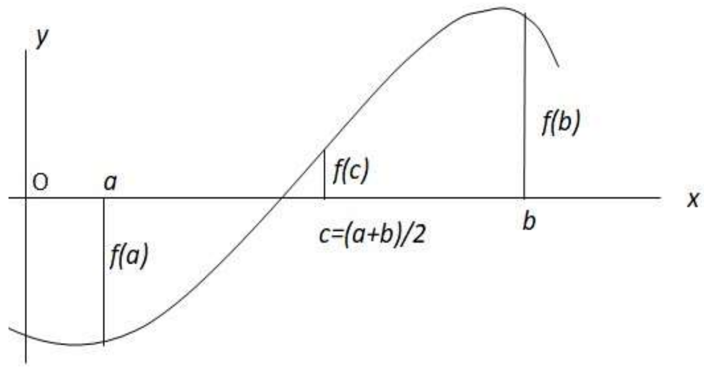
Solution of Equations

Bisection Method: The Bisection Method is applicable for numerically solving the equation $f(x) = 0$ for the real variable x, where $f(x)$ is a continuous function defined on an interval $[a, b]$ and where $f(a)$ and $f(b)$ have opposite signs. In this case, by the intermediate value theorem, the continuous function $f(x)$ must have at least one root in the interval (a, b).



At each step the method divides the interval $[a, b]$ in two equal subintervals $[a, c]$ and $[c, b]$ by computing the midpoint $c = \frac{a+b}{2}$ of the interval $[a, b]$. Unless c is itself a root (which is very unlikely, but possible) there are now only two possibilities: either $f(a)$ and $f(c)$ have opposite signs and there is a root between a and c, or $f(c)$ and $f(b)$ have opposite signs and there is a root between c and b. Explicitly, if $f(a)$ and $f(c)$ have opposite signs, then the method sets c as the new value for b, and if $f(b)$ and $f(c)$ have opposite signs then the method sets c as the new a. (If $f(c) = 0$ then c may be taken as the solution and the process stops.) In both cases, the new $f(a)$ and $f(b)$ have opposite signs, so the method is applicable to this smaller interval. In this way an interval that contains a zero of $f(x)$ is reduced in width by 50% at each step. The process is continued until the interval is sufficiently small, i.e., $f(c)$ is near to zero.

1 . Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution : We get

$$f(1) = 1^3 - 1 - 1 = -1 \quad \text{(-ve)}$$
$$f(2) = 2^3 - 2 - 1 = 5 \quad \text{(+ve)}$$

Hence there is a root between 1 & 2 so we take $x_0 = \frac{1+2}{2} = 1.5$

Now, $f(x_0) = f(1.5) = 1.5^3 - 1.5 - 1 = 0.875$ (+ve)

Hence there is a root between 1 & x_0. So we take $x_1 = \frac{1+x_0}{2} = \frac{1+1.5}{2} = 1.25$

We find, $f(x_1) = f(1.25) = (1.25)^3 - 1.25 - 1 = -0.296875$ (-ve)

Hence the root lies between x_0 & x_1. So we take $x_2 = \frac{x_0+x_1}{2} = \frac{1.5+1.25}{2} = 1.375$

We obtain, $f(x_2) = f(1.375) = (1.375)^3 - 1.375 - 1 = 0.2246$ (+ve)

Hence the root lies between x_1 & x_2. So we take $x_3 = \frac{x_1+x_2}{2} = \frac{1.25+1.375}{2} = 1.3125$

We obtain, $f(x_3) = f(1.3125) = (1.3125)^3 - 1.3125 - 1 = -0.051$ (-ve)

So we take $x_4 = \frac{x_2+x_3}{2} = 1.34375$

Now, $f(x_4) = 0.0826$ (+ve)

So let $x_5 = \frac{x_3+x_4}{2}$ =1.328125

Now, f(x_5)=0.0145, which is near to zero.

Hence the appropriate root of the given equation is 1.328125.

2 . Use bisection method determine the root of $f(x) = e^{-x} - x = 0$

Solution : we get, $f(0) = 1 - 0 = 1$ *(+ve)*

$f(1) = -0.63$ *(-ve)*

Hence the root lies between 0 & 1 so we take $x_0 = \frac{0+1}{2}$ = 0.5

Now, $f(x_0) = f(0.5) = 0.1065$ (+ve)

Hence the root lies between 1 & x_0. So we take $x_1 = \frac{1+x_0}{2} = \frac{1+.5}{2}$ =0.75

We find, $f(x_1) = f(.75) = -0.278$ (-ve)

Hence the root lies between x_0 & x_1, So we take $x_2 = \frac{x_0+x_1}{2} = \frac{.5+.75}{2}$ =.625

We obtain, $f(x_2) = f(.625) = -0.088$ (-ve)

Hence the root lies between x_0 & x_2. So we take $x_3 = \frac{x_0+x_2}{2} = \frac{.5+.625}{2}$ =.5625

We obtain, $f(x_3) = f(.5625) = 0.0073$, which is near to zero.

Hence the appropriate root of the given equation is 0.5625.

3 . Perform five interaction of the bisection method to determine the smallest positive real root of $f(x) = x^3 - 5x + 1 = 0$

Solution : we get $f(0) = 1$ (+ve)

$f(1) = -3$ (-ve)

Hence the root lies between 0 & 1, So we take $x_0 = \frac{0+1}{2}$ = 0.5

Now, $f(x_0) = f(0.5) = -01.375$ (-ve)

Hence the root lies between 0 & x_0, So we take $x_1 = \frac{0+x_0}{2} = \frac{0+.5}{2}$ =0.25

We find, $f(x_1) = f(.25) = -0.234$ (-ve)

Hence the root lies between 0 & x_1. So we take $x_2 = \frac{0+x_1}{2} = \frac{0+.25}{2}$ =.125

We obtain, $f(x_2) = f(.125) = 0.377$ (+ve)

Hence the root lies between x_1 & x_2, So we take $x_3 = \frac{x_1+x_2}{2}$ = .1875

We obtain, $f(x_3) = f(.1875) = 0.069$ (+ve)

Hence the root lies between x_1 & x_3. So we take $x_4 = \frac{x_1+x_3}{2} = \frac{.25+.1875}{2}$ =0.21875

We obtain, $f(x_4) = f(.21875) = -0.08$ (-ve)

Hence the root lies between x_3 & x_4. So we take $x_5 = \frac{x_3+x_4}{2}$ = 0.203125

We obtain, f(x_5) = f(.203125) = -0.007, which is near to zero.

Hence the smallest positive real root of the given equation is 0.20325

4. Use bisection method find x_3 of the equation $\sqrt{x} - cosx = 0$ on [0,1]

Let , $f(x) = \sqrt{x} - cosx$

Solution : we get $f(0) = -1$ (-ve)

$f(1) = 0.45970$ (+ve)

Hence the root lies between 0 & 1. So we take $x_1 = \frac{0+1}{2} = 0.5$

Now, $f(x_1) = f(0.5) = -0.17048$ (-ve)

Hence the root lies between 1 & x_1. So we take $x_2 = \frac{1+x_1}{2} = \frac{1+.5}{2} = 0.75$

We find, $f(x_2) = f(.75) = 0.13434$ (+ve)

Hence the root lies between x_1 & x_2 so we take $x_3 = \frac{x_1+x_2}{2} = \frac{.5+.75}{2} = .625$

Hence $x_3 = 0.625$

5. Use the bisection algorithm to find an approximation to $\sqrt{3}$.

Solution: Let , $x = \sqrt{3}$

or, $x^2 = 3$

or, $x^2 - 3 = 0$

therefore $\sqrt{3}$ is a root of $f(x) = x^2 - 3 = 0$

we get, $f(2) = 2^2 - 3 = 1$ (+ve)

So, there is a root between 1 and 2. Hence we take $x_1 = \frac{1+2}{2} = 1.5$

Now, $f(x_1) = f(1.5) = -0.75$ (-ve)

So we obtain $x_2 = \frac{2+x_1}{2} = \frac{2+1.5}{2} = 1.75$

We find, $f(x_2) = f(1.75) = 0.0625$ (+ve)

Hence the root lies between 1.5 & 1.75 so we take $x_3 = \frac{1.5+1.75}{2} = 1.625$

We obtain, $f(x_3) = f(1.625) = -0.3593$ (-ve)

Hence the root lies between x_2 & x_3. So we take $x_4 = \frac{x_2+x_3}{2} = 1.6875$

We obtain, $f(x_4) = f(1.6875) = -0.15234$ (-ve)

Hence the root lies between x_2 & x_4. So we take $x_5 = \frac{x_2+x_4}{2} = \frac{1.75+1.6875}{2} = 1.71875$

We obtain, $f(x_5) = f(1.71875) = -0.045899$ (-ve)

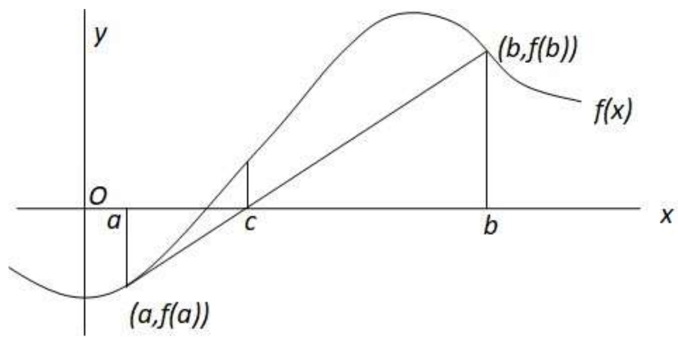
Hence the root lies between x_2 & x_5 so we take $x_6 = \frac{x_2+x_5}{2} = 1.734375$

We obtain, $f(x_6) = f(1.734375) = 0.008$, which is near to zero.

Hence 1.734375 is a root of x^2-3=0

$\therefore \sqrt{3} = 1.734375$

Method of False Position: The False Position Method is applicable for numerically solving the equation $f(x) = 0$ for the real variable x, where $f(x)$ is a continuous function defined on an interval $[a, b]$ and where $f(a)$ and $f(b)$ have opposite signs. In this case, by the intermediate value theorem, the continuous function $f(x)$ must have at least one zero in the interval (a, b).



At each step the method repless the curve $f(x)$ by the straight line joining the points $(a, f(a))$ and $(b, f(b))$. The equation of this line is

$$\frac{x-a}{a-b} = \frac{y-f(a)}{f(a)-f(b)}. \ldots (1)$$

If this line intersect the x axis at c (here $y = 0$), then we get the value of c by solving (1) for x with $y = 0$. Therefore $c = a - \frac{a-b}{f(a)-f(b)} f(a)$. Unless c is itself a root (which is very unlikely, but possible) there are now only two possibilities: either $f(a)$ and $f(c)$ have opposite signs and there is a root between a and c, or $f(c)$ and $f(b)$ have opposite signs and there is a root between c and b. Explicitly, if $f(a)$ and $f(c)$ have opposite signs, then the method sets c as the new value for b, and if $f(b)$ and $f(c)$ have opposite signs then the method sets c as the new a. (If $f(c) = 0$ then c may be taken as the solution and the process stops.) In both cases, the new $f(a)$ and $f(b)$ have opposite signs, so the method is applicable to this smaller interval. The process is continued until the interval is sufficiently small, i.e., $f(c)$ is near to zero.

1.Find a real root of $f(x) = x^3 - 2x - 5 = 0$ by method of false poaition.

Solution : we have

$$f(2) = -1 \quad \text{(-ve)} \quad [\text{Here } x_0 = 2]$$
$$f(3) = 16 \quad \text{(+ve)} \quad [\text{Here } x_1 = 3]$$

So, $x_2 = 2 - \frac{f(2)}{f(3)-f(2)}(3-2) = 2 - \frac{(-1)}{16-(-1)} = 2 + \frac{1}{17} = 2.0588$

Now, $f(x_2) = f(2.0588) = -0.386$ (-ve)

and hence the root lies between 2.0588 and 3.

So, x_3= 2 .0588 - $\frac{f(2.0588)}{f(3)-f(2.0588)}$(3-2.0588) = 2.0588 + $\frac{(.386)}{.6+0.386}$(3-2.0588) = 2.0812

We find, $f(x_3) = f(2.0812) = -0.1479$ (-ve)

Hence the root lies between 2.0812 and 3

So, x_4= 2 .0812 - $\frac{f(2.0812)}{f(3)-f(2.0812)}$(3-2 .0812) = 2 .0812 + $\frac{(.1479)}{16+0.1479}$(3-2.0812) = 2.0896

Now, $f(x_4) = f(2.0896) = -0.0551$ (-ve)

hence the root lies between 2.0896 and 3

So, x_5= 2.0896 - $\frac{f(2.0896)}{f(3)-f(2.0896)}$(3-2.0896) = 2.0588 + $\frac{(.0551)}{16+.0551}$(3-2.0896) = 2.0927

Hence the root of the given equation is 2.0927.

2. Solve x^3- x^2-1 = 0 by the method of false poaition.

Solution : Let, $f(x) = x^3 - x^2 - 1$

So, $f(1) = -1 < 0$

$f(2) = 3 > 0$

Hence the root lies between 1and 2.

So, x_2= 1- $\frac{f(1)}{f(2)-f(1)}$ (2-1) = 1 + $\frac{1}{3+1}$ × 1 = 1.25

We find, $f(x_2) = f(1.25) = -0.60938 < 0$

Hence the root lies between 1.25 and 2.

So, x_3= 1.25 - $\frac{f(1.25)}{f(2)-f(1.25)}$ (2-1.25) = 1.25 + $\frac{0.60938}{3+0.60938}$(.75) = 1.3766

Now, $f(x_3) = f(1.3766) = -0.28633 < 0.$

Hence the root lies between 1.3766 and 2 .

So, x_4= 1.3766 - $\frac{f(1.3766)}{f(2)-f(1.3766)}$ (2-1.3766) = 1.3766 + $\frac{0.28633}{3+0.28633}$(.6234) = 1.4304

Now, $f(x_4) = f(1.4304) = -0.11938 < 0$

Hence the root lies between 1.4304 and 2.

So, x_5= 1.4304 - $\frac{f(1.4304)}{f(2)-f(1.4304)}$ (2-1.4304) = 1.4304 + $\frac{0.11938}{3+0.11938}$(.5696) = 1.4522

Now, $f(x_5) = f(1.4522) = -0.046362 < 0$

hence the root lies between 1.4522 and 2.

