Otsu Thresholding

* Image segmentation → partitions the image into different segments or groups of pixel.

* Thresholding → simplest kind of image segmentation, because it partitions the image into two groups of pixel (white : foreground & black : background)

* **Local & Global thresholding :**

  Global → a single threshold (t) is used globally for whole image

  source : MATLAB Documentation

  Local → different threshold for different part of the image.

Otsu thresholding → Automatic global thresholding algorithm.

[Adaptive thresholding]

Histogram → It is representation of distribution of data

[Source: HBY Coding Academic]

* Otsu threshold grayscale histogram data to find the obtimal cut t^* that gives best seperation of the classes. [for image binarization → foreground & background] [The method assumes that the histogram of an image is <u>bimodal</u> (i.e. two classes).

<u>Algorithm</u> [automatically determine threshold]

<u>Step1</u> : Build a histogram for given image I.

<u>Step2</u> : for each threshold t in $[0, 255]$, pixels can be separated into two classes, C_1 & C_2.

<u>Step3</u> : Those pixels whose $p_i \leq t$ [i.e pixel intensity is less than threshold t] put into C_1, otherwise into C_2.

<u>Step4</u> : Calculate the probabilities of C_1 & C_2

$\left\{ \begin{array}{l} C_1 \to \text{Background} \\ C_2 \to \text{foreground} \end{array} \right\}$ or vice-versa

$$W_1 = \frac{(\#\text{ pixels in } C_1)}{\text{total pixel count}} \quad (\text{Probability of } C_1)$$

$$W_2 = \frac{(\#\text{ pixels in } C_2)}{\text{total pixel count}} \quad [\text{Probability of } C_2)$$

<u>Step5</u> : Calculate between class variance (V_b) & within class variance (V_w).

<u>step6</u> : Obtimal cut t^* corresponds to t where V_b is maximum or V_w is minimum.

$$\boxed{\mu_c = \frac{\sum_{i=0}^{c} p_i}{N_c}}$$

$$\text{variance}(\sigma^2) = \frac{\sum_{i=0}^{N} (p_i - \mu_c)}{N \times W_c}$$

} Higher the variance, more the spread

within class variance:

$$\boxed{V_w = W_{C_1} * \sigma_{C_1}^2 + W_{C_2} * \sigma_{C_2}^2}$$

(minimize)

Between class-variance

$$\boxed{V_b = W_{C_1} W_{C_2} (\mu_{C_1} - \mu_{C_2})^2}$$

(maximize)

(a) Total variance in one image $V_t = V_w + V_b$

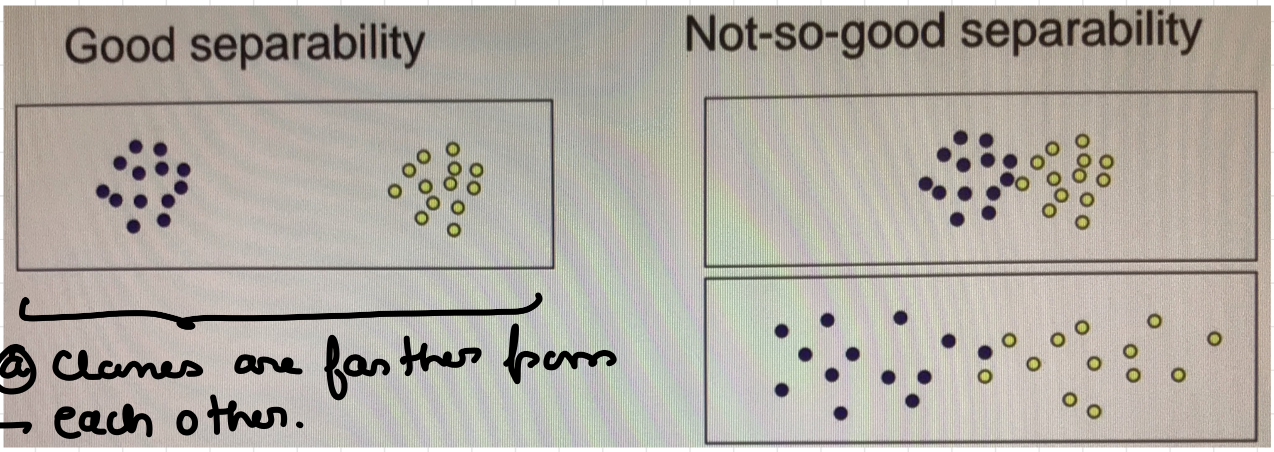
V_t cannot change therefore when V_w is minimum V_b will be maximum.

