, corresponds to the momentum in x-direction.

We do not have any such interpretation of D. Still, it would be interesting to find the eigenvalues and eigenvectors of D. We work in position basis and in one dimension, and take the eigenvalue to be d. Then

$$
\begin{aligned}
& 2it\frac{\partial\Psi}{\partial t} + ix\frac{\partial\Psi}{\partial x} + \frac{i}{2}\Psi = d\Psi \\
\implies & 2t\psi\frac{df}{dt} + xf\frac{d\psi}{dx} + \frac{1}{2}\psi f = -id\psi f \;\; [\Psi(t,x) = \psi(x)f(t)] \\
\implies & \frac{2t}{f}\frac{df}{dt} + \frac{x}{\psi}\frac{d\psi}{dx} = -\frac{1}{2} - id \\
\implies & \frac{2t}{f}\frac{df}{dt} = 2\alpha,\; \frac{x}{\psi}\frac{d\psi}{dx} = \beta,\; 2\alpha+\beta = -\frac{1}{2} - id \\
\implies & f(t) = C_1 t^{\alpha},\; \psi(x) = C_2 x^{\beta} \; (C_1 \text{ and } C_2 \text{ are arbitrary constants}) \\
\implies & \Psi(t,x) = Ct^{\alpha}x^{\beta},\; 2\alpha+\beta = -\frac{1}{2} - id.
\end{aligned}
\tag{1.13}
$$

For a particular time t, Ct^{α} can be taken to be the normalization constant. Since we have a time-dependent normalization constant, eigenstate of dilation generator is not a stationary state. The wave function can be exponential or oscillatory, depending on whether β is real or imaginary. In either case, the wave function is non-normalizable over the entire interval $(-\infty, \infty)$.

1.3.4 Generator of Galilean boost along x-axis

We next turn to the Galilean boost in x-direction,

$$(t, x, y, z) \mapsto (t, x + vt, y, z). \tag{1.14}$$

Schrödinger equation applies to the non-relativistic case ($v \ll c$), and should remain invariant under this transformation. The problem, however, is somewhat complicated, since this will involve the phase factor e^{ih} that we briefly mentioned in Eq. (1.7).

In the original coordinates (t, x), the free particle Schrödinger equation (in one dimension) is

$$-\frac{1}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\frac{\partial\Psi}{dt}. \tag{1.15}$$

Let us transform to the new coordinates (t', x') by a Galilean boost with velocity parameter v, i.e. $x' = x + vt,\ t' = t$. The wave function should transform as $\Psi'(t, x) = \Psi(t, x - vt)$ This wave function is normalized, but it does not satisfy the Schrödinger equation in the new frame. We try to find a wave function $\Psi_h(t, x) = \Psi(t, x - vt)e^{ih(t,x)}$ that satisfies the equation in transformed coordinates. We first find its derivatives:

$$\begin{aligned}
\frac{\partial \Psi_h}{\partial x'} &= \frac{\partial \Psi}{\partial x} e^{ih} + \Psi i e^{ih} \frac{\partial h}{\partial x} \\
\implies \frac{\partial^2 \Psi_h}{\partial x'^2} &= \frac{\partial^2 \Psi}{\partial x^2} e^{ih} + 2 \frac{\partial \Psi}{\partial x} i e^{ih} \frac{\partial h}{\partial x} + \Psi i^2 e^{ih} \left(\frac{\partial h}{\partial x}\right)^2 + \Psi i e^{ih} \frac{\partial^2 h}{\partial x^2} \\
&= \left(\frac{\partial^2 \Psi}{\partial x^2} + 2i \frac{\partial \Psi}{\partial x} \frac{\partial h}{\partial x} - \Psi \left(\frac{\partial h}{\partial x}\right)^2 + i\Psi \frac{\partial^2 h}{\partial x^2}\right) e^{ih}, \\
\frac{\partial \Psi_h}{\partial t'} &= \frac{\partial \Psi}{\partial t} e^{ih} + \frac{\partial \Psi}{\partial x}(-v) e^{ih} + \Psi i e^{ih} \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}(-v)\right) \\
&= \left(\frac{\partial \Psi}{\partial t} - v \frac{\partial \Psi}{\partial x} + i\Psi \frac{\partial h}{\partial t} - iv\Psi \frac{\partial h}{\partial x}\right) e^{ih}. \qquad (1.16)
\end{aligned}$$

Substituting in Schrödinger equation,

$$\begin{aligned}
&\therefore -\frac{1}{2m} \frac{\partial^2 \Psi_h}{\partial x'^2} = i \frac{\partial \Psi_h}{dt'} \\
\implies -\frac{1}{2m} &\left(\partial_x^2 \Psi + 2ih' \partial_x \Psi - h'^2 \Psi + ih'' \Psi\right) = i(\partial_t \Psi - v\partial_x \Psi + i\dot{h}\Psi - ivh'\Psi) \\
&\left[h' = \frac{\partial h}{\partial x},\ \dot{h} = \frac{\partial h}{\partial t}\right] \\
\implies -\frac{1}{2m}&(ih''\Psi + 2ih'\partial_x \Psi - h'^2 \Psi) = i(-v\partial_x \Psi + i\dot{h}\Psi - ivh'\Psi) \\
&\implies h'^2 \Psi = 2mvh'\Psi - 2m\dot{h}\Psi \\
&\text{and } h''\Psi + 2h'\partial_x \Psi = 2mv\partial_x \Psi. \qquad (1.17)
\end{aligned}$$

Assuming $h'' = 0$,

$$\begin{aligned}
h'^2 &= 2mvh' - 2m\dot{h} \\
\text{and } h' &= mv \\
\implies h &= mvx + h_t(t) \\
\text{and } m^2 v^2 &= 2m^2 v^2 - 2m\dot{h} \\
\implies \frac{dh_t}{dt} &= \frac{1}{2} m v^2 \\
\implies h_t &= \frac{1}{2} m v^2 t \\
\therefore h(t,x) &= \left(mvx + \frac{1}{2} m v^2 t\right). \qquad (1.18)
\end{aligned}$$

Also,

$$\begin{aligned}
\Psi(t, x - vt) &= \Psi(t,x) + v \frac{\partial \Psi}{\partial x}(-t) \\
&= \left(1 - vt \frac{\partial}{\partial x}\right) \Psi(t,x). \qquad (1.19)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi_h(t,x) &= e^{i\left(mvx + \frac{1}{2} m v^2 t\right)} (1 - vt\partial_x) \Psi(t,x) \\
&= (1 + iv(mx + it\partial_x))\Psi(t,x) + \mathcal{O}(v^2). \qquad (1.20)
\end{aligned}$$

Hence the generator of Galilean boost is $G = mx + it\frac{\partial}{\partial x} = mx - P_x t$.

Eigenvalues and eigenvectors of G

We find the eigenfunction of G in one-dimensional position space, taking the eigenvalue to be g. Then

$$mx\Psi + it\frac{\partial \Psi}{\partial x} = g\Psi. \tag{1.21}$$

Since this differential equation does not involve any derivative with respect to t, we can take time to be a constant $t \neq 0$. Then we have

$$\begin{aligned} it\frac{d\Psi}{dx} &= (g - mx)\Psi \\ \implies it\frac{d\Psi}{\Psi} &= (g - mx)dx \\ \implies it\ln\Psi &= gx - \frac{1}{2}mx^2 + \tilde{C}(t)\ [\tilde{C}(t) \text{ is an arbitrary function of } t] \\ \implies \Psi &= C(t)e^{-\frac{i}{t}(gx - \frac{1}{2}mx^2)}\ [C(t) = e^{-\frac{i}{t}\tilde{C}(t)}]. \end{aligned} \tag{1.22}$$

Then $|\Psi|^2 = |C(t)|^2$, i.e. the probability density function has no position dependence. The wave function is not normalizable over the interval $(-\infty, \infty)$ (like the energy eigenfunctions of scattering states), and we expect the set of eigenvalues to be continuous. $\Psi(t, x) \neq 0$ everywhere and for all g (unless $C(t) = 0$), and hence the wave function can not be confined to an interval. I could not find the spectrum of eigenvalues g, the function $C(t)$ or the physical significance of the eigenvalues or eigenvectors. At $t = 0$, we have $G = mx$, so an eigenfunction of G with eigenvalue g is nothing but a position eigenstate at g/m.

1.3.5 Generator of expansion

We finally come to the transformation

$$(t, \vec{x}) \mapsto \left(\frac{t}{1+\alpha t}, \frac{\vec{x}}{1+\alpha t}\right). \tag{1.23}$$

We will call this transformation 'expansion', which is the term used in Ref. [9]. This is the only remaining transformation that leaves the Schrödinger equation invariant. The generator of this transformation will be seen to involve a phase factor as well as a time dependent normalization factor.

$$\begin{aligned} (t, \vec{x}) &\mapsto \left(\frac{t}{1+\alpha t}, \frac{\vec{x}}{1+\alpha t}\right) \\ \implies \Psi'(t, \vec{x}) &\sim \Psi\left(\frac{t}{1-\alpha t}, \frac{\vec{x}}{1-\alpha t}\right)\ \text{[Without the normalization and phase factors]} \\ &= \Psi(t, \vec{x}) + \alpha\frac{\partial \Psi}{\partial t}(-t)(-t) + \sum_i \alpha\frac{\partial \Psi}{\partial x_i}(-x_i)(-t) + \mathcal{O}(\alpha^2) \\ &= \left(1 + \alpha\left(t^2\frac{\partial}{\partial t} + t\vec{x}\cdot\vec{\nabla}\right)\right)\Psi \end{aligned} \tag{1.24}$$

. To find A, we integrate the probability density:

$$\begin{aligned}
\int |\Psi'(t,x,y,z)|^2\,dx\,dy\,dz &= A^2 \int \left|\Psi\left(\frac{t}{1-\alpha t}, \frac{\vec{x}}{1-\alpha t}\right)\right|^2 dx\,dy\,dz \;[\vec{x}=(x,y,z)] \\
&= A^2(1-\alpha t)^3 \int \left|\Psi\left(\frac{t}{1-\alpha t}, \vec{x''}\right)\right|^2 dx''\,dy''\,dz'' \;\left[\vec{x''} = \frac{\vec{x}}{1-\alpha t}\right] \\
&= A^2(1-\alpha t)^3 \\
&= 1 \\
\implies A &= (1-\alpha t)^{-\frac{3}{2}} \\
&= 1 - \frac{3}{2}\alpha t + \mathcal{O}(\alpha^2).
\end{aligned} \tag{1.25}$$

In one dimension, $A = (1-\alpha t)^{-1/2}$.

All that is left now is to find the function h. For simplicity, we will work out the problem in one dimension and then give the 3D result by analogy. Let $\Psi'(t,x) = (1-\alpha t)^{-\frac{1}{2}} e^{ih(t,x)} \Psi\left(\frac{t}{1-\alpha t}, \frac{x}{1-\alpha t}\right)$ (in one dimension). Then we see that

$$\begin{aligned}
\frac{\partial \Psi'}{\partial t} =& \frac{\alpha}{2}(1-\alpha t)^{-\frac{3}{2}} e^{ih}\Psi + (1-\alpha t)^{-\frac{1}{2}} i e^{ih}\dot{h}\Psi \\
&+ (1-\alpha t)^{-\frac{1}{2}} e^{ih}\partial_t\Psi\left(\frac{1}{1-\alpha t} + \frac{\alpha t}{(1-\alpha t)^2}\right) + (1-\alpha t)^{-\frac{1}{2}} e^{ih}\partial_x\Psi \frac{\alpha x}{(1-\alpha t)^2} \\
=& \frac{\alpha}{2}(1-\alpha t)^{-\frac{3}{2}} e^{ih}\Psi + i(1-\alpha t)^{-\frac{1}{2}}\dot{h} e^{ih}\Psi + (1-\alpha t)^{-\frac{5}{2}} e^{ih}\partial_t\Psi \\
&+ (1-\alpha t)^{-\frac{5}{2}}\alpha x e^{ih}\partial_x\Psi, \\
\frac{\partial \Psi'}{\partial x} =& (1-\alpha t)^{-\frac{1}{2}} i e^{ih} h'\Psi + (1-\alpha t)^{-\frac{3}{2}} e^{ih}\partial_x\Psi, \\
\frac{\partial^2 \Psi'}{\partial x^2} =& (1-\alpha t)^{-\frac{1}{2}}(-1) e^{ih} h'^2\Psi + (1-\alpha t)^{-\frac{1}{2}} i e^{ih} h''\Psi + 2(1-\alpha t)^{-\frac{3}{2}} i e^{ih} h'\partial_x\Psi \\
&+ (1-\alpha t)^{-\frac{5}{2}} e^{ih}\partial_x^2\Psi \\
=& -(1-\alpha t)^{-\frac{1}{2}} h'^2 e^{ih}\Psi + i(1-\alpha t)^{-\frac{1}{2}} h'' e^{ih}\Psi + 2i(1-\alpha t)^{-\frac{3}{2}} h' e^{ih}\partial_x\Psi \\
&+ (1-\alpha t)^{-\frac{5}{2}} e^{ih}\partial_x^2\Psi.
\end{aligned} \tag{1.26}$$

