Null Structures in General Relativity

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Abstract

The Garfinkle-Vachaspati transform is studied as a way to generate new solutions of the Einstein equation from a known solution using a null Killing vector field, and its properties are studied. The transformation is applied on the D3 brane metric, to check if new singularities are introduced. A generalised form of the transformation with both null and spacelike Killing vectors is also discussed.

Contents

Chapter 1

The Garfinkle-Vachaspati transformation

Since the Einstein equations are nonlinear, it is not possible to generate new solutions of the metric from known solutions by taking linear combinations. However, Garfinkle and Vachaspati [1] showed if the known metric admits a null hypersurface orthogonal Killing vector, then there exists a transformation which can be used to generate new metrics satisfying the Einstein equations. The Garfinkle-Vachaspati (GV) transformation can be thought of as the addition of wave degrees of freedom, and is thus used to study oscillating sources. In this chapter, we will mainly follow the analysis given in [2] to study this transformation and its properties.

1.1 The action and equations of motion

Let us consider a D-dimensional spacetime with a metric and some fields. The action is taken to be of the following form:

$$S = \int \mathrm{d}^D x \sqrt{-g}\left(R(g_{\mu\nu}) - \frac{1}{2}\sum_a h_a(\phi_c)(\nabla\phi_a)^2 - \frac{1}{2}\sum_p f_p(\phi_c)F^2_{(p+1)}\right) = \int \mathrm{d}^D x\, \mathcal{L}, \tag{1.1}$$

where the corresponding Lagrangian density is

$$\mathcal{L} = \sqrt{-g}\left(R(g_{\mu\nu}) - \frac{1}{2}\sum_a h_a(\phi_c)\nabla_\mu\phi_a\nabla^\mu\phi_a - \frac{1}{2}\sum_p f_p(\phi_c)[F_{(p+1)}]_{\mu_1\ldots\mu_{p+1}}[F_{(p+1)}]^{\mu_1\cdots\mu_{p+1}}\right). \tag{1.2}$$

The metric is $g_{\mu\nu}$ with determinant g and corresponding Ricci scalar R. There are also a number of scalar fields ϕ_a (a labels the different fields and is not a vector index), and a number of $(p+1)$-form fields $F_{(p+1)}$ which are obtained as exterior derivatives of p-form fields $A_{(p)}$; $F_{(p+1)} = \mathrm{d}A_{(p)}$.

To obtain the equations of motion, we will work with the different terms in the action separately for convenience. Let us call $\mathcal{L}_1$ the term in the Lagrangian with the Ricci scalar, $\mathcal{L}_2$ the term with $(\nabla\phi_a)^2$, and $\mathcal{L}_3$ the term with F^2_{p+1}; the corresponding terms in the action are respectively S_1, S_2 and S_3. We note that the S_1 is simply the usual Einstein-Hilbert action [3], and we can write

$$\frac{\delta S_1}{\delta g^{\mu\nu}} = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \sqrt{-g}G_{\mu\nu}, \tag{1.3}$$

where $G_{\mu\nu}$ is the Einstein tensor. We note that the variation of the determinant of the metric may be obtained as

$$\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}(-g)^{-1/2}\,\delta\left(-\prod_i \lambda_i\right) \quad [\lambda_i \text{ are the eigenvalues of the matrix with entries } g_{\mu\nu}] \\
&= \frac{1}{2}\sqrt{-g}\sum_i \frac{\delta\lambda_i}{\lambda_i} \\
&= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\,\delta g_{\mu\nu} \quad [\lambda_i^{-1} \text{ are the eigenvalues of the matrix with entries } g^{\mu\nu}] \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu} \quad [\because \delta g^{\mu\nu} = -g^{\mu\lambda}g^{\nu\sigma}\,\delta g_{\lambda\sigma} \;\; [3]]
\end{aligned} \tag{1.4}$$

Thus we see that (taking variation with respect to the inverse metric $g^{\mu\nu}$),

$$\begin{aligned}
\delta\mathcal{L}_2 &= -\frac{1}{2}\,\delta\big(\sqrt{-g}\big)\sum_a h_a(\phi_c)(\nabla\phi_a)^2 - \frac{1}{2}\sqrt{-g}\sum_a h_a(\phi_c)\nabla_\mu\phi_a\nabla_\nu\phi_a\,\delta g^{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g}\sum_a h_a(\phi_c)\left(\nabla_\mu\phi_a\nabla_\nu\phi_a - \frac{1}{2}g_{\mu\nu}(\nabla\phi_a)^2\right)\delta g^{\mu\nu} \\
\implies \frac{\delta S_2}{\delta g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}\sum_a h_a(\phi_c)\left(\nabla_\mu\phi_a\nabla_\nu\phi_a - \frac{1}{2}g_{\mu\nu}(\nabla\phi_a)^2\right),
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
\delta\mathcal{L}_3 &= -\frac{1}{2}\sum_p f_p(\phi_c)\left(\delta\big(\sqrt{-g}\big)\,F^2_{(p+1)} + \sqrt{-g}\sum_{i=1}^{p+1}[F_{(p+1)}]^{\nu_1\ldots}{}_{\mu_i}{}^{\ldots\nu_{p+1}}[F_{(p+1)}]_{\nu_1\ldots\nu_i\ldots\nu_{p+1}}\,\delta g^{\mu_i\nu_i}\right) \\
&= -\frac{1}{2}\sqrt{-g}\sum_p f_p(\phi_c)\left((p+1)[F_{(p+1)}]_\mu{}^{\nu_2\ldots\nu_{p+1}}[F_{(p+1)}]_{\nu\nu_2\ldots\nu_{p+1}} - \frac{1}{2}g_{\mu\nu}F^2_{(p+1)}\right)\delta g^{\mu\nu} \\
\implies \frac{\delta S_3}{\delta g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}\sum_p f_p(\phi_c)\left((p+1)[F_{(p+1)}]_\mu{}^{\nu_2\ldots\nu_{p+1}}[F_{(p+1)}]_{\nu\nu_2\ldots\nu_{p+1}} - \frac{1}{2}g_{\mu\nu}F^2_{(p+1)}\right).
\end{aligned} \tag{1.6}$$

Combining, we can write the equation of motion for the metric (Einstein's equation) in mixed-index form as

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = \frac{1}{2}T^\mu{}_\nu, \tag{1.7}$$

where

$$\begin{aligned}
T^\mu{}_\nu = &\sum_a h_a(\phi_c)\left(\nabla^\mu\phi_a\nabla_\nu\phi_a - \frac{1}{2}\delta^\mu{}_\nu(\nabla\phi_a)^2\right) \\
&+ \sum_p f_p(\phi_c)\left((p+1)[F_{(p+1)}]^{\mu\nu_1\ldots\nu_p}[F_{(p+1)}]_{\nu\nu_1\ldots\nu_p} - \frac{1}{2}\delta^\mu{}_\nu F^2_{(p+1)}\right).
\end{aligned} \tag{1.8}$$

To find the other field equations, we now take variation with respect to ϕ_a to obtain

$$\begin{aligned}
\delta\mathcal{L} &= -\frac{1}{2}\left(\sqrt{-g}\sum_c \frac{\partial h_c}{\partial\phi_a}(\nabla\phi_c)^2\,\delta\phi_a + 2\sqrt{-g}h_a\partial^\mu\phi_a\,\delta(\partial_\mu\phi_a) + \sqrt{-g}\sum_p \frac{\partial f_p}{\partial\phi_a}F^2_{(p+1)}\,\delta\phi_a\right) \\
&= -\frac{1}{2}\left(\sqrt{-g}\sum_c \frac{\partial h_c}{\partial\phi_a}(\nabla\phi_c)^2 - 2\partial_\mu(\sqrt{-g}h_a\partial^\mu\phi_a) + \sqrt{-g}\sum_p \frac{\partial f_p}{\partial\phi_a}F^2_{(p+1)}\right)\delta\phi_a \\
\implies -2\frac{\delta S}{\delta\phi_a} &= \sqrt{-g}\sum_c \frac{\partial h_c}{\partial\phi_a}(\nabla\phi_c)^2 - 2\partial_\mu(\sqrt{-g}h_a\partial^\mu\phi_a) + \sqrt{-g}\sum_p \frac{\partial f_p}{\partial\phi_a}F^2_{(p+1)} = 0,
\end{aligned} \tag{1.9}$$

and similarly variation of a p-form field $A_{(p)}$ (where $F_{(p+1)} = \mathrm{d}A_{(p)}$) gives

$$\begin{aligned}
\delta\mathcal{L} &= -\sqrt{-g}f_p(\phi)\partial_{[\mu_1}[A_{(p)}]_{\mu_2\cdots\mu_{(p+1)}]}\,\delta\big(\partial^{[\mu_1}[A_{(p)}]^{\mu_2\cdots\mu_{(p+1)}]}\big) \\
&= \partial^{[\mu_1}(\sqrt{-g}f_p(\phi)[F_{(p+1)}]_{\mu_1\cdots\mu_{(p+1)}})\,\delta\big([A_{(p)}]^{\mu_2\cdots\mu_{(p+1)}]}\big) \\
\Longrightarrow \quad \frac{\delta S}{\delta[A_{(p)}]^{\mu_1\cdots\mu_p}} &= \partial^{\mu}(\sqrt{-g}f_p(\phi)[F_{(p+1)}]_{\mu\mu_1\cdots\mu_p}) = 0.
\end{aligned} \tag{1.10}$$

1.2 Properties of the null Killing vector field

Now, let us assume the existence of a solution $(g, \phi_a, A_{(p)})$ of the field equations which admits a vector field k^μ satisfying the following:

$$k^\mu k_\mu = 0, \qquad \text{(Null)} \tag{1.11}$$

$$\nabla_{[\mu}k_{\nu]} = k_{[\mu}\nabla_{\nu]}S, \qquad \text{(Hypersurface orthogonal)} \tag{1.12}$$

$$\nabla_{(\mu}k_{\nu)} = 0. \qquad \text{(Killing)} \tag{1.13}$$

While Eqs. (1.11) and (1.13) are the defining properties of null and Killing vectors respectively, it needs to be shown that Eq. (1.12) is a necessary and sufficient condition for a vector field to be hypersurface orthogonal. To prove necessity, let us first assume that k is orthogonal to the family of hypersurfaces given by $v =$ constant, i.e.

$$k_\mu = \alpha\partial_\mu v. \tag{1.14}$$

Then we see that

$$\begin{aligned}
\nabla_{[\mu}k_{\nu]} &= \frac{1}{2}(\alpha_{,\mu}v_{,\nu} + \alpha v_{;\nu\mu} - \alpha_{,\nu}v_{,\mu} - \alpha v_{;\mu\nu}) \\
&= \frac{1}{2}\Big(\frac{\alpha_{,\mu}}{\alpha}k_\nu - \frac{\alpha_{,\nu}}{\alpha}k_\mu\Big) \\
&= \frac{1}{2}(k_\mu\partial_\nu(-\ln\alpha) - k_\nu\partial_\mu(-\ln\alpha)) \\
&= k_{[\mu}\nabla_{\nu]}S, \quad \text{where} \quad \alpha = e^{-S}.
\end{aligned} \tag{1.15}$$

To prove sufficiency, we start with a vector field k^μ obeying Eq. (1.12). Then we see that

$$\begin{aligned}
k_{[\alpha;\beta}k_{\gamma]} &= k_\gamma\nabla_{[\beta}k_{\alpha]} + k_\alpha\nabla_{[\gamma}k_{\beta]} + k_\beta\nabla_{[\alpha}k_{\gamma]} \\
&= k_\gamma k_{[\beta}\nabla_{\alpha]}S + k_\alpha k_{[\gamma}\nabla_{\beta]}S + k_\beta k_{[\alpha}\nabla_{\gamma]}S \\
&= k_{[\gamma}k_{\beta]}\nabla_\alpha S + k_{[\alpha}k_{\gamma]}\nabla_\beta S + k_{[\beta}k_{\alpha]}\nabla_\gamma S \\
&= 0,
\end{aligned} \tag{1.16}$$

which implies that k^μ is hypersurface orthogonal, using Forbenius' theorem [4].

