Set cover

SET-COVER. Given a set U of elements, a collection S of subsets of U, and an integer k, are there $\leq k$ of these subsets whose union is equal to U?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The i^{th} piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$
$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$
$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$
$k = 2$

a set cover instance

Intractability: quiz 4

Given the universe U = { 1, 2, 3, 4, 5, 6, 7 } and the following sets, which is the minimum size of a set cover?

A. 1

B. 2

C. 3

D. None of the above.

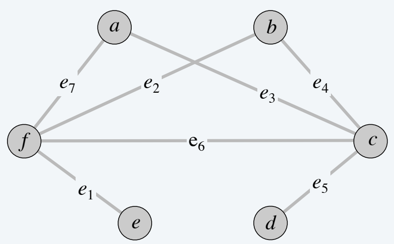
$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$S_a = \{ 1, 4, 6 \}$ $\quad$ $S_b = \{ 1, 6, 7 \}$
$S_c = \{ 1, 2, 3, 6 \}$ $\quad$ $S_d = \{ 1, 3, 5, 7 \}$
$S_e = \{ 2, 6, 7 \}$ $\quad$ $S_f = \{ 3, 4, 5 \}$

Vertex cover reduces to set cover

Theorem. VERTEX-COVER $\leq_P$ SET-COVER.
Pf. Given a VERTEX-COVER instance $G = (V, E)$ and k, we construct a SET-COVER instance (U, S, k) that has a set cover of size k iff G has a vertex cover of size k.

Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V$: $S_v = \{e \in E : e \text{ incident to } v\}$.



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$
$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$
$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

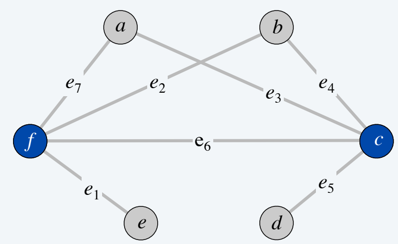
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Rightarrow$ Let $X \subseteq V$ be a vertex cover of size k in G.

- Then $Y = \{ S_v : v \in X \}$ is a set cover of size k. ▪

"yes" instances of VERTEX-COVER are solved correctly



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$
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$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

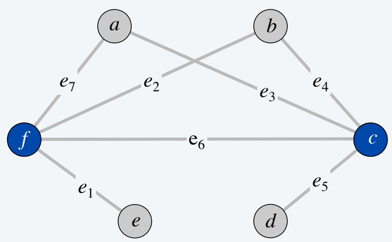
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Leftarrow$ Let $Y \subseteq S$ be a set cover of size k in (U, S, k).

- Then $X = \{ v : S_v \in Y \}$ is a vertex cover of size k in G. ▪

"no" instances of VERTEX-COVER are solved correctly



vertex cover instance
(k = 2)

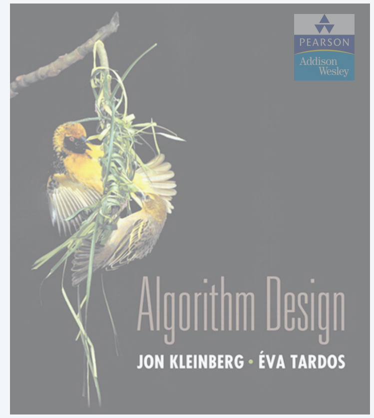
$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$\boxed{S_c = \{ 3, 4, 5, 6 \}}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $\boxed{S_f = \{ 1, 2, 6, 7 \}}$

set cover instance
(k = 2)



SECTION 8.2

8. INTRACTABILITY I

Satisfiability

Literal. A Boolean variable or its negation. $\quad x_i$ or $\overline{x_i}$

Clause. A disjunction of literals. $\quad C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF). A propositional formula Φ that is a conjunction of clauses. $\quad \Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals (and each literal corresponds to a different variable).

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

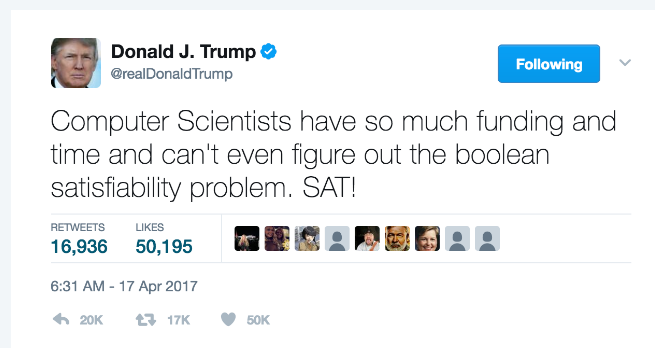
yes instance: x_1 = true, x_2 = true, x_3 = false, x_4 = false

Key application. Electronic design automation (EDA).

Satisfiability is hard

Scientific hypothesis. There does not exist a poly-time algorithm for 3-SAT.

P vs. NP. This hypothesis is equivalent to **P ≠ NP** conjecture.



https://www.facebook.com/pg/npcompleteteens

3-satisfiability reduces to independent set

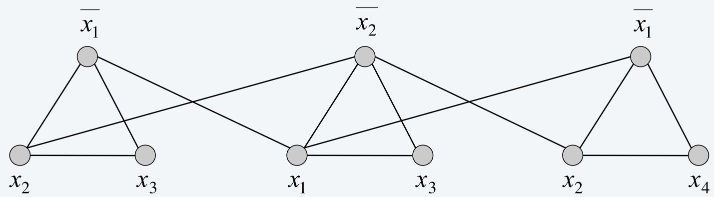
Theorem. 3-SAT $\le_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Construction.

- G contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3-satisfiability reduces to independent set

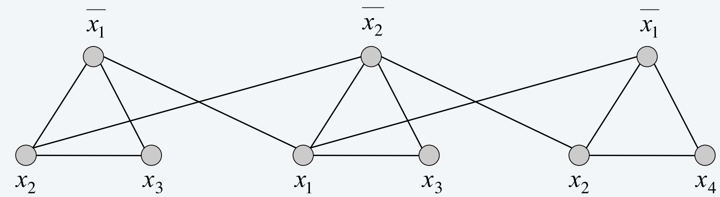
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Rightarrow$ Consider any satisfying assignment for Φ.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$. ▪

"yes" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3-satisfiability reduces to independent set

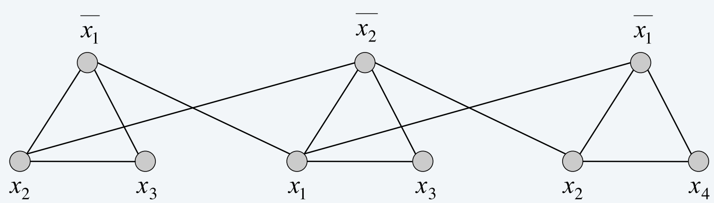
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Leftarrow$ Let S be independent set of size k.

- S must contain exactly one node in each triangle.
- Set these literals to *true* (and remaining literals consistently).
- All clauses in Φ are satisfied. ▪

"no" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\le_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\le_P$ INDEPENDENT-SET.

Transitivity. If $X \le_P Y$ and $Y \le_P Z$, then $X \le_P Z$.

Pf idea. Compose the two algorithms.

Ex. 3-SAT $\le_P$ INDEPENDENT-SET $\le_P$ VERTEX-COVER $\le_P$ SET-COVER.