So, x_6= 1.4522 - $\frac{f(1.4522)}{f(2)-f(1.4522)}$ (2-1.4522) = 1.4522 + $\frac{0.046362}{3+0.046362}$(.5478) = 1.4595

Now, $f(x_6) = f(1.4595) = -0.0212 < 0$

hence the root lies between 1.4595 and 2.

So, x_7= 1.4595 - $\frac{f(1.4595)}{f(2)-f(1.4595)}$ (2-1.4595) = 1.4595 + $\frac{0.0212}{3+0.0212}$(.5405) = 1.4633

Now, $f(x_7) = f(1.4633) = -0.00796$, which is near to zero.

Hence the root is , x_7=1.4633

Iteration Method :

1. Find a real root of the equation $f(x) = x^3 + x^2 - 1 = 0$ by iteration method .

Solution : we get

$f(0) = -1$ (-ve)

$f(1) = 1$ (+ve)

So, a root lies between 0 and 1 . Therefore we can take x_0=0.5

To find this root , we put the equation in the form $x = \varphi(x)$.

So $x^3 + x^2 - 1 = 0$

or, $x = \frac{1}{\sqrt{1+x}}$

So that $\varphi(x) = \frac{1}{\sqrt{1+x}}$ and $\varphi'(x) = \frac{1}{2(1+x)^{3/2}}$

we have $|\varphi'(x)| < 1$ for $x = x_0 = 0.5$

Hence the iteration method can be applied.

We get,

$x_1 = \varphi(x_0) = \varphi(.5)$= 0.81649

$x_2 = \varphi(x_1) = \varphi(.81649)$= .74196

$x_3 = \varphi(x_2) = \varphi(.74196)$=.75767

$x_4 = \varphi(x_3) = \varphi(.75767)$= .75427

$x_5 = \varphi(x_4) = \varphi(.75427)$= .75500

$x_6 = \varphi(x_5) = \varphi(.75500)$= .75485

$x_7 = \varphi(x_6) = \varphi(.75485)$= .75488

Hence the approximate value of the root is 0.75488.

2. Find the root of the equation $2x = cosx + 3$ correct to three decimal places by using iteration method .

Solution : The given equation can be put in the form, $x = \frac{1}{2}(cosx + 3)$

So that $\varphi(x) = \frac{1}{2}$(cosx+3)

and $\varphi'(x) = \frac{1}{2}(-sinx)$

we have $|\varphi'(x)| = |\frac{sinx}{2}| < 1$

Hence the iteration method can be applied .

Take , x_0=1

We get,

$x_1 = \varphi(x_0) = \varphi(1)$=1.7701

$x_2 = \varphi(x_1) = \varphi(1.7701)$= 1.40098

$x_3 = \varphi(x_2) = \varphi(1.40098)$=1.5845

$x_4 = \varphi(x_3) = \varphi(1.5845)$= 1.4931

$x_5 = \varphi(x_4) = \varphi(1.4931)$= 1.5388

$x_6 = \varphi(x_5) = \varphi(1.5388)$= 1.5160

$x_7 = \varphi(x_6) = \varphi(1.5160)$= 1.5274

$x_8 = \varphi(x_7) = \varphi(1.5274)$= 1.5217

$x_9 = \varphi(x_8) = \varphi(1.5217)$= 1.5245

$x_{10} = \varphi(x_9) = \varphi(1.5245)$= 1.5231

Hence the approximate value of the root is 1.5231.

3. Find a real root of $sinx = 10(x - 1)$ correct to four significant figures by using iteration method.

Solution : The equation can be written in the form ,

$10x - 10 = sinx$

or, $10x = 10 + sinx$

or, $x = \frac{1}{10}sinx + 1$

So that, $\varphi(x) = \frac{1}{10}$sinx+1

And $\varphi'(x) = \frac{-cos}{10}$

we have $|\varphi'(x)| < 1$

Hence the iteration method can be applied .

We get, $x_n = \varphi(x_{n-1})$

Choose , x_0=1

$x_1 = \varphi(x_0) = \varphi(1)$= 1.0841

$x_2 = \varphi(x_1) = \varphi(1.0841)$= 1.08834

$x_3 = \varphi(x_2) = \varphi(1.08834)$= 1.08859

$x_4 = \varphi(x_3) = \varphi(1.08859)$= 1.08860

$x_5 = \varphi(x_4) = \varphi(1.08860)$= 1.08860

Hence the approximate value of the root is 1.089.

4. Use the iteration method to find a real root of the following equations.

(a) $cosx = 3x - 1$

(b) x $= \frac{1}{(x+1)^2}$

(c) $x = (5-x)^{\frac{1}{3}}$

(d) $sinx = 10(x-1)$

(e) $e^{-x} = 10x$

(f) $x = cosecx$

(g) sin^2x=x^2-1

(h) $e^x = cotx$

Newton-Raphson Method:

Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$ that is, $f(x_0 + h) = 0$

Now expending $f(x_0 + h)$ by Taylor's series , we obtain,

$$f(x_0) + h\,f'(x_0) + \frac{h^2}{2!}\,f''(x_0) + \ldots\ldots\ldots\ldots = 0$$

Neglecting the second and higher order derivatives, we have

$$f(x_0) + h\,f'(x_0) = 0$$

or, $h = -\frac{f(x_0)}{f'(x_0)}$

A better approximation than x_0 is therefore given by x_1 when ,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are giveb by $x_2, x_3, . \ldots \ldots \ldots \ldots x_{n+1}$

Where , x_{n+1}=x_n - -$\frac{f(x_n)}{f'(x_n)}$

Which is Newton – Raphson formula.

1. The equation $x^3 + 2x^2 - 5x - 1 = 0$ has a root near to 1.4 . Use the Newton-Raphson method to find the root to four significant figures.

Solution : we have ,

$$f(x) = x^3 + 2x^2 - 5x - 1$$

therefore $f'(x) = 3x^2 + 4x - 5$

Given that , x_0=1.4

Now , $f(x_0) = -1.336$

$f'(x_0) = 1.6162$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.4 - \frac{-1.336}{6.48} = 1.6062$

Now, $f(x_1) = .2723$

$f'(x_1)$=9.1641

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.6062 - \frac{.2723}{9.1641} = 1.5765$

Now, $f(x_2) = .0059$

$f'(x_2)$=8.7615

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5765 - \frac{.0059}{8.7615} = 1.5757$

Now, $f(x_3) = .000003$ which is near to zero .

Hence the approximate value of the root is 1.5757

2. Use Newton's method to find a root of the equation $x^3 - 3x - 5 = 0$

Solution : we have , $f(x) = x^3 - 3x - 5 = 0$

$$f(2) = 2^3 - 3.2 - 5 = -3$$

$$f(3) = 3^3 - 3.3 - 5 = 13$$

So, there is a root between 2 & 3.

$\therefore$We choose x_0=2

Now, f'(x)= $3x^2$-3

$\therefore f(x_0) = f(2) = -3$

$f'(x_0)$= $f'(2)$=9

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-3}{9} = 2.3333$

Now,

$$f(x_1) = 0.7037$$

$$f'(x_1) = 13.333$$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.3333 - \frac{-.7037}{13.333} = 2.2805$

Now,

$$f(x_2) = f(2.2805) = 0.01935$$

$$f'(x_2) = f'(2.2805) = 12.6028$$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2805 - \frac{0.01935}{12.6028} = 2.2790$

Now, $f(x_3) = f(2.2790) = 0.000016$ Which is near to zero.

Hence the approximate value of the root is 2.2790.

2. Find a root of the equation $xsinx + cosx = 0$

Solution : we have , $f(x) = xsinx + cosx$

$$f'(x) = xcosx$$

Choose , x_0=π=3.1416

$\therefore f(x_0) = -1$

$f'(x_0)$= -3.1416

$\therefore x_1$=x_0 - $\frac{f(x_0)}{f'(x_0)}$ = 3.1416 - $\frac{-1}{-3.1416}$ =2.8233

Now,

$$f(x_1) = -0.06618$$

$$f'(x_1) = -2.6816$$

$\therefore x_2$=x_1 - $\frac{f(x_1)}{f'(x_1)}$ = 2.8233 - $\frac{-.06618}{-2.6816}$ =2.7985

Now, $f(x_2) = -0.00056$ Which is near to zero.

Hence the approximate value of the root is 2.7985.

4. $x^3 - 2x - 5 = 0$
5. $x^5 + 5x + 1 = 0$
6. $x^3 - 5x + 3 = 0$
7. $sinx = 1 - x$
8. $tanx = 4x$.

Newton's Interpolation

Established the Newton's forward difference interpolation formula.

Solution: Give the set of $(n+1)$ values, viz. $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots \dots \dots .(x_n, y_n)$, of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant.

i.e., Let $x_i = x_0 + ih \quad$ for $i = 1,2,3, \dots \dots \dots \dots .n$

Since $y(x)$ is a polynomial of the nth degree, it may be written as

$$y_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) +$$
$$\dots \dots \dots \dots \dots .. + a_n(x-x_0)(x-x_1)(x-x_2) \dots \dots \dots (x-x_{n-1}) \quad \dots\dots\dots(1)$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated values.

When $x = x_0$ then $y_0 = a_0$

When $x = x_1$ then $y_1 = a_0 + a_1(x - x_0)$

$$\Rightarrow y_1 = a_0 + a_1 h$$
$$\Rightarrow a_1 h = y_1 - a_0$$
$$\Rightarrow a_1 h = y_1 - y_0$$
$$\Rightarrow a_1 h = \Delta y_0$$
$$\therefore a_1 = \frac{\Delta y_0}{h}$$

When $x = x_2$ then $y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$$\Rightarrow y_2 = y_0 + a_1 2h + a_2 2h.h$$
$$\Rightarrow y_2 = y_0 + 2a_1 h + 2a_2 h^2$$
$$\Rightarrow y_2 = y_0 + 2\Delta y_0 + 2a_2 h^2$$
$$\Rightarrow 2a_2 h^2 = y_2 - y_0 - 2\Delta y_0$$
$$= y_2 - y_0 - 2(y_1 - y_0)$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$= \Delta y_1 - \Delta y_o$$
$$= \Delta^2 y_0$$
$$\therefore a_2 = \frac{\Delta^2 y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\, h^2}$$

Similarly we get

$$a_3 = \frac{\Delta^3 y_0}{3!\, h^3}$$

…………….

…………….

$$a_n = \frac{\Delta^n y_0}{n!\, h^n}$$

Putting the value of $a_0, a_1, a_2 \dots \dots \dots \dots \dots \dots .. a_n$ in (1) we get

$$y_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x-x_0)(x-x_1)$$
$$+ \frac{\Delta^3 y_0}{3!\ h^3}(x-x_0)(x-x_1)(x-x_2) + \cdots \dots \dots \dots \dots \dots$$

$$+\frac{\Delta^n y_0}{n!\,h^n}(x-x_0)(x-x_1)(x-x_2)\,\dots\dots\dots\dots(x-x_{n-1}) \qquad \dots\dots(2)$$

Sitting $x = x_0 + ph$

Then $\frac{x-x_0}{h} = p$

$$\frac{x-x_1}{h} = \frac{x-(x_0+h)}{h} = \frac{x-x_0}{h} - \frac{h}{h} = p-1$$

$$\frac{x-x_2}{h} = \frac{x-(x_0+2h)}{h} = \frac{x-x_0}{h} - \frac{2h}{h} = p-2$$

Similarly,

$$\frac{x-x_{n-1}}{h} = \frac{x-(x_0+(n-1)h)}{h} = \frac{x-x_0}{h} - \frac{(n-1)h}{h} = p-(n-1)$$

Putting this value's in (2) we get

$$y_n(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\dots(p-n+1)}{n!}\Delta^n y_0$$

Where p = $\frac{x-x_0}{h}$

Which is the Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabulated value?

1. Find the cubic polynomial which takes the following values.

$y(0) = 1, y(1) = 0, y(2) = 1$ and $y(3) = 10$

Hence obtain *y(4)*

Soln. the difference table is as follows:

x	y	Δ	Δ^2	Δ^3
0	**1**			
		-1		
1	**0**		**2**	
		1		**6**
2	**1**		**8**	
		9		
3	**10**			

Here *h=1,* $x_0 = 0$

Hence $p = \frac{x-x_0}{h} = \frac{x-0}{1} = x$

Therefore from Newton's forward interpolation formula we get

$y(x) = 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{3!}(6)$

$= 1 - x + x^2 - x + x(x^2 - 3x + 2)$

$= 1 - x + x^2 - x + x^3 - 3x^2 + 2x$

$= x^3 - 2x^2 + 1$

Which is the required polynomial and $y(4) = 4^3 - 2(4)^2 + 1$

$= 64 - 32 + 1$

$= 33$

2. The population of a country in the decennial census were as under .Estimate The population for the year 1895.

Year : x	1891	1901	1911	1921	1931
Population : y(In thousands)	46	66	81	93	101

Solution : We know that the Newton's forward difference interpolation formula is ,

$$y_n(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\dots \dots$$
$$+\frac{p(p-1)(p-2)+\dots\dots+(p-n+1)}{n!}\Delta^n y_0 \quad \dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

[Where , $p=\frac{x-x_0}{h}$]

Now, put

$h=10$, $x_0 = 1891$, $x = 1895$

$p = \frac{1895-189}{10} = 0.4$

The difference table is:

x	y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Hence (1) gives,

$$y(1895) = 46 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(-5) + \frac{0.4(0.4-1)(.4-2)}{3!}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 46 + 8 + \frac{0.4(-0.6)}{2}(-5) + \frac{0.4(-0.6)(-1.6)}{6}(2) + \frac{0.4(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$=54.85$ thousands

3. Given,

$$Sin45^{\circ} = 0.7071$$
$$Sin50^{\circ} = 0.7660$$
$$Sin55^{\circ} = 0.8192$$
$$Sin60^{\circ} = 0.8660$$

Find $Sin52^{\circ}$ by using Newton's forward interpolation formula.

