MA 203: Numerical Methods

Tutorial-2

Shardul Junagade
23110297

In [8]:
```
import numpy as np
import matplotlib.pyplot as plt
```

Question 1:

Suppose that a spherical droplet of a liquid evaporates at a rate that is proportional to its surface area.

$$\frac{dV}{dt} = -kA$$

where $V =$ volume (mm ^{3}), $t =$ time (min), $k =$ the evaporation rate (mm/min), and $A =$ surface area (mm ^{2}). Use Euler's method to compute the volume of the droplet from $t = 0$ to 10 min using a step size of 0.25 min. Assume that $k = 0.1$ mm/min and that the droplet initially has a radius of 3 mm. Assess the validity of your results by determining the radius of your final computed volume. Calculate the average evaporation rate (change in radius/time) and verify that it is consistent with the given evaporation rate.

Solution:

We will use Euler's method to approximate the volume of a spherical droplet as it evaporates over time. The evaporation rate is proportional to the surface area of the droplet. The differential equation governing this process is:

$$\frac{dV}{dt} = -kA$$

Constants and Initial Conditions:

- $k = 0.1$ mm/min (evaporation rate)
- $r_{\text{initial}} = 3$ mm (initial radius)

- $t_{\text{start}} = 0$ min (start time)
- $t_{\text{end}} = 10$ min (end time)
- $\Delta t = 0.25$ min (time step)

The initial volume V is calculated using the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r_{\text{initial}}^3$$

Euler's Method

Euler's method is used to iteratively compute the volume and radius of the droplet at each time step. The surface area A of the droplet is given by:

$$A = 4\pi r^2$$

The rate of change in volume $\frac{dV}{dt}$ is then calculated using the evaporation rate k and the surface area A:

$$\frac{dV}{dt} = -kA$$

The new volume V is then updated as:

$$V_{i+1} = V_i + \Delta t \cdot \left(\frac{dV}{dt}\right)_i$$

$$V_{i+1} = V_i - k \cdot 4\pi r_i^2 \cdot \Delta t$$

The new radius r is computed from the updated volume:

$$r_{i+1} = \left(\frac{3V_{i+1}}{4\pi}\right)^{1/3}$$

Average rate of evaporation

The average rate of evaporation can be calculated by dividing the radius change by the time change:

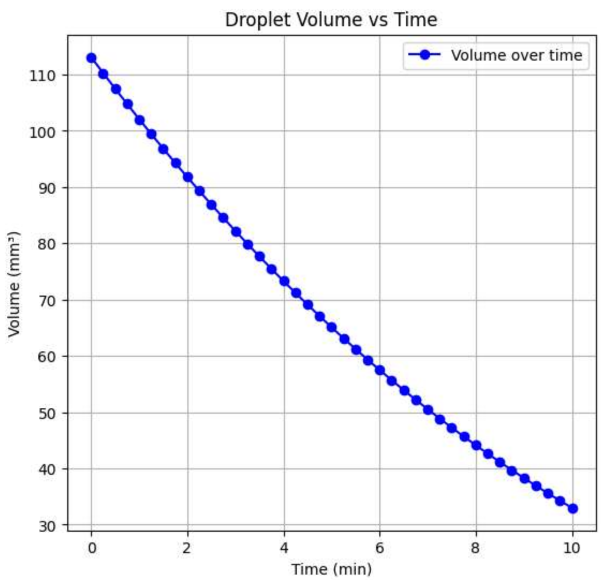
$$\text{Average rate of evaporation} = \frac{r_{\text{final}} - r_{\text{initial}}}{t_{\text{end}} - t_{\text{start}}}$$