Substituting in Schrödinger equation, we have

$$\begin{aligned}
&-\frac{1}{2m}\Big(-(1-\alpha t)^{-\frac{1}{2}} h'^2 e^{ih}\Psi + i(1-\alpha t)^{-\frac{1}{2}} h'' e^{ih}\Psi + 2i(1-\alpha t)^{-\frac{3}{2}} h' e^{ih}\partial_x\Psi \\
&+ (1-\alpha t)^{-\frac{5}{2}} e^{ih}\partial_x^2\Psi\Big) \\
&\quad= \frac{i\alpha}{2}(1-\alpha t)^{-\frac{3}{2}} e^{ih}\Psi - (1-\alpha t)^{-\frac{1}{2}}\dot{h} e^{ih}\Psi + i(1-\alpha t)^{-\frac{5}{2}} e^{ih}\partial_t\Psi + i(1-\alpha t)^{-\frac{5}{2}}\alpha x e^{ih}\partial_x\Psi \\
&\implies -\frac{1}{2m}\left(-(1-\alpha t)^2 h'^2\Psi + i(1-\alpha t)^2 h''\Psi + 2i(1-\alpha t) h'\partial_x\Psi + \partial_x^2\Psi\right) \\
&\quad= \frac{i\alpha}{2}(1-\alpha t)\Psi - (1-\alpha t)^2\dot{h}\Psi + i\partial_t\Psi + i\alpha x\partial_x\Psi \\
&\implies -\frac{1}{2m}\left(-(1-\alpha t)^2 h'^2\Psi + i(1-\alpha t)^2 h''\Psi + 2i(1-\alpha t) h'\partial_x\Psi\right) \\
&\quad= \frac{i\alpha}{2}(1-\alpha t)\Psi - (1-\alpha t)^2\dot{h}\Psi + i\alpha x\partial_x\Psi.
\end{aligned} \tag{1.27}$$

Equating the real and imaginary parts,

$$\begin{aligned}
\frac{1}{2m}(1-\alpha t)^2 h'^2 \Psi &= -(1-\alpha t)^2 \dot{h}\Psi \\
\implies h'^2 &= -2m\dot{h}, \\
-\frac{1}{2m}(1-\alpha t)^2 h''\Psi - \frac{1}{m}(1-\alpha t)h'\partial_x\Psi &= \frac{\alpha}{2}(1-\alpha t)\Psi + \alpha x\partial_x\Psi \\
\implies -\frac{1}{2m}(1-\alpha t)h'' &= \frac{\alpha}{2} \\
\text{and } -\frac{1}{m}(1-\alpha t)h' &= \alpha x.
\end{aligned} \tag{1.28}$$

Therefore, we get

$$\begin{aligned}
h(x,t) &= -\frac{m\alpha x^2}{2(1-\alpha t)} + h_t(t) \\
\therefore h'^2 &= -2m\dot{h} \\
\implies \frac{m^2\alpha^2x^2}{(1-\alpha t)^2} &= \frac{m^2\alpha^2x^2}{(1-\alpha t)^2} - 2mh_t'(t) \\
\implies h_t'(t) &= 0 \\
\implies h_t &= 0 \text{ (Since a constant phase factor has no significance, we can set it to 0).}
\end{aligned} \tag{1.29}$$

Finally, we are in a position to write the complete transformed wave function. Putting all terms together,

$$\begin{aligned}
\Psi'(t,x) &= e^{\frac{-im\alpha x^2}{2(1-\alpha t)}}(1-\alpha t)^{-\frac{1}{2}}\Psi\left(\frac{t}{1-\alpha t}, \frac{x}{1-\alpha t}\right) \text{ (in 1D)} \\
&= \left(1+i\alpha\frac{-mx^2}{2}\right)\left(1+i\alpha\frac{-it}{2}\right)\left(1+i\alpha\left(-it^2\partial_t - itx\partial_x\right)\right)\Psi + \mathcal{O}(\alpha^2) \\
&= \left[1+i\alpha\left(-i\left(t^2\partial_t + tx\partial_x + \frac{t}{2}\right) - \frac{mx^2}{2}\right)\right]\Psi + \mathcal{O}(\alpha^2) \\
&= (1+i\alpha A)\Psi.
\end{aligned} \tag{1.30}$$

Hence, the generator of this transformation in 3 dimensions is

$$A = -i\left(t^2\partial_t + t\vec{x}\cdot\vec{\nabla} + \frac{3t}{2}\right) - \frac{mx^2}{2}. \tag{1.31}$$

Thus we have found the generators of all symmetry transformations under which the free Schrödinger equation is invariant. We will show in the next chapter that this is indeed the case, and there exists no symmetry other than these. The group of all the transformations is called the maximal kinematical invariance group of the free Schrödinger equation, or the Schrödinger group. The translation, rotation, time translation and Galilean boost transformations together form a subgroup known as the Galilei group.

1.4 Commutators between the symmetry generators

One way to find the commutators between the generators is to use their explicit forms in the coordinate representation. We will show another way here, using the transformations themselves. We use the identity

$$e^{-i\epsilon_2 B}e^{-i\epsilon_1 A}e^{i\epsilon_2 B}e^{i\epsilon_1 A} = 1 + \epsilon_1\epsilon_2\,[A, B]\,. \tag{1.32}$$

As we will see in the next chapter, the transformations discussed above are the only possible symmetry transformations of the free particle Schrödinger equation. Any combination of symmetry transformation should also be a symmetry transformation, and should be generated by a linear combination of the generators we have studied. (In mathematical terms, we say that the symmetry operators form a group.) Therefore, we shall have

$$e^{-i\epsilon_2 B}e^{-i\epsilon_1 A}e^{i\epsilon_2 B}e^{i\epsilon_1 A} = 1 + i\sum_C \epsilon_C C + (?)I, \tag{1.33}$$

where the summation is over all generators C in the group $\{P_k, J_k, G_k, H, D, A\}$. Therefore, we should be able to find the commutator of any two generators as a linear combination of the generators. The multiple of identity $(?)I$ is because all operators are defined only up to an arbitrary phase factor (in Mathematics, this is called a ray representation). We shall deal with them later.

The evolution of commutators in this way is fairly simple. We will just show a few examples here.

Example 1. $[G_\alpha, H] = iP_\alpha + (?)I$

We operate the transformation corresponding to $e^{-i\epsilon_2 H}e^{-i\epsilon_1 G_1}e^{i\epsilon_2 H}e^{i\epsilon_1 G_1}$ on (t, x). We get

$$\begin{aligned}
(t, x) &\mapsto (t, x + \epsilon_1 t)\\
&\mapsto (t + \epsilon_2, x + \epsilon_1 t)\\
&\mapsto (t + \epsilon_2, x + \epsilon_1 t - \epsilon_1(t + \epsilon_2))\\
&\mapsto (t + \epsilon_2 - \epsilon_2, x + \epsilon_1 t - \epsilon_1(t + \epsilon_2)\\
&= (t, x - \epsilon_1\epsilon_2)\\
\implies\; I + \epsilon_1\epsilon_2\,[G_1, H] &= I + i\epsilon_1\epsilon_2 P_1 + (?)I\\
\implies\; [G_1, H] &= iP_1 + (?)I.\\
\text{In general,}\;\; [G_\alpha, H] &= iP_\alpha + (?)I.
\end{aligned} \tag{1.34}$$

Example 2. $[J_\alpha, G_\beta] = i\epsilon_{\alpha\beta\gamma}G_\gamma + (?)I$

We start with $[J_1, G_2]$. Operating the transformations,

$$\begin{aligned}
(t, x, y, z) &\mapsto (t, x, y + z\epsilon_1, z - y\epsilon_1)\\
&\mapsto (t, x, y + z\epsilon_1 + \epsilon_2 t, z - y\epsilon_1)\\
&\mapsto (t, x, y + z\epsilon_1 + \epsilon_2 t - \epsilon_1(z - y\epsilon_1), z - y\epsilon_1 + \epsilon_1(y + z\epsilon_1 + \epsilon_2 t))\\
&= (t, x, y + \epsilon_2 t + \epsilon_1^2 y, z + \epsilon_1^2 z + \epsilon_1\epsilon_2 t)\\
&\mapsto (t, x, (1 + \epsilon_1^2 y), (1 + \epsilon_1^2)z + \epsilon_1\epsilon_2 t)\\
&= (t, x, y, z + \epsilon_1\epsilon_2 t) \text{ (Up to the first order in } \epsilon_1)\\
\therefore [J_1, G_2] &= iG_3 + (?)I.\\
\text{In general,}\;\; [J_\alpha, G_\beta] &= i\epsilon_{\alpha\beta\gamma}G_\gamma + (?)I.
\end{aligned} \tag{1.35}$$

Example 3. $[A, H] = iD + (?)I$

Proceeding as before (in 1D for convenience),

$$\begin{aligned}
(t,x) &\mapsto \left(\frac{t}{1+\epsilon_1 t}, \frac{x}{1+\epsilon_1 t}\right) \\
&\mapsto \left(\frac{t}{1+\epsilon_1 t} + \epsilon_2, \frac{x}{1+\epsilon_1 t}\right) \\
&= \left(\frac{t+\epsilon_2+\epsilon_1\epsilon_2 t}{1+\epsilon_1 t}, \frac{x}{1+\epsilon_1 t}\right) \\
&\mapsto \left(\frac{\frac{t+\epsilon_2+\epsilon_1\epsilon_2 t}{1+\epsilon_1 t}}{1-\epsilon_1\frac{t+\epsilon_2+\epsilon_1\epsilon_2 t}{1+\epsilon_1 t}}, \frac{\frac{x}{1+\epsilon_1 t}}{1-\epsilon_1\frac{t+\epsilon_2+\epsilon_1\epsilon_2 t}{1+\epsilon_1 t}}\right) \\
&= \left(\frac{t+\epsilon_2+\epsilon_1\epsilon_2 t}{1-\epsilon_1\epsilon_2-\epsilon_1^2\epsilon_2 t}, \frac{x}{1-\epsilon_1\epsilon_2-\epsilon_1^2\epsilon_2 t}\right) \\
&\mapsto \left(\frac{t+\epsilon_2+\epsilon_1\epsilon_2 t}{1-\epsilon_1\epsilon_2-\epsilon_1^2\epsilon_2 t} - \epsilon_2, \frac{x}{1-\epsilon_1\epsilon_2-\epsilon_1^2\epsilon_2 t}\right) \\
&= \left(\frac{t+\epsilon_1\epsilon_2 t+\epsilon_1\epsilon_2^2+\epsilon_1^2\epsilon_2^2 t}{1-\epsilon_1\epsilon_2-\epsilon_1^2\epsilon_2 t}, \frac{x}{1-\epsilon_1\epsilon_2-\epsilon_1^2\epsilon_2 t}\right) \\
&= \left(\frac{(1+\epsilon_1\epsilon_2)t}{1-\epsilon_1\epsilon_2}, \frac{x}{1-\epsilon_1\epsilon_2}\right) \text{ (Neglecting higher order terms)} \\
&= \left((1+\epsilon_1\epsilon_2)^2 t, (1+\epsilon_1\epsilon_2)x\right) \text{ (Up to first order in } \epsilon_1) \\
&= \left(e^{2\epsilon_1\epsilon_2} t, e^{\epsilon_1\epsilon_2} x\right) \text{ (Up to first order in } \epsilon_1)
\end{aligned}$$

$$\therefore [A, H] = iD + (?)I. \tag{1.36}$$

All other commutators may be similarly determined. We will list them after evaluating the multiples of identity.

Next, we will find the multiples of identity. There are three forms of $(?)I$ that we shall encounter:

1. Those that are found to be zero from Jacobi identity,
2. Those that are arbitrary but may be eliminated by redefining the generators by a phase factor, and
3. Those that can not be removed and physically significant.

The Jacobi identity is a relation satisfied by the commutators. If we have three operators A, B and C, then the identity states that

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = O. \tag{1.37}$$

We first show some examples where $(?)I$ may be removed by invoking Jacobi identity.