Solution: We know that the Newton's forward difference interpolation formula is,

$$y_n(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\dots \dots \dots+\frac{p(p-1)(p-2)+\dots\dots+(p-n+1)}{n!}\Delta^n y_0$$

… ...(1)

[Where, $p = \frac{x - x_0}{h}$]

We have,

x	45°	50°	55°	60°
y	0.7071	0.7660	0.8192	0.8660

The difference table is as followed:

x	y	Δ	Δ^2	Δ^3
45°	0.7071			
		0.0589		
50°	0.7660		-0.0057	
		0.0532		-0.0007
55°	0.8192		-0.0064	
		0.0468		
60°	0.8660			

Here, $h = 5^\circ$, $x_0 = 45^\circ$, $x = 52^\circ$, $p = \frac{52^\circ - 45^\circ}{5^\circ} = 1.4$

So, from (1) we get,

$$y(52^\circ) = 0.7071 + 1.4(0.0589) + \frac{1.4(1.4-1)}{2!}(-0.0057) + \frac{1.4(1.4-1)(1.4-2)}{3!}(-0.0007)$$
$$= 0.7071 + 0.08246 - 0.001567 + 0.0000392$$
$$= 0.7880032$$

Thus, $sin 52^\circ = 0.788$ (approximately).

Established the Newton's backward difference interpolation formula.

Solution: Given the set of $(n+1)$ values viz. $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots \dots . (x_n, y_n)$ of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated values, let the values of x be equidistant. i.e.,

Let $x_i = x_0 + ih$ for $i = 1,2,3, \dots \dots \dots \dots . n$

Since $y(x)$ is a polynomial of the nth degree, it may be written as

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \dots \dots \dots + a_n (x - x_n)(x - x_{n-1})(x - x_{n-2}) \dots \dots \dots (x - x_1) \dots \dots \dots \dots (1)$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated values.

When $x = x_n$ then $y_n = a_0$

When $x = x_{n-1}$ then $y_{n-1} = a_0 + a_1(x_{n-1} - x_n)$

$\Rightarrow y_{n-1} = y_n + a_1(x_{n-1} - x_n)$

$\Rightarrow a_1 (x_n - x_{n-1}) = y_n - y_{n-1}$

$\Rightarrow a_1 h = y_n - y_{n-1}$

$\Rightarrow a_1 = \frac{\nabla y_n}{h}$

When $x = x_{n-2}$ then $y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$

$\Rightarrow y_{n-2} = y_n - a_1 (x_n - x_{n-2}) + a_2(x_n - x_{n-2})(x_{n-1} - x_{n-2})$

$\Rightarrow y_{n-2} = y_n - a_1 2h + a_2 2h.h$

$\Rightarrow y_{n-2} = y_n - 2a_1 h + 2a_2 h^2$

$\Rightarrow 2a_2 h^2 = y_{n-2} + 2a_1 h - y_n$

$= y_{n-2} - 2(y_n - y_{n-1}) - y_n$

$= y_{n-2} + 2 y_n - 2y_{n-1} - y_n$

$= y_{n-2} + y_n - 2y_{n-1}$

$= y_n - y_{n-1} - y_{n-1} + y_{n-2}$

$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$

$= \nabla y_n - \nabla y_{n-1}$

$= \nabla^2 y_n$

$\therefore a_2 = \frac{\nabla^2 y_n}{2!h^2}$

Similarly we get,

$$a_3 = \frac{\nabla^3 y_n}{h^3.3!}$$

....

....

$$a_n = \frac{\nabla^n y_n}{h^n.n!}$$

Putting the value of $a_0, a_1, \dots \dots \dots \dots \dots \dots .. a_n$ *in* (1) *we get*

$y_n(\mathrm{x}) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{h^2.3!}(x - x_n)(x - x_{n-1})$

$+ \frac{\nabla^3 y_n}{h^3.4!}(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots \dots \dots \dots \dots \dots ....$

$+ \frac{\nabla^n y_n}{h^n.n!}(x - x_n)(x - x_{n-1})(x - x_{n-2}) \dots \dots \dots \dots ..(x - x_1)$ .... (2)

Sitting $x = x_n + ph$

Then $\frac{x - x_n}{h} = p$

$$\frac{x - x_{n-1}}{h} = \frac{x - (x_n - h)}{h} = \frac{x - x_n}{h} + \frac{h}{h} = p + 1$$

$$\frac{x - x_{n-2}}{h} = \frac{x - (x_n - 2h)}{h} = \frac{x - x_n}{h} + \frac{2h}{h} = p + 2$$

Similarly,

$$\frac{x - x_1}{h} = \frac{x - (x_n - (n-1)h)}{h} = \frac{x - x_n}{h} + \frac{(n-1)h}{h} = p + (n - 1)$$

Putting this value's in (2) we get

$y_n(x) =$

$$y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \cdots \ldots\ldots + \frac{p(p+1)(p+2)\ldots.(p+n-1)}{n!}\nabla^n y_n$$

Where p = $\frac{x-x_n}{h}$

Which is the Newton's backward difference interpolation formula and is useful for interpolation near the end of a set of the value?

4. The population of a country in the decennial census where as under .Estimate the population for the year 1925.

Year : x	1891	1901	1911	1921	1931
Population : y (In thousands)	46	66	81	93	101

Solution: Here the interpolation is desired at the end of the table and so we use Newton's backward differences interpolation formula.

We know that the Newton's backward difference interpolation formula is,

$y_n(x) =$

$$y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \cdots \ldots\ldots + \frac{p(p+1)(p+2)\ldots.(p+n-1)}{n!}\nabla^n y_n$$

Now, put

$$h = 10 \quad , \quad x_0 = 1931\ , \quad x = 1925$$

$$p = \frac{1931-1925}{10} = -0.6$$

The difference table is :

x	y	∇	∇^2	∇^3	∇^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Hence (1) gives,

$$y(1925) = 101 + (-0.6)8 + \frac{-0.6(-0.6+1)}{2!}(-4) + \frac{-0.6(-0.6+1)(-0.6+2)}{3!}(-1)$$

$$+ \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(-3)$$

$$= 96.84 \text{ thousands}$$

5. The table below gives the value of tan(x) for $0.10 \leq x \leq 0.30$

X	0 .10	0.15	0.20	0.255	0.30
Y=$tan(x)$	0.1003	0.1511	0.2027	0.2553	0.3093

Find tan 0.26

Solution. We know Newton's backward difference interpolation formula is,

$$y_n(x) = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \cdots \ldots \ldots$$
$$+ \frac{p(p+1)(p+2)\ldots.(p+n-1)}{n!}\nabla^n y_n \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where p=$\frac{x-x_n}{h}$

Here x=0.26, $x_n = 0.30,$ $h = 0.05$

$$p = \frac{0.26 - 0.30}{0.05} = -0.8$$

The difference table is as follows,

x	Y	∇	∇^2	∇^3	∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.25553		0.0014		
		0.0540			
0.30	0.3093				

Here(1) gives

$y(0.26) =$

$$0.3093 - 0.8(0.0540) + \frac{-0.8(-0.8+1)}{2!}(0.0014) + \frac{-0.8(-0.8+1)(-0.8+2)}{3!}(0.0004) + \frac{-0.8(-0.8+1)(-0.8+2)(-0.8+3)}{4!}(0.0002)$$

= 0.2662

Interpolation with unequal Intervals

Discus about Divided Differences.

Let $f(x_0)$,$f(x_1)$,……………$f(x_n)$be the entries corresponding to the arguments $x_0, x_1, \dots \dots \dots \dots \dots \dots . x_n$where the intervals $x_1 - x_0, x_2 - x_1, x_n - x_{n-1}$ may not be equal.Then the first divided difference of $f(x)$ for the arguments x_0, x_1 is defined as $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ and is denoted by $f(x_{0,}x_1)$, that is, $f(x_{0,}x_1)$=$\frac{f(x_1)-f(x_0)}{x_1-x_0}$

Similarly the other first divided differences of $f(x)$ for the arguments $x_1, x_2; x_2, x_3; \dots \dots \dots \dots; x_{n-1}, x_n$ are

$f(x_{1,}x_2)$=$\frac{f(x_2)-f(x_1)}{x_2-x_1}$

$f(x_{2,}x_3)$=$\frac{f(x_3)-f(x_2)}{x_3-x_2}$

………………………………..

$f(x_{n-1,}x_n)$=$\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$

The second divided difference of $f(x)$ for the three arguments x_0, x_1 and x_2 is defined as

$f(x_{0,}x_1, x_2)$=$\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$

The nth divided difference is given by $f(x_0, x_1, x_2, \dots \dots x_n) = \frac{f(x_1,x_2,\dots.x_n)-f(x_0,x_1,\dots\dots x_{n-1})}{x_n-x_0}$

Note: If two of the arguments coincide the divided difference can be given a meaning assigned by taking the limit. Thus

$$f(x_0, x_0) = \lim_{\epsilon\to 0}(x_0, x_0 + \epsilon)$$

$$=\lim_{\epsilon\to 0} \frac{f(x_0+\epsilon)-f(x_0)}{(x_0+\epsilon)-x_0}$$

$$=\lim_{\epsilon\to 0} \frac{f(x_0+\epsilon)-f(x_0)}{\epsilon}$$

$$=f'(x_0)$$

Similarly $f(\underbrace{x_0, x_0, \dots\dots x_0}_{(r+1)\text{ arguments}}) = \frac{1}{r!} f^{(r)}(x_0)$

Theorem: The divided difference are symmetrical in all its arguments, That is independent of the order of the arguments.

<u>Proof:</u> We have

$$f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \quad = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f(x_1, x_0)$$

$$= \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0}$$

$=\sum \frac{f(x_0)}{x_0-x_1}$, Showing that $f(x_0, x_1)$ is symmetrical in x_0, x_1,

Again $f(x_0, x_1 x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$

$= \frac{1}{x_2-x_0}\ [\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}]$

$= \frac{1}{x_2-x_0}\ [\frac{f(x_2)}{x_2-x_1} - \frac{f(x_1)}{x_1-x_0} - \frac{f(x_1)}{x_1-x_0} + \frac{f(x_0)}{x_1-x_0}]$

$= \frac{1}{x_2 - x_0}\ [\frac{f(x_2)}{x_2 - x_1} - f(x_1)(\frac{1}{x_2 - x_1} - \frac{1}{x_1 - x_0}) + \frac{f(x_0)}{x_1 - x_0}]$

$= \frac{1}{x_2-x_0}\ [\frac{f(x_2)}{x_2-x_1} - f(x_1)\frac{x_1-x_0+x_2-x_1}{(x_2-x_1)(x_1-x_0)} + \frac{f(x_0)}{x_1-x_0}]$

$= \frac{1}{x_2-x_0}\ [\frac{f(x_2)}{x_2-x_1} - \frac{f(x_1)(x_2-x_0)}{(x_2-x_1)(x_1-x_0)} + \frac{f(x_0)}{x_1-x_0}]$

$= \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{f(x_1)}{(x_1-x_2)(x_1-x_0)} + \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$

$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$

$=\sum \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$,Showing that $f(x_0, x_1, x_2)$ is symmetrical in x_0, x_1, x_2

Let us assume similar symmetrical expressions for the (n-1)th divided differences. That is, let us assume that

$$f(x_0, x_1, \dots \dots x_{n-1}) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots \dots \dots \dots (x_0 - x_{n-1})} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots \dots \dots \dots (x_1 - x_{n-1})} + \cdots \dots \dots \dots \dots + \frac{f(x_{n-1})}{(x_{n-1} - x_0)(x_{n-1} - x_1) \dots \dots \dots \dots (x_{n-1} - x_{n-2})}$$

$$= \sum \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots \dots \dots \dots (x_0 - x_{n-1})}$$

And similar expressions for the other (n-1)th divided difference.

Then $f(x_0, x_1, \dots \dots x_n) = \frac{f(x_0, x_1, \dots \dots x_{n-1}) - f(x_1, x_2, \dots \dots x_n)}{x_0 - x_n}$

$= \frac{1}{x_0-x_n} [\{\frac{f(x_0)}{(x_0-x_1)\dots\dots(x_0-x_{n-1})} + \frac{f(x_1)}{(x_1-x_0)\dots\dots(x_1-x_{n-1})} + \frac{f(x_{n-1})}{(x_{n-1}-x_0)\dots\dots(x_{n-1}-x_{n-2})}\} - \{\frac{f(x_1)}{(x_1-x_2)\dots\dots(x_1-x_n)} + \frac{f(x_2)}{(x_2-x_1)\dots\dots(x_2-x_n)} + \cdots \dots + \frac{f(x_n)}{(x_n-x_1)\dots\dots(x_n-x_{n-1})}\}]$

$= \frac{f(x_0)}{(x_0-x_1)\dots\dots(x_0-x_n)} + \frac{f(x_1)}{(x_1-x_0)\dots\dots(x_1-x_n)} + \frac{f(x_n)}{(x_n-x_0)\dots\dots(x_n-x_{n-1})}$

$= \sum \frac{f(x_0)}{(x_0-x_1)\dots\dots(x_0-x_n)}$

showing that the nth divided difference $f(x_0, x_1, \dots \dots x_n)$ is also symmetrical in $x_0, x_1, \dots \dots x_n$ and thus completing the proof of the theorem by mathematical induction.

Theorem: The nth divided differences of a polynomial of the nth degree are constant.