Equations (1.12) and (1.13) together also give

$$\nabla_\mu k_\nu = \frac{1}{2}(k_\mu\nabla_\nu S - k_\nu\nabla_\mu S), \tag{1.17}$$

which we will often find useful. Also, the Riemann tensor contracts with k as

$$\begin{aligned}
k_\alpha R^\alpha{}_{\beta\mu\nu} &= \nabla_\nu\nabla_\mu k_\beta - \nabla_\mu\nabla_\nu k_\beta \\
&= \frac{1}{2}\nabla_\nu(k_\mu\nabla_\beta S - k_\beta\nabla_\mu S) - \frac{1}{2}\nabla_\mu(k_\nu\nabla_\beta S - k_\beta\nabla_\nu S) \\
&= k_{[\mu}\nabla_{\nu]}\nabla_\beta S - \frac{1}{2}k_{[\mu}\nabla_{\nu]}S\nabla_\beta S.
\end{aligned} \tag{1.18}$$

Contracting Eq. (1.12) with k^μ, we get

$$\begin{aligned}
k^\mu \nabla_{[\mu} k_{\nu]} &= k^\mu k_{[\mu} \nabla_{\nu]} S \\
\implies -2k^m u \nabla_\nu k_\mu &= -k_\nu k^\mu \nabla_\mu S && \text{(Using Eqs. (1.11) and (1.13))} \\
\implies \nabla_\nu (k^\mu k_\mu) &= -k_\nu \mathcal{L}_k S \\
\implies \mathcal{L}_k S = k^\mu \partial_\mu S &= 0. && (k \text{ is null but non-zero}) \qquad (1.19)
\end{aligned}$$

Thus we see that S has a vanishing Lie derivative with respect to k. We assume that this holds for the matter fields ϕ_a and $F_{(p+1)}$ as well, i.e.

$$\mathcal{L}_k \phi_a = k^\mu \partial_\mu \phi_a = 0, \tag{1.20}$$

$$\mathcal{L}_k F_{(p+1)} = (\mathrm{d} i_k + i_k \mathrm{d}) F_{(p+1)} = \mathrm{d} i_k F_{(p+1)} = 0. \tag{1.21}$$

The last equation uses $\mathcal{L}_k = \mathrm{d} i_k + i_k \mathrm{d}$ (where i_k denotes the interior product with k and d the exterior derivative), and the Bianchi identity $\mathrm{d}F_{(p+1)} = \mathrm{d}^2 A_{(p)} = 0$. An additional assumption is that the $(p+1)$-form fields satisfy a transversality constraint with respect to k,

$$i_k F_{(p+1)} = k \wedge \theta_{(p-1)}, \tag{1.22}$$

where the k on the right hand side denotes the one-form field $k_\mu \, \mathrm{d}x^\mu$ corresponding to the vector field $k = k^\mu \partial_\mu$. We also note that $(i_k)^2 F_{(p+1)} = -k \wedge i_k \theta_{(p-1)} = 0$, which implies that $i_k \theta_{(p-1)} = 0$.

A plane wave metric admits a null covariantly constant (i.e. gradient and Killing) vector field [5]. The vector k in this case is null and Killing and hypersurface orthogonal (though not necessarily a gradient field), and thus any solution having such a vector field may be interpreted as gravitational waves. Since k is Killing, we can take $k = \partial_u$ (i.e. v parametrizes the integral curves of k), where u is a cyclic coordinate: $g_{\mu\nu,u} = 0$. The vanishing Lie derivatives of the matter fields imply that they are independent of u as well.

1.3 Properties of the Garfinkle-Vachaspati transformation

The Garfinkle-Vachaspati technique involves defining a new metric as

$$g'_{\mu\nu} = g_{\mu\nu} + e^S \Psi k_\mu k_\nu, \tag{1.23}$$

Where the indices of k have been lowered using the original metric. It will be seen that the configuration $(g', \phi_a, A_{(p)})$, with the new metric and the original matter fields, will be a solution of the equations of motion if Ψ satisfies certain conditions as will be shown later. Thus, the transformation given in Eq. (1.23), with appropriately chosen Ψ, can be used to obtain new solutions to the field equations from a known solution.

The metric transformation clearly does not affect the condition that k is null. The hypersurface orthogonality is also preserved, as Eq (1.12) can be written in terms of partial derivatives instead of covariant derivatives (the terms with Christoffel symbols in the left hand side cancel out), and thus does not depend on the metric. We also impose the condition that the scalar field Ψ, like all the other fields, has a vanishing Lie derivative with respect to k; $\mathcal{L}_k \Psi = k^\mu \partial_\mu \Psi = 0$. This gives $\mathcal{L}_k g'_{\mu\nu} = 0$, i.e. the vector field is Killing with respect to the new metric as well. In the following, we will sometimes write $e^S \Psi$ as κ for convenience, where $\mathcal{L}_k \kappa = 0$.

1.3.1 Matter field equations are unchanged

Let us first check that the transformed field configuration satisfies the matter field equations. First of all, we note that

$$g'^{\mu\nu} = g^{\mu\nu} - e^{S}\Psi k^{\mu}k^{\nu}, \tag{1.24}$$

which can be verified as follows:

$$\begin{aligned} g'_{\mu\nu}g'^{\nu\lambda} &= (g_{\mu\nu} + \kappa k_{\mu}k_{\nu})(g^{\nu\lambda} - \kappa k^{\nu}k^{\lambda}) \\ &= \delta_{\mu}{}^{\lambda} + \kappa k_{\mu}k^{\lambda} - \kappa k_{\mu}k^{\lambda} - \kappa^2(k^{\nu}k_{\nu})k_{\mu}k^{\lambda} \\ &= \delta_{\mu}{}^{\lambda}. \end{aligned} \tag{1.25}$$

To check that the determinant of the metric remains unchanged by the transformation, we write Eq. (1.23) as $g'_{\mu\nu} = g_{\mu\lambda}(\delta^{\lambda}{}_{\nu}+\kappa k^{\lambda}k_{\nu})$. Thus, if we write $\mathbf{g}$ to be the matrix whose elements are $g_{\mu\nu}$, then the transformation may be written in matrix form as $\mathbf{g}' = \mathbf{g}(\mathbf{I}+\mathbf{K})$, where $K^{\lambda}{}_{\nu} = \kappa k^{\lambda}k_{\nu}$. Therefore, $g' = g\det(\mathbf{I}+\mathbf{K})$. Let us now define $\det(\mathbf{I}+\mathbf{K}) = x$. Now we see that $[\mathbf{K}^2]^{\mu}{}_{\nu} = \kappa^2 k^{\mu}k_{\lambda}k^{\lambda}k_{\nu} = 0$, i.e. $\mathbf{K}$ is nilpotent. Therefore, $(\mathbf{I}+\mathbf{K})^2 = \mathbf{I}+2\mathbf{K}$, and if we suppose that $(\mathbf{I}+\mathbf{K})^m = \mathbf{I}+m\mathbf{K}$ for some m, then

$$(\mathbf{I}+\mathbf{K})^{m+1} = (\mathbf{I}+\mathbf{K})^m(\mathbf{I}+\mathbf{K}) = (\mathbf{I}+m\mathbf{K})(\mathbf{I}+\mathbf{K}) = \mathbf{I}+(m+1)\mathbf{K}. \tag{1.26}$$

So we can say that by the principle of mathematical induction, $(\mathbf{I}+\mathbf{K})^n = \mathbf{I}+n\mathbf{K}$ for any natural number n. Thus, $\det(\mathbf{I}+n\mathbf{K}) = \det(\mathbf{I}+\mathbf{K})^n = x^n$. But for matrices of size $D\times D$, $\det(\mathbf{I}+n\mathbf{K})$ should be a polynomial in n having degree at most D. The only values of x for which x^n is a polynomial function in n of finite degree are 0 and 1. But if $\det(\mathbf{I}+\mathbf{K}) = 0$, then there exists some non-zero vector $\mathbf{v}$ such that $(\mathbf{I}+\mathbf{K})\mathbf{v} = 0$, i.e. $\mathbf{K}\mathbf{v} = -\mathbf{v}$. Then $\mathbf{K}^2\mathbf{v} = \mathbf{v} = \mathbf{0}$ since $\mathbf{K}$ is nilpotent, which is a contradiction. Thus we can say that $\det(\mathbf{I}+\mathbf{K}) = 1$, and thus $g' = g$.

To show that the matter field equations are not affected by the transformation, it remains to be proven that raising all indices of a $(p+1)$-form field gives the same expression with respect to the new metric as before. To show this, we first verify that this is true for a 2-form, using the transversality condition stated before. Then we see that

$$\begin{aligned} [F'_{(2)}]^{\mu_1\mu_2} &= [F_{(2)}]^{\mu_1\mu_2} - \kappa k^{\mu_1}k^{\nu_1}[F_{(2)}]_{\nu_1}{}^{\mu_2} - \kappa k^{\mu_2}k^{\nu_2}[F_{(2)}]^{\mu_1}{}_{\nu_2} + \kappa^2 k^{\mu_1}k^{\nu_1}k^{\mu_2}k^{\nu_2}[F_{(2)}]_{\nu_1\nu_2} \\ &= [F_{(2)}]^{\mu_1\mu_2} - \kappa\theta_{(0)}k^{\mu_1}k^{\mu_2} + \kappa\theta_{(0)}k^{\mu_2}k^{\mu_1} + \kappa^2\theta_{(0)}k^{\mu_1}k^{\mu_2}k^{\nu_2}k_{\nu_2} \\ &\quad [k^{\nu_1}[F_{(2)}]_{\nu_1\nu_2} = \theta_{(0)}k_{\nu_2}, \quad k^{\nu_2}[F_{(2)}]_{\nu_1\nu_2} = -\theta_{(0)}k_{\nu_1} \quad \text{by transversality condition}] \\ &= [F_{(2)}]^{\mu_1\mu_2}. \end{aligned} \tag{1.27}$$