Example 1. $[P_\alpha, H] = O$

$$\begin{aligned}
[[J_2, P_3], H] + [[P_3, H], J_2] + [[H, J_2], P_3] &= O \\
\implies i[P_1, H] + [O + (?)I, J_2] + [O + (?)I, P_3] &= O \\
\implies [P_\alpha, H] &= O.
\end{aligned} \tag{1.38}$$

Example 2. $[P_\alpha, P_\beta] = O$

$$\begin{aligned} [[J_2, P_3], P_2] + [[P_3, P_2], J_2] + [[P_2, J_2], P_3] &= O \\ \implies i[P_1, P_2] + [O + (?)I, J_2] + [O + (?)I, P_3] &= O \\ \implies [P_\alpha, P_\beta] &= O. \end{aligned} \tag{1.39}$$

Example 3. $[G_\alpha, G_\beta] = O$

$$\begin{aligned} [[J_2, G_3], G_2] + [[G_3, G_2], J_2] + [[G_2, J_2], G_3] &= O \\ \implies i[G_1, G_2] + [O + (?)I, J_2] + [O + (?)I, G_3] &= O \\ \implies [G_\alpha, G_\beta] &= O. \end{aligned} \tag{1.40}$$

Example 4. $[J_\alpha, H] = O$

$$\begin{aligned} [[J_2, J_3], H] + [[J_3, H], J_2] + [[H, J_2], J_3] &= O \\ \implies i[J_1, H] + [O(?) + I, J_2] + [O + (?)I, J_3] &= O \\ \implies [J_\alpha, H] &= O. \end{aligned} \tag{1.41}$$

Example 5. $[G_\alpha, H] = iP_\alpha$

$$\begin{aligned} [[J_1, G_2], H] + [[G_2, H], J_1] + [[H, J_1], G_2] &= O \\ \implies i[G_3, H] + i[P_2, J_1] + [O, G_2] &= O \\ \implies [G_3, H] &= iP_3. \\ \text{In general, } [G_\alpha, H] &= iP_\alpha. \end{aligned} \tag{1.42}$$

Example 6. $[D, G_\alpha] = iG_\alpha$

$$\begin{aligned} [[J_1, G_2], D] + [[G_2, D], J_1] + [[D, J_1], G_2] &= O \\ \implies i[G_3, D] - i[G_2, J_1] + O &= O \\ \implies [D, G_3] &= [J_1, G_2] \\ \implies [D, G_3] &= iG_3. \\ \text{In general, } [D, G_\alpha] &= iG_\alpha. \end{aligned} \tag{1.43}$$

In the last two examples, we have used that $[J_1, P_2] = iP_3$ and $[J_1, G_2] = iG_3$ (without any multiple of identity). These are obtained by redefining the vector operators. This will be discussed in the following.

Now we deal with cases where the generators are redefined to eliminate $(?)I$.

Example 1. $[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}J_\gamma$

We know that

$$[J_\alpha, J_\beta] = -[J_\beta, J_\alpha]. \tag{1.44}$$

Therefore, the multiple of identity in the expression of $[J_\alpha, J_\beta]$ will be a number depending on the third index γ. This is because specifying two indices is the same as specifying the third index, and interchanging the order of the indices will only change the sign of the constant. It is antisymmetric in α and β. And because we have only α and β in the left hand side and only γ in the right hand side, we may also consider it to be antisymmetric in all three indices without loss of generality, i.e. we can introduce the Levi-Civita symbol $\epsilon_{\alpha\beta\gamma}$. Finally, because the commutator of two Hermitian operators is anti-Hermitian, the constant will be purely imaginary. Thus we take

$$[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}J_\gamma + i\epsilon_{\alpha\beta\gamma}b_\gamma \text{ for some } b_\gamma \in \mathbb{R}. \tag{1.45}$$

Redefining J_α as $J_\alpha + b_\alpha I$ for $\alpha = 1, 2, 3$,

$$[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}J_\gamma. \tag{1.46}$$

Since we have only added multiples of identity to J_α, this will simply introduce constant phase factors to the transformed state vectors. This does not change the Physics at all.

Example 2. $[J_\alpha, G_\beta] = i\epsilon_{\alpha\beta\gamma}G_\gamma$

Similar arguments will work for $[J_\alpha, G_\beta]$ and $[J_\alpha, P_\beta]$. However, here we do not have the anticommutation relation, and so some more work is required. We show here for $[J_\alpha, G_\beta]$; $[J_\alpha, P_\beta]$ can be worked out similarly.

From Jacobi identity,

$$\begin{aligned}
[[J_1, J_2], G_3] + [[G_3, J_1], J_2] + [[J_2, G_3], J_1] &= O \\
\implies i[J_3, G_3] + i[G_2, J_2] + i[G_1, J_1] &= O \\
\implies [J_3, G_3] &= [J_1, G_1] + [J_2, G_2] \\
\text{Similarly, } [J_1, G_1] &= [J_2, G_2] + [J_3, G_3] \\
[J_2, G_2] &= [J_3, G_3] + [J_1, G_1] \\
\implies [J_1, G_1] = [J_2, G_2] = [J_3, G_3] &= O.
\end{aligned} \tag{1.47}$$

Again,

$$\begin{aligned}
[[J_3, J_1], G_3] + [[G_3, J_3], J_1] + [[J_1, G_3], J_3] &= O \\
\implies i[J_2, G_3] + [O, J_2] - i[G_2, J_3] &= O \\
\implies [J_2, G_3] = [G_2, J_3] &= -[G_3, J_2] \\
\text{In general, } [J_\alpha, G_\beta] &= -[J_\beta, G_\alpha].
\end{aligned} \tag{1.48}$$

The rest of the reasoning is the same as for $[J_\alpha, J_\beta]$.

$[J_\alpha, P_\beta]$ can be found in a similar manner.

Example 3. $[A, H] = iD$

Let us take

$$[A, H] = iD + icI,\ c \in \mathbb{R} \tag{1.49}$$

Here, the multiple of identity can be removed simply by redefining A as $A + cI$. The logic is same as before; the problem is simpler as these are scalar operators and we need not worry about the components. Similarly, we can redefine A and H to get $[D, A] = 2iA$ and $[D, H] = -2iH$.

It is interesting to note that, we have had to redefine all of the generators to eliminate $(?)I$. There is no freedom left to further redefine any generator. If we had taken only the Galilei group (which does not include the transformations generated by A and D), then H would remain arbitrary.

Finally, we come to the scenario where the multiple of identity can not be removed. This occurs only in one case: in the commutator $[G_\alpha, P_\beta]$. From Jacobi identity, we see that

$$\begin{aligned}
[[J_1, G_2], P_1] + [[G_2, P_1], J_1] + [[P_1, J_1], G_2] &= O \\
\implies i[G_3, P_1] + [O + (?)I, J_1] + [O, G_2] &= O \\
\implies [G_3, P_1] &= O. \\
\text{In general, } [G_\alpha, P_\beta] &= O \text{ for } \alpha \neq \beta
\end{aligned} \tag{1.50}$$

$$\begin{aligned}
[[J_1, G_2], P_3] + [[G_2, P_3], J_1] + [[P_3, J_1], G_2] &= O \\
\implies i[G_3, P_3] + [O, J_1] + i[P_2, G_2] &= O \\
\implies [G_2, P_2] &= [G_3, P_3] \\
&= [G_1, P_1] \text{ (Similarly)} \\
&= iMI \text{ (Say)}. \\
\therefore [G_\alpha, P_\alpha] &= i\delta_{\alpha\beta} MI.
\end{aligned} \tag{1.51}$$

M can not be found from the theory of symmetries, and can only be found experimentally. In fact, it turns out to be the mass of the particle (we have taken $\hbar = 1$ here).

Finally, we list the commutators between all the symmetry generators:

$$\begin{array}{ccc}
[J_i, J_j] = i\epsilon_{ijk} J_k & [K_i, K_j] = O & [P_i, P_j] = O \\
[J_i, P_j] = i\epsilon_{ijk} P_k & [G_i, P_j] = iM\delta_{ij} & [P_i, H] = O \\
[J_i, G_j] = i\epsilon_{ijk} G_k & [G_i, H] = iP_i & [J_i, H] = O \\
[D, J_i] = O & [D, G_i] = iG_i & [D, P_i] = -iP_i \\
[A, J_i] = O & [A, G_i] = O & [A, P_i] = -iG_i \\
[D, H] = 2iH & [A, H] = iD & [D, A] = 2iA
\end{array} \tag{1.52}$$

Many generators correspond to physical observables. The generator of translator corresponds to momentum, that of rotation to angular momentum, and the generator of time translation, called the Hamiltonian, corresponds to the energy of the system. A detailed discussion can be found in Ref. [1].

1.5 Symmetries and conservation laws; Noether's theorem

Let $U = e^{isK}$ be a continuous unitary transformation operator with the Hermitian generator K. H is the generator of time evolution, with the time evolution operator $S = e^{itH}$. Now we see that for small t and s,

$$\begin{aligned}
U(s)HU(s)^{-1} &= (I + isK)H(I - isK) = H - is[H, K], \\
S(t)KS(t)^{-1} &= (I + itH)K(I - itH) = K + it[H, K].
\end{aligned} \tag{1.53}$$

$$\therefore UHU^{-1} = H \iff [H, K] = O \iff SKS^{-1} = K. \tag{1.54}$$

The first equation states that the Hamiltonian is invariant under the transformation. Since H generates time-evolution, this means that U represents a symmetry of the dynamics of the system. The last statement is that the operator K is invariant under time evolution, i.e. the expected value of its corresponding observable is a constant of motion. The equivalence of the two statements implies that *for every symmetry of the dynamical laws of the system, we will have a conserved quantity.* This is Noether's theorem.

For example, a system which is symmetric under translation in space has momentum conserved. A system having rotational symmetry has its angular momentum constant. The concepts of symmetry and conservation will again be mentioned in the contexts of gauge transformations and classical symmetries.

Chapter 2

The Schrödinger Group

In the previous chapter, we discussed the spacetime transformation symmetries and their generators. But this gives rise to two questions. Firstly, how did we know that these transformations will preserve Schrödinger equation? While some of them, such as space and time translations, can be understood intuitively, others such as the transformation $(t, x) \mapsto \left(\frac{t}{1+\alpha t}, \frac{x}{1+\alpha t}\right)$ can not be recognized as symmetries so easily. Secondly, how do we know that these are the only possible symmetries of the dynamics of a free particle? These questions motivate us to find the maximal kinematical invariance group of the Schrödinger equation, i.e. the group of all transformations that preserve the dynamics of a non-relativistic free particle. For this chapter, we refer to Ref. [9].

We will be only considering continuous symmetries here. We know that any continuous space-time transformation can be represented by a unitary operator in Hilbert space, which in turn will have Hermitian operators as generators. As we have seen previously, the generators are usually obtained by Taylor expanding the transformed wave function up to first order. Thus we consider the generator (in co-ordinate basis) to be a linear combination of first derivatives with respect to space and time co-ordinates. We take an arbitrary transformation generator G to be of the form

$$G = ia\partial_0 + ib_k\partial_k + ic, \tag{2.1}$$

where a, b_k and c are functions of position and time. Repeated indices are summed over. The Schrödinger equation is (taking $\hbar = 1$)

$$\begin{aligned} -\frac{1}{2m}\nabla^2\Psi &= i\partial_0\Psi \\ \implies \Delta\Psi &= 0 \end{aligned} \tag{2.2}$$

where $\Delta = \frac{1}{2m}\nabla^2 + i\partial_0$ is the Schrödinger operator. If the wave function Ψ satisfies the equation and the transformation generated by G is a symmetry, then the transformed wave function $(1+\epsilon G)\Psi$ (ϵ is an infinitesimal parameter) should also satisfy the equation. Therefore,

$$\begin{aligned} &\Delta\Psi = 0 \implies \Delta G\Psi = 0 \\ &\therefore [\Delta, G] = i\lambda\Delta, \end{aligned} \tag{2.3}$$

where λ is a function of position and time. Expanding G and Δ, we get

$$\left[\frac{1}{2m}\nabla^2 + i\partial_0, a\partial_0 + b_k\partial_k + c\right] = \frac{\lambda}{2m}\nabla^2 + i\lambda\partial_0$$

$$\begin{aligned}\implies \frac{1}{2m}&\left((\nabla^2 a)\partial_0 + 2(\partial_k a)\partial_k\partial_0 + (\nabla^2 b_k)\partial_k + 2(\partial_j b_k)\partial_j\partial_k\right)\\ &+\frac{1}{2m}\left((\nabla^2 c) + 2(\partial_k c)\partial_k\right) + i\left((\partial_0 a)\partial_0 + (\partial_0 b_k)\partial_k + (\partial_0 c)\right) = \frac{\lambda}{2m}\nabla^2 + i\lambda\partial_0. \end{aligned} \tag{2.4}$$

This gives us the following partial differential equations:

$$\frac{1}{2m}\nabla^2 a + i\partial_0 a = i\lambda, \tag{2.5}$$

$$\vec{\nabla} a = 0, \tag{2.6}$$

$$\frac{1}{2m}(\nabla^2 b_k + 2\partial_k c) + i\partial_0 b_k = 0 \ (\forall k \in \{1,2,3\}), \tag{2.7}$$

$$2\partial_k b_k = \lambda \ (\forall k \in \{1,2,3\},\ k \text{ is not summed over}), \tag{2.8}$$

$$\partial_j b_k + \partial_k b_j = 0 \ (j,k \in \{1,2,3\}, j \neq k), \tag{2.9}$$

$$\frac{1}{2m}\nabla^2 c + i\partial_0 c = 0. \tag{2.10}$$

We have to solve these equations to obtain a, $\vec{b}$, c and λ. It is difficult to solve the equations generally, so we will make a number of intuitively reasonable assumptions regarding the functions. Firstly, we expect the functions to be analytic, and will work out the problem by considering all functions to be polynomials.