Proof: First consider the function $f(x) = x^n$.The first divided differences of this function are given by

$f(x_r, x_{r+1}) = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r}$

$=\frac{x_{r+1}^n - x_r^n}{x_{r+1}-x_r}$

$=x_{r+1}^{n-1} + x_r x_{r+1}^{n-2} + \cdots \dots \dots . + x_{r+1} x_r^{n-2} + x_r^{n-1}$

Which is a homogeneous expression of degree n-1 in x_{r+1} and x_r.The second divided differences are given by

$$f(x_r, x_{r+1}, x_{r+2}) = \frac{f(x_r, x_{r+1}) - f(x_{r+1}, x_{r+2})}{x_r - x_{r+2}}$$

$=\frac{f(x_{r+1},x_{r+2}) - f(x_r,x_{r+1})}{r_{r+2}-x_r}$

$= \frac{1}{(x_{r+2}-x_r)}[f(x_{r+1}, x_{r+2}) - f(x_r, x_{r+1})]$

$= \frac{1}{x_{r+2}-x_r}[\frac{f(x_{r+2})-f(x_{r+1})}{x_{r+2}-x_{r+1}} - \frac{f(x_{r+1})-f(x_r)}{x_{r+1}-x_r}]$

$= \frac{1}{x_{r+2}-x_r}[\frac{x_{r+2}^n - x_{r+1}^n}{x_{r+2}-x_{r+1}} - \frac{x_{r+1}^n - x_r^n}{x_{r+1}-x_r}]$

$= \frac{1}{x_{r+2}-x_r}[(x_{r+2}^{n-1} + x_{r+1}.x_{r+2}^{n-2} + \cdots \dots \dots \dots + x_{r+1}^{n-2}x_{r+2} + x_{r+2}^{n-1}) - (x_{r+1}^{n-1} + x_r.x_{r+1}^{n-2} +$

$\dots \dots \dots \dots + x_r^{n-2}x_{r+1} + x_r^{n-1})]$

$= \frac{x_{r+1}^{n-1} - x_r^{n-1}}{x_{r+2}-x_r} + x_{r+1}\frac{x_{r+2}^{n-2} - x_r^{n-2}}{x_{r+2}-x_r} + x_{r+1}^{n-2}\frac{x_{r+2}-x_r}{x_{r+2}-x_r}$

$= (x_{r+2}^{n-2} + x_r x_{r+2}^{n-3} + \cdots + x_r^{n-2}) + x_{r+1}(x_{r+2}^{n-3} + \cdots + x_r^{n-3}) + \cdots \dots \dots \dots \dots . + x_{r+1}^{n-2}$

Which is a homogeneous expression of degree n-2 in $x_r, x_{r+1}, x_{r+2}$by induction it can be shown that $f(x_r, x_{r+1}, \dots \dots \dots \dots \dots ., x_{r+m})$ is a homogeneous expression of degree n-m.

In particular, The nth divided difference of $f(x) = x^n$ is an expression of degree zero, it is a constant, and is therefore independent of the values of $x_r, x_{r+1}, \dots \dots \dots x_{r+n}$

Since the nth divided difference of x^n are constant, Therefore the divided differences of x^n of order greater than n will all be zero.

If $f(x) = ax^{\mathrm{n}}$, where a is a constant, Then the nth divided difference of $f(x) = a$ (The nth divided difference of x^n)

Which is a constant.

Hence if $f(x) = a_0x^n + a_1x^{n-1} + \cdots \dots \dots \dots \dots + a_n$ be a polynomial of degree n, Then the nth divided differences of all the terms except a_0x^n will be zero. and so the nth divided difference of the whole polynomial will be constant.

Newton's interpolation formula for unequal intervals.

Or. Newton's interpolation formula for divided difference.

Let $f(x_0), f(x_1), \dots \dots \dots \dots \dots f(x_n)$ be the values of $f(x)$ corresponding to the arguments $x_0, x_1, \dots \dots \dots \dots \dots . x_n$, not necessarily equally spaced. From the definition of divided differences,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

Or. $f(x) = f(x_0) + (x - x_0)f(x, x_0)$ (1)

Also $f(x,x_0,x_1)=\frac{f(x,x_0)-f(x_0,x_1)}{x-x_1}$

$\Rightarrow f(x,x_0)=f(x_{0,}x_1)+\ldots\ldots(x-x_1)f(x,x_0,x_1)$………………….. (2)

Similarly

$f(x,x_0,x_1)=f(x_0,x_1,x_2)+(x-x_2)f(x,x_0,x_1x_2)$ ……………… (3)

…………………………………………………..

$f(x,x_0,x_1,\ldots\ldots.x_{n-1})=f(x_0,x_1,\ldots..x_n)+(x-x_n)f(x,x_0,x_1,\ldots\ldots x_n)$ …….. (4)

Multiplying the equation (2) by $(x-x_0)$ (3) by $(x-x_0)(x-x_1)$ and so on and finally the equation (4) by $(x-x_0).(x-x_1)\ldots\ldots.(x-x_{n-1})$ and adding to the equation (1), we have

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+\cdots\ldots\ldots\ldots$$
$$+(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})f(x_{0,}x_1,\ldots\ldots x_n)+R_n$$

Where R_n is given by

$R_n=(x-x_0)(x-x_1)\ldots\ldots\ldots\ldots(x-x_n)f(x,x_0,x_1,\ldots\ldots\ldots\ldots\ldots\ldots x_n)$ Assuming that $f(x)$ is a polynomial of degree n, then $f(x,x_0,\ldots\ldots\ldots x_n)$

Vanishes, so that

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+\cdots\ldots\ldots+$$
$$(x-x_0)(x-x_1)\ldots\ldots\ldots.(x-x_{n-1})f(x_0,x_1,\ldots\ldots x_n)$$

This formula is called Newton's divided difference interpolation formula.

Ex 1. Fit a polynomial which passes through the points (1,4), (3,7), (4,8) and (6,11) by using Newton's divided difference formula. Also find the value of y at x=2.

Soln. The difference table is as follows.

x	$f(x)$	1st diff	2nd diff	3rd diff
1	4			
		$\frac{7-4}{3-1}=1.5$		
3	7		$\frac{1-1.5}{4-1}=-.167$	
		$\frac{8-7}{4-3}=1$		$\frac{.167+.167}{6-1}=.067$
4	8			
		$\frac{11-8}{6-4}=1.5$	$\frac{1.5-1}{6-3}=.167$	
6	11			

Here $f(1)=4, f(1,3)=1.5, f(1,3,4)=-.167\,, f(1,3,4,6)=.067$

Therefore from Newton's divided difference formula we get

$$f(x) = f(1) + (x-1)f(1.3) + (x-1)(x-3)f(1,3,4)$$
$$+ (x-1)(x-3)(x-4)f(1,3,4,6)$$
$$= 4 + 1.5(x-1) - .167(x-1)(x-3) + .067(x-1)(x-3)(x-4)$$

Which is the required polynomial

Now put $x = 2$, we get

$$y = f(2)$$
$$= 4 + 1.5(2-1) - .167(2-1)(2-3) + .067(2-1)(2-3)(2-4)$$
$$=5.799 \quad \text{Ans.}$$

Ex 2. If $f(x) = \frac{1}{x^2}$, find $f(a,b), f(a,b,c)\ and\ f(a,b,c,d)$

Soln **:** We have

$$f(a,b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a} = \frac{\frac{a^2-b^2}{a^2.b^2}}{b-a} = \frac{(a+b)(a-b)}{a^2.b^2(b-a)} = -\frac{a+b}{a^2.b^2} \quad \ldots\ldots\ldots\ldots (1)$$

Again $f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a}$

$$= \frac{-\frac{b+c}{b^2.c^2}+\frac{a+b}{a^2 \cdot b^2}}{c-a}$$
$$= \frac{1}{c-a} \cdot \frac{c^2(a+b)-a^2(b+c)}{a^2b^2c^2}$$
$$= \frac{1}{c-a} \cdot \frac{(c^2-a^2)b+c^2a-a^2c}{a^2b^2c^2}$$
$$= \frac{1}{c-a} \cdot \frac{(c-a)\{(c+a)b+ca\}}{a^2b^2c^2} = \frac{ab+bc}{a^2b^2c^2}$$

Similarly $f(a,b,c,d) = -\frac{abc+bcd+acd+a}{a^2b^2c^2d^2}$

Ex 3. Show that $\underset{yz}{\Delta^2}x^3 = x + y + z$

Soln: $f(x) = x^3$

Now $\underset{y}{\Delta}f(x) = f(x,y) = \frac{f(y)-f(x)}{y-x}$

$$= \frac{y^3-x^3}{y-x}$$
$$= y^2 + xy + x^2$$

Similarly $\underset{y}{\Delta}f(z) = z^2 + yz + y^2$

Now $\underset{yz}{\Delta^2}f(x) = \underset{yz}{\Delta^2}x^3$

$$= f(x,y,z)$$
$$= \frac{f(y,z)-f(x,y)}{z-x}$$
$$= \frac{y^2+yz+z^2-x^2-xy-\ {}^{2}}{z-x}$$

$$=\frac{z^2-x^2+yz-xy}{z-x}$$
$$=\frac{(z-x)(z+x+y)}{z-x}$$
$$=x+y+z \qquad \underline{\text{proof.}}$$

Lagranges Interpolation formula for unequal intervals:

Let $f(x)$ denote a polynomial of the nth degree which takes the values $y_0, y_1, \ldots\ldots\ldots\ldots\ldots . y_n$ when x has the values $x_0, x_1, x_2, \ldots\ldots\ldots . . x_n$ respectively. Then the (n+1)th differences of this polynomial are zero. Hence

$f(x, x_0, x_1, \ldots\ldots\ldots\ldots . x_n)=0$ ……………………………… (1)

But we have

$$f(x, x_0, x_1, \ldots\ldots\ldots\ldots . x_n)$$
$$=\frac{y}{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}$$
$$+\frac{y_0}{(x_0-x)(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}$$
$$+\frac{y_1}{(x_1-x)(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}+\cdots\ldots\ldots\ldots\ldots$$
$$+\frac{y_n}{(x_n-x)(x_n-x_0)\ldots\ldots(x_n-x_{n-1})}$$

$\therefore$ using (1) , we get

$$\Rightarrow \frac{y}{(x-x_0)(x-x_1)\ldots\ldots\ldots\ldots(x-x_n)}+\frac{y_0}{(x_0-x)(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots\ldots(x_0-x_n)}+\frac{y_1}{(x_1-x)(x_1-x_0)(x_1-x_2)\ldots\ldots\ldots(x_1-x_n)}+\cdots\ldots\ldots\ldots\ldots+\frac{y_n}{(x_n-x)(x_n-x_0)(x_n-x_1)\ldots\ldots\ldots\ldots(x_n-x_{n-1})}=0$$

$$\Rightarrow \frac{y}{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}=\frac{y_0}{(x-x_0)(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}+\frac{y_1}{(x-x_1)(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{y_n}{(x-x_n)(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}$$

$$\Rightarrow y=\frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x-x_0)(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} y_0+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x-x_1)(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} y_1+\ldots\ldots\ldots\ldots+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x-x_n)(x_n-x_0)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})} y_n$$

$$\therefore y=\frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} y_0+\frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} y_1+\ldots\ldots\ldots\ldots+\frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} y_n$$

Which is Lagranges formula .

Ex 4. Apply Lagranges Formula Calculate $f(1.5)$ by using the following table.

	x_0	x_1	x_2	x_3	x_4	x_5
x	1.00	1.20	1.40	1.60	1.80	2.00
$f(x)$	.2420	.1942	.1497	.1109	.0790	.0540

Soln. Using Lagrange formula we get,

$$f(1.5) = \frac{(1.5-1.2)(1.5-1.4)(1.5-1.6)(1.5-1.8)(1.5-2)}{(1-1.2)(1-1.4)(1-1.6)(1-1.8)(1-2)}(.2420)$$

$$+\frac{(1.5-1)(1.5-1.4)(1.5-1.6)(1.5-1.8)(1.5-2)}{(1.2-1)(1.2-1.4)(1.2-1.6)(1.2-1.8)(1.2-2)}(.1942)$$

$$+\frac{(1.5-1)(1.5-2)(1.5-1.6)(1.5-1.8)(1.5-2)}{(1.4-1)(1.4-1.2)(1.4-1.6)(1.4-1.8)(1.4-2)}(.1497)$$

$$+\frac{(1.5-1)(1.5-1.2)(1.5-1.4)(1.5-1.8)(1.5-2)}{(1.6-1)(1.6-1.2)(1.6-1.4)(1.6-1.8)(1.6-2)}(.1109)$$

$$+\frac{(1.5-1)(1.5-1.2)(1.5-1.4)(1.5-1.6)(1.5-2)}{(1.8-1)(1.8-1.2)(1.8-1.4)(1.8-1.6)(1.8-2)}(.0790)$$

$$\frac{(1.5-1)(1.5-1.2)(1.5-1.4)(1.5-1.6)(1.5-1.8)}{(2-1)(2-1.2)(2-1.4)(2-1.6)(2-1.8)}(.0540)$$

$$= .002835937 - .018964843 + .087714843$$

$$-.064980468 - .007714843 + .000632812$$

$$= .129484375 \quad \text{Ans.}$$

Gauss's Forward Difference Interpolation Formula:

We consider the following difference table in which the central ordinate is taken for convenience as y_0 corresponding to x = x_0.

x	y	Δ	Δ^2	Δ^3	Δ^4
x_{-3}	y_{-3}				
		Δy_{-3}			
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		
		Δy_{-2}		$\Delta^3 y_{-3}$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		
		Δy_2			
x_3	y_3				

The differences used in this formula lie on the line shown in the table.
The formula is, therefore, of the form.

$$y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \cdots \ldots\ldots\ldots \quad (1)$$

Where G_1,G_2,G_3………….. have to be determined, the y_p on the left side can be expressed in terms of y_0, Δy_0 and higher order differences of y_0 as follows.

$y_p = E^p y_0 = (1+\Delta)^p y_0$

$=y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{2!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \ldots$

Similarly the right side of (1) can also expressed in terms of y_0, Δy_0 and higher order differences.

Now $\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0$

$=\Delta^2(1+\Delta)^{-1}y_0$

$=\Delta^2(1-\Delta+\Delta^2-\Delta^3+\cdots\ldots\ldots)y_0$

$\therefore \Delta^2 y_{-1}=\Delta^2 y_0-\Delta^3 y_0+\Delta^4 y_0-\Delta^5 y_0+\cdots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore \Delta^3 y_{-1}=\Delta^3 y_0-\Delta^4 y_0+\Delta^5 y_0-\Delta^6 y_0+\cdots\ldots\ldots\ldots\ldots\ldots\ldots$

$\Delta^4 y_{-2}=\Delta^4 E^{-2} y_0$

$=\Delta^4(1+\Delta)^{-2}y_0$

$=\Delta^4(1-2\Delta+3\Delta^2-4\Delta^3+\cdots\ldots\ldots)y_0$

$=\Delta^4 y_0-2\Delta^5 y_0+3\Delta^6 y_0-4\Delta^7 y_0+\cdots\ldots\ldots$

Hence (1) give the identity

$$y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots\ldots\ldots\ldots\ldots$$

$= y_0+G_1\Delta y_0+G_2(\Delta^2 y_0-\Delta^3 y_0+\Delta^4 y_0-\Delta^5 y_0+\cdots\ldots\ldots)+G_3(\Delta^3 y_0-\Delta^4 y_0+\Delta^5 y_0-\Delta^6 y_0+\cdots\ldots\ldots)+G_4(\Delta^4 y_0-2\Delta^5 y_0+3\Delta^6 y_0-4\Delta^7 y_0+\cdots\ldots)+\cdots\ldots$ (2)

Equating the coefficients of

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots\ldots\ldots\ldots\ldots\ldots$ *on bothsides of* (2), *we obtain*

$$G_1=p$$

$$G_2=\frac{p(p-1)}{2!}$$

$$-G_2+G_3=\frac{p(p-1)(p-2)}{3!}$$

$$\therefore G_3=\frac{p(p-1)(p-2)}{6}+\frac{p(p-1)}{2}$$

$$=\frac{(p+1)p(p-1)}{3!}$$

Similarly $G_4=\frac{(p+1)p(p-1)(p-2)}{4!}$ etc.