(b) We know that there are two classes in an image, which are background and foreground.

If V_b is small that means they are not very far apart from each other (it is not good result).

As the result V_b should be maximum.

" Thresholding giving the best seperation of classes in gray levels would be the best threshold."



(a) Classes are farther from each other.

(b) Element in each class is most concentrated.

V_b ensure it — V_w ensure it

Higher class separability → better thresholding

Between Variance: (V_b)

$$V_c = \frac{\sum_{p_i \in C} (p_i - \mu_c)^2}{N \times W_c} = \frac{\sum_{p_i \in C} (p_i^2 - 2p_i\mu_c + \mu_c^2)}{N \times W_c}$$

$$\Rightarrow \frac{\sum_{p_i \in C} p_i^2 - \sum_{p_i \in C} 2p_i\mu_c + \sum_{p_i \in C} \mu_c^2}{N \times W_c}$$

$$\Rightarrow \frac{\sum_{p_i \in C} p_i^2}{N \times W_c} - \frac{2\mu_c \cdot \sum p_i}{W_c \times N} + \frac{\mu_c^2 \cdot \cancel{N \times W_c}}{\cancel{N \times W_c}}$$

$$\Rightarrow \frac{\sum_{p_i \in C} p_i^2}{N \times W_c} - 2\mu_c \left(\frac{\sum p_i}{N \times W_c}\right) + \mu_c^2$$

$$\Rightarrow \frac{\sum_{p_i \in C} p_i^2}{N \times W_c} - 2\mu_c \cdot \mu_c + \mu_c^2$$

$$\boxed{V_c \Rightarrow \frac{\sum_{p_i \in C} p_i^2}{N \times W_c} - \mu_c^2}$$

Now, [within variance]

$$V_w = W_{c_1} \times V_{c_1} + W_{c_2} \times V_{c_2}$$

$$= W_{c_1} \times \left(\frac{\sum_{p_i \in C_1} p_i^2}{N \times W_{c_1}} - \mu_{c_1}^2\right) + W_{c_2}\left(\frac{\sum_{p_i \in C_2} p_i^2}{N \times W_{c_2}} - \mu_{c_2}^2\right)$$

$$\boxed{V_w = \frac{\sum_{p_i \in C_1} p_i^2}{N} - W_{C_1} \times \mu_{C_1}^2 + \frac{\sum_{p_i \in C_2} p_i^2}{N} - W_{C_2} \times \mu_{C_2}^2}$$

[Total variance]

$$V_T = \frac{\sum_{i=1}^{N} (p_i - \mu_T)^2}{N}$$

following previous expension

$$\boxed{V_T = \frac{\sum_{i=1}^{N} p_i^2}{N} - \mu_T^2}$$

[Between variance]

$$\Rightarrow \quad V_T = V_b + V_w$$

$$\Rightarrow \quad V_b = V_T - V_w$$

$$\Rightarrow \quad V_b = \left(\frac{\sum p_i^2}{N} - \mu_T^2 \right) - \left(\frac{\sum_{p_i \in C_1} p_i^2}{N} - W_{C_1} \times \mu_{C_1}^2 + \frac{\sum_{p_i \in C_2} p_i^2}{N} - W_{C_2} \times \mu_{C_2}^2 \right)$$

$$\Rightarrow V_b = W_{C_1} \times \mu_{C_1}^2 + W_{C_2} \times \mu_{C_2}^2 - \mu_T^2$$

$$\Rightarrow V_b = W_{C_1} \times \mu_{C_1}^2 + W_{C_2} \times \mu_{C_2}^2 - 2\mu_T^2 + \mu_T^2$$

Here, $\quad W_{C_1} + W_{C_2} = 1 \quad \& \quad \mu_T = W_{C_1} \times \mu_{C_1} + W_{C_2} \times \mu_{C_2}$