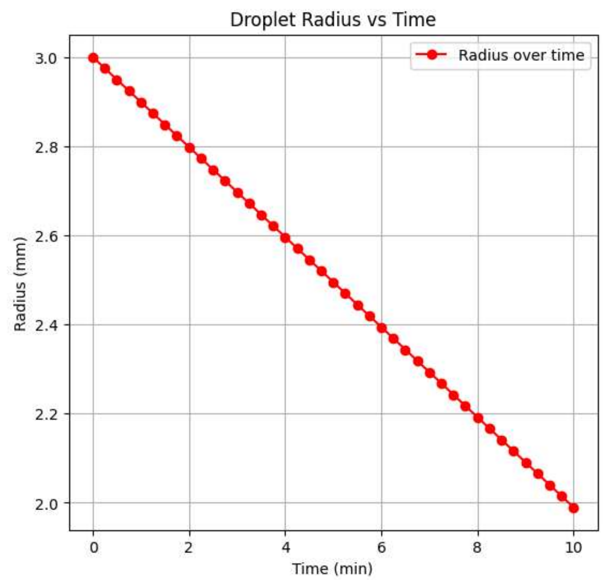
In [9]:
```
# Constants
k = 0.1                 # Evaporation rate (mm/min)
r_initial = 3           # Initial radius (mm)
t_start = 0             # Start time (min)
```

```python
t_end = 10            # End time (min)
dt = 0.25             # Time step (min)
# Initial Conditions
V = (4/3) * np.pi * r_initial**3
r = r_initial

V_list = [V]
t_list = [t_start]
r_list = [r_initial]

# Euler's Method
t = t_start
while t < t_end:
        A = 4 * np.pi * r**2
        dV = -k * A
        V += dV * dt
        r = (3 * V / (4 * np.pi))**(1/3)
        t += dt

        print(f"Time: {t:.2f} min, Volume: {V:.4f} mm³, Radius: {r:.4f} mm")
        V_list.append(V)
        r_list.append(r)
        t_list.append(t)


# Final Results
final_volume = V_list[-1]
final_radius = r_list[-1]
print(f"\nFinal volume: {final_volume:.4f} mm³")
print(f"Final radius: {final_radius:.4f} mm")

# Calculate Average Evaporation Rate (Change in radius/time)
avg_rate = (r_initial - final_radius) / (t_end - t_start)
print(f"Average evaporation rate: {avg_rate:.4f} mm/min")
print(f"Given evaporation rate: {k:.4f} mm/min")

# Plot Results
fig, ax1 = plt.subplots(1, 2, figsize=(14, 6))
# Plot Volume vs Time
ax1[0].plot(t_list, V_list, label='Volume over time', color='blue', marker='o')
ax1[0].set_xlabel('Time (min)')
ax1[0].set_ylabel('Volume (mm³)')
ax1[0].set_title('Droplet Volume vs Time')
ax1[0].legend()
ax1[0].grid(True)
# Plot Radius vs Time
ax1[1].plot(t_list, r_list, label='Radius over time', color='red', marker='o')
ax1[1].set_xlabel('Time (min)')
ax1[1].set_ylabel('Radius (mm)')
ax1[1].set_title('Droplet Radius vs Time')
```

```python
ax1[1].legend()
ax1[1].grid(True)
plt.show()
```

```
Time: 0.25 min, Volume: 110.2699 mm³, Radius: 2.9748 mm
Time: 0.50 min, Volume: 107.4898 mm³, Radius: 2.9496 mm
Time: 0.75 min, Volume: 104.7566 mm³, Radius: 2.9244 mm
Time: 1.00 min, Volume: 102.0700 mm³, Radius: 2.8991 mm
Time: 1.25 min, Volume: 99.4294 mm³, Radius: 2.8739 mm
Time: 1.50 min, Volume: 96.8347 mm³, Radius: 2.8487 mm
Time: 1.75 min, Volume: 94.2852 mm³, Radius: 2.8235 mm
Time: 2.00 min, Volume: 91.7807 mm³, Radius: 2.7983 mm
Time: 2.25 min, Volume: 89.3208 mm³, Radius: 2.7730 mm
Time: 2.50 min, Volume: 86.9050 mm³, Radius: 2.7478 mm
Time: 2.75 min, Volume: 84.5330 mm³, Radius: 2.7226 mm
Time: 3.00 min, Volume: 82.2043 mm³, Radius: 2.6973 mm
Time: 3.25 min, Volume: 79.9186 mm³, Radius: 2.6721 mm
Time: 3.50 min, Volume: 77.6754 mm³, Radius: 2.6469 mm
Time: 3.75 min, Volume: 75.4745 mm³, Radius: 2.6216 mm
Time: 4.00 min, Volume: 73.3153 mm³, Radius: 2.5964 mm
Time: 4.25 min, Volume: 71.1975 mm³, Radius: 2.5711 mm
Time: 4.50 min, Volume: 69.1206 mm³, Radius: 2.5459 mm
Time: 4.75 min, Volume: 67.0844 mm³, Radius: 2.5206 mm
Time: 5.00 min, Volume: 65.0884 mm³, Radius: 2.4954 mm
Time: 5.25 min, Volume: 63.1321 mm³, Radius: 2.4701 mm
Time: 5.50 min, Volume: 61.2152 mm³, Radius: 2.4449 mm
Time: 5.75 min, Volume: 59.3374 mm³, Radius: 2.4196 mm
Time: 6.00 min, Volume: 57.4981 mm³, Radius: 2.3944 mm
Time: 6.25 min, Volume: 55.6971 mm³, Radius: 2.3691 mm
Time: 6.50 min, Volume: 53.9338 mm³, Radius: 2.3438 mm
Time: 6.75 min, Volume: 52.2080 mm³, Radius: 2.3185 mm
Time: 7.00 min, Volume: 50.5192 mm³, Radius: 2.2933 mm
Time: 7.25 min, Volume: 48.8670 mm³, Radius: 2.2680 mm
Time: 7.50 min, Volume: 47.2510 mm³, Radius: 2.2427 mm
Time: 7.75 min, Volume: 45.6708 mm³, Radius: 2.2174 mm
Time: 8.00 min, Volume: 44.1261 mm³, Radius: 2.1921 mm
Time: 8.25 min, Volume: 42.6164 mm³, Radius: 2.1669 mm
Time: 8.50 min, Volume: 41.1414 mm³, Radius: 2.1416 mm
Time: 8.75 min, Volume: 39.7006 mm³, Radius: 2.1163 mm
Time: 9.00 min, Volume: 38.2936 mm³, Radius: 2.0910 mm
Time: 9.25 min, Volume: 36.9200 mm³, Radius: 2.0657 mm
Time: 9.50 min, Volume: 35.5795 mm³, Radius: 2.0403 mm
Time: 9.75 min, Volume: 34.2717 mm³, Radius: 2.0150 mm
Time: 10.00 min, Volume: 32.9961 mm³, Radius: 1.9897 mm

Final volume: 32.9961 mm³
Final radius: 1.9897 mm
Average evaporation rate: 0.1010 mm/min
Given evaporation rate: 0.1000 mm/min
```