Hence (1) becomes

$$y_p=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_{-1}+\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} +\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}+\cdots\ldots\ldots\ldots$$

Which is Gauss's forward formula.

EX 1. Use Gauss's forward formula to find f(x) when x=3.6 from the data below

X	2.5	3.0	3.5	4.0	4.5
Y	24.145	22.043	20.225	18.644	17.644

Solution:

We know the gauss's forward formula is,

$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \ldots \ldots \ldots \ldots)$ (1)

Here we take $x_0 = 3.5$, x=3.6, h=0.5

P=$\frac{x-x_0}{h} = \frac{3.6-3.5}{0.5} = 0.2$

The difference table is as follows

X	Y	Δ	Δ^2	Δ^3	Δ^4
2.5	24.145				
		-2.012			
3.0	22.043		.284		
		-1.818		-.47	
3.5	20.225		.237		.391
		-1.581		.344	
4.0	18.644		.581		
		-1.000			
4.5	17.644				

Therefore $y_0 = 20.225$, $\Delta y_0 = -1.581$,$\Delta^2 y_{-1} = .237, \Delta^3 y_{-1} = .344$, $\Delta^4 y_{-2} = .391$

Hence from (1) we get

$y_{.2} = y(3.6)$

$$= 20.225 + .2(-1.581) + \frac{.2(.2-1)}{2}(.237) + \frac{(1+.2)(.2)(.2-1)}{3!}(.344) + \frac{(1+.2)(.2)(.2-1)(.2-2)}{4!}(.391)$$

= 19.8844624 (answer)

Gauss's backward difference interpolation formula

We consider the following difference table in which the central ordinate is taken for convenience as y_0 corresponding to x=x_0

X	Y	Δ	Δ^2	Δ^3	Δ^5
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

The differences used in this formula lie on the line shown in the table. The formula is, therefore of the form

$y_p = y_0 + G_1^{'}\Delta y_{-1} + G_2^{'}\Delta^2 y_{-1} + G_3^{'}\Delta^3 y_{-2} + G_4^{'}\Delta^4 y_{-2} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots..(1)$

Where $G_1^{'}, G_2^{'}, G_3^{'}, G_4^{'}, \ldots\ldots\ldots\ldots\ldots.. \mathit{have\ to\ be\ determined.}$

y_p on the left side of (1) can be expressed in terms of y_0, Δy_0 ,and higher order differences of y_0 as follows

$y_p = E^p y_0 = (1+\Delta)^p y_0$

$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{2!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \ldots\ldots\ldots\ldots$

Similarly the right side of (1) can also expressed in terms of y_0, Δy_0 and higher order differences.

Now $\Delta y_{-1} = \Delta E^{-1} y_0$

$= \Delta(1+\Delta)^{-1} y_0$

$= \Delta(1 - \Delta + \Delta^2 - \Delta^3 + \cdots \ldots\ldots\ldots..) y_0$

$\therefore \Delta y_{-1} = \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$\therefore \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$\therefore \Delta^3 y_{-2} = \Delta^3 E^{-2} y_0$

$= \Delta^3 (1+\Delta)^{-2} y_0$

$= \Delta^3 (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \cdots \ldots\ldots\ldots..) y_0$

$\therefore \Delta^3 y_{-2} = \Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - 4\Delta^6 y_0 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$\therefore \Delta^4 y_{-2} = \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Here (1) Gives the identity

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \ldots\ldots$$

$$= y_0 + G_1'(\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \cdots \ldots\ldots) + G_2'(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \cdots \ldots\ldots\ldots) + G_3'(\Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - \cdots \ldots) + G_4'(\Delta^4 y_0 - \Delta^5 y_0 + \Delta^6 y_0 - \cdots \ldots.) \ldots \ldots\ldots (2)$$

Equating the coefficient of

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots\ldots\ldots\ldots\ldots\ldots$ *on bothsides of* (2), *we obtain*

$$G_1' = p$$

$$-G_1' + G_2' = \frac{p(p-1)}{2!}$$

$$\therefore G_2' = \frac{p(p-1)}{2!} + G_1' = \frac{p(p-1)}{2} + p$$

$$= \frac{p(p+1)}{2!}$$

Similarly $G_3' = \frac{(p+1)p(p-1)}{3!}$ And $G_4' = \frac{(p+2)(P+1)p(p-1)}{4!}$ etc.

Hence (1) becomes,

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \ldots\ldots\ldots$$

Which is Gauss's backward formula.

Ex 3. Interpolate by means of Gauss's backward formula the population for the year 1936, Given the following table.

Year:	1901	1911	1921	1931	1941	1951
Population:(thousands)	12	15	20	27	39	52

Solution: We know the Gauss's backward formula is,

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \ldots\ldots \quad \ldots\ldots(1)$$

Here we take $x_0 = 1931$, x=1936, $h = 10$

$$\therefore p = \frac{x - x_0}{h} = \frac{1936 - 1931}{10} = \frac{5}{10} = 0.5$$

The difference table is given below:

x	Y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1901	12					
		3				
1911	15		2			
		5		0		
1921	20		2		3	
		7		3		-10
1931	27		5		7	
		12		-4		
1941	39		1			
		13				
1951	52					

Therefore $y_0 = 27, \quad \Delta y_{-1} = 7, \quad \Delta^2 y_{-1} = 5$
$\Delta^3 y_{-2} = 3, \quad \Delta^4 y_{-2} = -7, \quad \Delta^5 y_{-3} = -10$

Hence from (1) we get,
$y_p = y_{0.5} = y(1936)$
$=27+0.5(7) + \frac{(0.5+1)0.5}{2!}(5) + \frac{(0.5+1)(0.5)(0.5-1)}{3!}(3) + \frac{(0.5+2)(0.5+1)(0.5)(0.5-1)}{4!}(-7) + \frac{(0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)}{5!}(-10)$
=27+3.5+1.875-0.1875+0.2734-0.1172
=32.3437 (Thousand) Ans.

Stirling's Formula:
We know the Gauss's forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

And the Gauss's backward formula is,

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots(2)$$

Now taking the mean of (1) and (2) we obtain

$$y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Which is called the Stirling's formula.

Ex 5. Use Stirling's formula to find $f(35)$ where $f(20) = 512, f(30) = 439, f(40) = 346, f(50)=243$

Solution: We know the Stirling's formula is

$$y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \ldots\ldots(1)$$

The difference table is as follows.

X	Y	Δ	Δ^2	Δ^3
20	512			
		-73		
30	439		-20	
		-93		10
40	346		-10	
		-103		
50	243			

Here we take $x_0 = 30,\ \ x = 35,\ \ h = 10, p = (x - x_0)/h = (35-30)/10 = 0.5$

Therefore from (1)

$$y_{0.5} = f(35) = 439 + 0.5\frac{(-73-93)}{2} + \frac{0.5^2}{2}(-20) + \frac{0.5\{0.5^2-1\}}{3!}.\frac{10+0}{2}$$

=395

Bessel's formula:

This formula uses the differences as shown in the following table where brackets mean that the average of the values have to be taken.

x	y	Δ	Δ^2	Δ^3	Δ^4
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$
		Δy_{-2}		$\Delta^3 y_{-3}$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$

Here Bessel's formula can be assumed in the form,

$$y_p = \frac{y_0 + y_1}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \ldots \ldots$$

$$y_p = y_0 + (B_1 + 1/2)\Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \ldots \ldots \quad \ldots\ldots\ldots\ldots(1)$$

Where $B_1, B_2, \ldots \ldots \ldots$ Have to be determined.The y_p of the left side of (1) can be expressed in terms of $y_0, \Delta y_0$ and higher order differences of y_0 as follows.

$$y_p = E^p y_0 = (1 + \Delta)^p y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \ldots$$

Similarly the right side of (1) can also be expressed in terms of $y_0, \Delta y_0$ and higher order differences as follows.

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0 = \Delta^2 (1 + \Delta)^{-1} y_0$$

$$= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \cdots \ldots) y_0$$

$$\therefore \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots \ldots \ldots \ldots \ldots$$

$$\therefore \Delta^3 y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \cdots \ldots \ldots \ldots \ldots$$

$$\Delta^4 y_{-1} = \Delta^4 y_0 - \Delta^5 y_0 + \Delta^6 y_o - \cdots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\Delta^4 y_{-2} = \Delta^4 E^{-2} y_0 = \Delta^4 (1 + \Delta)^{-2} y_0$$

$$= \Delta^4 (1 - 2\Delta + 3\Delta^2 - \cdots \ldots \ldots \ldots) y_0$$

$$= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - \cdots \ldots \ldots \ldots$$

Here (1) Gives the identity

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \dots ..$$

$$= y_0 + (B_{1+}1/2)\Delta y_0 + \frac{B_2}{2}(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \cdots \dots \dots . + \Delta^2 y_0) + B_3(\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \cdots \dots \dots \dots) + \frac{B_4}{2}(\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - \cdots \dots \dots \dots + \Delta^4 y_0 - \Delta^5 y_0 + \Delta^6 y_0 - \cdots)$$

$$= y_0 + \left(B_1 + \frac{1}{2}\right)\Delta y_0 + \frac{B_2}{2}(2\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \cdots \dots \dots) + B_3(\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \cdots \dots .) + \frac{B_4}{2}(2\Delta^4 y_0 - 3\Delta^5 y_0 + 4\Delta^6 y_0 - \cdots \dots \dots .) \dots \dots \dots \dots \dots . (2)$$

Equating the coefficient of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_{0,}$.................etc on both sides of (2) ,we obtain.

P=$B_1 + \frac{1}{2}$ $\quad or, B_1 = p - \frac{1}{2}$

$$\frac{p(p-1)}{2!} = B_2 \quad or, B_2 = \frac{p(p-1)}{2!}$$

$$\Rightarrow \frac{p(p-1)(p-2)}{3!} = -\frac{B_2}{2} + B_3$$

$$\Rightarrow B_3 = \frac{p(p-1)(p-2)}{6} + \frac{p(p-1)}{4}$$

$$= \frac{p(p-1)}{2}\left[\frac{p-2}{3} + \frac{1}{2}\right]$$

$$= \frac{p(p-1)}{12}\left[\frac{2p-4+3}{6}\right]$$

$$= \frac{p(p-1)(2p-1)}{12} = \frac{p(p-1)(p-\frac{1}{2})}{3!}$$

Similarly $B_4 = \frac{(p+1)p(p-1)(p-2)}{4!}$ etc

Thus (1) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \dots \dots ..$$

Which is the Bessel's formula,

Ex. The following table gives seadings of the temperature (0^0) secorded of given time (t).

t	2	3	4	5	6
θ	61.87	54.08	47.03	40.65	34.88

Using Bessel's formula find θ at t=4.3

Solution: We know the Bessel's formula is ,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \ldots\ldots$$

The difference table is as follows,

t	$\theta = f(t)$	Δ	Δ^2	Δ^3	Δ^4
2	61.87				
		-7.79			
3	54.08		0.74		
		-7.05		-0.07	
$4(t_0)$	47.03		0.67		(0.01)
		-6.38		-0.06	
5	40.65		0.61		
		-5.77			
6	34.88				

Here $t_0 = 4$, $t = 4.3$, h=1 .

$\therefore p = \frac{t-t_0}{h} = \frac{4.3-4}{1} = 0.3$

Thus from (1) we have ,

$\theta = f(t) = f(4.3) = y_{0.3}$

$=47.03+0.3(-6.38)+\frac{0.3(0.3-1)}{2!}\frac{0.67+0.61}{2} + \frac{0.3(0.3-1)\left(0.3-\frac{1}{2}\right)}{3!}(-0.06)$

$=45.04838 \cong 45.05$ (Ans).

Numerical Differentiation

Defn: The process of calculating the derivatives of a function by means of the set of given values of the function is called numerical differentiation.

Differentiation formula by using Newton's forward interpolation formula:

Consider Newton's forward difference formula

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots \quad (1)$$

Where $x = x_0 + hp$

$$p = \frac{x-x_0}{h}$$

$$\frac{dp}{dx} = \frac{1}{h}$$

Now differentiating (1) w. r. to x we get

$$y'(x) = \frac{dy}{dp}.\frac{dp}{dx}$$

$$= \frac{dp}{dx}.\frac{dy}{dp}$$

$$= \frac{1}{h}(\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3+18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\ldots) \quad (2)$$

Again differentiating (2) w.r.to x we get,

$$y'' = \frac{1}{h^2}(\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2+36p}{4!}\Delta^3 y_0 + \ldots\ldots) \ldots\ldots\ldots (3)$$

Formula (2) and (3) can be used for computing the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for non-tabular values of x. The formula takes a simpler form by using $x=x_0$ we obtain,

$p=0$, Hence from (2), we get

$$[y']_{x=x_0} = \frac{1}{h}[\Delta y_0 + \frac{-1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0 + \frac{-6}{4!}\Delta^4 y_0 + \ldots\ldots]$$

$$Or, [y']_{x=x_0} = \frac{1}{h}[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots] \ldots\ldots\ldots (4)$$

Again from (3) we obtain,

$$[y'']_{x=x_0} = \frac{1}{h^2}[\Delta^2 y_0 + \frac{-6}{3!}\Delta^3 y_0 + \frac{22}{4!}\Delta^3 y_0 + \ldots\ldots]$$

$$Or, [y'']_{x=x_0} = \frac{1}{h^2}[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\ldots] \ldots\ldots\ldots (5)$$

Equation (2) and (4) gives the 1st derivative and (3) and (5) give the 2nd derivative. Similarly formula for higher derivatives may be obtained by successive differentiation.