This gives us an idea of how the raised-indices components appear in general. The terms of second and higher order in κ vanish because of the transversality and null conditions, and the first order terms cancel out. For a general $(p+1)$-form, this can be seen as

$$\begin{aligned} [F'_{(p+1)}]^{\mu_1\cdots\mu_{p+1}} - [F_{(p+1)}]^{\mu_1\cdots\mu_{p+1}} &= -\kappa\sum_{i=1}^{p+1} k^{\mu_i}k^{\nu_i}[F_{(p+1)}]^{\mu_1\cdots}{}_{\nu_i}{}^{\cdots\mu_{p+1}} \\ &= -\kappa k^{\mu_1}k^{[\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} - \kappa\sum_{i=2}^{p+1}(-1)^{i-1}k^{\mu_i}k^{[\mu_1}[\theta_{(p-1)}]^{\mu_2\cdots\underline{\mu_i}\cdots\mu_{p+1}]} \\ &\quad [\text{Underlined index is omitted}] \\ &= -\kappa(p+1)k^{[\mu_1}k^{\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} \\ &= 0, \end{aligned} \tag{1.28}$$

where the penultimate step is obtained by noting that

$$
\begin{aligned}
&(p+1)k^{[\mu_1}k^{\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} \\
&\qquad = k^{\mu_1}k^{[\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} + \sum_{i=2}^{p+1}(-1)^{-p(i-1)}k^{\mu_i}k^{[\mu_{i+1}}[\theta_{(p-1)}]^{\mu_{i+2}\cdots\mu_{p+1}\mu_1\cdots\mu_{i-1}]} \\
&\qquad = k^{\mu_1}k^{[\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} + \sum_{i=2}^{p+1}(-1)^{-p(i-1)+(i-1)(p+1)}k^{\mu_i}k^{[\mu_1}[\theta_{(p-1)}]^{\mu_2\cdots\underline{\mu_i}\cdots\mu_{p+1}]} \\
&\qquad = k^{\mu_1}k^{[\mu_2}[\theta_{(p-1)}]^{\mu_3\cdots\mu_{p+1}]} + \sum_{i=2}^{p+1}(-1)^{i-1}k^{\mu_i}k^{[\mu_1}[\theta_{(p-1)}]^{\mu_2\cdots\underline{\mu_i}\cdots\mu_{p+1}]}. \qquad (1.29)
\end{aligned}
$$

Thus we see that the field equation (1.10) is satisfied by the new field configuration. Similarly, the fact that $\mathcal{L}_k\phi_a = 0$ implies that $g'^{\mu\nu}\partial_\nu\phi_a = g^{\mu\nu}\partial_\nu\phi_a$, and thus Eq. (1.9) is also satisfied by the transformed metric with the original matter fields.

1.3.2 Effect on Einstein equations; transformation of Ricci tensor

We now need to find the suitable condition on Ψ to ensure that Eq. (1.7) is also satisfied. Clearly, the previously obtained identities show that the expression (1.8) of the stress-energy tensor is not affected by the transformation, and hence the right hand side of Eq. (1.7) is unchanged. Let us now find how the Ricci tensor is transformed by the metric transformation (1.23). Let ∇ and ∇' represent the covariant derivatives with respect to the metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ respectively, and $\Gamma^\mu{}_{\alpha\beta}$ and $\Gamma'^\mu{}_{\alpha\beta}$ be the corresponding Christoffel symbols, where $\Omega^\mu{}_{\alpha\beta} = \Gamma'^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\alpha\beta}$. Then we see that (following the derivation in Ref. [6])

$$
\begin{aligned}
&\nabla'_\alpha g'_{\mu\nu} = 0 \\
\implies &\nabla_\alpha(g_{\mu\nu} + \kappa k_\mu k_\nu) - \Omega^\beta{}_{\alpha\mu}g'_{\beta\nu} - \Omega^\beta{}_{\alpha\nu}g'_{\mu\beta} = 0 \\
\implies &\Omega^\beta{}_{\alpha\mu}g'_{\beta\nu} + \Omega^\beta{}_{\alpha\nu}g'_{\mu\beta} = \nabla_\alpha(\kappa k_\mu k_\nu). \quad [\nabla_\alpha g_{\mu\nu} = 0] \\
\text{Similarly,}\quad &\Omega^\beta{}_{\nu\alpha}g'_{\beta\mu} + \Omega^\beta{}_{\nu\mu}g'_{\alpha\beta} = \nabla_\nu(\kappa k_\alpha k_\mu), \\
&\Omega^\beta{}_{\mu\nu}g'_{\beta\alpha} + \Omega^\beta{}_{\mu\alpha}g'_{\nu\beta} = \nabla_\mu(\kappa k_\nu k_\alpha). \\
\therefore\ &\Omega^\beta{}_{\mu\nu}g'_{\alpha\beta} = \frac{1}{2}(\nabla_\mu(\kappa k_\nu k_\alpha) + \nabla_\nu(\kappa k_\alpha k_\mu) - \nabla_\alpha(\kappa k_\mu k_\nu)) \\
\implies &\Omega^\alpha{}_{\mu\nu} = \frac{1}{2}(\nabla_\mu(\kappa k_\nu k^\alpha) + \nabla_\nu(\kappa k^\alpha k_\mu) - \nabla^\alpha(\kappa k_\mu k_\nu)). \qquad (1.30)
\end{aligned}
$$

Now we see that from the null, Killing and hypersurface orthogonality conditions of the vector field and the fact that the Lie derivative of κ with respect to k^μ is zero,

$$
\begin{aligned}
\Omega^\mu{}_{\mu\nu} &= \frac{1}{2}(\nabla_\mu(\kappa k_\nu k^\mu) + \nabla_\nu(\kappa k^\mu k_\mu) - \nabla^\mu(\kappa k_\mu k_\nu)) = 0, \qquad (1.31) \\
k^\mu\Omega^\alpha{}_{\mu\nu} &= k^\mu\frac{1}{2}(\nabla_\mu(\kappa k_\nu k^\alpha) + \nabla_\nu(\kappa k^\alpha k_\mu) - \nabla^\alpha(\kappa k_\mu k_\nu)) = 0, \\
k_\alpha\Omega^\alpha{}_{\mu\nu} &= k_\alpha\frac{1}{2}(\nabla_\mu(\kappa k_\nu k^\alpha) + \nabla_\nu(\kappa k^\alpha k_\mu) - \nabla^\alpha(\kappa k_\mu k_\nu)) = 0 \\
\implies k^\mu\nabla_\alpha\Omega^\alpha{}_{\mu\nu} &= -\Omega^\alpha{}_{\mu\nu}\nabla_\alpha k^\mu = 0, \qquad (1.32) \\
\Omega^\alpha{}_{\mu\beta}\Omega^\beta{}_{\alpha\nu} &= \frac{1}{4}(\nabla_\mu(\kappa k_\beta k^\alpha) + \nabla_\beta(\kappa k^\alpha k_\mu) - \nabla^\alpha(\kappa k_\mu k_\beta))\big(\nabla_\alpha(\kappa k_\nu k^\beta) + \nabla_\nu(\kappa k^\beta k_\alpha) - \nabla^\beta(\kappa k_\alpha k_\nu)\big) \\
&= 0. \qquad (1.33)
\end{aligned}
$$

Thus it can be seen that

$$
\begin{aligned}
R'_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\
&= \Gamma'^{\alpha}{}_{\mu\nu,\alpha} - \Gamma'^{\alpha}{}_{\mu\alpha,\nu} + \Gamma'^{\alpha}{}_{\beta\alpha}\Gamma'^{\beta}{}_{\mu\nu} - \Gamma'^{\alpha}{}_{\beta\nu}\Gamma'^{\beta}{}_{\mu\alpha} \\
&= R_{\mu\nu} + \Omega^{\alpha}{}_{\mu\nu,\alpha} - \Omega^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} + \Omega^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} + \Omega^{\alpha}{}_{\beta\alpha}\Omega^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} - \Omega^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} \\
&\quad - \Omega^{\alpha}{}_{\beta\nu}\Omega^{\beta}{}_{\mu\alpha} - \Gamma^{\beta}{}_{\nu\alpha}\Omega^{\alpha}{}_{\mu\beta} + \Gamma^{\beta}{}_{\alpha\nu}\Omega^{\alpha}{}_{\mu\beta} \\
&= R_{\mu\nu} + \nabla_{\alpha}\Omega^{\alpha}{}_{\mu\nu} \\
\implies R'^{\mu}{}_{\nu} &= R^{\mu}{}_{\nu} - e^{S}\Psi k^{\mu}k^{\lambda}R_{\lambda\nu} + \nabla_{\alpha}\big(g^{\mu\lambda}\Omega^{\alpha}{}_{\lambda\nu}\big). \qquad (1.34)
\end{aligned}
$$