It is clear from Eq. (2.6) that a can not be a function of position. a only depends on time t, and we write it as

$$a = b + 2\delta t - \alpha t^2 + \text{higher order terms}. \tag{2.11}$$

We write the coefficients this way for later convenience. We write up to order of t^2, for we will see later that the higher order terms will vanish. From 2.5, we get

$$\lambda = \partial_0 a = 2\delta - 2\alpha t + \text{higher order terms}. \tag{2.12}$$

c is a scalar, and it can depend on the position vector $\vec{x}$ either through terms of the form $\vec{c_2}\cdot\vec{x} = c_{2i}x_i$ (linear) or the term $x^2 = \sum x_i^2$ (quadratic). Higher powers of these terms are possible, but we rarely come across such terms in Physics. It is thus a reasonable guess to take c to be of the form

$$c = c_0 + c_1 x^2 + c_{2i}x_i + c_{3j}t^j + c_{4j}t^j x^2 + c_{5ij}x_i t^j \text{ (Summed over } i \in \{1,2,3\},\ j \in \mathbb{N}). \tag{2.13}$$

Now, Eq. (2.10) gives

$$\frac{1}{2m}(2c_1 \times 3 + 2c_{4j}t^j \times 3) + i(jc_{3j}t^{j-1} + jc_{4j}t^{j-1}x^2 + jc_{5ij}t^{j-1}x_i) = 0 \tag{2.14}$$

Comparing the coefficients,

$$\frac{3c_1}{m} + ic_{31} = 0, \tag{2.15}$$

$$\frac{3c_{4j}}{m} + i(j+1)c_{3\,j+1} = 0, \tag{2.16}$$

$$c_{4j} = c_{5ij} = 0 \tag{2.17}$$

$$\implies c_{3j} = 0\ \forall j > 1. \tag{2.18}$$

Therefore, c can be written as

$$c = c_0 + c_1 x^2 + c_{2i} x_i + \frac{3ic_1}{m} t. \tag{2.19}$$

$\vec{b}$ is a vector, and let us consider a component b_k. Since λ is independent of $\vec{x}$, Eq. (2.8) shows that b_k can be only up to the first order in x_k. Similarly, Eq. (2.9) shows b_k is at most linear in x_i ($i \neq k$). Finally, Eq. (2.7) implies that b_k is linear in t. Thus we write b_k as

$$b_k = a_k + b_{k1} x_k + \sum_{i \neq k} b_{k2i} x_i + v_k t + b_{k4} x_k t + \sum_{i \neq k} b_{k5i} x_i t. \tag{2.20}$$

From Eq. (2.8),

$$\begin{aligned} 2(b_{k1} + b_{k4} t) &= 2\delta - 2\alpha t + \text{higher order terms in } t \\ \implies b_{k1} &= \delta, \qquad (2.21) \\ b_{k4} &= -\alpha. \qquad (2.22) \end{aligned}$$

Also, we find that all higher order terms in λ are zero. Therefore, $\lambda = \partial_0 a$ is linear and a is quadratic in t.

Now we look at Eq. (2.9). We get

$$\begin{aligned} b_{k2j} + b_{k5j} t + b_{j2k} + b_{j5k} t &= 0 \\ \implies b_{k2j} &= -b_{j2k}, \qquad (2.23) \\ b_{k5j} &= -b_{j5k}. \qquad (2.24) \end{aligned}$$

Let us define

$$\begin{matrix} b_{122} = -b_{221} = -r_3, & b_{123} = -b_{321} = r_2, & b_{223} = -b_{322} = -r_1, \\ b_{152} = -b_{251} = -\rho_3, & b_{153} = -b_{351} = \rho_2, & b_{253} = -b_{352} = -\rho_1. \end{matrix} \tag{2.25}$$

Now b_k can be written as

$$b_k = a_k + \delta x_k + ((\vec{r} + t\vec{\rho}) \times \vec{x})_k + v_k t - \alpha x_k t. \tag{2.26}$$

Finally, we work with Eq. (2.7) to get

$$\frac{2c_1 x_k + c_{2k}}{m} + i(v_k - \alpha x_k + (\vec{\rho} \times \vec{x})_k) = 0. \tag{2.27}$$

Equating the terms,

$$\begin{aligned} \frac{2c_1}{m} - i\alpha &= 0 \\ \implies c_1 &= \frac{mi\alpha}{2}, \qquad (2.28) \\ \frac{c_{2k}}{m} + iv_k &= 0 \\ \implies c_{2k} &= -imv_k, \qquad (2.29) \\ \vec{\rho} &= 0 \qquad (2.30) \end{aligned}$$

Now we can write down the functions $a(t, \vec{x})$, $b(t, \vec{x})$, $c(t, \vec{x})$ and $\lambda(t, \vec{x})$ as:

$$a = -\alpha t^2 + 2\delta t + b, \tag{2.31}$$

$$\vec{b} = (-\alpha t + \delta)\vec{x} + \vec{r} \times \vec{x} + \vec{v}t + \vec{a}, \tag{2.32}$$

$$c = \frac{-3\alpha t}{2} + im(\frac{\alpha}{2}x^2 - \vec{v} \cdot \vec{x}) + c_0, \tag{2.33}$$

$$\lambda = -2\alpha t + 2\delta, \tag{2.34}$$

where α, δ, b, $\vec{r}$, $\vec{v}$, $\vec{a}$, c_0 are real constants.

The constant c_0 will just contribute a constant phase factor e^{ic_0} in the transformation operator, so its value has no physical significance. We have seen before that the dilation generator requires a phase constant of $\frac{3}{2}$ to make the transformed Hamiltonian normalized. Hence we usually take $c_0 = \frac{3\delta}{2}$, for δ is the dilation parameter (as we shall soon see).

If we now take the transformation generator $G = i(a\partial_0 + b_k\partial_k + c)$ and group the terms in terms of the parameters, we obtain

$$G = \alpha A + \delta D + bH + v_k G_k - a_k P_k - r_k J_k, \tag{2.35}$$

where

$$A = -i\left(t^2\partial_0 + tx_k\partial_k + \frac{3}{2}t\right) - \frac{m}{2}x^2 \qquad \text{(Expansion by } \alpha) \tag{2.36}$$

$$D = i\left(2t\partial_0 + x_k\partial_k + \frac{3}{2}\right) \qquad \text{(Dilation by } e^\delta = d) \tag{2.37}$$

$$H = i\partial_0 \qquad \text{(Time translation by } b) \tag{2.38}$$

$$G_k = it\partial_k + mx_k \qquad \text{(Galilean boost by } \vec{v}) \tag{2.39}$$

$$P_k = -i\partial_k \qquad \text{(Translation by } \vec{a}) \tag{2.40}$$

$$J_k = -i(\vec{x} \times \vec{\nabla})_k \qquad \text{(Rotation by } \vec{r}) \tag{2.41}$$

The expression of J_k is because $i(\vec{r} \times \vec{x})_j\partial_j = i\epsilon_{jki}r_k x_i\partial_j = r_k(i\epsilon_{kij}x_i\partial_j)$. The transformations corresponding to these generators have already been discussed.

The transformations generated by $\{P_k, J_k, G_k, H, D, A\}$ are the only transformations that leave the Schrödinger equation invariant, and they form the maximum kinematical invariance group of the Schrödinger equation or the Schrödinger group. The subgroup generated by $\{P_k, J_k, G_k, H\}$ is called the Galilei group.

If we take $(d, \alpha, b, \vec{a}, \vec{v}, R)$ to denote rotation by R followed by Gailean boost by $\vec{v}$, followed by translation by $\vec{a}$, time translation by b, expansion by α and finally dilation by d, then the most general transformation is given as

$$(d, \alpha, b, \vec{a}, \vec{v}, R)(t, \vec{x}) = \left(\frac{d^2(t+b)}{1+\alpha(t+b)}, \frac{d(R\vec{x} + \vec{v}t + \vec{a})}{1+\alpha(t+b)}\right). \tag{2.42}$$

It interesting to note that the *discrete* transformation Σ, $\Sigma(t, \vec{x}) = \left(-\frac{1}{t}, \frac{\vec{x}}{t}\right)$, is obtained as a limiting case of this: $\Sigma = \lim_{\epsilon \to 0}\left(\epsilon, \epsilon, -\frac{1}{\epsilon}, 0, 0, 1\right)$. Also, $\Sigma^2 = \Pi$, the parity operator. We will study Π in more detail in the chapter on discrete symmetries.

Chapter 3

Gauge symmetry

We have only considered space-time symmetries till now. But there is another kind of symmetry which is not related to the spacetime coordinates of the system, but to the *field* in which it is present. This is called gauge symmetry. For this chapter, we will mostly refer to Refs. [2, 3, 10].

3.1 Particle in electromagnetic field

Till now, we have considered a free particle. Let us now introduce an electric field $\vec{E}$ and a magnetic field $\vec{B}$. These fields can be derived from the scalar potential Φ and the vector potential $\vec{A}$ as:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial A}{\partial t}, \tag{3.1}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{3.2}$$

Φ and $\vec{A}$ are not unique. In fact, we see that the transformation

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}, \tag{3.3}$$

$$\vec{A'} = \vec{A} + \vec{\nabla}\Lambda \tag{3.4}$$

gives us the same $\vec{E}$ and $\vec{B}$. Such a transformation in the potentials is called a gauge transformation.

Let us now work with Lorentz force law.

$$
\begin{aligned}
\vec{F} &= q(\vec{E} + \vec{v} \times \vec{B}) && (3.5)\\
\implies \frac{d\vec{p}}{dt} &= q\left(-\vec{\nabla}\Phi - \frac{\partial A}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right)\\
&= q\left(-\vec{\nabla}\Phi - \frac{\partial A}{\partial t} + \vec{\nabla}(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \vec{\nabla})\vec{A}\right)\\
&\left[\vec{\nabla}(\vec{v} \cdot \vec{A}) = \vec{v} \cdot (\vec{\nabla} \times \vec{A}) + (\vec{v} \cdot \vec{\nabla})\vec{A}\right]\\
&= q\left(-\vec{\nabla}\Phi - \frac{d\vec{A}}{dt} + \vec{\nabla}(\vec{v} \cdot \vec{A})\right)\\
&\left[\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A}\right]\\
\implies \frac{d}{dt}(\vec{p} + q\vec{A}) &= -\vec{\nabla}(q\Phi - q\vec{v} \cdot \vec{A}). && (3.6)
\end{aligned}
$$

This suggests that in the Schrödinger Hamiltonian, we should replace $\vec{p}$ with $\vec{p} - q\vec{A}$ and add a potential term $q\Phi$:

$$
H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi. \tag{3.7}
$$

We can get the equations of motion in Heisenberg picture. The velocity is obtained as

$$
\begin{aligned}
\frac{dx_i}{dt} &= -i\,[x_i, H]\\
&= -i\left[x_i, \frac{1}{2m}(p^2 - q\vec{p} \cdot \vec{A} - q\vec{A} \cdot \vec{p} + q^2A^2) + q\Phi\right]\\
&= \frac{p_i - qA_i}{m}. && (3.8)
\end{aligned}
$$

We define $\vec{\Pi} = m\frac{d\vec{x}}{dt} = \vec{p} - q\vec{A}$, called the kinematic momentum. Then $H = \frac{\Pi^2}{2m} + q\Phi$. The commutators between the components of $\vec{\Pi}$ are as follows:

$$
\begin{aligned}
[\Pi_1, \Pi_2] &= [p_1 - qA_1, p_2 - qA_2]\\
&= -q\,[p_1, A_2] - q\,[A_1, p_2]\\
&= iq\frac{\partial A_2}{\partial x_1} - iq\frac{\partial A_1}{\partial x_2}\\
&= iq(\vec{\nabla} \times \vec{A})_3\\
&= iqB_3.\\
\text{In general, } [\Pi_i, \Pi_j] &= iq\epsilon_{ijk}B_k && (3.9)\\
\left[\Pi_1, \Pi_2^2\right] &= [\Pi_1, \Pi_2]\,\Pi_2 + \Pi_2\,[\Pi_1, \Pi_2]\\
&= iqB_3\Pi_2 - iq\Pi_2B_3\\
&= -iq(\vec{\Pi} \times \vec{B})_1.\\
\text{In general, } \left[\Pi_i, \Pi_j^2\right] &= -iq\epsilon_{ijk}q(\vec{\Pi} \times \vec{B})_i && (3.10)
\end{aligned}
$$

Therefore, the equation of motion is

$$
\begin{aligned}
m\frac{d^2x_1}{dt^2} &= \frac{d\Pi_1}{dt} \qquad (3.11)\\
&= -i\,[\Pi_1, H]\\
&= -\frac{i}{2m}\left[\Pi_1, \Pi^2\right] - iq\,[\Pi_1, \Phi]\\
&= \frac{1}{2m}q(\vec{\Pi}\times\vec{B} - \vec{B}\times\vec{\Pi})_1 - iq\,[p_i, \Phi]\\
&= \frac{1}{2m}q(\vec{\Pi}\times\vec{B} - \vec{B}\times\vec{\Pi})_1 - q\frac{\partial\Phi}{\partial x_1}\\
&= q\left(\vec{E} + \frac{1}{2}(\vec{v}\times\vec{B} - \vec{B}\times\vec{v})\right)_1 \qquad \left[\vec{v} = \frac{d\vec{x}}{dt} = \frac{\vec{\Pi}}{m}\right]\\
\implies m\frac{d^2\vec{x}}{dt^2} &= q\left(\vec{E} + \frac{1}{2}(\vec{v}\times\vec{B} - \vec{B}\times\vec{v})\right). \qquad \text{(Lorentz force law)} \qquad (3.12)
\end{aligned}
$$

We can use the gauge transformation $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda,\ \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}$. Then the transformed Hamiltonian is $H' = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi'$. If we find a unitary operator U such that $U\Psi$ satisfies the Schrödinger equation with the transformed Hamiltonian, then we can say that the system is invariant under gauge transformation. We claim that $U = e^{iq\Lambda(\vec{x},t)}$ satisfies this condition.