Ex. Find the 1st and 2nd derivatives of the function tabulated below at the point x=3.

x	3.0	3.2	3.4	3.6	3.8	4.0
y=f(x)	-14	-10.032	-5.296	.256	6.672	14

Soln: The difference table is as below.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
3.0	-14.00					
		3.968				
3.2	-10.032		.768			
		4.736		.048		
3.4	-5.296		.816		0	
		5.552		.048		0
3.6	.256		.864		0	
		6.416		.048		
3.8	6.672		.912			
		7.328				
4.0	14.00					

We know the differentiation formulas by using the Newton's forward formula are

$y' = \frac{1}{h}[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...]$ . . . . (1)

$y'' = \frac{1}{h^2}[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - ....]$ . . . . (2)

Here, $x_0 = 3.0, \quad h = .2, \quad y_0 = -14.00$

$\Delta y_0 = 3.968, \quad \Delta^2 y_0 = .768, \quad \Delta^3 y_0 = .048, \quad \Delta^4 y_0 = 0$

Hence from (1) we get,

$$[y']_{x=3.0} = \frac{1}{.2}\left[3.968 - \frac{1}{2}(.768) + \frac{1}{3}(.048)\right]$$

=18

And from (2) we get,

$$[y'']_{x=3.0} = \frac{1}{(.2)^2}[.786 - .048 + 0]$$

=18

Ex. Find the 1st and 2nd derivatives of the function tabulated below at the point x=1.1

Soln: Since the derivatives are required at x=1.1, which is near the beginning of the table, we shall use Newton's Forward formula.

The difference formula is as below

x	y	Δ	Δ^2	Δ^3	Δ^4
1.0	0				
		.1280			
1.2	.1280		.280		
		.4160		.0480	
1.4	.5440		.3360		0
		.7520		.0480	
1.6	1.2960		.3840		0
		1.1360		.0480	
1.8	2.4320		.4320		
		1.5680			
2.0	4.00				

We know Newton's Forward difference formula is

$$y(x)= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots \quad (1)$$

Where $p= \frac{x-x_0}{h}$

$$\frac{dp}{dx} = \frac{1}{h}$$

Differentiating (1) w.r.to x we get,

$$y'(x) = \frac{1}{h}[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \ldots\ldots] \quad (2)$$

Again differentiating (2) w. r. to x

$$y''(x) = \frac{1}{h^2}[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{12p^2-36p+}{24}\Delta^4 y_0 + \ldots\ldots..] \ldots\ldots.. (3)$$

Here, x_0= 1.0, h= .2, x=1.1

$$p = \frac{x-x_0}{x} = \frac{1.1-1.0}{.2} = \frac{.1}{.2} = .5$$

Hence from (2) we get,

$$y'(1.1) = \frac{1}{.2}[.1280 + \frac{2(.5)-1}{2}(.2880) + \frac{3(.5)^2-6(.5)+2}{6}(.0480)$$

$$= .630$$

Again from (3) we get,

$$y''(1.1) = \frac{1}{(.2)^2}\left[.2880 + \frac{6(.5)-6}{6}(.0480)\right]$$

=0.660

<u>Derivative using Newton's backward difference formula:</u>

Consider Newton's backward formula:

y(x)=$y_n + k\nabla y_n + \frac{k(k+1)}{2!}\nabla^2 y_n + \frac{k(k+1)(k+2)}{3!}\nabla^2 y_n + \frac{k(k+1)(k+2)(k+3)}{4!}\nabla^4 y_n +$.............(1)

Where x= $x_n + kh$, $k = \frac{x-x_n}{h}$

$\frac{dk}{dx} = \frac{1}{h}$

Now differentiating (1) w.r.to x we get

$y'(x) = \frac{dy}{dk}.\frac{dk}{dx} = \frac{1}{h}\frac{dy}{dk}$

$= \frac{1}{h}[\nabla y_n + \left(k+\frac{1}{2}\right)\nabla^2 y_n + \left(\frac{k^2}{2} + k + \frac{1}{3}\right)\nabla^3 y_n + (\frac{k^3}{6} + \frac{3}{4}k^2 + \frac{11}{12}k + \frac{1}{4})\nabla^4 y_n +$....] . . . . (2)

Again differentiating (2) w. r. to x

$y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (k+1)\nabla^3 y_n + (\frac{1}{2}k^2 + \frac{3}{2}k + \frac{11}{12}\right)\nabla^4 y_n +$........] . . . . (3)

Formula (2) and (3) can be used for computing the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively for non-tabular value of x.

For tabular value of x, the formula takes a simpler form for by using x= x_n i.e. k=0.

Then (2) and (3) becomes

$f'(x_n) = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n +$........] . . . . . . (4)

$f''(x_n) = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n +$........] . . . . (5)

(4) and (5) gives 1st and 2nd derivative.

Ex: Find $f'(2.2)$ and $f''(2.2)$ from the following table.

x	1.4	1.6	1.8	2.0	2.2
f(x)	4.0552	4.9530	6.0496	7.3891	9.0250

Soln: The difference table is as below:

x	y=f(x)	∇	∇^2	∇^3	∇^4
1.4	4.0552				
		.8918			
1.6	4.9530		.1988		
		1.0966		.0441	

1.8	6.0496		.2429		.0094
		1.3395		.0535	
2.0	7.3891		.2964		
		1.6359			
2.2	9.0250				

Here $x_n = 2.2,\ y_n = 9.0250,\ \ \nabla f_n = 1.6359,\ \nabla^2 y_n = .2964,\ \nabla^3 y_n = .0535,$ $\nabla^4 y_n = .0094,$ h=.02

We know

$f'(x_n) = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + ...]$

$f'(2.2) = \frac{1}{.2}\left[1.6359 + \frac{1}{2}(.2264) + \frac{1}{3}(.0535) + \frac{1}{4}(.0094)\right] = 9.021416667$

Again

$f''(x_n) = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...]$

$f''(2.2) = \frac{1}{(.2)^2}\left[.2964 + .0535 + \frac{11}{12}(.0094)\right] = 8.962916667$

Derivative based on Stirling's formula:

We know the Stirling's formula is

f(x)=$f_0 + p\frac{\Delta f_0 + \Delta f_{-1}}{2} + \frac{p^2}{2}\Delta^2 f_{-1} + \frac{p(p^2-1)}{3!}\ \frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 f_{-2}$+........(1)

Where p = $\frac{x-x_0}{h}$ i.e. $\frac{dp}{dx} = \frac{1}{h}$

Differentiation (10 w. r. to x, we get

$f'(x) = \frac{1}{h}[\frac{\Delta f_0 + \Delta f_{-1}}{2} + p\Delta^2 f_{-1} + \frac{3p^2-1}{3!}\ \frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2} + \frac{4p^3-2p}{4!}\Delta^4 f_{-2}$+......] . . . .(2)

Again differentiating (2) w. r. to x

$f''(x) = \frac{1}{h^2}[\Delta^2 f_{-1} + p\frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2} + \frac{12p^2-2}{4!}\Delta^4 f_{-2}$+.........] . . . . . . (3)

For the point x=x_0 we get

p=0 Thus (2) and (3) becomes

$f'(x_0) = \frac{1}{h}[\frac{\Delta f_0 + \Delta f_{-1}}{2} - \frac{1}{6}\ \frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2}$ +........... (4)

$f''(x_0) = \frac{1}{h^2}[\Delta^2 f_{-1} - \frac{1}{2}\ \Delta^4 f_{-2}$+............. (5)

(2) and (4) gives the 1st derivative and (3) and (5) gives the 2nd derivative.

Ex: Given the table

x	0	1	2	3	4
f(x)	6.9897	7.4036	7.7815	8.1281	8.4510

Find $f'(2) and\ f''(2)$ using appropriate formula.

Soln: Here we use the derivative which is obtained from Stirling's formula.

The table is

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
0	6.9897				
		.4139			
1	7.4036		-.0366		
		.3779		.0047	
2	7.7815		-.0313		.0025
		.3466		.0076	
3	8.1281		-.0237		
		.3224			
4	8.4510				

Here x_0=2, $h = 1$

$f_0 = 7.7815,\quad \Delta f_0 = .3466,\ \Delta f_{-1} = .3779,$

$\Delta^2 f_{-1} = -.0313,\ \Delta^3 f_{-1} = .0076,\ \Delta^3 f_{-2} = .0047,\ \Delta^4 f_{-2} = .0025$

We know that

$$f'(x_0) = \frac{1}{h}[\frac{\Delta f_0+\Delta f_{-1}}{2} - \frac{1}{3!}\ \frac{\Delta^3 f_{-2}+\Delta^3 f_{-1}}{2}+........]$$

$$f'(2) = \frac{1}{1}[\frac{.3779 + .3466}{2} - \frac{1}{6}\ \frac{.0047 + .0076}{2}]$$

= .361225

And $f''(x_0) = \frac{1}{h^2}[\Delta^2 f_{-1} - \frac{1}{12}\Delta^4 f_{-2}+........]$

$$f''(2) = \frac{1}{1^2}[-.0313 - \frac{1}{2}(.0025)]$$

= -.03151

Ex: Given the data

x	0	1	2	4
f(x)	5	14	41	98

Use the Lagrange's interpolation formula to find $f'(0.9)$.

Soln: We know the Lagrange's formula is

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)x_3-x_2)}f(x_3)$$

$$=\frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(5)+\frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(14)$$
$$+\frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(41)+\frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(98)$$
$$=\frac{x^3-7x^2+14x-8}{-8}(5)+\frac{x^3-6x^2+8x}{3}(14)$$
$$+\frac{x^3-5x^2+4x}{-4}(41)+\frac{x^3-3x^2+2x}{24}(98)$$
$$=\left(-\frac{5}{8}+\frac{14}{3}-\frac{41}{4}+\frac{98}{24}\right)x^3+\left(\frac{35}{8}-28+\frac{205}{4}-\frac{3}{24}\right)x^2+$$
$$\left(-\frac{35}{4}+\frac{112}{3}-41+\frac{98}{12}\right)x+5$$
$$\therefore f(x)=-\frac{17}{8}x^3+\frac{123}{8}x^2-\frac{17}{4}x+5$$
$$f'(x)=\frac{-51}{8}x^2+\frac{123}{4}x-\frac{17}{4}$$
$$f'(.9)=\frac{-51}{8}(.9)^2+\frac{123}{4}(.9)-\frac{17}{4}$$
$$=18.26125$$

Ex: Use the following data to find $f'(5)$

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Soln: Here we use Newton's divided difference formula.

The table is

x	$f(x)$	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference
0	4				

		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

We know the Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \dots\dots$$

We can take $x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4$

Therefore $f(x_0) = 4,\ \ f(x_0, x_1) = 11,\ \ f(x_0, x_1, x_2) = 7,\ \ f(x_0, x_1, x_2, x_3) = 1$

$\therefore f(x) = 4 + (x - 0)11 + (x - 0)(x - 2)7 + (x - 0)(x - 2)(x - 3)1$

$= 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$

$= x^3 + 2x^2 + 3x + 4$

So

$f'(x) = 3x^2 + 4x + 3$

$f'(5) = 3(5)^2 + 4(5) + 3$

$= 98$

Numerical Integration

***A General Quadrature Formula for Equidistance Ordinates.**

Let a set of data points (x_0, y_0), (x_1, y_1), (x_2, y_2), ... (x_n, y_n) of a function $y = f(x)$ be given, where *f(x)* is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y dx \qquad (1)$$

Let the interval [a,b] be divided into n equal subintervals each length h such that $a = x_0 < x_1 < x_2 < ..... < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence (1) becomes

$$I = \int_{x_0}^{x_n} y dx$$

Approximating *y* by Newton's forward difference interpolation formula, we obtain

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ... \right] dx$$

Since $x = x_0 + ph$

$dx = hdp$

$x = x_0 \Rightarrow p = 0$

$x = x_n \Rightarrow p = n$ and hence the above integral becomes

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ... \right] dp$$

Which gives on simplification

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + ... \right].$$

....(2)

This is the general formula.

***Trapezoidal Rule.**

Let a set of data points (x_0, y_0), (x_1, y_1), (x_2, y_2), ... (x_n, y_n) of a function $y = f(x)$ be given, where *f(x)* is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y dx \qquad (1)$$

Let the interval [a,b] be divided into n equal subintervals each length h such that $a = x_0 < x_1 < x_2 < ..... < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence (1) becomes

$$I = \int_{x_0}^{x_n} y dx$$

Approximating y by Newton's forward difference interpolation formula, we obtain

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ... \right] dx$$

Since $x = x_0 + ph$

$dx = hdp$

$x = x_0 \Rightarrow p = 0$

$x = x_n \Rightarrow p = n$ and hence the above integral becomes

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ... \right] dp$$

Which gives on simplification

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + ... \right].$$

....(2)

Putting $n=1$ in (2) and neglecting 2nd and higher order differences, we get

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right]$$

$$= h\left[y_0 + \frac{y_1 - y_0}{2} \right]$$

$$= h\left[\frac{y_0 + y_1}{2} \right]$$

Similarly $\int_{x_1}^{x_2} y\,dx = h\left[\frac{y_1 + y_2}{2} \right]$

$$\int_{x_2}^{x3} y\,dx = h\left[\frac{y_2 + y_3}{2} \right]$$

....

$$\int_{x_{n-1}}^{x_n} y\,dx = h\left[\frac{y_{n-1} + y_n}{2} \right]$$

Adding those n integrals, we get

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \dots\dots + y_{n-1}) + y_n\right]$$

Which is known as trapezoidal rule.

Ex. Calculate the value of the integral $\int_4^{5.2} \ln x\,dx$ by Trapezoidal rule.