Now,

$$
\begin{aligned}
k^{\lambda}R_{\lambda\nu} &= k^{\lambda}R^{\alpha}{}_{\lambda\alpha\nu} = -k_{\lambda}R^{\lambda\alpha}{}_{\alpha\nu} \\
&= -\frac{1}{2}k_{\alpha}\nabla_{\nu}\nabla^{\alpha}S + \frac{1}{2}k_{\nu}\nabla^{2}S + \frac{1}{4}k_{\alpha}\nabla_{\nu}S\nabla^{\alpha}S - \frac{1}{4}k_{\nu}\nabla_{\alpha}S\nabla^{\alpha}S \qquad \text{[Using Eq. (1.18)]} \\
&= \frac{1}{2}\nabla_{\nu}k_{\alpha}\nabla^{\alpha}S + \frac{1}{2}k_{\nu}\nabla^{2}S - \frac{1}{4}k_{\nu}\nabla_{\alpha}S\nabla^{\alpha}S \qquad [\because \mathcal{L}_{k}S = 0] \\
&= \frac{1}{2}k_{\nu}\nabla^{2}S, \qquad (1.35)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{\alpha}(g^{\mu\lambda}\Omega^{\alpha}{}_{\lambda\nu}) &= \frac{1}{2}\nabla_{\alpha}(\nabla^{\mu}(\kappa k^{\alpha}k_{\nu}) + \nabla_{\nu}(\kappa k^{\alpha}k^{\mu}) - \nabla^{\alpha}(\kappa k^{\mu}k_{\nu})) \\
&= \frac{1}{2}\nabla_{\alpha}\bigg(k^{\alpha}k_{\nu}\nabla^{\mu}\kappa + \frac{1}{2}\kappa k_{\nu}(k^{\mu}\nabla^{\alpha}S - k^{\alpha}\nabla^{\mu}S) + \frac{1}{2}\kappa k^{\alpha}(k^{\mu}\nabla_{\nu}S - k_{\nu}\nabla^{\mu}S)\bigg) \\
&\quad + \frac{1}{2}\nabla_{\alpha}\bigg(k^{\alpha}k^{\mu}\nabla_{\nu}\kappa + \frac{1}{2}\kappa k^{\mu}(k_{\nu}\nabla^{\alpha}S - k^{\alpha}\nabla_{\nu}S) + \frac{1}{2}\kappa k^{\alpha}(k_{\nu}\nabla^{\mu}S - k^{\mu}\nabla_{\nu}S)\bigg) \\
&\quad - \frac{1}{2}\nabla_{\alpha}\bigg(k^{\mu}k_{\nu}\nabla^{\alpha}\kappa + \frac{1}{2}\kappa k_{\nu}(k^{\alpha}\nabla^{\mu}S - k^{\mu}\nabla^{\alpha}S) + \frac{1}{2}\kappa k^{\mu}(k^{\alpha}\nabla_{\nu}S - k_{\nu}\nabla^{\alpha}S)\bigg) \\
&= \frac{1}{2}\nabla_{\alpha}(k^{\alpha}k_{\nu}\nabla^{\mu}\kappa + k^{\alpha}k^{\mu}\nabla_{\nu}\kappa - k^{\mu}k_{\nu}\nabla^{\alpha}\kappa + \kappa(2k^{\mu}k_{\nu}\nabla^{\alpha}S - k^{\alpha}k_{\nu}\nabla^{\mu}S - k^{\alpha}k^{\mu}\nabla_{\nu}S)) \\
&= \frac{1}{2}(k^{\alpha}k_{\nu}\nabla_{\alpha}\nabla^{\mu}\kappa + k^{\alpha}k^{\mu}\nabla_{\alpha}\nabla_{\nu}\kappa - k^{\mu}k_{\nu}\nabla^{2}\kappa) - \frac{1}{4}k^{\mu}(k_{\alpha}\nabla_{\nu}S - k_{\nu}\nabla_{\alpha}S)\nabla^{\alpha}\kappa \\
&\quad - \frac{1}{4}k_{\nu}(k_{\alpha}\nabla^{\mu}S - k^{\mu}\nabla_{\alpha}S)\nabla^{\alpha}\kappa + k^{\mu}k_{\nu}\nabla_{\alpha}\kappa\nabla^{\alpha}S + \frac{1}{2}\kappa k^{\mu}(k_{\alpha}\nabla_{\nu}S - k_{\nu}\nabla_{\alpha}S)\nabla^{\alpha}S \\
&\quad + \frac{1}{2}\kappa k_{\nu}(k_{\alpha}\nabla^{\mu}S - k^{\mu}\nabla_{\alpha}S)\nabla^{\alpha}S + \kappa k^{\mu}k_{\nu}\nabla^{2}S - \frac{1}{2}k^{\alpha}k_{\nu}\nabla_{\alpha}\kappa\nabla^{\mu}S - \frac{1}{2}k^{\alpha}k_{\nu}\kappa\nabla_{\alpha}\nabla^{\mu}S \\
&\quad - \frac{1}{2}k^{\alpha}k^{\mu}\nabla_{\alpha}\kappa\nabla_{\nu}S - \frac{1}{2}\kappa k^{\alpha}k^{\mu}\nabla_{\alpha}\nabla_{\nu}S \\
&= -\frac{1}{2}e^{S}\Psi k^{\mu}k_{\nu}\nabla_{\alpha}S\nabla^{\alpha}S + k^{\mu}k_{\nu}\nabla_{\alpha}(e^{S}\Psi)\nabla^{\alpha}S + e^{S}\Psi k^{\mu}k_{\nu}\nabla^{2}S - \frac{1}{2}k^{\mu}k_{\nu}\nabla^{2}(e^{S}\Psi) \\
&= \frac{1}{2}k^{\mu}k_{\nu}(-e^{S}\Psi\nabla_{\alpha}S\nabla^{\alpha}S + 2e^{S}\Psi\nabla_{\alpha}S\nabla^{\alpha}S + 2e^{S}\nabla_{\alpha}\Psi\nabla^{\alpha}S + 2e^{S}\Psi\nabla^{2}S \\
&\quad - \nabla_{\alpha}(e^{S}\Psi\nabla^{\alpha}S + e^{S}\nabla^{\alpha}\Psi)) \\
&= \frac{1}{2}k^{\mu}k_{\nu}(e^{S}\Psi\nabla_{\alpha}S\nabla^{\alpha}S + 2e^{S}\nabla_{\alpha}\Psi\nabla^{\alpha}S + 2e^{S}\Psi\nabla^{2}S - e^{S}\Psi\nabla_{\alpha}S\nabla^{\alpha}S - e^{S}\nabla_{\alpha}\Psi\nabla^{\alpha}S \\
&\quad - e^{S}\Psi\nabla^{2}S - e^{S}\nabla_{\alpha}S\nabla^{\alpha}\Psi - e^{S}\nabla^{2}\Psi) \\
&= \frac{1}{2}k^{\mu}k_{\nu}(e^{S}\Psi\nabla^{2}S - e^{S}\nabla^{2}\Psi). \qquad (1.36)
\end{aligned}
$$

Therefore, the change in the Ricci tensor is obtained as

$$R'^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2} e^{S} k^{\mu} k^{\nu} \nabla^2 \Psi. \tag{1.37}$$

Thus, Eq. (1.24) can be used to generate solutions for the given Lagrangian, provided that Ψ satisfies $k^{\mu}\partial_{\mu}\Psi = 0$, and the Laplace equation $\nabla^2\Psi = 0$ in the background of the original metric $g_{\mu\nu}$.

1.4 Curvature invariants are unchanged under the Garfinkle-Vachaspati transformation

It was seen before that the Ricci tensor can be made to remain unchanged under a Garfinkle-Vachaspati transformation if the associated function Ψ satisfies the Laplace equation. The Ricci scalar, however, can be seen to be invariant under the transformation even without imposing that condition. In fact, it will be seen in this section that *any* scalar constructed from the metric, the Riemann tensor, the matter fields, and their covariant derivatives will not change upon the application of a GV transformation. This means that the existence of plane wave terms in the metric can not be detected from any curvature scalar, and any curvature singularity introduced in the transformed metric can be found only from the Riemann tensor components in a parallelly propagated frame and not from any scalar [3, 7].

Let us now formally describe the above statement. We have a spacetime with a metric g and some scalar or $(p+1)$-form matter fields, and there exists a vector k which is null, hypersurface orthogonal and Killing with respect to the original metric. The Lie derivative of any matter field is zero with respect to k. We transform the metric as

$$g'_{\mu\nu} = g_{\mu\nu} + \kappa k_{\mu} k_{\nu}, \tag{1.38}$$

where the scalar κ satisfies $\mathcal{L}_k \kappa = 0$.

We have already seen some of the identities satisfied by the vector field k in the previous section. We will need some more identities to prove the invariance of the curvature scalars. First of all, the commutator of covariant derivatives of k is given by the Riemann tensor as

$$\begin{aligned}
& & \nabla_{\alpha}\nabla_{\beta}k_{\mu} - \nabla_{\beta}\nabla_{\alpha}k_{\mu} &= -R_{\nu\mu\alpha\beta}k^{\nu}. \\
&\text{Permuting indices,} & \nabla_{\mu}\nabla_{\alpha}k_{\beta} - \nabla_{\alpha}\nabla_{\mu}k_{\beta} &= -R_{\nu\beta\mu\alpha}k^{\nu}. \\
&\text{Adding, using Eq. (1.13) and the Bianchi identity,} & 2\nabla_{\alpha}\nabla_{\beta}k_{\mu} + \nabla_{\beta}\nabla_{\mu}k_{\alpha} - \nabla_{\mu}\nabla_{\beta}k_{\alpha} &= R_{\nu\alpha\beta\mu}k^{\nu} \\
& & \implies 2\nabla_{\alpha}\nabla_{\beta}k_{\mu} + R_{\alpha\nu\beta\mu}k^{\nu} &= R_{\nu\alpha\beta\mu}k^{\nu} \\
& & \implies \nabla_{\alpha}\nabla_{\beta}k_{\mu} &= R_{\nu\alpha\beta\mu}k^{\nu}.
\end{aligned} \tag{1.39}$$

The Lie derivative of a $(0,q)$ tensor field T with respect to a vector field v is defined as [2]

$$\mathcal{L}_v T_{\nu_1\ldots\nu_q} = v^{\mu}\nabla_{\mu}T_{\nu_1\ldots\nu_q} + T_{\mu\nu_2\ldots\nu_q}\nabla_{\nu_1}v^{\mu} + \cdots + T_{\nu_1\ldots\nu_{q-1}\mu}\nabla_{\nu_q}v^{\mu}. \tag{1.40}$$

We can see that the Lie derivative operator with respect to the vector field k commutes with the covariant derivative as follows:

$$
\begin{aligned}
\nabla_\lambda \mathcal{L}_k T_{\nu_1\ldots\nu_q} &= \nabla_\lambda (k^\mu \nabla_\mu T_{\nu_1\ldots\nu_q}) + \nabla_\lambda (T_{\mu\nu_2\ldots\nu_q} \nabla_{\nu_1} k^\mu) + \cdots + \nabla_\lambda (T_{\nu_1\ldots\nu_{q-1}\mu} \nabla_{\nu_q} k^\mu) \\
&= \nabla_\lambda k^\mu \nabla_\mu T_{\nu_1\ldots\nu_q} + k^\mu \nabla_\lambda \nabla_\mu T_{\nu_1\ldots\nu_q} + \nabla_{\nu_1} k^\mu \nabla_\lambda T_{\mu\nu_2\ldots\nu_q} + \cdots + \nabla_{\nu_q} k^\mu \nabla_\lambda T_{\nu_1\ldots\nu_{q-1}\mu} \\
&\quad + T_{\mu\nu_2\ldots\nu_q} \nabla_\lambda \nabla_{\nu_1} k^\mu + \cdots + T_{\nu_1\ldots\nu_{q-1}\mu} \nabla_\lambda \nabla_{\nu_q} k^\mu \\
&= \big(k^\mu \nabla_\lambda \nabla_\mu T_{\nu_1\ldots\nu_q} + R_{\rho\lambda\nu_1}{}^\mu k^\rho T_{\mu\nu_2\ldots\nu_q} + \cdots + R_{\rho\lambda\nu_q}{}^\mu k^\rho T_{\nu_1\ldots\nu_{q-1}\mu} \big) \\
&\quad + \nabla_\lambda k^\mu \nabla_\mu T_{\nu_1\ldots\nu_q} + \nabla_{\nu_1} k^\mu \nabla_\lambda T_{\mu\nu_2\ldots\nu_q} + \cdots + \nabla_{\nu_q} k^\mu \nabla_\lambda T_{\nu_1\ldots\nu_{q-1}\mu} \\
&= \big(k^\mu \nabla_\lambda \nabla_\mu T_{\nu_1\ldots\nu_q} + k^\mu R_{\nu_1}{}^\rho{}_{\mu\lambda} T_{\rho\nu_2\ldots\nu_q} + \cdots + k^\mu R_{\nu_q}{}^\rho{}_{\mu\lambda} T_{\nu_1\ldots\nu_{q-1}\rho} \big) \quad [\text{Swapping } \mu \leftrightarrow \rho] \\
&\quad + \nabla_\lambda k^\mu \nabla_\mu T_{\nu_1\ldots\nu_q} + \nabla_{\nu_1} k^\mu \nabla_\lambda T_{\mu\nu_2\ldots\nu_q} + \cdots + \nabla_{\nu_q} k^\mu \nabla_\lambda T_{\nu_1\ldots\nu_{q-1}\mu} \\
&= k^\mu \nabla_\mu \nabla_\lambda T_{\nu_1\ldots\nu_q} + \nabla_\lambda k^\mu \nabla_\mu T_{\nu_1\ldots\nu_q} + \nabla_{\nu_1} k^\mu \nabla_\lambda T_{\mu\nu_2\ldots\nu_q} + \cdots + \nabla_{\nu_q} k^\mu \nabla_\lambda T_{\nu_1\ldots\nu_{q-1}\mu} \\
&= \mathcal{L}_k \nabla_\lambda T_{\nu_1\ldots\nu_q}.
\end{aligned}
\tag{1.41}
$$

While we showed this for a $(0, q)$ tensor (all lower indices), it can be generalised to a tensor with some upper indices, since $\nabla_\mu g^{\alpha\beta} = \mathcal{L}_k g^{\alpha\beta} = 0$. It ca be also be proven that the Lie derivative of the Riemann tensor with respect to k is zero. Since $\mathcal{L}_k$ and the covariant derivative commute, we can say that any tensor obtained by taking covariant derivatives of S, κ or the Riemann tensor also has a vanishing Lie derivative with respect to k.