Results:

- The final volume of the droplet is $\mathbf{32.9961}$ **mm³**, the final radius is $\mathbf{1.9897}$ **mm**.
- The average evaporation rate is $\mathbf{0.1010}$ **mm/min**, which is consistent with the given evaporation rate of $\mathbf{0.1000}$ **mm/min**.

Question 2:

Use zero-through third-order Taylor series expansions to predict $f(3)$ for

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$

Using a base point at $x = 1$. Compute the true percent relative error ϵ_t for each approximation.

Solution:

The Taylor series expansion of a function $f(x)$ about a base point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where $f'(x)$, $f''(x)$, and $f'''(x)$ are the first, second, and third derivatives of $f(x)$, respectively.

Given $f(x) = 25x^3 - 6x^2 + 7x - 88$, we can compute the first, second, and third derivatives as follows:

1. First derivative:

$$f'(x) = 75x^2 - 12x + 7$$

2. Second derivative:

$$f''(x) = 150x - 12$$

3. Third derivative:

$$f'''(x) = 150$$

Using the base point $x = 1$, we can compute the function and its derivatives at $x = 1$:

- $f(1) = 25(1)^3 - 6(1)^2 + 7(1) - 88 = 25 - 6 + 7 - 88 = -62$
- $f'(1) = 75(1)^2 - 12(1) + 7 = 75 - 12 + 7 = 70$
- $f''(1) = 150(1) - 12 = 150 - 12 = 138$
- $f'''(1) = 150$

Zero-through third-order approximations:

Now, we can use the Taylor series expansions to predict $f(3)$ for zero-through third-order approximations:

$$\begin{aligned}
\text{Zero-order approximation:} \quad & f(3) \approx f(1) = -62 \\
\text{First-order approximation:} \quad & f(3) \approx f(1) + f'(1)(3-1) \\
& \quad = -62 + 70 \cdot 2 = -62 + 140 \\
& \quad = 78 \\
\text{Second-order approximation:} \quad & f(3) \approx f(1) + f'(1)(3-1) + \frac{f''(1)}{2!}(3-1)^2 \\
& \quad = -62 + 70 \cdot 2 + \frac{138}{2} \cdot 4 \\
& \quad = -62 + 140 + 276 \\
& \quad = 354 \\
\text{Third-order approximation:} \quad & f(3) \approx f(1) + f'(1)(3-1) + \frac{f''(1)}{2!}(3-1)^2 + \frac{f'''(1)}{3!}(3 - \\
& \quad = -62 + 70 \cdot 2 + \frac{138}{2} \cdot 4 + \frac{150}{6} \cdot 8 \\
& \quad = -62 + 140 + 276 + 200 \\
& \quad = 554
\end{aligned}$$