Soln. Divide the interval [4, 5.2] into six equal parts. So that h=(5.2-4)/6=0.2 and the value of $\ln x$ for each point of sub division are given below:

n	*x*	*y=ln x*
0	4.0	1.3862944
1	4.2	1.4350845
2	4.4	1.4816045
3	4.6	1.5260563
4	4.8	1.5686159
5	5.0	1.6094379
6	5.2	1.6486586

We have from the trapezoidal rule

$$\int_{x_0}^{x_6} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6\right]$$

$$\therefore \int_4^{5.2} \ln x dx = \frac{0.2}{2}\begin{bmatrix}1.3862944 + 2(1.4350845 + 1.4816045 \\ +1.5260563 + 1.5686159 + 1.6094379) + 1.6486586\end{bmatrix}$$

$= 0.1(18.276551)$

$= 1.8276551$

Ex. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Find the area bounded by the curve, the x-axis and the lines *x=1, x=4*

Soln. In order to find the required area we shall compute the value of the integral

$$I = \int_1^4 y\,dx$$

Here *n=6*, therefore *h=(4-1)/6=0.5*

We have from the trapezoidal rule

$$\int_{x_0}^{x_6} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6\right]$$

$$\therefore \int_4^{5.2} y\,dx = \frac{0.5}{2}\left[2+2(2.4+2.7+2.8+3+2.6)+2.1\right]$$
$$= 7.775$$

***Simpson's 1/3 Rule.**

Let a set of data points (x_0, y_0), (x_1, y_1), (x_2, y_2), ... (x_n, y_n) of a function $y = f(x)$ be given, where *f(x)* is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y dx \qquad (1)$$

Let the interval [a,b] be divided into n equal subintervals each length h such that $a = x_0 < x_1 < x_2 < ..... < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence (1) becomes

$$I = \int_{x_0}^{x_n} y dx$$

Approximating *y* by Newton's forward difference interpolation formula, we obtain

$$I = \int_{x_0}^{x_n}\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ...\right]dx$$

Since $x = x_0 + ph$

$dx = hdp$

$x = x_0 \Rightarrow p = 0$

$x = x_n \Rightarrow p = n$ and hence the above integral becomes

$$I = h\int_0^n\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ...\right]dp$$

Which gives on simplification

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + ...\right].$$

....(2)

Putting *n=2* in (2) and neglecting 3rd and higher order differences, we get

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0) \right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right]$$

Similarly $\int\limits_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4 \right]$

$$\int\limits_{x_4}^{x_6} y\,dx = \frac{h}{3}\left[y_4 + 4y_5 + y_6 \right]$$

…………………………..

$$\int\limits_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$$

Adding those integrals, we get

$$\int\limits_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[\begin{array}{l} y_0 + 4(y_1 + y_3 + y_5 + \ldots\ldots + y_{n-1}) \\ + 2(y_2 + y_4 + y_6 + \ldots..y_{n-2}) + y_n \end{array} \right]$$

Which is known as Simpson's 1/3 Rule, or simply Simpson's rule. It should be noted that this rule requires the division of the whole range into an even number of subintervals of width h.

<u>Ex.</u> Calculate the value of the integral $\int\limits_{4}^{5.2} \ln x\,dx$ by Simpson's 1/3 Rule.

<u>Soln.</u> Divide the interval [4, 5.2] into six equal parts. So that h=(5.2-4)/6=0.2 and the value of $\ln x$ for each point of sub division are given below:

n	*x*	*y=ln x*
0	4.0	1.3862944
1	4.2	1.4350845
2	4.4	1.4816045
3	4.6	1.5260563
4	4.8	1.5686159
5	5.0	1.6094379
6	5.2	1.6486586

We have from Simpson's 1/3 rule

$$\int_{x_0}^{x_6} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 +) + 2(y_2 + y_4) + y_6\right]$$

$$\therefore \int_4^{5.2} \ln x dx$$

$$= \frac{0.2}{3}\begin{bmatrix} 1.3862944 + 4(1.4350845 + 1.5260563 + 1.6094379) \\ + 2(1.4816045 + 1.5686159) + 1.6486586 \end{bmatrix}$$

$$= \frac{0.2}{3}(27.417709)$$

$$= 1.8278472$$

Ex. Find $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 1/3 Rule. Hence obtain the approximate value of π

.

Soln. Divide the the range of integration [0, 1] into 6 equal parts. So that h=(1-0)/6=1/6.

the value of $y = \frac{1}{1+x^2}$ for each point of sub division are given below:

n	x	y
0	0	1
1	1/6	0.9729729
2	2/6	0.9
3	3/6	0.8
4	4/6	0.6923076
5	5/6	0.5901639
6	1	0.5

By Simpson's 1/3 Rule we get

$$\int_{x_0}^{x_6} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 +) + 2(y_2 + y_4) + y_6\right]$$

$$\therefore \int_0^1 \frac{dx}{1+x^2}$$

$$= \frac{1/6}{3}\begin{bmatrix} 1 + 4(0.9729729 + 0.8 + 0.5901639) \\ + 2(0.9 + 0.6923076) + 0.5 \end{bmatrix}$$

$$= \frac{1}{18}(14.137163)$$

$$= 0.7853979$$

But $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \pi / 4$

Therefore $\pi/4 = 0.7853979$

$\therefore \pi = 4(0.7853979) = 3.1415916$

Ex. Evaluate $\int_{0.5}^{0.7} x^{1/2}e^{-x}dx$ by using Simpson's 1/3 Rule.

Soln. Divide the range of integration [0.5, 0.7] into 4 equal parts. So that h=(0.7-0.5)/4=.05. the value of $y = x^{1/2}e^{-x}$ for each point of sub division are given below:

n	x	y
0	0.50	0.4288818
1	0.55	0.4278774
2	0.60	0.4251076
3	0.65	0.4208867
4	0.70	0.4154730

By Simpson's 1/3 Rule we get

$$\int_{x_0}^{x_4} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2(y_2) + y_6\right]$$

$$\therefore \int_0^1 \mathrm{x}^{1/2}e^{-x}dx$$

$$= \frac{0.05}{3}\begin{bmatrix} 0.4288818 + 4(0.4278774 + 0.4208867) \\ + 2(0.4251076) + 0.4154730 \end{bmatrix}$$

$$= 0.0848271$$

Ex. Evaluate $\int_{0.2}^{1.4} (\sin x - \ln x + e^x)dx$ by using Simpson's 1/3 Rule.

Soln. Divide the the range of integration [0.2, 1.4] into 12 equal parts. So that h=(1.4-0.2)/12=.1. the value of $y = \sin x - \ln x + e^x$ for each point of sub division are given below:

n	x	y		
0	0.2	3.02950		
1	0.3	2.84935	2.84935	
2	0.4	2.79753		2.79753
3	0.5	2.82130	2.82130	
4	0.6	2.89759		2.89759
5	0.7	3.01464	3.01464	
6	0.8	3.16604		3.16604

7	0.9	3.34829	3.34829	
8	1.0	3.55975		3.55975
9	1.1	3.80007	3.80007	
10	1.2	4.06984		4.06984
11	1.3	4.37050	4.37050	
12	1.4	4.70418		
sum			20.2042	16.4908

By Simpson's 1/3 Rule we get

$$\int_{x_0}^{x_6} y\,dx = \frac{h}{3}\begin{bmatrix} y_0 + 4(y_1 + y_3 + y_5 + y_7 + y_9 + y11) \\ + 2(y_2 + y_4 + y_6 + y_8 + y_{10}) + y_{12} \end{bmatrix}$$

$$\therefore \int_{0.2}^{1.4} (\sin x - \ln x + e^x)dx$$

$$= \frac{0.1}{3}[3.02950 + 4(20.2042) + 2(16.4908) + 4.70418]$$

$$= 4.051059$$

***Simpson's 3/8 Rule.**

Let a set of data points (x_0, y_0), (x_1, y_1), (x_2, y_2), ... (x_n, y_n) of a function $y = f(x)$ be given, where *f(x)* is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y dx \qquad (1)$$

Let the interval [a,b] be divided into n equal subintervals each length h such that $a = x_0 < x_1 < x_2 < ..... < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence (1) becomes

$$I = \int_{x_0}^{x_n} y dx$$

Approximating *y* by Newton's forward difference interpolation formula, we obtain

$$I = \int_{x_0}^{x_n}\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ... \right]dx$$

Since $x = x_0 + ph$

$dx = hdp$

$x = x_0 \Rightarrow p = 0$

$x = x_n \Rightarrow p = n$ and hence the above integral becomes

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \dots\dots\dots\dots \right] dp$$

Which gives on simplification

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \dots\dots\dots\dots \right].$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Putting n=3 in (2) and neglecting 4th and higher order differences, we get

$$\int_{x_0}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 \right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly $\displaystyle\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$

$$\int_{x_6}^{x_9} y\,dx = \frac{3h}{8}[y_6 + 3y_7 + 3y_8 + y_9]$$

…………………………..

$$\int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding those integrals, we get

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[\begin{array}{l} y_0 + 3(y_1 + y_2 + y_4 + y_5 \dots\dots + y_{n-2} + y_{n-1}) \\ + 2(y_3 +_6 + \dots\dots y_{n-3}) + y_n \end{array}\right]$$

Which is known as Simpson's 3/8 Rule.

Ex. Calculate the value of the integral $\int_4^{5.2} \ln x\,dx$ by Simpson's 3/8 Rule.

Soln. Divide the interval [4, 5.2] into six equal parts. So that h=(5.2-4)/6=0.2 and the value of $\ln x$ for each point of sub division are given below:

n	*x*	*y=ln x*
0	4.0	1.3862944
1	4.2	1.4350845
2	4.4	1.4816045

3	4.6	1.5260563
4	4.8	1.5686159
5	5.0	1.6094379
6	5.2	1.6486586

We have from Simpson's 3/8 rule

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) + y_6\right]$$

$$\therefore \int_{4}^{5.2} \ln x dx$$

$$= \frac{3\times 0.2}{8}\begin{bmatrix} 1.3862944 \\ +3(1.4350845 + 1.4816045 + 1.5686159 + 1.6094179) \\ +2(1.5260563) + 1.6486586 \end{bmatrix}$$

$$= \frac{0.6}{8}(24.371294)$$

$$= 1.827847$$

Ex. Calculate the value of the integral $\int_0^1 e^{-x^2}\, dx$ by Simpson's 3/8 Rule.

Soln. Divide the interval [0, 1] into 10 equal parts. So that h=(1-0)/10=0.1 the value of e^{-x^2} for each point of sub division are given below:

n	x	$y=e^{-x^2}$
0	0.0	1.00000
1	0.1	0.99005
2	0.2	0.96079
3	0.3	0.91393
4	0.4	0.85214
5	0.5	0.77880
6	0.6	0.69768
7	0.7	0.61263
8	0.8	0.52729
9	0.9	0.44486
10	1.0	0.36788

We have from Simpson's 3/8 rule

$$\int_{x_0}^{x_{10}} y\,dx = \frac{3h}{8}\begin{bmatrix} y_0 + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8) \\ + 2(y_3 + y_6 + y_9) + + y_{10} \end{bmatrix}$$

$$\therefore \int_0^1 e^{-x^2}\,dx$$

$$= \frac{3\times 0.1}{8}\begin{bmatrix} 1.00000 \\ + 3(0.99005 + 0.96079 + 0.85214 + 0.77880 + 0.61263 + 0.52729) \\ + 2(0.91393 + 0.69768 + 0.44486) + 0.36788 \end{bmatrix}$$

$$= \frac{0.3}{8}\left[1 + 3(4.7222) + 2(2.05647) + 0.36788\right]$$

$$= 0.73677825$$

System of linear equation

#Gausses scidal method :

Consider the system of equations

$a_{11}x + a_{12}y + a_{13}z = b_1$
$a_{21}x + a_{22}y + a_{23}z = b_2$
$a_{31}x + a_{32}y + a_{33}z = b_3$

Now, we solve the 1st equation for x in terms of 'y' and 'z'

$$x = \frac{b_1 - a_{12}x - a_{13}z}{a_{11}} \qquad (1)$$

Similarly from the 2nd & 3rd equation

$$y = \frac{b_3 - a_{21}x - a_{23}z}{a_{22}} \qquad (2)$$

And

$$z = \frac{b_3 - a_{31}x - a_{32}y}{a_{33}} \qquad (3)$$

Next, we make a first approximation to the solution which we call x_1 , y_1 & z_1 (Usually we select zeros for the 1st approximation to the solution) .

These 1st approximation are then used in equation (1) to compute a new approximate value for x

$$x_2 = \frac{b_1 - a_{12}y_1 - a_{13}z_1}{a_{11}} \qquad (4)$$
$$y_2 = \frac{b_2 - a_{21}x_2 - a_{23}z_1}{a_{22}} \qquad (5)$$
$$z_2 = \frac{b_3 - a_{31}x_2 - a_{32}y_2}{a_{33}} \qquad (6)$$

Proceeding in this way we get the general equation for the Gausses scidal iterations are

$$x_{n+1} = \frac{b_1 - a_{12}y_n - a_{13}z_n}{a_{11}}$$
$$y_{n+1} = \frac{b_2 - a_{21}x_{n+1} - a_{23}z_n}{a_{22}}$$
$$z_{n+1} = \frac{b_3 - a_{31}x_{n+1} - a_{32}y_{n+1}}{a_{33}}$$

Ex: Solve the system by Gausses scidal method.

$$2x + y = 4$$
$$x - y = -1$$

Solution : Our initial estimation of the solution

$$x_1 = 0 \ , \ y_1 = 0$$

Iteration 1: $x_2 = \frac{4-y_1}{2} = \frac{4-0}{2} = 2$

$y_2 = \frac{-1-x_2}{-1} = \frac{-1-2}{-1} = 3$

Iteration 2: $x_3 = \frac{4-y_2}{2} = \frac{4-3}{2} = 0.5$

$y_3 = \frac{-1-x_3}{-1} = \frac{-1-\frac{1}{2}}{-1} = \frac{3}{2} = 1.5$

Iteration 3 : $x_4 = \frac{4-y_3}{2} = \frac{4-1.5}{2} = 1.25$

$y_4 = \frac{-1-x_4}{-1} = \frac{-1-1.25}{-1} = 2.25$

Thus the approximate solution is x $= 1.25$ & $y = 2.25$

Ex: Solve the system

$27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$

Solution: Solving the equations for the unknown we can rewrite the equations

$$x = \frac{1}{27}(85 - 6y + z) \quad (1)$$
$$y = \frac{1}{15}(72 - 6x - 2z) \quad (2)$$
$$z = \frac{1}{54}(110 - x - y) \quad (3)$$

Our initial estimations are

$$x_0 = 0 \ , \ y_0 = 0 \ \& \ z_0 = 0$$

1st iteration:

$x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = \frac{85}{27} = 3.15$

$y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = \frac{1}{15}\{72 - (6 \times 3.15)\} = 3.54$

$z_1 = \frac{1}{54}(110 - y_1 - x_1) = 1.91$

2nd Iteration: $x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = \frac{1}{27}\{85 - (6 \times 3.54) + 1.91\} = 2.43$

$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = \frac{1}{15}\{72 - (6 \times 2.43) - (2 \times 1.91)\} = 3.57$