1.4.1 Contraction of k with a tensor

To show the invariance of the curvature scalar under k, we first show that the contraction of any tensor constructed from the Riemann tensor, any scalar whose lie derivative with respect to k vanishes (such as S and κ) and their covariant derivatives can be written as a linear combination of terms, each of which can be written as the product of k (or its corresponding one-form) and a tensor of lower rank. Thus, if T is a $(0, p + q + 1)$ tensor constructed in the above way, then we have

$$
k^\mu T_{\nu_1\ldots\nu_p\mu\lambda_1\ldots\lambda_q} = \sum_{i=1}^{p} k_{\nu_i} \theta^{(i)}_{\nu_1\ldots\underline{\nu_i}\ldots\nu_p\lambda_1\ldots\lambda_q} + \sum_{i=1}^{q} k_{\lambda_i} \theta^{(p+i)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q}, \tag{1.42}
$$

where the underlined index is omitted. $\theta(1), \ldots, \theta^{(p+q)}$ are $p + q$ tensors of rank $p + q - 1$.

To prove this, let us start from the simplest cases. The only tensors of rank 1 that can be constructed in the stated way are of the form $\nabla_\mu B$, where the scalar B can be S or κ. Its contraction with k^μ is simply the lie derivative of B, which is zero. The contraction of k with the Riemann tensor is known from Eq. (1.18). One way to construct tensors of other rank is by taking covariant derivatives of the above tensors. We can show that the contraction property holds for such cases using the principle of mathematical induction on the index of the tensor. Let us suppose that Eq. 1.42 holds for all tensors of rank q or less. A tensor of rank $q + 1$ may be constructed by taking the covariant derivative of T. Now we see that

$$\begin{aligned}
k^{\mu}\nabla_{\nu_1}T_{\mu\nu_2\ldots\nu_q} &= \nabla_{\nu_1}(k^{\mu}T_{\mu\nu_2\ldots\nu_q}) - T_{\mu\nu_2\ldots\nu_q}\nabla_{\nu_1}k^{\mu} \\
&= \sum_{i=2}^{q}\nabla_{\nu_1}(k_{\nu_i}\theta^{(i)}_{\nu_2\ldots\underline{\nu_i}\ldots\nu_q}) - \frac{1}{2}T_{\mu\nu_2\ldots\nu_q}(k_{\nu_1}\nabla^{\mu}S - k^{\mu}\nabla_{\nu_1}S) \\
&= \sum_{i=2}^{q}k_{\nu_i}\nabla_{\nu_1}\theta^{(i)}_{\nu_2\ldots\underline{\nu_i}\ldots\nu_q} + \frac{1}{2}\sum_{i=2}^{q}(k_{\nu_1}\nabla_{\nu_i}S - k_{\nu_i}\nabla_{\nu_1}S)\theta^{(i)}_{\nu_2\ldots\underline{\nu_i}\ldots\nu_q} - \frac{1}{2}k_{\nu_1}T_{\mu\nu_2\ldots\nu_q}\nabla^{\mu}S \\
&\quad + \frac{1}{2}\nabla_{\nu_1}S\sum_{i=2}^{q}k_{\nu_i}\theta^{(i)}_{\nu_2\ldots\underline{\nu_i}\ldots\nu_q} \\
&= \frac{1}{2}k_{\nu_1}\left(\sum_{i=2}^{q}\nabla_{\nu_i}S\theta^{(i)}_{\nu_2\ldots\underline{\nu_i}\ldots\nu_q} - T_{\mu\nu_2\ldots\nu_q}\nabla^{\mu}S\right) + \sum_{i=2}^{q}k_{\nu_i}\left(\nabla_{\nu_1}\theta^{(i)}_{\nu_2\ldots\underline{\nu_i}\ldots\nu_q}\right) \\
&= \sum_{i=1}^{q}k_{\nu_i}\bar{\theta}^{(i)}_{\nu_1\ldots\underline{\nu_i}\ldots\nu_q}, \qquad (1.43)
\end{aligned}$$

which is in the desired form. The other situation that may arise is $k^{\mu}\nabla_{\mu}T_{\nu_1\ldots\nu_q}$. In this case, we use the fact that $\mathcal{L}_kT = 0$ to obtain

$$\begin{aligned}
k^{\mu}\nabla_{\mu}T_{\nu_1\ldots\nu_q} &= -T_{\mu\nu_2\ldots\nu_q}\nabla_{\nu_1}k^{\mu} - \cdots - T_{\nu_1\ldots\nu_{q-1}\mu}\nabla_{\nu_q}k^{\mu} \\
&= -\frac{1}{2}T_{\mu\nu_2\ldots\nu_q}k_{\nu_1}\nabla^{\mu}S - \cdots - \frac{1}{2}T_{\nu_1\ldots\nu_{q-1}\mu}k_{\nu_q}\nabla^{\mu}S + \frac{1}{2}T_{\mu\nu_2\ldots\nu_q}k^{\mu}\nabla_{\nu_1}S \\
&\quad + \cdots + \frac{1}{2}T_{\nu_1\ldots\nu_{q-1}\mu}k^{\mu}\nabla_{\nu_q}S \\
&= \frac{1}{2}\sum_{i=1}^{q}k_{\nu_i}\left(\sum_{j\neq i}\theta^{(i,j)}_{\nu_1\ldots\underline{\nu_i}\ldots\underline{\nu_j}\ldots\nu_q}\nabla_{\nu_j}S - T_{\nu_1\ldots\nu_{i-1}\mu\nu_{i+1}\ldots\nu_q}\nabla^{\mu}S\right) \\
&= \sum_{i=1}^{q}k_{\nu_i}\bar{\theta}^{(i)}_{\nu_1\ldots\underline{\nu_i}\ldots\nu_q}, \qquad (1.44)
\end{aligned}$$

$$\text{where} \quad k^{\mu}T_{\nu_1\ldots\nu_{j-1}\mu\nu_{j+1}\ldots\nu_q} = \sum_{\substack{i=1\\ i\neq j}}^{q}k_{\nu_i}\theta^{(i,j)}_{\nu_1\ldots\underline{\nu_i}\ldots\underline{\nu_j}\ldots\nu_q}. \qquad (1.45)$$

Thus we see that any tensor constructed from the Riemann tensor, scalars with vanishing Lie derivative with respect to k, and their covariant derivatives (called a 'primary tensor') satisfies the contraction property. To prove that contractions of two or more such primary tensors also satisfies this, we need to check that the 'secondary tensors' (i.e. the θ tensors obtained after the contraction) satisfy this property as well. It can be seen that the secondary tensors obtained from contracting ∇B or the Riemann tensor with k satisfy the contraction property, and they also have zero Lie derivative with respect to k. Let us assume that the θ tensors coming from the contraction of T with k satisfy these two properties. Then, from the above equations, we see that the secondary tensors $\bar{\theta}$ appearing from the contraction of ∇T with k consist of terms such as the covariant derivative of θ, product of θ with ∇S, and contraction of T with ∇S. It can be checked that each of these terms satisfy the contraction and vanishing Lie derivative properties as well, since $\mathcal{L}_kS = \mathcal{L}_kT = \mathcal{L}_k\theta = 0$ and $[\mathcal{L}_k, \nabla] = 0$.

Now we are in a position to check that the contraction property holds for contractions of primary tensors. Let $T^{(1)}$ and $T^{(2)}$ be two primary tensors. Then we have

$$\begin{aligned}
k^\mu T^{(1)}{}_{\mu\nu_1\ldots\nu_p\lambda_1\ldots\lambda_q} T^{(2)}{}^{\lambda^1\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} &= \sum_{i=1}^{p} k_{\nu_i}\theta^{(i)}_{\nu_1\ldots\underline{\nu_i}\ldots\nu_p\lambda_1\ldots\lambda_q} T^{(2)}{}^{\lambda_1\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \\
&\quad + \sum_{i=1}^{q} k_{\lambda_i}\theta^{(p+i.1)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q} T^{(2)}{}^{\lambda_1\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \quad [\text{if } q>0] \\
&= (\ldots) + \sum_{i=1}^{q} \theta^{(p+i.1)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q} \sum_{j=1}^{r} k_{\omega_j}\theta^{(i,q+j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q}{}_{\omega_1\ldots\underline{\omega_j}\ldots\omega_r} \\
&\quad + \sum_{i=1}^{q} \theta^{(p+i.1)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\lambda_q} \sum_{\substack{j=1\\ j\neq i}}^{q} k^{\lambda_j}\theta^{(i,j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \quad [\text{if } q>1] \\
&= (\ldots) + \sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q} \theta^{(i,j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \sum_{l=1}^{p} k_{\nu_l}\theta^{(p+i,1,l)}_{\nu_1\ldots\underline{\nu_l}\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q} \\
&\quad + \sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q} \theta^{(i,j,2)}{}^{\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\lambda_q}{}_{\omega_1\ldots\omega_r} \sum_{\substack{l=1\\ l\neq i,j}}^{q} k_{\lambda_l}\theta^{(p+i,1,p+l)}_{\nu_1\ldots\nu_p\lambda_1\ldots\underline{\lambda_i}\ldots\underline{\lambda_j}\ldots\underline{\lambda_l}\ldots\lambda_q} \quad [\text{if } q>2] \\
&= \ldots, \qquad (1.46)
\end{aligned}$$

where the terms already in the desired form have not been written in subsequent steps for the sake of simplicity. Since $T^{(1)}$, $T^{(2)}$ as well as the θ tensors satisfy the contraction property and q is finite, the above steps can be carried out till no contraction with k remains in any term. Thus the contraction identity is seen to be valid for contractions of two primary tensors, which can be generalised to contractions of any number of primary tensors.

We also check that the Levi-Civita tensor ϵ also satisfies the contraction property. To do this, we first use the Killing condition to say there exists a u such that $k^\mu = \delta^\mu{}_u$ and $g_{\mu\nu,u} = 0$, and the hypersurface orthogonal condition to state the existence of v such that $k_\mu = e^{-S}\partial_\mu v$. We choose a coordinate system with u and v as two coordinates. Then we have

$$\begin{aligned}
k^\mu \epsilon_{\mu\alpha\ldots\beta} &= \delta^\mu{}_u \epsilon_{\mu\alpha\ldots\beta} \\
&= \epsilon_{u\alpha\ldots\beta}. \qquad (1.47)
\end{aligned}$$

Since ϵ is completely antisymmetric, every non-zero term in the above must have v as a free index, we can thus write $i_k\epsilon = k \wedge \theta$, where θ is a $(D-2)$-form. Hence, the Levi-Civita tensor also satisfies the contraction property.