Firstly, we find the required relation between the original and transformed Hamiltonians:

$$
\begin{aligned}
H|\Psi\rangle &= i\frac{\partial|\Psi\rangle}{\partial t},\\
H'U|\Psi\rangle &= i\frac{\partial(U|\Psi\rangle)}{\partial t}\\
\implies H'U|\Psi\rangle &= iU\frac{\partial|\Psi\rangle}{\partial t} + i\frac{\partial U}{\partial t}|\Psi\rangle\\
&= UH|\Psi\rangle + i\frac{\partial U}{\partial t}|\Psi\rangle\\
\implies H'U|\Psi\rangle &= UHU^\dagger U|\Psi\rangle + i\frac{\partial U}{\partial t}U^\dagger U|\Psi\rangle\\
\implies H' &= UHU^\dagger + i\frac{\partial U}{\partial t}U^\dagger. \qquad (3.13)
\end{aligned}
$$

If we can show that this holds for the transformation stated here, the claim will be proved.

Now, we see that

$$U\vec{x}U^\dagger = \vec{x}, \qquad [U \text{ is a function of position and hence } [U,\vec{x}]{=}0] \tag{3.14}$$

$$\begin{aligned} U(\vec{p}-q\vec{A})U^\dagger &= U\vec{p}U^\dagger - q\vec{A} && \left[\left[U,\vec{A}\right]=0\right] \\ &= \left([U,\vec{p}] + \vec{p}U\right)U^\dagger - q\vec{A} \\ &= i(\vec{\nabla}U)U^\dagger + \vec{p} - q\vec{A} \\ &= -q\vec{\nabla}\Lambda + \vec{p} - q\vec{A} \\ &= \vec{p} - q\vec{A}', && [\because U = e^{iq\Lambda} \implies \vec{\nabla}U = iqe^{iq\Lambda}\vec{\nabla}\Lambda = iq(\vec{\nabla}\Lambda)U] \end{aligned} \tag{3.15}$$

$$\begin{aligned} i\frac{\partial U}{\partial t}U^\dagger &= -qe^{iq\Lambda}\frac{\partial \Lambda}{\partial t}U^\dagger \\ &= -q\frac{\partial \Lambda}{\partial t} \end{aligned} \tag{3.16}$$

And thus

$$\begin{aligned} UHU^\dagger + i\frac{\partial U}{\partial t}U^\dagger &= U\left(\frac{1}{2m}(\vec{p}-q\vec{A})^2 + q\Phi\right)U^\dagger + i\frac{\partial U}{\partial t}U^\dagger \\ &= \frac{1}{2m}(\vec{p}-q\vec{A}')^2 + q\Phi - q\frac{\partial \Lambda}{\partial t} \\ &= H'. \end{aligned} \tag{3.17}$$

Therefore, we see that introduction of the phase factor $e^{iq\Lambda}$ makes the system invariant under gauge transformation. Also, the position $\vec{x}$ and kinematical momentum $\vec{\Pi}$ of the particle are invariant, even though the canonical momentum $\vec{p}$ is not.

3.2 General gauge symmetry; non-abelian gauge

In the previous section, we started with gauge invariance of the electromagnetic field, and worked out the gauge symmetry of the state vector. But we could have also done the opposite. If we start with a symmetry $|\Psi\rangle \mapsto |\Phi'\rangle = e^{iq\Lambda}|\Psi\rangle$, it can be shown that this implies the existence of a field (the electromagnetic field, in this case) obeying gauge freedom. This is the idea of the gauge principle, which states that *every continuous symmetry of nature is a local symmetry.* Any symmetry transformation would be manifested as a change in the sate vectors by a phase factor, $|\Psi\rangle \mapsto e^{i\theta}|\Psi\rangle$. It is a local symmetry, if θ is a function of position and time ($\theta = \theta(t,x,y,z)$), i.e. the transformation of the wave function is different at different points in space and time. We shall see that this holds only if we introduce a gauge field with properties similar to the electromagnetic field.

From Noether's theorem, we know that every continuous symmetry corresponds to a conserved quantity. This gives a correspondence between conservation law, symmetry and gauge field, shown in Fig. 3.1. For example, the gauge symmetry of the wave function corresponds to the conservation of charge, as well as the existence of electromagnetic field satisfying gauge invariance.

We can make a general study of gauge symmetries. The situation turns out to be complicated for cases such as energy or momentum conservation, and we will instead

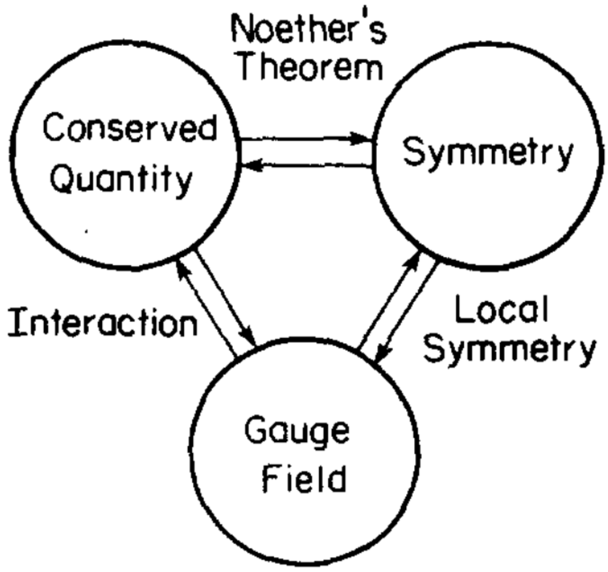


Figure 3.1: The logical pattern of a gauge theory. (Reproduced from Ref. [10])

take symmetries in the internal degrees of freedom of the system, such as isospin or colour. Let the symmetry operator be of the form $U = e^{i\alpha(\vec{r},t)}$, which is analogous to the electromagnetic case. In that case, the group of transformations was $U(1)$ (the group of one dimensional unitary operators), which is abelian. In general, this need not be the case. For example, the symmetry corresponding to isospin conservation is $SU(2)$, the group of unitary 2×2 matrices with determinant 1, and it is non-abelian.

In the case of electromagnetism, we a scalar field Φ and a vector field $\vec{A}$, satisfying

$$U(-i\vec{\nabla} - q\vec{A})U^\dagger = -i\vec{\nabla} - q\vec{A}', \ U\left(i\frac{\partial}{\partial t} - q\Phi\right)U^\dagger = i\frac{\partial}{\partial t} - q\Phi' \tag{3.18}$$

Similarly for the general gauge symmetry, the requirement that the dynamics of the system are invariant under $U = e^{i\alpha}$ is satisfied only if we introduce a scalar field g and a vector field $\vec{G}$ such that

$$U(-i\vec{\nabla} - \vec{G})U^\dagger = -i\vec{\nabla} - \vec{G}', \ U\left(i\frac{\partial}{\partial t} - g\right)U^\dagger = i\frac{\partial}{\partial t} - g' \tag{3.19}$$

If U belongs to $SU(2)$, g and $\vec{G}$ are 2×2 matrices in the isospin space. We get the following transformation equations for the gauge fields:

$$
\begin{aligned}
U(-i\vec{\nabla} - \vec{G})U^\dagger &= -i\vec{\nabla} - \vec{G}' \\
\implies -i\vec{\nabla} - iU(\vec{\nabla}U^\dagger) - U\vec{G}U^\dagger &= -i\vec{\nabla} - \vec{G}' \\
\implies \vec{G}' &= U\vec{G}U^\dagger + iU(\vec{\nabla}U^\dagger) \\
&= (1+i\alpha)\vec{G}(1-i\alpha) + i(1+i\alpha)(-i\vec{\nabla}\alpha) \\
&\text{[Upto first order]} \\
&= \vec{G} + i\left[\alpha, G\right] + \vec{\nabla}\alpha \text{ [Upto first order]} \\
U\left(i\frac{\partial}{\partial t} - g\right)U^\dagger &= i\frac{\partial}{\partial t} - g' \\
\implies i\frac{\partial}{\partial t} + iU\frac{\partial U}{\partial t} - UgU^\dagger &= i\frac{\partial}{\partial t} - g' \\
\implies g' &= UgU^\dagger - iU\frac{\partial U}{\partial t} \\
&= (1+i\alpha)g(1-i\alpha) - i(1+i\alpha)\left(-i\frac{\partial\alpha}{\partial t}\right) \\
&\text{[Upto first order]} \\
&= g + i\left[\alpha, g\right] - \frac{\partial\alpha}{\partial t}
\end{aligned}
$$

These are similar to the previously derived gauge transformation equations for electromagnetic field. The commutator terms appear as the group of transformations is non-abelian here.

Further analysis depends on the particular group being considered. In case of the group $SU(2)$ (group of all 2×2 unitary matrices of determinant 1), the generators are traceless 2×2 Hermitian matrices, which are spanned by the three matrices τ_1, τ_2, τ_3 satisfying

$$[\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k \tag{3.20}$$

Then we can say

$$\alpha = \frac{1}{2}\sum \alpha_i\tau_i \tag{3.21}$$

and

$$g = \frac{1}{2}\sum g_i\tau_i,\ \vec{G} = \frac{1}{2}\sum \vec{G}_i\tau_i \tag{3.22}$$

In this case, we can say, component by component, that

$$
\begin{aligned}
\vec{G}_i' &= \vec{G}_i + \vec{\nabla}\alpha_i + \epsilon_{ijk}\alpha_j\vec{G}_k \\
g_i' &= g_i - \frac{\partial\alpha_i}{\partial t} + \epsilon_{ijk}\alpha_j g_k
\end{aligned}
$$

These are the gauge field transformation equations for a general gauge. This theory was proposed by Yang and Mills, and is of great significance in Quantum Field Theory.

Chapter 4

Classical symmetries and Conservation Laws

Till now, we have been concerned with symmetries in Quantum Mechanics. In this chapter, we will briefly discuss symmetries and conservation laws in the context of classical particles and fields, in terms of the Lagrangian formulation. We mostly refer to Refs. [4, 5, 6].

4.1 Symmetries in the dynamics of a point mass

We know that particle motion follows the principle of least action. We define the Lagrangian of a particle as $L(q_i, \dot{q}_i, t) = T - V$ (T and V are the kinetic and potential energies respectively), and the action as $S = \int_{t_1}^{t_2} L\ dt$. ($(q_1, q_2, \ldots, q_n)$ are the generalized coordinates of the system, and n is the number of degrees of freedom.) Let the co-ordinates be subjected to a variation $q_i \mapsto q_i' = q_i + \epsilon_i$. The variation vanishes at the initial and final times, $\epsilon_i(t_1) = \epsilon_i(t_2) = 0$. The least action principle (Hamilton's principle) states that the the corresponding first order variation in the action, δS, will be zero.

Now we see that

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \Big(L(q_i', \dot{q_i'}, t) - L(q_i, \dot{q}_i, t) \Big)\ dt \\
&= \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\epsilon_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\epsilon}_i \right)\ dt \\
&= \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\epsilon_i\ dt + \frac{\partial L}{\partial \dot{q}_i}\epsilon_i \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \epsilon_i\ dt \\
&= \sum_i \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \epsilon_i\ dt\ [\because\ \epsilon_i(t_1) = \epsilon_i(t_2) = 0]
\end{aligned}$$

Since this holds for any arbitrary set of functions ϵ_i satisfying the boundary conditions, we get the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{4.1}$$

For the Lagrangian $L(q_i, \dot{q}_i, t) = \sum_i \frac{m}{2}\dot{q}_i^2 - V(q_i)$ (q_i are the Cartesian coordinates of the particle, potential is independent of time or velocity), this reduces to Newton's law, $m\ddot{q}_i = -\frac{\partial V}{\partial q_i}$.