Now,

$$\Rightarrow W_{C_1}\mu_{C_1}^2 + W_{C_2}\mu_{C_2}^2 - 2\left(W_{C_1}\mu_{C_1} + W_{C_2}\mu_{C_2}\right) \cdot \mu_T + \left(W_{C_1} + W_{C_2}\right)\mu_T^2$$

$$\Rightarrow W_{C_1}\mu_{C_1}^2 + W_{C_2}\mu_{C_2}^2 - 2W_{C_1}\mu_{C_1}\mu_T - 2W_{C_2}\mu_{C_2} \cdot \mu_T + W_{C_1}\mu_T^2 + W_{C_2}\mu_T^2$$

$$\Rightarrow W_{c_1}\left(\mu_{c_1}^2 - 2\mu_{c_1}\mu_T + \mu_T^2\right) + W_{c_2}\left(\mu_{c_2}^2 - \mu_{c_2}\mu_T + \mu_T^2\right)$$

$$V_b \Rightarrow \boxed{W_{c_1}\left(\mu_{c_1} - \mu_T\right)^2 + W_{c_2}\left(\mu_{c_2} - \mu_T\right)^2}$$

here, $\mu_T = W_{c_1}\mu_{c_1} + W_{c_2}\mu_{c_2}$

now,

$$V_b \Rightarrow W_{c_1}\left(\mu_{c_1} - \left(W_{c_1}\mu_{c_1} + W_{c_2}\mu_{c_2}\right)\right)^2 + W_{c_2}\left(\mu_{c_2} - \left(W_{c_1}\mu_{c_1} + W_{c_2}\mu_{c_2}\right)\right)^2$$

$$\Rightarrow W_{c_1}\left(\mu_{c_1} - \left((1 - W_{c_2})\mu_{c_1} + W_{c_2}\mu_{c_2}\right)\right)^2 + W_{c_2}\left(\mu_{c_2} - \left(W_{c_1}\mu_{c_1} + (1 - W_{c_1})\mu_{c_2}\right)\right)^2$$

$$\Rightarrow W_{c_1}\left(\mu_{c_1} - \left(\mu_{c_1} - \mu_{c_1}W_{c_2} + W_{c_2}\mu_{c_2}\right)\right)^2 + W_{c_2}\left(\mu_{c_2} - \left(W_{c_1}\mu_{c_1} + \mu_{c_2} - W_{c_1}\mu_{c_2}\right)\right)^2$$

$$\Rightarrow W_{c_1}\left(\cancel{\mu_{c_1}} - \cancel{\mu_{c_1}} + \mu_{c_1}W_{c_2} - W_{c_2}\mu_{c_2}\right)^2 + W_{c_2}\left(\cancel{\mu_{c_2}} - W_{c_1}\mu_{c_1} - \cancel{\mu_{c_2}} + W_{c_1}\mu_{c_2}\right)^2$$

$$\Rightarrow W_{c_1}\left(W_{c_2}\left(\mu_{c_1} - \mu_{c_2}\right)\right)^2 + W_{c_2}\left(W_{c_1}\left(\mu_{c_2} - \mu_{c_1}\right)\right)^2$$

$$\Rightarrow W_{c_1}W_{c_2}^2\left(\mu_{c_1} - \mu_{c_2}\right)^2 + W_{c_2}W_{c_1}^2\left(\mu_{c_2} - \mu_{c_1}\right)^2$$

$$\Rightarrow \left(W_{c_1}W_{c_2}^2 + W_{c_2}W_{c_1}^2\right)\underline{\left(\mu_{c_1} - \mu_{c_2}\right)^2}$$

$$(a-b)^2 = a^2 + b^2 - 2ab \iff (b-a)^2 = b^2 + a^2 - 2ab$$

$$\Rightarrow W_{c_1} W_{c_2} \left(\frac{W_{c_2} + W_{c_1}}{1} \right) (\mu_{c_1} - \mu_{c_2})^2$$

$$\boxed{V_b = W_{c_1} W_{c_2} (\mu_{c_1} - \mu_{c_2})^2}$$ //

Observation:

(a) Large mean difference → large V_b (between variance)

(b) When, mean diff is constant then maximum V_b can only be reach by $\boxed{W_{c_1} = W_{c_2}}$ (i.e. equal no. of background & fore ground pixel).