1.4.2 Transformation of the Riemann tensor

We have seen before that the change in the Christoffel symbol is given by

$$\begin{aligned}
\Omega^{\mu}{}_{\alpha\beta} &= \frac{1}{2}(\nabla_\alpha(\kappa k^\mu k_\beta) + \nabla_\beta(\kappa k^\mu k_\alpha) - \nabla^\mu(\kappa k_\alpha k_\beta)) \\
&= \frac{1}{2}(k^\mu k_\beta \nabla_\alpha \kappa + k^\mu k_\alpha \nabla_\beta \kappa - k_\alpha k_\beta \nabla^\mu \kappa) \\
&\quad + \frac{\kappa}{4}(k_\alpha k_\beta \nabla^\mu S - k^\mu k_\beta \nabla_\alpha S + k^\mu k_\alpha \nabla_\beta S - k^\mu k_\beta \nabla_\alpha S + k_\beta k_\alpha \nabla^\mu S - k^\mu k_\alpha \nabla_\beta S + k^\mu k_\beta \nabla_\alpha S) \\
&\quad + \frac{\kappa}{4}(-k^\mu k_\alpha \nabla_\beta S - k^\mu k_\beta \nabla_\alpha S + k_\alpha k_\beta \nabla^\mu S - k_\alpha k^\mu \nabla_\beta S + k_\alpha k_\beta \nabla^\mu S) \\
&= \frac{1}{2}(k^\mu k_\beta \nabla_\alpha \kappa + k^\mu k_\alpha \nabla_\beta \kappa - k_\alpha k_\beta \nabla^\mu \kappa) - \frac{\kappa}{2}(k^\mu k_\beta \nabla_\alpha S + k^\mu k_\alpha \nabla_\beta S - 2k_\alpha k_\beta \nabla^\mu S).
\end{aligned} \tag{1.48}$$

The shift in the Riemann tensor is thus given by

$$\begin{aligned}
R'^{\alpha}{}_{\beta\gamma\delta} &= \Gamma'^{\alpha}{}_{\beta\delta,\gamma} - \Gamma'^{\alpha}{}_{\beta\gamma,\delta} + \Gamma'^{\alpha}{}_{\mu\gamma}\Gamma'^{\mu}{}_{\beta\delta} - \Gamma'^{\alpha}{}_{\mu\delta}\Gamma'^{\mu}{}_{\beta\gamma} \\
&= R^{\alpha}{}_{\beta\gamma\delta} + \nabla_\gamma \Omega^{\alpha}{}_{\beta\delta} - \nabla_\delta \Omega^{\alpha}{}_{\beta\gamma} + \Omega^{\alpha}{}_{\mu\gamma}\Omega^{\mu}{}_{\beta\delta} - \Omega^{\alpha}{}_{\mu\delta}\Omega^{\mu}{}_{\beta\gamma}.
\end{aligned} \tag{1.49}$$

But we see that

$$\begin{aligned}
\Omega^{\alpha}{}_{\mu\gamma}\Omega^{\mu}{}_{\beta\delta} &= \left(\frac{1}{2}(k^\alpha k_\gamma \nabla_\mu \kappa + k^\alpha k_\mu \nabla_\gamma \kappa - k_\mu k_\gamma \nabla^\alpha \kappa) - \frac{\kappa}{2}(k^\alpha k_\gamma \nabla_\mu S + k^\alpha k_\mu \nabla_\gamma S - 2k_\mu k_\gamma \nabla^\alpha S)\right) \\
&\quad \left(\frac{1}{2}(k^\mu k_\delta \nabla_\beta \kappa + k^\mu k_\beta \nabla_\delta \kappa - k_\beta k_\delta \nabla^\mu \kappa) - \frac{\kappa}{2}(k^\mu k_\delta \nabla_\beta S + k^\mu k_\beta \nabla_\delta S - 2k_\beta k_\delta \nabla^\mu S)\right) \\
&= -\frac{1}{4}k^\alpha k_\beta k_\gamma k_\delta \nabla_\mu \kappa \nabla^\mu \kappa + \frac{3\kappa}{4}k^\alpha k_\beta k_\gamma k_\delta \nabla_\mu \kappa \nabla^\mu S \quad - \frac{\kappa^2}{2}k^\alpha k_\beta k_\gamma k_\delta \nabla_\mu S \nabla^\mu S,
\end{aligned} \tag{1.50}$$

which is symmetric under the interchange of γ and δ. Thus, the Riemann tensor is

$$R'^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} + \nabla_\gamma \Omega^{\alpha}{}_{\beta\delta} - \nabla_\delta \Omega^{\alpha}{}_{\beta\gamma}, \tag{1.51}$$

and the expression with all covariant indices will be

$$\begin{aligned}
R'_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + g_{\alpha\mu}\nabla_\gamma \Omega^{\alpha}{}_{\beta\delta} - g_{\alpha\mu}\nabla_\delta \Omega^{\alpha}{}_{\beta\gamma} + \kappa k_\alpha \mu R^{\mu}{}_{\beta\gamma\delta} \\
&= R_{\alpha\beta\gamma\delta} + \nabla_\gamma \left(\frac{1}{2}(k_\alpha k_\delta \nabla_\beta \kappa + k_\alpha k_\beta \nabla_\delta \kappa - k_\beta k_\delta \nabla_\alpha \kappa) - \frac{\kappa}{2}(k_\alpha k_\delta \nabla_\beta S + k_\alpha k_\beta \nabla_\delta S - 2k_\beta k_\delta \nabla_\alpha S)\right) \\
&\quad - \nabla_\delta \Omega^{\alpha}{}_{\beta\gamma}.
\end{aligned} \tag{1.52}$$

Since every term in $\Omega^{\alpha}{}_{\beta\delta}$ as well as $\Omega^{\alpha}{}_{\beta\gamma}$ is bilinear in k, we use Eq. (1.17) to show that the shift in the Riemann tensor is also bilinear in k. It can also be shown that the shift any tensor obtained by taking any number of covariant derivatives of the Riemann tensor is also at least bilinear in k.

Since the determinant of the metric is unchanged under the transformation, the Levi-Civita tensor also remains the same.

1.4.3 Invariance of curvature scalars

An arbitrary curvature scalar will be a product of some number of tensors obtained from the Riemann tensor and its covariant derivatives and Levi-Civita tensors, contracted with the sufficient number of inverse metrics:

$$\mathcal{I}' = \prod_{j=1}^{N} T'_{\nu_1^j \ldots \nu_{q_j}^j} \prod_{k=1}^{K} \epsilon'_{\lambda_1^k \ldots \lambda_D^k} \prod_{l=1}^{M} g'^{\alpha_l \beta_l}, \quad \text{where} \quad \sum_{j=1}^{N} q_j + KD = 2M. \tag{1.53}$$

Now we know that $T' = T + \chi$, where χ is at least bilinear in k, $\epsilon' = \epsilon$, and $g^{\alpha\beta} = g^{\alpha\beta} - \kappa k^\alpha k^\beta$. So we see that

$$\begin{aligned}\mathcal{I}' &= \prod_{j=1}^{N} \Big(T_{\nu_1^j \dots \nu_{q_j}^j} + \chi_{\nu_1^j \dots \nu_{q_j}^j} \Big) \prod_{k=1}^{K} \epsilon_{\lambda_1^k \dots \lambda_D^k} \prod_{l=1}^{M} \big(g^{\alpha_l \beta_l} - \kappa k^{\alpha_l} k^{\beta_l} \big) \\ &= \mathcal{I} + \mathcal{J}, \end{aligned} \tag{1.54}$$

where $\mathcal{I}$ is the same curvature scalar for the original metric, and $\mathcal{J}$ consists of all the other terms. Since T satisfies the contraction property and χ is at least bilinear in k, we cause the fact that k is null to prove that $\mathcal{J} = 0$, i.e. the curvature scalar is unchanged.

1.5 Neutral oscillating string

As an example [8] of the GV transformation, let us consider the following metric g and fields ϕ and $B_{\mu\nu}$ in $D = 10$ dimensions,

$$\mathrm{d}s^2 = -e^{2\phi}\, \mathrm{d}u\, \mathrm{d}v + \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}\,, \tag{1.55}$$

$$e^{-2\phi} = 1 + \frac{Q}{r^{D-4}}, \tag{1.56}$$

$$B_{uv} = \frac{1}{2}\big(e^{2\phi} - 1\big). \tag{1.57}$$

The above can be thought of as the metric corresponding to a neutral straight string. We can see that both the vectors ∂_u and ∂_v are null (since $g_{uu} = g_{vv}$), hypersurface orthogonal (since $g_{u\mu} = -e^{2\phi}\partial_\mu v$ and $g_{v\mu} = -e^{2\phi}\partial_\nu u$, so $S = -2\phi$ in both cases) and Killing (since $g_{\mu\nu,u} = g_{\mu\nu,v} = 0$). Applying the GV technique with the vector $k = \partial_u$, we get

$$g' = g + e^S \Psi k \otimes k = g + e^{-S}\Psi(v, \vec{x})\, \mathrm{d}v \otimes \mathrm{d}v\,, \tag{1.58}$$

which gives the new metric

$$\mathrm{d}s^2 = -e^{2\phi}(\mathrm{d}u\, \mathrm{d}v - \Psi(v, \vec{x})\, \mathrm{d}v^2) + \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}\,, \tag{1.59}$$

leaving the fields ϕ and $B_{\mu\nu}$ unchanged. It can be seen that Ψ is independent of u by noting that $\mathcal{L}_k \Psi = 0$. Also, since $\partial_u \Psi = 0$ and $g_{\mu v}$ is non-zero only for $\mu = u$, the condition that $\nabla^2 \Psi = 0$ reduces to $\partial^2 \Psi = 0$, i.e. Ψ satisfies the Laplace equation in the flat transverse space given by x^i. The solution of this may be written in terms of the $(D-2)$-dimensional spherical harmonics Y_ℓ as

$$\Psi(v, \vec{x}) = \sum_{\ell \geq 0} \big(a_\ell(v) r^\ell + b_\ell(v) r^{-D+4-\ell} \big) Y_\ell. \tag{1.60}$$

It can be seen [8] that only the terms of order r^1 correspond to string sources, and only such terms are kept in the expression of $\Psi(v, \mathbf{x})$ to obtain

$$\Psi(v, \vec{x}) = \vec{f}(v) \cdot \vec{x}. \tag{1.61}$$

The metric now does not appear to be asymptotically flat. However, this can be remedied by changing the coordinates as

$$\begin{aligned} v &= v', \\ u &= u' - 2\dot{\vec{F}} \cdot \vec{x}' + 2\dot{\vec{F}} \cdot \vec{F} - \int^{v'} \dot{F}^2\, \mathrm{d}v''\,, \\ \vec{x} &= \vec{x}' - \vec{F}, \end{aligned} \tag{1.62}$$

where $\vec{f}(v) = -2\ddot{\vec{F}}(v)$ and $\dot{F}^2 = \dot{\vec{F}} \cdot \dot{\vec{F}}$, where the dot denotes derivative with respect to v. The metric in the new coordinates becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -e^{2\phi}(\mathrm{d}u\,\mathrm{d}v - \vec{f} \cdot \vec{x}\,\mathrm{d}v^2) + \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} \\
&= -e^{2\phi}\Big((\mathrm{d}u' - 2\ddot{\vec{F}} \cdot \vec{x}\,\mathrm{d}v' - 2\dot{\vec{F}} \cdot \mathrm{d}\vec{x} + 2\ddot{\vec{F}} \cdot \vec{F}\,\mathrm{d}v' + 2\dot{\vec{F}} \cdot \dot{\vec{F}}\,\mathrm{d}v' - \dot{F}^2\,\mathrm{d}v')\,\mathrm{d}v' + 2\ddot{\vec{F}} \cdot (\vec{x}' - \vec{F})\,\mathrm{d}v'^2\Big) \\
&\quad + (\mathrm{d}\vec{x}' - \dot{\vec{F}}\,\mathrm{d}v') \cdot (\mathrm{d}\vec{x}' - \dot{\vec{F}}\,\mathrm{d}v') \\
&= -e^{2\phi}(\mathrm{d}u' - 2\dot{\vec{F}} \cdot \mathrm{d}\vec{x} + \dot{F}^2\,\mathrm{d}v')\,\mathrm{d}v' + \mathrm{d}\vec{x}' \cdot \mathrm{d}\vec{x}' - 2\dot{\vec{F}} \cdot \mathrm{d}x'\,\mathrm{d}v' + \dot{F}^2\,\mathrm{d}v'^2 \\
&= -e^{2\phi}\,\mathrm{d}u\,\mathrm{d}v + 2(e^{2\phi} - 1)\dot{\vec{F}} \cdot \mathrm{d}\vec{x}\,\mathrm{d}v - \Big(2e^{2\phi}\ddot{\vec{F}} \cdot \vec{F} + (e^{2\phi} - 1)\dot{F}^2\Big)\,\mathrm{d}v^2\,,
\end{aligned} \tag{1.63}$$

which is asymptotically flat.