If the particle Lagrangian L is independent of a co-ordinate q_j (i.e. the Physics is invariant under transformation of q_j), then the Euler-Lagrange equation gives

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) &= \frac{\partial L}{\partial q_j} = 0 \\ \implies \frac{dp_j}{dt} &= 0 \end{aligned}$$

where $p_j = \frac{\partial L}{\partial \dot{q}_j}$ is called the momentum conjugate to the co-ordinate q_j. Thus we see that every transformation symmetry corresponds to a conserved quantity, which is Noether's theorem.

4.2 Lagrangian formulation of fields

We also have a Lagrangian formulation of fields. Here, we consider a scalar field $\phi(x, y, z, t)$. Since the field is extended over a region, we will have a Lagrangian density $\mathcal{L}$ instead of the Lagrangian L as for a point particle. The action will be $S = \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu) d^4x$. Here $(x^0, x^1, x^2, x^3) = (t, x, y, z)$.

We now subject both the field variable ϕ and the co-ordinates x to a variation, which vanishes at the initial and final times t_1 and t_2.

$$\begin{aligned} x^\mu \mapsto x'^\mu &= x^\mu + \delta x^\mu \\ \phi(x) \mapsto \phi'(x) &= \phi(x) + \delta\phi(x) \end{aligned}$$

It is to be noted that $\delta\phi$ defined here is due to the variation in the field function ϕ at the same point in space-time. The total variation in ϕ due to the variations $\delta\phi$ and δx^μ is given by

$$\begin{aligned} \Delta\phi &= \phi'(x') - \phi(x) \\ &= \phi'(x') - \phi(x') + \phi(x') - \phi(x) \\ &= \delta\phi + (\partial_\mu \phi)\delta x^\mu \end{aligned}$$

The variation in the action is

$$\delta S = \int \mathcal{L}(\phi', \partial_\mu \phi', x'^\mu)\ d^4x' - \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\ d^4x \tag{4.2}$$

We note that the functional form of the Lagrangian does not change. Now, $d^4x' = J(x'/x)d^4x$, where $J(x'/x) = \det\left(\frac{\partial x'^\mu}{\partial x^\lambda}\right)$ is the Jacobian of the transformation $x \mapsto x'$. We see that $x'^\mu = x^\mu + \delta x^\mu \implies \frac{\partial x'^\mu}{\partial x^\lambda} = \delta^\mu_\lambda + \partial_\lambda(\delta x^\mu)$. So $J(x'/x) = \det\left(\frac{\partial x'^\mu}{\partial x^\lambda}\right) = 1 + \partial_\mu(\delta x^\mu)$ up to first order in the variations. Therefore,

$$\begin{aligned} \delta S &= \int (\mathcal{L} + \delta\mathcal{L})(1 + \partial_\mu(\delta x^\mu))\ d^4x - \int \mathcal{L}\ d^4x \\ &= \int (\delta\mathcal{L} + \mathcal{L}\partial_\mu(\delta x^\mu))\ d^4x \end{aligned}$$

where $\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu$.

Again, $\phi' = \phi + \delta\phi \implies \partial_\mu\phi' = \partial_\mu\phi + \partial_\mu(\delta\phi) \implies \delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$. Hence

$$\delta S = \int_R \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \partial_\mu(\mathcal{L}\delta(x^\mu))\right] \, d^4x \tag{4.3}$$

where R is a region of space-time. Here, the third term is a divergence and can be converted to a surface integral by the 4-dimensional Gauss's theorem. The second term may be rewritten as

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi \tag{4.4}$$

where the first part is a divergence. Then we get

$$\delta S = \int_R \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi \; d^4x + \int_{\partial R}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \mathcal{L}\delta x^\mu\right] \, d\sigma_\mu \tag{4.5}$$

But the variations vanish at the boundary surface: $\delta\phi = 0$, $\delta x^\mu = 0$ on ∂R. So the surface integral vanishes. Thus, the least action principle $\delta S = 0$ holds on any arbitrary region R if and only if

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0 \tag{4.6}$$

This is the Euler-Lagrange equation for the field ϕ.

We will also have a Noether's theorem for fields. Let us consider a transformation in x^μ and ϕ which leaves the action integral unchanged. The change in S is obtained as before, but R is here an arbitrary region and the δx^μ and $\delta\phi$ do not vanish on ∂R. Then the surface term in the expression of δS becomes

$$\delta S_{\text{surface}} = \int_{\partial R}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\delta\phi + (\partial_\nu\phi)\delta x^\nu) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}\right)\delta x^\nu\right] d\sigma_\mu \tag{4.7}$$

where we have added and subtracted a term. We recall that the total variation in ϕ is $\Delta\phi = \delta\phi + (\partial_\nu\phi)\delta x^\nu$. We define $\theta^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$. Then

$$\delta S_{\text{surface}} = \int_{\partial R}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \theta^\mu_\nu\delta x^\nu\right] \, d\sigma_\mu \tag{4.8}$$

Now let us consider the action S to be invariant under a group of transformations on x^μ and ϕ characterised by the infinitesimal parameter $\delta\omega$, expressed as

$$\delta x^\nu = X^\nu_\lambda\delta\omega^\lambda, \;\; \Delta\phi = \Phi_\lambda\delta\omega^\lambda \tag{4.9}$$

We substitute this expression in the expression for $\delta S_{\text{surface}}$ derived below. Symmetry demands that $\delta S = 0$. If we assume that both the original and the transformed fields obeyed the Euler-Lagrange equations, then the volume integral term is anyway zero. Thus we get

$$\int_{\partial R}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Phi_\lambda - \theta^\mu_\nu X^\nu_\lambda\right]\delta\omega^\lambda \; d\sigma_\mu = 0 \tag{4.10}$$

Since $\delta\omega^\lambda$ is arbitrary, we have

$$\int_{\partial R} J^\mu_\lambda \; d\sigma_\mu = 0 \tag{4.11}$$

where $J^\mu_\lambda = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Phi_\lambda - \theta^\mu_\nu X^\nu_\lambda$. This can be converted to a four dimensional volume integral by Gauss's theorem as $\int_R \partial_\mu J^\mu_\lambda \; d^4x = 0$. Since R is an arbitrary region, we get

$$\partial_\mu J^\mu_\lambda = 0 \tag{4.12}$$

J^μ_λ is thus a divergence-less current. We now define

$$Q_\nu = \int_V J^0_\nu d^3x \tag{4.13}$$

where the integral is over the 3-volume V, with $t =$ constant.

Now, we can integrate the equation $\partial_\mu J^\mu_\nu = 0$ over V, we obtain the following expression:

$$\int_V \partial_0 J^0_\nu \; d^3x + \int_V \partial_i J^i_\nu \; d^3x = 0 \tag{4.14}$$

The second term is a 3-dimensional volume integral, and it can be converted to a surface integral by the 3 dimensional Gauss theorem.

$$\frac{d}{dt}Q_\nu + \int_{\partial V} J^i d\sigma_i = 0 \tag{4.15}$$

This is the continuity equation for the quantity Q_ν. If the surface is far enough away, this term vanishes, leaving only the first integral.

$$\frac{d}{dt}\int J^0_\nu \; d^3x = \frac{dQ_\nu}{dt} = 0 \tag{4.16}$$

This is Noether's theorem for fields, and Q_ν is the conserved charge.

4.3 Example of classical field symmetry: spacetime translation

In this case, the transformations are

$$\delta x^\mu = \epsilon^\mu, \; \Delta\phi = 0 \implies X^\mu_\nu = \delta^\mu_\nu, \; \Phi_\mu = 0 \tag{4.17}$$

Then $J^\mu_\nu = -\theta^\mu_\nu$, and the conservation law is $\frac{d}{dt}\int \theta^0_\nu d^3x = 0$.

Now we find the physical significance of the conserved quantity $P_\nu = \int \theta^0_\nu d^3x$. Firstly, we use the definition of θ^μ_ν to get

$$\begin{aligned}\int \theta^0_0 d^3x &= \int \left(\frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi - \mathcal{L}\right) d^3x \\ &= \int \left(\frac{\partial \mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L}\right) d^3x\end{aligned}$$

We recall that, in the mechanics of a particle, the Hamiltonian (which is the energy in most cases) is expressed as

$$H = \sum_i p_i \dot{q}_i - L = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{4.18}$$

The obtained expression is equivalent to the Hamiltonian, for a continuous field. Hence, the conserved quantity $P_0 = \int \theta^0_0 d^3x$ is the energy of the field.

We know that the energy and momentum of a system form a 4-vector. We have obtained the conserved 4-vector P_ν, where P_0 is the energy. Thus it can be said that P_i is the momentum of the field. We have thus obtained that a system which is invariant under translations in space and time has constant energy and momentum, as expected.

Chapter 5

Discrete Symmetries

Till now, we have only considered continuous symmetries, i.e. transformations which depend on a continuous parameter. In such cases, the system may be transformed to the final state via a large number of infinitesimal transformations. But there are symmetries where this need not hold, an example being the parity transformation $(t, x) \mapsto (t, -x)$. Such symmetries are called *discrete* symmetries, and these will be the focus of this chapter. We will follow Refs. [1, 3].

Because the identity operator is linear and unitary and all continuous symmetry operators can be varied continuously from the identity, they are always unitary. Discrete transformation operators however, may be unitary or anti-unitary (by Wigner's theorem), and we need to be careful regarding this.

5.1 Space inversion (Parity)

The operator corresponding to space inversion is called the parity operator, denoted by Π. By definition, this reverses the sigh of both the position and momentum operators, but leaves angular momentum unchanged:

$$\begin{aligned}
\Pi\vec{Q}\Pi^{-1} &= -\vec{Q} \\
\Pi\vec{P}\Pi^{-1} &= -\vec{P} \\
\Pi\vec{J}\Pi^{-1} &= \vec{J}
\end{aligned}$$

To check that Π is linear, we use the Heisenberg uncertainty principle:

$$\begin{aligned}
P_kQ_k - Q_kP_k &= i \\
\implies \Pi P_kQ_k\Pi^{-1} - \Pi Q_kP_k\Pi^{-1} &= \Pi i\Pi^{-1} \\
\implies (\Pi P_k\Pi^{-1})(\Pi Q_k\Pi^{-1}) - (\Pi Q_k\Pi^{-1})(\Pi P_k\Pi^{-1}) &= \Pi i\Pi^{-1} \\
\implies (-P_k)(-Q_k) - (-Q_k)(-P_k) &= \Pi i\Pi^{-1} \\
\implies P_kQ_k - Q_kP_k &= \Pi i\Pi^{-1} \\
\implies \Pi i\Pi^{-1} &= i
\end{aligned}$$

Hence Π is linear, and therefore unitary, $\Pi^\dagger\Pi = I$.

Since two consecutive space inversions produce no change to the system, $|\Psi\rangle$ and $\Pi^2|\Psi\rangle$ must refer to the same state. We have $\Pi^2|\Psi\rangle = e^{i\theta}|\Psi\rangle$. Thus, $\Pi^2 = e^{i\theta}I$.

We now study the action of parity on state vectors and wave functions. Let $|\vec{x}\rangle$ be a position eigenstate with eigenvalue $\vec{x}$; $Q_j|\vec{x}\rangle = x_j|\vec{x}\rangle$. Now we see that

$$Q_j\Pi|\vec{x}\rangle = -\Pi Q_j|\vec{x}\rangle = -\Pi x_j|\vec{x}\rangle = -x_j\Pi|\vec{x}\rangle \tag{5.1}$$

Therefore, $\Pi|\vec{x}\rangle$ is the same as the position eigenstate $|-\vec{x}\rangle$ up to a phase factor: $\Pi|\vec{x}\rangle = e^{i\phi}|-\vec{x}\rangle$. Hence, $\Pi^2|\vec{x}\rangle = e^{2i\phi}|\vec{x}\rangle = e^{i\theta}|\vec{x}\rangle$, i.e. $2\phi = \theta$. θ and ϕ are arbitrary, and we can take them to be zero. Then $\Pi|\vec{x}\rangle = |-\vec{x}\rangle$, and $\Pi^2 = I$.

The effect of parity on a wave function is given as

$$\Pi\Psi(x) = \langle x|\Pi|\Psi\rangle = \langle -x|\Psi\rangle = \Psi(-x) \tag{5.2}$$

Let $|\phi\rangle$ be an eigenstate of Π with eigenvalue c. Then

$$\begin{aligned} \Pi|\phi\rangle &= c|\phi\rangle \\ \implies \Pi^2|\phi\rangle &= c^2|\phi\rangle \\ \implies c^2 &= 1 \\ \implies c &= \pm 1 \end{aligned}$$

A parity eigenstate with eigenvalue 1 is said to have even parity, while one with eigenvalue -1 has odd parity.

