SDE Examples.

1. $dX(t) = AX(t)dt + h\,dW(u)$.

$X(t) = e^{At}X(0) + \int_0^t e^{A(t-u)} h\,dW(u)$

$$\frac{d\mathbb{E}[\phi]}{dt} = \mathbb{E}\left[\frac{\partial\phi}{\partial t}\right] + \sum_i \mathbb{E}\left[\frac{\partial\phi}{\partial x_i} f_i(\underline{x}, t)\right] \quad \Leftarrow dx = f(x,t)dt + L(x,t)d\beta$$

$$+ \frac{1}{2}\sum_{i,j} \mathbb{E}\left[\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)\left[L(x,t)Q\,L^T(x,t)\right]_{ij}\right]$$

take $\phi = X(t)$, $f(x,t) = A\,x(t)$ $\quad L(x,t) = h$

$$\frac{dm_t}{dt} = A\,m_t \quad \Rightarrow \quad m_t = \exp(A(t-t_0))\,m_{t_0} = \exp(At)\,m_0.$$

take $\phi = (X(t) - m_t)^2$.

$$\frac{dP_t}{dt} = 2\mathbb{E}\{(X(t)-m_t)\,A\,X(t)\} + \frac{1}{2} \times 2 \times h^2 q$$

$$= 2\mathbb{E}\{A\,X(t)\,(X(t)-m_t)\} + h^2 q.$$

$$= 2A\mathbb{E}\{(X(t)-m_t)(X_t - m_t)\} + h^2 q$$

$$= 2A\,P_t + h^2 q.$$

$$\Rightarrow P_t = \exp(2At)P_0 + \int_0^t \exp(2A(t-\tau))\,h^2 q\,d\tau$$

$$= \exp(2At)P_0 + \frac{h^2 q}{2A}\left(\exp(2At) - \cancel{\exp(2A\cdot 0)}^{\,1}\right)$$

$$p(X(t)|X(s)) = N(X(t)\,|\,m_{t|s}, P_{t|s})$$

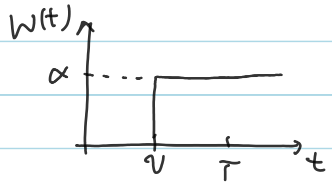
$$m_{t|s} = \exp(A(t-s))\,\cancel{m_s}^{\,X(s)}$$

$$P_{t|s} = \exp(2A(t-s))\,\cancel{P_s}^{\,0} + \frac{h^2 q}{2A}(\exp(2A(t-s)) - 1)$$

$$= \frac{h^2 q}{2A}(\exp(2A(t-s)) - 1)$$

Why $A<0 \Rightarrow$ mean is not exploding

2. $W(t) = \alpha \mathbb{1}[v, +\infty)$ $\quad dW = \alpha \delta(t-v)$



$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-u)} \alpha \delta(u-v) h$$

$$= e^{At} X(0) + \alpha h e^{A(t-v)}$$

3. $W(t) = tb$ $\quad dW = b\,dt$

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-u)} h b\, du.$$

$$= e^{At} X(0) + \frac{1}{A} h b (e^{At} - 1)$$

4. $W(t) = \sum_{i=1}^{I} u_i \mathbb{1}(V_i < t)$ $\quad dW = \sum_{i=1}^{I} u_i \delta(t - V_i)$

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-u)} h \sum_{i=1}^{I} u_i \delta(t - V_i)$$

$$= e^{At} X(0) + h \sum_{i=1}^{I} u_i e^{A(t-V_i)} \mathbb{1}(V_i < t)$$

5. Lévy process $\quad (A, v, \gamma)$

$\Rightarrow$ Brownian motion cov, Lévy measure, 'drift'

4) $u_i \sim N(0,1)$

$V_i \sim U[0,T]$

compound Poisson process

$v(dx) = I/T$, $A = \gamma = 0$.

Lévy measure : # of jumps / unit time

1) Brownian motion

$\gamma = 0$. $A = 1$ for standard brownian motion.

no jumps $v = 0$.

2) not infinitely divisible.

3) pure drift. $A = v = 0$, $\gamma = b$.

6. $A = \begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix}$

$$m(t|s) = \exp(A(t-s))\,x(s)$$
$$= \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix}(t-s)\right) x(s)$$

I for std.

$$P(t|s) = \int_s^t \exp(A(t-\tau))\, \underbrace{h\,Q\,h^T}\, \exp(A(t-\tau))^T d\tau.$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \int_s^t$$

$$X(t) = \exp(At)X(0) + \int_0^t \exp(A(t-u))\,h\,dW(u)$$
$$= \exp(At)X(0) + \int_0^t \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix}(t-u)\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} dW(u)$$

$$\exp\left(\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix} t\right) = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots$$
$$= I + t\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} 0 & \theta \\ 0 & \theta^2 \end{bmatrix}$$
$$+ \frac{t^3}{3!}\begin{bmatrix} 0 & \theta^2 \\ 0 & \theta^3 \end{bmatrix} + \cdots$$
$$= \begin{bmatrix} 1 & \frac{1}{\theta}(\exp(t\theta)-1) \\ 0 & \exp(t\theta) \end{bmatrix}$$

$$X(T) = \exp(At)\,X(0) + \int_0^t \begin{bmatrix} \frac{1}{\theta}(\exp(t-u)\theta - 1 + \theta) \\ \exp((t-u)\theta) \end{bmatrix} dW(u)$$

Calculate matrix exp using . SVD.

$$\exp At = \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix} t\right)$$

$$\overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix}} = Q_1 \Sigma Q_2^t$$

Q_1: eigenvectors of AA^t
Q_2: eigenvectors of A^tA.

$$AA^t = \begin{bmatrix} 1 & \theta \\ \theta & \theta^2 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ \theta \end{bmatrix} \quad \lambda = \theta^2+1$$

$$A^tA = \begin{bmatrix} 0 & 0 \\ 0 & 1+\theta^2 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda = \theta^2+1.$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{\theta^2+1}} & 0 \\ \frac{\theta}{\sqrt{\theta^2+1}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\theta^2+1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \exp At = \begin{bmatrix} \frac{1}{\sqrt{\theta^2+1}} & 0 \\ \frac{\theta}{\sqrt{\theta^2+1}} & 0 \end{bmatrix} \begin{bmatrix} \exp(\sqrt{\theta^2+1}\, t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\exp\sqrt{\theta^2+1}\, t}{\sqrt{\theta^2+1}} & 0 \\ \frac{\theta \exp\sqrt{\theta^2+1}\, t}{\sqrt{\theta^2+1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\exp\sqrt{\theta^2+1}\, t}{\sqrt{\theta^2+1}} \\ 0 & \frac{\theta \exp\sqrt{\theta^2+1}\, t}{\sqrt{\theta^2+1}} \end{bmatrix}$$