The above analysis can be extended for multi-string solutions as well, giving

$$\Psi(v, \vec{x}) = \vec{f}(v) \cdot \vec{x}, \tag{1.64}$$

$$e^{-2\phi(\vec{x})} = 1 + \sum_i \frac{Q}{|\vec{x} - \vec{x}_i|^{D-4}}, \tag{1.65}$$

which can be then made asymptotically flat by making the appropriate coordinate transformation as before. It is interesting to note that even though the equations of motion are in general are nonlinear, the multi-string solutions can be simply written as a linear combination of single-string solutions in this case, as both $e^{-2\phi}$ and Ψ follow the Laplace equation, which is linear.

Chapter 2

Garfinkle-Vachaspati transformation of a D3 brane

2.1 Properties of a D3 brane metric

We now turn our attention to the D3 brane, which is a $(3+1)$-dimensional object embedded in a 10-dimensional spacetime. Its metric is given as

$$\mathrm{d}s^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \eta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j + \left(1 + \frac{L^4}{r^4}\right)^{1/2} (\mathrm{d}r^2 + r^2\, \mathrm{d}{\Omega_5}^2). \tag{2.1}$$

The D3 brane metric is a solution for the action

$$S = \int \mathrm{d}^{10}x\, \sqrt{-g} \left(R(g) - \frac{1}{2} f F_{(4)}^2\right), \tag{2.2}$$

which is a special case of the action given Eq. (1.1) for 10 dimensions with no scalar field and one 4-form field $F_{(4)} = \mathrm{d}C$, where the 3-form potential C is given as $C_{0123} = g_{\mathrm{s}}^{-1}\left(1 - (1 + L^4/r^4)^{-1}\right)$ [9], g_{s} being the string coupling constant.

We first make the transformation

$$r = \frac{L\rho}{(1-\rho^4)^{1/4}}, \qquad \rho = \left(1 + \frac{L^4}{r^4}\right)^{-1/4}, \tag{2.3}$$

so that the metric in the new coordinates becomes

$$\mathrm{d}s^2 = \rho^2 \eta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j + L^2 \rho^{-2} \left(\frac{\mathrm{d}\rho^2}{(1-\rho^4)^{5/2}} + \frac{\rho^2}{(1-\rho^4)^{1/2}}\, \mathrm{d}{\Omega_5}^2\right). \tag{2.4}$$

The metric components diverge at $\rho = 0$, but that is not a true singularity as the metric can be made regular by choosing a different set of coordinates (as will be shown later). It is, however, a Killing horizon, as all the Killing vectors ∂_i (corresponding to translations along x^i) become null at $\rho = 0$. We see that the metric is symmetric under the transformation $\rho \mapsto -\rho$, and thus the inside region of the horizon is a copy of the outside region. D3 brane metric thus has no singularity.

At ρ close to 0 ($\rho \ll 1$), the metric approaches that of $AdS_5 \times S^5$. While the $L^2\, \mathrm{d}{\Omega_5}^2$ term in the metric obviously corresponds to a sphere of radius L, it can be seen that the other terms correspond to a 5-dimensional anti-de Sitter space by going over to the embedding coordinates

$$\begin{gathered} X^i = \rho x^i \quad (i \in \{1,2,3\}), \quad X^5 = \rho x^0, \\ X^0 = \frac{L}{2\rho}\left(1 + \rho^2 + \frac{\rho^2}{L^2}(\vec{x}^2 - (x^0)^2)\right), \quad X^4 = \frac{L}{2\rho}\left(1 - \rho^2 + \frac{\rho^2}{L^2}(\vec{x}^2 - (x^0)^2)\right). \end{gathered} \tag{2.5}$$

These coordinates satisfy the constraint

$$(X^0)^2 + (X^5)^2 - \sum_{i=1}^{4}(X^i)^2 = \rho^2\left((x^0)^2 - \sum_{i=1}^{3}(x^i)^2\right) + \frac{L^2}{4\rho^2}4\rho^2\left(1 + \frac{\rho^2}{L^2}(\vec{x}^2 - (x^0)^2)\right) = L^2, \qquad (2.6)$$

with the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}\big(X^0\big)^2 - \mathrm{d}\big(X^5\big)^2 + \sum_{i=1}^{4}\mathrm{d}\big(X^i\big)^2 . \qquad (2.7)$$

2.2 Nonsingular coordinates of the D3 brane metric

The D3 metric is devoid of any singularity, even though it does have a horizon at $\rho = 0$. However, in the coordinates chosen, the metric components diverge at $\rho \to 0$. This can be avoided by choosing a different set of coordinates. Let us take the coordinates

$$v = x^1 + x^0 = -\frac{1}{V}, \qquad u = x^1 - x^0 = U + \frac{1}{VW^2} + \frac{(X^2)^2 + (X^3)^2}{V}, \qquad \rho = LVW, \qquad x^{2,3} = \frac{X^{2,3}}{V}. \qquad (2.8)$$

For any timelike or null geodesic going towards $\rho \to 0$, v approaches infinity. This is equivalent to the Schwarzschild metric, where any timelike or null geodesic going as $r \to 2GM$ has $t \to \infty$ [3]. In the new coordinates, this will correspond to $V \to 0$. ρ appears only in the definition $\rho = LVW$, where W remains finite as $\rho \to 0$ and $V \to 0$. Then the metric becomes

$$\begin{aligned}
\mathrm{d}s^2 &= \rho^2\Big(\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2\Big) + \frac{L^2}{\rho^2}\left(\frac{\mathrm{d}\rho^2}{(1-\rho^4)^{5/2}} + \frac{\rho^2}{(1-\rho^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\right)\\
&= L^2V^2W^2\left(\mathrm{d}U - \frac{\mathrm{d}V}{V^2W^2} - \frac{2\,\mathrm{d}W}{VW^3} + \frac{2X^2\,\mathrm{d}X^2 + 2X^3\,\mathrm{d}X^3}{V} - \frac{(X^2)^2 + (X^3)^2}{V^2}\,\mathrm{d}V\right)\frac{\mathrm{d}V}{V^2}\\
&\quad + L^2V^2W^2\left(\frac{\mathrm{d}(X^2)^2 + \mathrm{d}(X^3)^2}{V^2} + \frac{(X^2)^2 + (X^3)^2}{V^4}\,\mathrm{d}V^2 - \frac{2X^2\,\mathrm{d}X^2 + 2X^3\,\mathrm{d}X^3}{V^3}\,\mathrm{d}V\right)\\
&\quad + \frac{1}{V^2W^2}\left(\frac{L^2(V^2\,\mathrm{d}W^2 + W^2\,\mathrm{d}V^2 + 2VW\,\mathrm{d}V\,\mathrm{d}W)}{(1-L^4V^4W^4)^{5/2}} + \frac{L^2V^2W^2}{(1-L^4V^4W^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\right)\\
&= L^2W^2\left(\mathrm{d}U - \frac{\mathrm{d}V}{V^2W^2} - \frac{2\,\mathrm{d}W}{VW^3}\right)\mathrm{d}V + L^2W^2\Big(\mathrm{d}\big(X^2\big)^2 + \mathrm{d}\big(X^3\big)^2\Big)\\
&\quad + \frac{1}{V^2W^2}\left(\frac{L^2(V^2\,\mathrm{d}W^2 + W^2\,\mathrm{d}V^2 + 2VW\,\mathrm{d}V\,\mathrm{d}W)}{(1-L^4V^4W^4)^{5/2}} + \frac{L^2V^2W^2}{(1-L^4V^4W^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\right)\\
&= L^2W^2\Big(\mathrm{d}U\,\mathrm{d}V + \mathrm{d}\big(X^2\big)^2 + \mathrm{d}\big(X^3\big)^2\Big) + \frac{L^2\,\mathrm{d}W^2}{W^2(1-L^4V^4W^4)^{5/2}} + \frac{L^2}{(1-L^4V^4W^4)^{1/2}}\,\mathrm{d}{\Omega_5}^2\\
&\quad + \left(\sum_{r=1}^{\infty}\frac{L^{4r+2}V^{4r-2}W^{4r-1}}{2^r r!}\prod_{k=0}^{r-1}(5+2k)\right)(W\,\mathrm{d}V^2 + 2V\,\mathrm{d}V\,\mathrm{d}W). \qquad [\rho < 1]
\end{aligned} \qquad (2.9)$$

This is seen to be finite when $V \to 0$. Thus, the new coordinates show that $\rho = 0$ is not a singularity of the metric; in fact, no singularity exists for this metric.