If the Hamiltonian H is invariant under parity, $\Pi H\Pi = H \implies [\Pi, H] = O$, then any nondegenerate eigenstate $|n\rangle$ of H ($H|n\rangle = E_n|n\rangle$) is also a parity eigenstate. This is because the states $|m_\pm\rangle = \frac{1}{2}(1 \pm \Pi)|n\rangle$ are parity eigenstates (of even and odd parity respectively), as well as energy eigenstates with the same energy E_n. Since the state is nondegenerate, they can not be different states. Therefore, $|n\rangle$ is a parity eigenvalue and is identical to one of the states $|m_\pm\rangle$, while the other one will be the zero vector.

5.2 Time Reversal

The term "time reversal" is somewhat misleading, since time is a parameter and replacing $t \to -t$ has no physical significance. The term also does not imply any idea of "time flowing backward". The correct picture is that of motion reversal; i.e. reversal of the linear and angular momenta while leaving the position unchanged. Thus, the time reversal operator T satisfies the following by definition:

$$\begin{aligned} T\vec{Q}T^{-1} &= \vec{Q} \\ T\vec{P}T^{-1} &= -\vec{P} \\ T\vec{J}T^{-1} &= -\vec{J} \end{aligned}$$

These relations immediately show that T is an antiunitary operator. We take the Heisenberg uncertainty principle:

$$Q_kP_k - P_kQ_k = i \tag{5.3}$$

Operating both sides of the equation with T on the left and T^{-1} on the right, we get

$$\begin{aligned} TQ_kP_kT^{-1} - TP_kQ_kT^{-1} &= TiT^{-1} \\ \implies TQ_kT^{-1}TP_kT^{-1} - TP_kT^{-1}TQ_kT^{-1} &= TiT^{-1} \\ \implies Q_k(-P_k) - (-P_k)Q_k &= TiT^{-1} \\ \implies -(Q_kP_k - P_kQ_k) &= TiT^{-1} \\ \implies -i &= TiT^{-1} \end{aligned}$$

This can hold only if T is antilinear. We can also draw the same conclusion from other commutation relations, such as $[J_i, J_k]$ and $[J_i, P_k]$. As we had mentioned in the first chapter, a symmetry operator can only be unitary or antiunitary. Since T is antilinear, in must be antiunitary.

5.2.1 Properties of antilinear and antiunitary operators

An *antilinear* operator A on a vector space defined on the field $\mathbb{C}$ of complex numbers is one that satisfies the following relation:

$$A(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1^* A|\psi_1\rangle + c_2^* A|\psi_2\rangle \tag{5.4}$$

Where c_1 and c_2 are complex scalars, and $|\psi_1\rangle$ and $|\psi_2\rangle$ are vectors. Clearly, its action on a scalar is to take its complex conjugate:

$$Ac = c^* A \tag{5.5}$$

An antilinear operator A is called *antiunitary* if it is invertible, and

$$A|u\rangle = |u'\rangle \text{ and } A|v\rangle = |v'\rangle \implies \langle u'|v'\rangle = \langle u|v\rangle^* \tag{5.6}$$

We usually do not define the action of an antilinear operator on the right of a bra vector. This is because the bra vectors are linear functionals; and for such a functional $\langle u|$ and a linear operator A, we could define the linear functional $\langle u|A$ as $(\langle u|A)|v\rangle = \langle u|(A|v\rangle)$. But $\langle u|A$ defined this way will not be a linear functional if A is antilinear. So we refrain from defining the right action of antilinear operators. Also, in such cases $\langle u|A|v\rangle$ will mean $\langle u|(A|v\rangle)$.

5.2.2 Time reversal symmetry of the Schrödinger equation

Let us consider the Schrödinger equation

$$H|\Psi(t)\rangle = i\partial_t|\Psi(t)\rangle \tag{5.7}$$

We suppose that the Hamiltonian H is invariant under time reversal; $THT^{-1} = H$. Then we operate T on the equation to get

$$\begin{aligned} THT^{-1}T|\Psi(t)\rangle &= Ti\partial_t|\Psi(t)\rangle \\ \implies HT|\Psi(t)\rangle &= -i\partial_t T|\Psi(t)\rangle \end{aligned}$$

Thus we see that $T|\Psi(t)\rangle$ does not satisfy the Schrödinger equation. But we can make it satisfy the equation by replacing the dummy variable t with $-t$, i.e. $T|\Psi(-t)\rangle$ satisfies the equation.

Let us try to find the explicit form of T. This would depend on the basis chosen. First, we work with position basis. Then the Schrödinger equation becomes

$$\left(-\frac{1}{2m}\nabla^2 + V(x)\right)\Psi(x,t) = i\partial_t\Psi(x,t) \tag{5.8}$$

Taking the complex conjugate, we get

$$\left(-\frac{1}{2m}\nabla^2 + V^*(x)\right)\Psi(x,t) = -i\partial_t\Psi(x,t) \tag{5.9}$$

If we demand that the Hamiltonian be invariant under complex conjugation ($V^* = V$), then we see that $\Psi^*(x, -t)$ satisfies the Schrödinger equation. This is analogous to the fact that if H is invariant under time reversal and $|\Psi(t)\rangle$ satisfies Schrödinger equation, then so does $T|\Psi(-t)\rangle$. This suggests that we identify the time reversal operator in coordinate space by the position basis complex conjugation operator K, where by definition $K\Psi(x,t) = \Psi^*(x,t)$. We also see that this gives the correct transformations of the position ($\vec{Q} = \vec{x}$), momentum ($\vec{P} = -i\vec{\nabla}$) and orbital angular momentum ($\vec{L} = \vec{x} \times (-i\vec{\nabla})$) operators.

If we express a state $|\Psi\rangle$ as $|\Psi\rangle = \int d^3x|\vec{x}\rangle\langle\vec{x}|\Psi\rangle = \int \Psi(\vec{x})|\vec{x}\rangle d^3x$, then the effect of T will be $T|\Psi\rangle = \int d^3x T\Psi(\vec{x})|\vec{x}\rangle = \int d^3x \Psi^*(\vec{x})|\vec{x}\rangle$, where $T|\vec{x}\rangle = |\vec{x}\rangle$.

In momentum representation, $|\Psi\rangle = \int \Psi(\vec{p})|\vec{p}\rangle d^3p$, where $|\vec{p}\rangle$ is a momentum eigenstate. Then we get $T|\Psi\rangle = \int \Psi^*(p)T|\vec{p}\rangle d^3p$, where $T|\vec{p}\rangle$ is obtained as

$$\begin{aligned} T|\vec{p}\rangle &= T\int |\vec{x}\rangle\langle\vec{x}|\vec{p}\rangle d^3x \\ &= \int Te^{i\vec{p}\cdot\vec{x}}(2\pi)^{-\frac{3}{2}}|\vec{x}\rangle d^3x \\ &= \int e^{-i\vec{p}\cdot\vec{x}}(2\pi)^{-\frac{3}{2}}|\vec{x}\rangle d^3x \\ &= \int |\vec{x}\rangle\langle\vec{x}|\vec{-p}\rangle d^3x \\ &= |-\vec{p}\rangle \end{aligned}$$

Thus, time reversal operator on momentum space is given by $T|\Psi\rangle = \int \Psi^*(\vec{p})|-\vec{p}\rangle d^3p$.

5.2.3 Time reversal squared

From the definition of time reversal, it is clear that two successive applications of T will leave the system unchanged. Therefore the vectors $|\Psi\rangle$ and $T^2|\Psi\rangle$ describe the same state, and should be equal up to a phase factor.

$$T^2|\Psi\rangle = c|\Psi\rangle,\ |c| = 1 \tag{5.10}$$

Now we see that

$$T^2(T|\Psi\rangle) = T(T^2|\Psi\rangle) = T(c|\Psi\rangle) = c^*T|\Psi\rangle \tag{5.11}$$

Thus,

$$T^2(|\Psi\rangle + T|\Psi\rangle) = c|\Psi\rangle + c^*T|\Psi\rangle \tag{5.12}$$

Again, If we replace $|\Psi\rangle$ with $|\Psi\rangle + T|\Psi\rangle$ in the original equation, we get

$$T^2(|\Psi\rangle + T|\Psi\rangle) = c'(|\Psi\rangle + T|\Psi\rangle),\ |c'| = 1 \tag{5.13}$$

Comparing the above two equations, we get

$$c = c^* = c' \tag{5.14}$$

Thus, c is real and $|c| = 1$, which implies that $c = \pm 1$. Thus,

$$T^2|\Psi\rangle = \pm|\Psi\rangle \tag{5.15}$$

Thus we see that, even though two successive time reversals give back the initial state, T^2 is not the identity operator. There exist vectors $|\phi\rangle$ such that $T^2|\phi\rangle = -|\phi\rangle$. This is analogous to the fact that the operator corresponding to rotation by 2π along any axis, $R(2\pi)$, is not the identity operator.

5.2.4 Kramers degeneracy

Time-reversal symmetry sometimes increases the degree of degeneracy of energy eigenstates. Let $|\Psi\rangle$ be an eigenstate of the Hamiltonian H ($H|\Psi\rangle = E|\Psi\rangle$), and H be time reversal invariant, $TH = HT$. Then we see that

$$HT|\Psi\rangle = TH|\Psi\rangle = TE|\Psi\rangle = ET|\Psi\rangle \tag{5.16}$$

(Note that H is hermitian, so its eigenvalue E is real.) Thus, $|\Psi\rangle$ and $T|\Psi\rangle$ are two eigenstates with the same energy.

Now, we have two possibilities: either $|\Psi\rangle$ and $T|\Psi\rangle$ represent the same state, or they are degenerate states. If we assume the former, then $T|\Psi\rangle = a|\Psi\rangle$, $|a| = 1$. Then

$$T^2|\Psi\rangle = T(T|\Psi\rangle) = T(a|\Psi\rangle) = a^*T|\Psi\rangle = a^*a|\Psi\rangle = |a|^2|\Psi\rangle = |\Psi\rangle \tag{5.17}$$

But we have seen that there also exist states that satisfy $T^2|\Psi\rangle = -|\Psi\rangle$. Clearly, $|\Psi\rangle$ and $T|\Psi\rangle$ can not be linearly dependent in this case. Thus we get two degenerate states with energy E. This is Kramers theorem: any system having time reversal invariant Hamiltonian for which $T^2|\Psi\rangle = -|\Psi\rangle$ has only degenerate energy levels.

While the degenerate levels are often trivial such as spin states, the theorem shows that degeneracy will be seen for any time reversal invariant Hamiltonian, however complicated it be. The degeneracy can be lifted by applying an external magnetic field, since this adds terms to the Hamiltonian which are not time reversal invariant.

Chapter 6

Lie Groups and Lie Algebras

The set G of all symmetry transformations on a system satisfies the following properties:

1. Any two such transformations when operated one after another give another transformation within that set ($g \in G,\ h \in G \implies gh \in G$),

2. The set operations are *associative*, i.e. if we have more than two operations, it does not matter how we group them ($(gh)j = g(hj)$ for $g, h, j \in G$),

3. The set includes the identity transformation, which when operated before or after any other transformation causes no change ($\exists e \in G$ such that $ge = eg = g \forall g \in G$),

4. Every transformation g has an inverse transformation g^{-1}, which when composed with g gives the identity ($\forall g \in G, \exists g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$).

Such a mathematical structure is called a *group*, and the concept of groups is often used in Physics. A trivial example is the set of all translations T_a in one dimension, $T_a(x) = x + a$. The first two properties clearly hold. The identity is T_0 ($T_0(x) = x$), and the inverse of T_a is T_{-a} (a translation by the same distance in the opposite direction).

The group of symmetry transformations that we have considered is a special kind of group, called *Lie group*. The elements of such groups can be expressed as functions of a finite number of continuously varying parameters. The translations $T(a)$ that we mentioned before, for example, depend on a single real parameter a that can take any real value. There may be groups with more than one independent parameters, as we shall see. We have referred to Ref. [7, 8].

If we have two elements $g(a)$ and $g(b)$ of a Lie group where $a = (a_1, \ldots, a_n),\ b = (b_1, b_2, \ldots, b_n)$, then their product can be expressed as $g(a)g(b) = g(c)$, where $c = (c_1, \ldots, c_n)$. Then $c_i = \phi_i(a, b)$, where ϕ_i are differentiable functions of the variables $(a_1 \ldots, a_n, b_1, \ldots, b_n)$. Similarly, the inverse is expressed as $g(a)^{-1} = g(\bar{a})$, where $\bar{a}_i = \psi_i(a)$, ψ_i is differentiable.

We have seen that the continuous transformations have corresponding generators, which are much more convenient to work with than the transformation operators themselves. The generators form a vector space, so we can take their linear combination. We have also seen that the commutators of the generators play an important role in the study of transformations. This motivates us to define another mathematical structure called the *Lie algebra*.