$z_2 = \frac{1}{54}(110 - y_2 - x_2) = \frac{1}{54}(110 - 3.57 - 2.43) = 1.93$

3rd iteration:

$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = \frac{1}{27}\{85 - (6 \times 3.57) + 1.93\} = 2.43$

$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = \frac{1}{15}\{72 - (6 \times 2.43) - (2 \times 1.93)\} = 3.57$

$z_3 = \frac{1}{54}(110 - y_3 - x_2) = \frac{1}{54}(110 - 3.57 - 2.43) = 1.93$

Thus the approximate result is $x = 2.43, y = 3.57 \ \& \ z = 1.93$

<u>#Gausses elimination method:</u> (Forward elimination and backward substitution)

Consider the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = c_1 \quad (1)$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n = c_2 \quad (2)$$

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$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n = c_n \quad (3)$$

The first phase is designed to reduce the set of equation to an upper triangular system.
The initial step in the process in to divide equation (1) (pivot equation)by the constant a_{11} , (the pivote coefficient)

$$x_1 + \frac{a_{12}}{a_{11}}x_2 + \dots\dots . + \frac{a_{1n}}{a_{11}}x_n = \frac{c_1}{a_{11}} \quad (4)$$

The above process is called normalizations.
Next multiply the normalized equation (4) by the first coefficient of second equation, we get

$a_{21}x_1 + (\, a_{21}\frac{a_{12}}{a_{11}})x_2 + \;\ldots\ldots\ldots\ldots. +(\, a_{21}\frac{a_{1n}}{a_{11}})x_n = \; a_{21}\frac{c_1}{a_{11}}$ (5)

The 1st unknown can be eliminated from the 2nd equation by substituting equation (5) from (2)

$\left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)x_2 + \;\ldots\ldots\ldots\ldots.. +(a_{2n} - a_{21}\frac{a_{1n}}{a_{11}})x_n = \; c_2 - a_{21}\frac{c_1}{a_{11}}$

$\Rightarrow \; a'_{22}x_2 + \;\ldots\ldots\ldots\ldots\ldots\ldots + \; a'_{2n}x_n = \; c'_2$

Repeating the process for the remaining equation, the result become in the following modified system.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \;\ldots\;\ldots\ldots\ldots\ldots\ldots. +a_{1n}x_n = c_1$ (6)

$a'_{22}x_2 + a'_{23}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots. +a'_{2n}x_n = c'_2$ (7)

$a'_{32}x_2 + a'_{33}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots. +a'_{3n}x_n = c'_3$ (8)

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$a'_{n2}x_2 + a'_{n3}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots. +a'_{nn}x_n = c'_n$ (9)

Now repeat the above process in order to eliminate the 2nd unknown from equation (8) through (9), we get

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \;\ldots\ldots\ldots\ldots\ldots.. +a_{1n}x_n = c_1$

$a'_{22}x_2 + a'_{23}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots + a'_{2n}x_n = c'_2$

$a''_{33}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots + a''_{3n}x_n = c''_3$

$a''_{n3}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots + a''_{nn}x_n = c''_n$

Where the double prime indicates that the eliminates have been changed or modified twice .The procedure can be continued using the remaining pivot equations, we get

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \;\ldots\ldots\ldots\ldots\ldots.. +a_{1n}x_n = c_1$ (10)

$a'_{22}x_2 + a'_{23}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots + a'_{2n}x_n = c'_2$ (11)

$a''_{33}x_3 + \;\ldots\ldots\ldots\ldots\ldots\ldots + a''_{3n}x_n = c''_3$ (12)

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$$a_{nn}^{(n-1)}x_n = c_n^{(n-1)} \quad (13)$$

From (13), we *get* $\quad x_n = \frac{c_n^{(n-1)}}{a_{nn}^{(n-1)}}$ (14)

Remaining x's can be represented by the following formula

$$x_i = \frac{c_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \quad (15)$$

[where *i* = *n-1 ,n-2 ,………., 3 , 2 , 1*]

Ex: Use Gausses elimination method to solve

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$ (1)

$0.1x_1 + 7x_2 - 0.3x_3 = \; -19.3$ (2)

$0.3x_3 + 0.2x_2 + 10x_3 = 71.4$ (3)

Carry six significant figures during computation .

Solution: Now normalized the equation (1) by dividing the pivot coefficient 3 we get

$x_1 - 0.333333x_2 - 0.666666x_3 = 2.61667$ (4)

Next, multiplying equation (4) by 0.1 and subtracting the result from (2) we get

$7.00333x_2 - 0.293333x_3 = -19.5617$ (5)

Now to eliminate x_1 multiplying equation (4) by 0.3 and subtracting the result from (3) we get the new system

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$ (6)

$7.00333x_2 - 0.293333x_3 = -19.5617$ (7)

$-0.190000x_2 + 10.0200x_3 = 70.6156$ (8)

Now normalized the equation (7) dividing by 7.00333 we get

$x_2 - 0.0418848x_3 = -2.79320$ (9)

Then multiplying the equation (9) by -0.190000 and subtracting the result from (8)

$3x_1 - 0.1x_2 - 0.3x_3 = 7.85$

$7.00333x_2 - 0.293333\, x_3 = -19.5617$

$10.0120x_3 = 70.0843$

We can now solve the equations by backward substitution

$x_3 = 7.00003$

$x_2 = -2.50000$

$x_1 = 3.00000$

Gausses Jordan method :

Let the system of equation be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

Then the augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{bmatrix}$$

By any number of row operation the above matrix is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & c_1' \\ 0 & 1 & 0 & c_2' \\ 0 & 0 & 1 & c_3' \end{bmatrix}$$

Thus the solution is $x_1 = c_1'$

$x_2 = c_2'$

$x_3 = c_3'$

Ex: Solve the following system of equation by Gausses Jordan method .

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + 4z = 1$$

Solution: The augmented matrix of the above system is

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & -5 \end{bmatrix} \; [\mathrm{R}_2' = \mathrm{R}_2 - \mathrm{R}_1],\; [\mathrm{R}_3' = \mathrm{R}_3 - \mathrm{R}_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -9 \end{bmatrix} [\mathrm{R}_3' = \mathrm{R}_3 - \mathrm{R}_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 15 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix} [\mathrm{R}_1' = \mathrm{R}_1 - \mathrm{R}_3],\; [\mathrm{R}_2' = \mathrm{R}_2 - 2\mathrm{R}_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix} \; [\mathrm{R}_1' = \mathrm{R}_1 - \mathrm{R}_2]$$

Thus the solutions are $x = -7$

$y = 22$

$z = -9$

Curve Fitting

Suppose we have m observations $(x_1, y_1), (x_2, y_2). . . . . .(x_m, y_m)$ of two variables x and y and it is requested to fit a curse of the type

$$y = a + bx + cx^2 + ... + kx^n \qquad . . . .(1)$$

Now we have to determine the constants a, b . . . . . k such that it represents the curve of best fit of that degree. In case m=n, we get in general a unique set of values satisfying the given systems of equations. But if m>n, then by substituting the different values of x and y in equation (1) we get m equations and we want to find only n constants. Hence there may be no such solution to satisfy all m equations. So we try to obtain those values of a, b . . . . k which may give the best fit i.e. which may satisfy all the equations as nearly as possible.

In such a case the principle of least squares asserts a suitable method.

Putting $x_1, x_2 . . . . x_m$ for x in equation (1), we have

$$Y_1 = a + bx_1 + cx_1^2 + ... + kx_1^n$$
$$Y_2 = a + bx_2 + cx_2^2 + ... + kx_2^n$$
$$. . . . .$$
$$Y_m = a + bx_m + cx_m^2 + ... + kx_m^n$$

The quantities $Y_1, Y_2 . . . . Y_m$ are the expected values of y corresponding to $x = x_1$, $x = x_2 . . . . x = x_m$ respectively. The values $y_1, y_2, ...y_m$ are the observed values of y corresponding to the given values $x_1, x_2, ... x_m$ of x. In general the expected values are different from the observed values.

Let $R_r = y_r – Y_r$. For different values of r these differences are called residuals.

The quantity $R_1^2 + R_2^2 + ... + R_m^2$ provides a measure of the 'goodness of fit' of the curse to the given data.

If it is small, the fit is good, if is large the fit is bad.

Now set

$$U = \sum R_i^2$$
$$= \sum(y_r – Y_r)^2$$
$$= \sum(y_r – a – bx_r – cx_r^2 - . . . .)^2$$

By this principle of least square the constants a, b, c . . . . . k are chosen in such a manner that sum of square of residuals in minimum. For maximum or minimum of U, we must have

$$\frac{\partial U}{\partial a} = 0 = \frac{\partial U}{\partial b} = \frac{\partial U}{\partial c} =. . . . .= \frac{\partial U}{\partial k}$$

After simplifying these relations, we have

$$\sum y = ma + b\sum x + ... + k\sum x^n$$
$$\sum xy = a\sum x + b\sum x^2 . . . . + k\sum x^{n+1}$$
$$\sum x^2y = a\sum x^2 + b\sum x^3 . . . . + k\sum x^{n+2}$$
$$. . . . .$$
$$\sum x^ny = a\sum x^n + b\sum x^{n+1} . . . . + k\sum x^{2n}$$

These equations are *(n+1)* in number and can be solved as simultaneous equations to give the values of the constants *a, b . . . . . k.* These equations are called normal equations.

If we calculate the second order partial derivatives and the values $a, b \ldots\ldots k$ are put in these derivatives. They give a positive value of the function.
So U is minimum.
Particular case.
If $n = 1$, then the curve to be fitted is a straight line $y = a + bx$ and the normal equations are

$$\sum y = ma + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

If $n = 2$, then the curve to be fitted is a second degree parabola $y = a + bx + cx^2$ and the normal equations are

$$\sum y = ma + b\sum x + c\sum x^2$$
$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$
$$\sum x^2 y = a\sum x^2 + b\sum x^3 + k\sum x^4$$

Ex. Fit a straight line to the following data regarding x as the independent variable:

x:	0	1	2	3	4
y:	1	1.8	3.3	4.5	6.3

Solution. Let the straight line to be fitted to the given data be
$y = a + bx$. Then the normal equations are

$$\sum y = ma + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Now

	x	y	xy	x^2
	0	1	0	0
	1	1.8	1.8	1
	2	3.3	6.6	4
	3	4.5	13.5	9
	4	6.3	25.2	16
Total	10	16.9	47.1	30

In this case $m = 5, \sum x = 10, \sum y = 16.9, \sum xy = 47.1, \sum x^2 = 30$
Substituting these values in the normal equations, we have

$$16.9 = 5a + 10b$$

And $$47.1 = 10a + 30b$$

Solving the above equations, we obtain $a = 0.72, \quad b = 1.33.$
Hence the fitted line is

$$y = 0.72 + 1.33x \text{ (answer)}$$

Ex: Fit a second degree parabola to the following data

x:	0	1	2	3	4
y:	1	5	10	22	38

Solution: Let the parabola to be fitted to the given data be $y = a + bx + cx^2$.
Then the normal equations are $\sum y = ma + b\sum x + c\sum x^2$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$
$$\sum x^2 y = a\sum x^2 + b\sum x^3 + k\sum x^4$$

Now

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	5	1	1	1	5	5
2	10	4	8	16	20	40
3	22	9	27	81	66	198
4	38	16	64	256	152	608
10	76	30	100	354	243	851

Substituting the values of $\sum x$, $\sum y$ . . . . . etc in the normal equations, we get

$$76 = 5a + 10b + 30c$$
$$243 = 10a + 30b + 100c$$
$$851 = 30a + 100b + 354c$$

Solving those equations simultaneously, we get

$$a = 1.43$$
$$b = 0.24$$
$$c = 2.21$$

Hence the parabola is

$$y = 1.43 + 0.24x + 2.21x^2 \text{ (answer)}$$

Ex. The weights of a calf taken at weekly intervals are given below. Fit straight line using the method of least squares and calculate the rate of growth per week.

Age(x):	1	2	3	4	5	6	7	8	9	10
Weight (y):	52.2	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Solution: Here x and y denote the variables age and weight respectively.
Let the least square line is $y = a + bx$. Then the normal equations are

$$\sum y = ma + b\sum x$$
$$\sum xy = a\sum x + b\sum x2$$

Now

x	y	x^2	xy
1	52.2	1	52.2
2	58.7	4	117.4
3	65.0	9	195.0
4	70.2	16	280.8
5	75.4	25	377.0
6	81.1	36	486.6
7	87.2	49	610.4
8	95.5	64	764.0
9	102.2	81	919.8
10	108.4	100	1084.0
55	795.9	385	4887.2

Substituting these values in the normal equations, we get

$$795.9 = 10a + 55b$$
$$4887.2 = 55a + 385b$$

Solving above equations, we get

$$a = 45.61$$
$$b = 6.18$$

∴ The line is

$$y = 45.61 + 6.18x \text{ (answer)}$$

Again we get

$$\frac{dy}{dx} = 6.18$$

∴ the rate of growth per week is 6.18 units. (Answer)

Ex. Fit an exponential curve of the form $y = ab^x$ to the following data:

x:	1	2	3	4	5	6	7	8
y:	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution. Here $y = ab^x$

$$\Longrightarrow ln\, y = ln\, a + ln\, b.x$$

Let

$$Y = ln\, y$$
$$A = ln\, a$$
$$B = ln\, b$$
$$\therefore Y = A + Bx$$

The normal equations are

$$\sum Y = mA + B\sum x$$
$$\sum xY = A\sum x + B\sum x2$$

x	y	Y = lny	xY	x^2
1	1.0	0.0000	0.0000	1
2	1.2	0.0792	0.1548	4
3	1.8	0.2553	0.7659	9
4	2.5	0.3979	1.5916	16
5	3.6	0.5563	2.7815	25
6	4.7	0.6721	4.0326	36
7	6.6	0.8195	5.7365	49
8	9.1	0.9590	7.6720	64
36	30.5	3.7393	22.7385	204

Now

∴ From normal equations

$$3.7393 = 8A + 36B$$

$$22.7385 = 36A + 204B$$

Solving these, we get

$$A = 1.8336, B = 0.1406$$

$$\therefore a = eA = e1.8336 = 0.68$$

$$b = eB = e0.1406 = 1.38$$

∴ The required curve is

$$y = ab^x$$

$$y = 0.68 \times 1.38^x$$