2.3 Garfinkle-Vachaspati transformation of the D3 brane metric

For the D3 brane, it can be easily seen that the vector ∂_u is null (since $g_{uu} = 0$), hypersurface orthogonal (since $g_{u\mu} = \rho^2 \partial_\mu v$) and Killing (since $g_{\mu\nu,u} = 0$). Therefore, we can apply a GV transformation on the metric with this vector. The associated function Ψ will be independent of u (as $\mathcal{L}_{\partial_u}\Psi = 0$) and will satisfy the Laplace equation $\nabla^2\Psi = 0$. For simplicity, we assume for now that Ψ is also independent of the coordinates x^2 and x^3 and is thus a function of only v and the coordinates of the scaled $\mathbb{R}^6$ submanifold. The metric can be written as $\mathbf{g} = H^{-1/2}\boldsymbol{\eta}_{\mathbb{R}^{3+1}} \otimes H^{1/2}\mathbf{g}_{\mathbb{R}^6}$, where $H = 1 + L^4/r^4$. Therefore, its determinant will be $g = H^{-2}(-1) \times H^3 g_{\mathbb{R}^6} = -H g_{\mathbf{R}^6}$, where $g_{\mathbf{R}^6}$ is the determinant of the 6-dimensional Euclidean space in spherical coordinates, and the determinant of the Minkowski metric in cartesian coordinates is -1. Then we see that

$$\begin{aligned}
\nabla^2\Psi &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) \\
&= \frac{1}{H^{1/2}\sqrt{g_{\mathbb{R}^6}}}\partial_i(H^{1/2}\sqrt{g_{\mathbb{R}^6}}H^{-1/2}(g_{\mathbb{R}^6})^{ij}\partial_j\Psi) \qquad [i,j \text{ are coordinates of } \mathbb{R}^6] \\
&= H^{-1/2}\frac{1}{\sqrt{g_{\mathbb{R}^6}}}\partial_i(\sqrt{g_{\mathbb{R}^6}}(g_{\mathbb{R}^6})^{ij}\partial_j\Psi) \\
&= H^{-1/2}\nabla^2_{\mathbb{R}^6}\Psi = 0, \qquad (2.10)
\end{aligned}$$

i.e. Ψ satisfies the Laplace equation in the flat transverse space. This can be solved similar to the example of the neutral oscillating string [8], and we thus obtain $\Psi(v,\vec{r}) = \vec{f}(v)\cdot\vec{r}$, where $\vec{r}$ is the position vector in $\mathbb{R}^6$. Therefore, the GV transformed D3 brane metric is

$$\mathrm{d}s^2 = H^{-1/2}(\mathrm{d}u\,\mathrm{d}v + \vec{f}(v)\cdot\vec{r}\,\mathrm{d}v^2 + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2) + H^{1/2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2). \tag{2.11}$$

To make the metric asymptotically flat, we apply the coordinate transformation

$$u \to u - \vec{r}\cdot\int^v \mathrm{d}v'\,\vec{f}(v'). \tag{2.12}$$

Then the metric becomes

$$\mathrm{d}s^2 = H^{-1/2}\left(\mathrm{d}u\,\mathrm{d}v - \mathrm{d}v\,\mathrm{d}\vec{r}\cdot\int^v \mathrm{d}v'\,\vec{f}(v') + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2\right) + H^{1/2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2), \tag{2.13}$$

which is asymptotically flat.

Writing $\vec{r} = r\hat{n}$, such that $\mathrm{d}r^i = r\,\mathrm{d}n^i + n^i\,\mathrm{d}r$ we can rewrite the metric as

$$\begin{aligned}
\mathrm{d}s^2 = H^{-1/2}&\left(\mathrm{d}u\,\mathrm{d}v - r\,\mathrm{d}v\,\mathrm{d}n^i\int^v \mathrm{d}v'\,f_i(v') - n_i\,\mathrm{d}v\,\mathrm{d}r\int^v \mathrm{d}v'\,f_i(v') + \mathrm{d}\big(x^2\big)^2 + \mathrm{d}\big(x^3\big)^2\right) \\
&+ H^{1/2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2). \qquad (2.14)
\end{aligned}$$

It is to be noted that the 6 variables n^i are not independent coordinates as $\sum_i (n^i)^2 = 1$. They can be related to the 5 angular coordinates of $\mathbb{R}^6$.

Let us define $f_i(v) = \ddot{F}_i(v)$, where the dots denote differentiation with respect to v. We see that the GV transformation has introduced two additional terms in the metric. Since we have already seen that the original metric is regular at the horizon $r = 0$, we only need to check if the new terms diverge at $r = 0$ or not. The first term can be written in terms of the $\{U, V, W, \ldots\}$ coordinates as

$$-L^2V^2W^2\frac{L^2VW}{(1-L^4V^4W^4)^{1/4}}\frac{\mathrm{d}V}{V^2}\,\mathrm{d}n^i\,\dot{F}_i(v) = -\frac{L^4VW^3}{(1-L^4V^4W^4)^{1/4}}\dot{F}_i(v)\,\mathrm{d}V\,\mathrm{d}n^i\,, \tag{2.15}$$

which remains finite as $V \to 0$. The second term is

$$-L^2V^2W^2n_i\frac{\mathrm{d}V}{V^2}\frac{L^2(V\,\mathrm{d}W+W\,\mathrm{d}V)}{(1-L^4V^4W^4)^{5/4}}\dot{F}_i(v) = -L^4W^2n^i\dot{F}_i(v)\frac{V\,\mathrm{d}V\,\mathrm{d}W+W\,\mathrm{d}V^2}{(1-L^4V^4W^4)^{5/4}}, \tag{2.16}$$

which is also regular as $v \to 0$. Thus we have a coordinates in which the metric components are regular at the horizon. However, the derivatives of the metric components may diverge as $V \to 0$, so this is only a C^0 extension. We will try to find a C^1 extension of this metric and then use the method described in [7] to see if a C^2 solution exists.

Chapter 3

Extended Garfinkle-Vachaspati transformation

The Garfinkle-Vachaspati transformation discussed before involves a null hypersurface orthogonal Killing vector. But there have been ideas [10] to generalize it to transformations of the form

$$g'_{\mu\nu} = g_{\mu\nu} + Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu), \tag{3.1}$$

where k is null, l is spacelike, and the two vectors are orthogonal with respect to the original metric. We also impose the condition that both the vectors are hypersurface orthogonal and Killing, which had not been assumed in [10]. We can now do the analysis in a manner similar to that followed in [2] to check how such a transformation would affect the Physics of the problem.

First of all, let us find the inverse metric $g'^{\mu\nu}$. We expect to be of the form

$$g'^{\mu\nu} = g^{\mu\nu} + \tilde{H}k^\mu k^\nu + \tilde{K}(k^\mu l^\nu + l^\mu k^\nu) + \tilde{G}l^\mu l^\nu. \tag{3.2}$$

Then we see that

$$\begin{aligned}
g'_{\mu\nu}g'^{\nu\lambda} &= (g_{\mu\nu} + Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu))\Big(g^{\nu\lambda} + \tilde{H}k^\nu k^\lambda + \tilde{K}(k^\nu l^\lambda + l^\nu k^\lambda) + \tilde{G}l^\nu l^\lambda\Big) \\
&= \delta_\mu{}^\lambda + \tilde{H}k_\mu k^\lambda + \tilde{K}(k_\mu l^\lambda + l_\mu k^\lambda) + \tilde{G}l_\mu l^\lambda + Hk_\mu k^\lambda + K(k_\mu l^\lambda + l_\mu k^\lambda) + K\tilde{K}(l_\nu l^\nu)k_\mu k^\lambda \\
&\quad + K\tilde{G}(l_\nu l^\nu)k_\mu l^\lambda \\
&= \delta_\mu{}^\lambda + (\tilde{H} + H + K\tilde{K}l^2)k_\mu k^\lambda + (\tilde{K} + K + K\tilde{G}l^2)k_\mu l^\lambda + (\tilde{K} + K)l_\mu k^\lambda + \tilde{G}l_\mu l^\lambda = \delta_\mu{}^\lambda \\
\implies 0 &= (\tilde{H} + H + K\tilde{K}l^2)k_\mu k^\lambda + (\tilde{K} + K + K\tilde{G}l^2)k_\mu l^\lambda + (\tilde{K} + K)l_\mu k^\lambda + \tilde{G}l_\mu l^\lambda.
\end{aligned} \tag{3.3}$$

Contracting both sides with l^μ, we get

$$(\tilde{K} + K)l^2 k^\lambda + \tilde{G}l^2 l^\lambda = 0. \tag{3.4}$$

This upon contraction with l_λ gives $\tilde{G} = 0$, which in turn implies $\tilde{K} = -K$. Then we obtain $(\tilde{H} + H - K^2 l^2)k_\mu k^\lambda = 0 \implies \tilde{H} = -H + l^2K^2$. Thus, the transformed inverse metric is given by

$$g'^{\mu\lambda} = g^{\mu\lambda} - (H - l^2K^2)k^\mu k^\nu - K(k^\mu l^\nu + l^\mu k^\nu). \tag{3.5}$$

The inverse metric is thus seen to have a term of second order in the coefficient K. Another way of writing this is given in [10]. The change in the metric is written as $h_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu} = Hk_\mu k_\nu + K(k_\mu l_\nu + l_\mu k_\nu)$. Then the inverse metric transforms as $g'^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu$.

We can also see that

$$g'_{\mu\nu}k^\mu = g_{\mu\nu}k^\mu, \tag{3.6}$$
$$\text{and} \quad g'_{\mu\nu}l^\mu = g_{\mu\nu}l^\mu + Kl^2k_\nu. \tag{3.7}$$

So we see that the raising and lowering of indices of l gives different results for the original and transformed metrics. However, the orthogonality of k and l in the original metric imply that the norms of both the vector fields, as well as their orthogonality, are preserved under the transformation. In the present analysis, including the we will raise and lower indices using the original metric; use of the transformed metric will be denoted by primes such as $l'_\nu = g'_{\mu\nu}l^\mu$.

Let us check if the above transformation changes the metric. The following analysis closely resembles that done for the usual GV transformation, with only minor modifications. We start by writing

$$g'_{\mu\nu} = g_{\mu\lambda}(\delta^\lambda{}_\nu + h^\lambda{}_\nu), \tag{3.8}$$

which can be written as a matrix equation $\mathbf{g'} = \mathbf{g}(\mathbf{I} + \mathbf{H})$, where $\mathbf{g}$ is the matrix with entries $g_{\mu\nu}$, $\mathbf{I}$ the identity matrix and $\mathbf{H}$ the matrix with entries $h^\mu{}_\nu = Hk^\mu k_\nu + K(k^\mu l_\nu + l^\mu k_\nu)$. Then we see that $[\mathbf{H}^2]^\mu{}_\nu = K^2l^2k^\mu k_\nu$ and $[\mathbf{H}^3]^\mu{}_\nu = 0$, i.e. $\mathbf{H}$ is nilpotent of index 3. So we can say that

$$(\mathbf{I} + \mathbf{H})^n = \mathbf{I} + n\mathbf{H} + \frac{n(n-1)}{2}\mathbf{H}^2 \qquad [n \in \mathbb{N}]. \tag{3.9}$$

Let us now take $\det(\mathbf{I} + \mathbf{H}) = x$. Then, taking the determinant of both sides od the above equation gives that

$$x^n = \det\left(\mathbf{I} + n\mathbf{H} + \frac{n(n-1)}{2}\mathbf{H}^2\right), \tag{3.10}$$

where the left hand side is an exponential function in n, and the right hand side is a polynomial in n of degree at most $2D$ for a D-dimensional manifold. This can hold for every natural number n only if x is 0 or 1. But $x = 0$ implies the existence of a column vector $\mathbf{v} \neq \mathbf{0}$ such that $(\mathbf{I} + \mathbf{H})\mathbf{v} = \mathbf{0} \implies \mathbf{H}\mathbf{v} = -\mathbf{v}$. Then $\mathbf{H}^2\mathbf{v} = -\mathbf{H}\mathbf{v} = \mathbf{v}$, and $\mathbf{H}^3 v = -\mathbf{v} = \mathbf{0}$, which contradicts the condition that $\mathbf{v} \neq \mathbf{0}$. Thus we have $\det(\mathbf{I} + \mathbf{H}) = 1$, i.e. the determinant of the metric is unchanged under the generalised GV transformation.

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