True Percent Relative Error:

The true value of $f(3)$ is:

$$
\begin{aligned}
f(3) &= 25(3)^3 - 6(3)^2 + 7(3) - 88 \\
&= 25 \cdot 27 - 6 \cdot 9 + 21 - 88 \\
&= 675 - 54 + 21 - 88 \\
&= 554
\end{aligned}
$$

The true percent relative error ϵ_t for each approximation is given by:

$$
\epsilon_t = \left| \frac{f(3) - f_{\text{approx}}(3)}{f(3)} \right| \times 100\%
$$

where $f_{\text{approx}}(3)$ is the predicted value of $f(3)$ using the respective approximation.

The true percent relative errors for each approximation are:

1. Zero-order approximation:

$$
\epsilon_t = \left| \frac{554 - (-62)}{554} \right| \times 100\% = \left| \frac{616}{554} \right| \times 100\% = 111.19\%
$$

2. First-order approximation:

$$
\epsilon_t = \left| \frac{554 - 78}{554} \right| \times 100\% = \left| \frac{476}{554} \right| \times 100\% = 85.92\%
$$

3. Second-order approximation:

$$
\epsilon_t = \left| \frac{554 - 354}{554} \right| \times 100\% = \left| \frac{200}{554} \right| \times 100\% = 36.10\%
$$

4. Third-order approximation:

$$
\epsilon_t = \left| \frac{554 - 554}{554} \right| \times 100\% = 0\%
$$

Results:

The predicted values of $f(3)$ using zero-through third-order Taylor series approximations are:

1. Zero-order approximation: $f(3) \approx \textbf{-62}$
2. First-order approximation: $f(3) \approx \mathbf{78}$
3. Second-order approximation: $f(3) \approx \mathbf{354}$
4. Third-order approximation: $f(3) \approx \mathbf{554}$

The true percent relative errors for each approximation are:

1. Zero-order approximation: $\mathbf{111.19\backslash\%}$

2. First-order approximation: $\mathbf{85.92\%}$
3. Second-order approximation: $\mathbf{36.10\%}$
4. Third-order approximation: $\mathbf{0\%}$

Question 3:

The N-th Taylor polynomial for $f(x) = \log x$ expanded about $x_0 = 1$ is

$$P_N = \sum_{i=1}^{N} \frac{(-1)^{(i+1)}}{i}(x-1)^i$$

and the value of $\log 1.5$ to eight decimal places is 0.40546511. Write an algorithm to determine the minimal value of N required for

$$|\log 1.5 - P_N(1.5)| < 10^{-5},$$

without using the Taylor polynomial remainder term.

Solution:

We are given the N-th Taylor polynomial for $f(x) = \log x$ expanded about $x_0 = 1$ as:

$$P_N = \sum_{i=1}^{N} \frac{(-1)^{(i+1)}}{i}(x-1)^i$$

We are also given the value of $\log 1.5$ to eight decimal places as 0.40546511.

Our goal is to determine the minimal value of N required for

$$|\log 1.5 - P_N(1.5)| < 10^{-5}$$

We can approximate the value of $\log 1.5$ using the Taylor polynomial P_N and check the error until it is less than the tolerance 10^{-5}. We increment the value of N and calculate the N-th term of the Taylor polynomial until the error is less than the tolerance.

In [10]:
```
x = 1.5
log_1_5 = 0.40546511
err_tol = 1e-5

N = 0
P_n = 0
while True:
```

```python
        N += 1
        ith_term = ((-1)**(N+1) / N) * (x - 1)**N
        P_n += ith_term
        err = abs(log_1_5 - P_n)
        # print(f"{err:.8f}")
        if err < err_tol:
                break

print(f"Minimum N required = {N}")
print(f"Approximated value:     P_N(1.5) = {P_n:.8f}")
print(f"Actual value:           log(1.5) = {log_1_5:.8f}")
print(f"Error: {err:.8f}")
```

```
Minimum N required = 12
Approximated value:     P_N(1.5) = 0.40545869
Actual value:           log(1.5) = 0.40546511
Error: 0.00000642
```

Thus, the minimal value of N required for $|\log 1.5 - P_N(1.5)| < 10^{-5}$ is **12**.