Validating the Multidimensional Item Response Theory Model

Shariar Kabir

shariar1405076@gmail.com

1 Introduction

A central challenge in Multidimensional Item Response Theory (MIRT) models is the issue of *identifiability*. The latent ideological space is invariant to rotation, reflection, and translation, meaning that an infinite number of parameter sets $(\boldsymbol{\theta}, \boldsymbol{\alpha}, \beta)$ can produce the identical likelihood. Without proper constraints, this leads to unstable parameter estimates and prevents meaningful interpretation of the dimensions. This appendix documents our iterative approach to resolving this issue. The success of each identification strategy is evaluated by its ability to produce stable and externally valid ideal points, which we assess by correlating our estimates with the widely-used DW-NOMINATE [2] and IDEAL scores [1].

2 Model Specification

The core of our model is a two-parameter logistic D-dimensional IRT, where the probability of a candidate j voting "yes" on item k is given by the *liklihood function*:

$$\begin{aligned} P(y_{jk} = 1) &= \text{logit}^{-1}\left(\boldsymbol{\theta}_j \cdot \boldsymbol{\alpha}_k - \beta_k\right) \\ y_{jk} &\sim Bernoulli(logit^{-1}(\sum_{d=1}^{D} \theta_{jd}\alpha_{kd} - \beta_k) \end{aligned} \tag{1}$$

where $\boldsymbol{\theta}_j \in \mathbb{R}^D$ is the vector of ideal points for candidate j, $\boldsymbol{\alpha}_k \in \mathbb{R}^D$ is the vector of discrimination parameters for bill k, and $\beta_k \in \mathbb{R}$ is the difficulty parameter of the item.

Additionally, two $D \times D$ correlation matrices Ω_θ and Ω_α is used to represent how different ideological dimensions (e.g., economic and social conservatism) correlate across candidates. We used the following priors: $\theta_j \sim \mathcal{N}_D(0_D, \Omega_\theta)$, $\alpha_k \sim \mathcal{N}_D(0_D, \Omega_\alpha)$, $\Omega_\theta \sim LKJ(1)$ and $\beta_k \sim \mathcal{N}(0, 10)$.

The LKJ distribution is used for Ω_θ, because it provides a principled prior over the space of the correlation matrix. The distribution $LKJ(\eta)$ has density: $p(\Omega) \propto |\Omega|^{\eta-1}$ where $|\Omega|$ is the determinant. For $\eta = 1$, this yields a uniform distribution over correlation matrices, ensuring no a priori bias toward any particular correlation structure while maintaining proper normalization. We used an improper uniform prior for Ω_α over the space of valid correlation matrices.

3 Evolution of Identification Strategy

We explored three identification strategies to resolve the rotational invariance of the latent space.

3.1 Identification via Priors (No Fixed References)

Our initial model specification attempted to achieve identification solely through the specification of priors, particularly on the correlation matrices for the ideal points ($\boldsymbol{\Omega}_\theta$) and discrimination parameters ($\mathbf{\Omega}_\alpha$). While this approach allowed the model to converge, it proved insufficient for resolving the rotational ambiguity. The model identified one strong dimension that correlated highly with the same dimension of DW-NOMINATE, but frequently failed to produce another stable dimension. Consequently, the correlation on the other dimension was typically low and statistically insignificant (p-value $\sim$ 0.19), as the model could freely rotate the latent space without penalty. As shown at the bottom of Figure 1, this results in wide standard deviations (the shaded bars), indicating high uncertainty and model instability.

3.2 Two-Point Constraint Identification

Following the precedent set by the IDEAL model by Clinton et al. [1], we introduced constraints by fixing the positions of reference candidates with well-established, opposing ideologies to anchor the primary dimension. For instance, a strongly liberal candidate was fixed at $\theta_{j_1,1} = -2.0$ and a strongly conservative candidate was fixed at $\theta_{j_2,1} = 2.0$. This successfully stabilized the first dimension, yielding consistently high correlations with DW-NOMINATE and IDEAL. However, the second dimension remained unidentified. While the plane of candidate ideal points was fixed relative to the two reference points, it could still rotate around the axis connecting them, leading to highly variable and unreliable estimates for the second dimension across repeated trials. This can be seen in the middle of Figure 1 where Dim. 1 becomes highly correlated and very stable (small bars), but Dim. 2 remains uncertain.

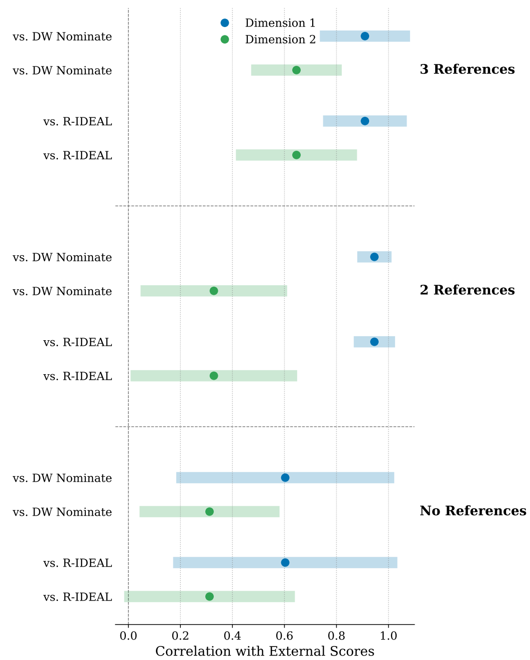

Figure 1: Model Performance by Identification Strategy.

3.3 Three-Point Constraint Identification

To achieve full model identification in two dimensions, a third reference point is necessary to fix the rotational freedom of the plane. Our final model specification implements this by constraining a third candidate, chosen to be orthogonal to the primary liberal-conservative dimension. The final constraints were set as follows: $\theta_{j_1,1} = -2.0$ (Liberal), $\theta_{j_2,1} = 2.0$ (Conservative), $\theta_{j_3,1} = 0.0$ and $\theta_{j_3,2} = 2.0$ (Orthogonal). This three-point constraint fully locks the orientation of the latent space. As evidenced by the correlation plots in the main report, this specification produces stable and substantively meaningful results. The model now achieves consistently high correlations on *both* dimensions when validated against DW-NOMINATE and the IDEAL model scores. This can be seen at the top of Figure 1, where both dimensions show strong and stable correlations.

This confirms that a three-point identification constraint is crucial for recovering a valid and reliable two-dimensional ideological space from legislative voting data.

References

[1] J. Clinton, S. Jackman, and D. Rivers. The statistical analysis of roll call data. *American Political Science Review*, 98(2):355–370, 2004.

[2] K. T. Poole and H. Rosenthal. A spatial model for legislative roll call analysis. *American journal of political science*, pages 357–384, 1985.