A (real or complex) Lie algebra is a vector space L (on the field of real or complex numbers), with an operator called Lie bracket ($[,] : L \times L \to L$) defined on it. The Lie bracket satisfies the following conditions:

1. The Lie bracket is *bilinear*, i.e. for u, v and w in L and a and b in $\mathbb{R}$ or $\mathbb{C}$, we have

$$[au + bv, w] = a\,[u, w] + b\,[v, w] \tag{6.1}$$

2. It is antisymmetric, i.e. $[u, v] = -[v, u]$, and

3. It satisfies the Jacobi identity,

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0 \tag{6.2}$$

It is clear that the commutator of operators, $[A, B] = AB - BA$, satisfies all these conditions. Even though that is the most common situation encountered, the Lie bracket does not have to be of that form. In fact, the product need not be defined at all in L. One example of a different Lie bracket is for the algebra of smooth real valued functions of two variables q and p, where the Lie bracket is the Poisson bracket, $[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$.

Now we can appreciate that while the transformation operators themselves form a Lie group, their generators form a Lie algebra.

6.1 Examples of Lie groups and Lie algebras

The general treatment of Lie groups and algebras is somewhat complicated. The most commonly encountered Lie groups in physics are matrix Lie groups, and we will discuss some such groups.

Let us first consider a matrix Lie group with n real parameters, where the group elements are represented by $A(a)$ (where $a = (a_1 \dots a_n)$ are the parameters). Also, we take $A(0) = I$. Now, for small a, we can Express $A(a)$ as

$$A(a) = I + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial a_i}\right|_{a=0} a_i + \mathcal{O}(a^2) = I + \sum_{i=1}^{n} a_i M_i \tag{6.3}$$

where $M_i = \left.\frac{\partial g}{\partial a_i}\right|_{a=0}$.

Thus we get n matrices, $M_1, \dots, M_n$. The vector space spanned by these matrices forms a Lie algebra, and it is called the matrix Lie algebra of the Lie group. The commutator is defined to be the Lie bracket. Thus, if we take any group element close to the identity, it will be of the form

$$I + \text{an element of the Lie algebra} \tag{6.4}$$

. This will be helpful in finding the Lie algebras of matrix Lie groups.

Now, let us discuss some common examples. We will define the groups, find the number of independent real parameters in each case, and obtain the Lie algebras. We may also verify some of the mathematical properties in some cases.

6.1.1 General linear groups, $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$

$GL(n, \mathbb{R})$ is the set of all invertible $n \times n$ matrices with real entries,

$$GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) | A^{-1} \text{ exists}\} \tag{6.5}$$

We know that the product of two invertible matrices is invertible ($B^{-1}A^{-1}AB = I \implies (AB)^{-1} = B^{-1}A^{-1}$), matrix multiplication is associative and the identity matrix is invertible. The existence of inverse is given in the definition. Also, the elements are functions of the n^2 real entries. Therefore, $GL(n, \mathbb{R})$ is a Lie group.

We have seen that if we take a group element close to the identity and express it up to the first order, then we get the elements of the Lie algebra. Here, we express an element of $GL(n, \mathbb{R})$ as $I + \epsilon$, where ϵ is an $n \times n$ real matrices. We have only one condition here ($I + \epsilon$ is invertible $\implies \det(I + \epsilon) \neq 0$), which is always true for small ϵ. Therefore, the Lie algebra of $GL(n, \mathbb{R})$ is the space of all real matrices of dimension $n \times n$.

For $GL(n, \mathbb{C})$, the situation is similar. Here, each element is a complex number, which has a real and an imaginary part. Therefore, this group has $2n^2$ real parameters, and its Lie algebra is the set of all $n \times n$ complex matrices. The Lie bracket will be the commutator for all matrix groups.

6.1.2 Special linear groups, $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$

$SL(n, \mathbb{F})$ is the group of all real $n \times n$ matrices with entries from $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), with determinant 1;

$$SL(n, \mathbb{F}) = \{A \in GL(n, \mathbb{F}) | \det(A) = 1\} \tag{6.6}$$

For $SL(n, \mathbb{R})$, we have n^2 real entries with one constraint equation, i.e. $n^2 - 1$ independent parameters. Similarly for $SL(n, \mathbb{C})$, the number of independent parameters is $2(n^2 - 1)$.

It can be shown that for any triangulable matrix A, $\det(e^A) = e^{\operatorname{tr} A}$. Thus, the Lie algebra will be the space of all traceless matrices with entries from $\mathbb{R}$ (for $SL(n, \mathbb{R})$) or $\mathbb{C}$ (for $SL(n, \mathbb{C})$). It is easily seen that the commutator of traceless matrices is traceless.

6.1.3 Unitary group, $U(n)$ and special unitary group, $SU(n)$

$U(n)$ is the group of $n \times n$ matrices A with complex entries, satisfying $A^\dagger A = I$. We know that $A^\dagger A = I$ and $B^\dagger B = I$ gives $(AB)^\dagger AB = B^\dagger A^\dagger AB = I$. The other group properties are easy to verify.

Unitary matrices are important in quantum mechanics. As we have seen before, transformations on state vectors defined by unitary matrices preserve the inner product. Most transformation operators are represented by unitary matrices (except a few such as time reversal), and so almost any group of transformations should be a subgroup of $U(n)$.

In this case, we again have $2n^2$ real parameters, but they are related by the unitarity condition. We note that

$$(A^\dagger A)^\dagger = A^\dagger A \tag{6.7}$$

i.e. $A^\dagger A$ is Hermitian. Thus, if we denote the elements of $A^\dagger A$ by u_{ij}, then $u_{ji} = u_{ij}^*$. Thus, each of the diagonal entries is real, and the condition that every diagonal element is 1 gives us n equations. On each side of the diagonal, we have $n(n-1)/2$ entries. But the entries on one side of the diagonal are just complex conjugates of those on the other side; and the equations $a_{ij} = 0$ and $a_{ij}^* = 0$ ($i \neq j$) are the same. Thus, we get $n(n-1)/2$ complex equations, i.e. $n(n-1)$ real equations, from the off-diagonal terms of $A^\dagger A = I$. In total, we have $2n^2$ real parameters related by $n + n(n-1) = n^2$ equations. Therefore, the number of independent parameters is n^2.

We again express a group element as $I + \epsilon$. But now we have the condition $(I + \epsilon)^\dagger (I + \epsilon) = I$, which gives (up to the first order) $\epsilon + \epsilon^\dagger = O$, i.e. the Lie algebra of $U(n)$ is the set of $n \times n$ anti-Hermitian matrices. This is an n^2 dimensional vector space, as expected. Also, we see that if $A^\dagger = -A$ and $B^\dagger = -B$, then

$$[A, B]^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB = -[A, B] \tag{6.8}$$

i.e. the commutator of two anti-Hermitian matrices is also an anti-Hermitian matrix. Hence the property of Lie bracket is satisfied.

In Quantum Mechanics, we insert i as a coefficient to make the generators hermitian. We note that the set of Hermitian matrices is not a Lie algebra, since the commutator of two such matrices is anti-hermitian.

For a matrix A in $U(n)$, $\det(A^\dagger A) = |\det A|^2 = \det I = 1$, so $\det A = e^{i\delta},\ \delta \in \mathbb{R}$. In $SU(n)$, we impose the condition $\det A = 1$, i.e. $\delta = 0$. This extra condition makes the number of parameters in $SU(n)$ to be $(n^2 - 1)$. The Lie algebra of $SU(n)$ will be the the space of traceless anti-Hermitian matrices.

6.1.4 Orthogonal group, $O(n)$ and special orthogonal group, $SO(n)$

$O(n)$ is the set of $n \times n$ real matrices A that are orthogonal, $A^T A = I$. We can easily prove that this indeed forms a group.

An $n \times n$ matrix has n^2 real entries, but they are related by the orthogonality condition. The matrix $A^T A$ is symmetric, $(A^T A) = A^T (A^T)^T = A^T A$. Thus only the n diagonal entries and $n(n-1)/2$ entries above the diagonal are distinct. This makes the number of indepedent equations obtained from the orthogonality condition $n(n-1)/2 + n = n(n+1)/2$. Hence the number of independent parameters is $n^2 - n(n+1)/2 = n(n-1)/2$.

The Lie algebra is again obtained from taking $A = I + \epsilon$ as usual. We get

$$A^T A = (I + \epsilon)^T (I + \epsilon) = I + \epsilon^T + \epsilon = I \implies \epsilon^T = -\epsilon \tag{6.9}$$

i.e. the Lie algebra of $O(n)$ is the set of $n \times n$ real antisymmetric matrices.

For an orthogonal matrix A, $\det(A^T A) = \det(A^T)\det(A) = (\det(A))^2 = \det(I) = 1 \implies \det(A) = \pm 1$. The group manifold of $O(n)$ has two subsets (components) corresponding to $\det(A) = 1$ and -1. The two components are not connected, i.e. there can be no path joining any point in one component to any point in the other that lies entirely in the group manifold. The component for $\det(A) = 1$ contains the identity. Choosing $\det(A) = 1$ does not reduce the number of independent parameters; it just restricts the group manifold to the component containing the identity. Thus, $SO(n)$ also has $n(n-1)/2$ independent components. Its Lie algebra is also the space of antisymmetric matrices, since the condition of unit determinant in the Lie group translates to the condition of tracelessness in the Lie algebra, and antisymmetric matrices are anyway traceless.

$O(n)$ and $SO(n)$ are important as coordinate transformations that preserve the norm. a vector x in n-dimensional Euclidean space $\mathbb{R}^n$ is labelled by its coordinates, $x = (x_1, \ldots, x_n)$. Its norm is defined as $\|x\|^2 = \sum_i x_i^2 = x^T x$. If we define a transformation $\mathbb{R}^n$ as $x \mapsto x' = Ax$ (A is an $n \times n$ real matrix), then we see that the norm is preserved if and only if A is orthogonal:

$$x'^T x = (Ax)^T (Ax) = x^T A^T A x = x^T x \iff A^T A = I \tag{6.10}$$

Therefore, the group of transformations preserving the norm is $O(n)$. It includes both rotations and inversions ($x_i \mapsto -x_i$). If we take only rotations, then the group is $SO(n)$.

6.1.5 Pseudo-orthogonal group, $O(p, q)$ and special pseudo-orthogonal group, $SO(p, q)$

Let p and q be two positive integers, and we define the $(p + q) \times (p + q)$ matrix η as

$$\eta = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \tag{6.11}$$

where I_p and I_q are the $p \times p$ and $q \times q$ unit matrices respectively. We note that $\eta^T = \eta$ and $\eta^2 = I$.

We define $O(p, q)$ to be the set of all real $(p + q) \times (p + q)$ matrices satisfying $A^T \eta A = \eta$. To show that this is a group, we see that for $A, B \in O(p, q)$, we have $(AB)^T \eta (AB) = B^T A^T \eta AB = B^T \eta B = \eta$, i.e. $AB \in O(p, q)$. The identity matrix clearly belongs to $O(p, q)$. Also, $A^T \eta A = \eta \implies \eta A^T \eta A = \eta^2 = I$, so $A^{-1} = \eta A^T \eta$. Therefore, $A^{-1^T} \eta A^{-1} = (\eta A^T \eta)^T \eta (\eta A^T \eta) = \eta A \eta^3 A^T \eta = \eta A \eta A^T \eta = \eta A A^{-1} = \eta$, since $\eta^2 = I$. Hence $O(p, q)$ is a group.

Each element A has $(p+q)^2$ real entries. But we have the condition $A^T \eta A = \eta$, which gives $(p + q)(p + q + 1)/2$ real equations (Note that $(A^T \eta A)^T = A^T \eta A$). Thus we have $(p + q)(p + q - 1)/2$ independent parameters.

To find the Lie algebra, we once again take $A = I + \epsilon$. Then

$$(I + \epsilon)^T \eta (I + \epsilon) = \eta + (\epsilon^T \eta + \eta \epsilon) = \eta \implies \epsilon^T = -\eta \epsilon \eta \tag{6.12}$$

Thus the Lie algebra of $O(p, q)$ is $\mathfrak{o}(p, q) = \{A \in M_{(p+q)\times(p+q)}(\mathbb{R}) : A^T = -\eta A \eta\}$. The matrices will be of the form

$$\begin{bmatrix} A_{p\times p} & B_{p\times q} \\ C_{q\times p} & D_{q\times q} \end{bmatrix}, \; A^T = -A, \; D^T = -D, \; B^T = C \tag{6.13}$$

The group $O(1, 3)$ is of particular importance in Physics, called the Lorentz group. In Special Relativity, we define a co-ordinate 4-vector in space-time as (x^0, x^1, x^2, x^3), where $x^0 = ct$ (t is time and c is the speed of light), and (x^1, x^2, x^3) is the position vector. The length invariant in this case is defined to be $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x^T \eta x$, where

$$x = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \; \eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6.14}$$

Then if we consider a transformation $x \mapsto x' = Ax$, then the invariant length is conserved if $A^T \eta A = \eta$. Therefore, $O(1, 3)$ is the group of Lorentz transformations.

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