Chalmers University of Technology

SSY281 - Model predictive control

Assignment 4

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1 MPT and polyhedral sets

a)

The H-representation of the polyhedron:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

is :

$$H_{rep} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The V-representation of the polyhedron is :

$$V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Where the rows of V defines the vertices, i.e. three vertices.

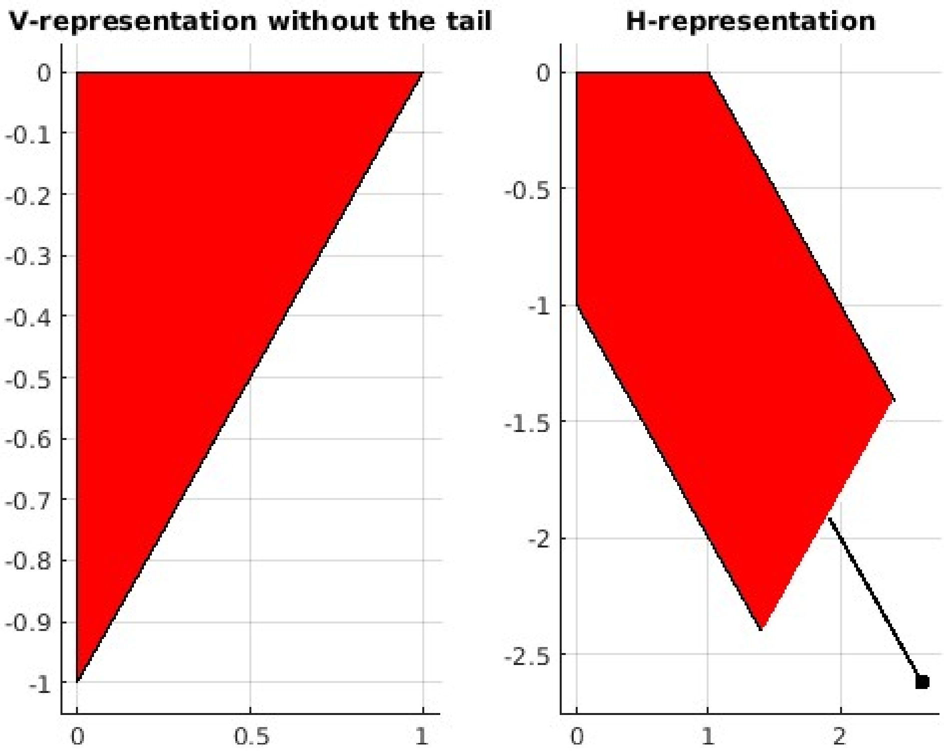

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Figure 1: V and H representation of the polyhedron

Figure 1 shows the plot for the different representation. The main difference is, the V-representation does not contain the tail. Which makes the plot to the right describe a polytope. The vertices in the V-representation has been calculated by taking the intersection between the half-spaces defined by A and b.

b)

The following figure shows the result of different polyhedron operations:

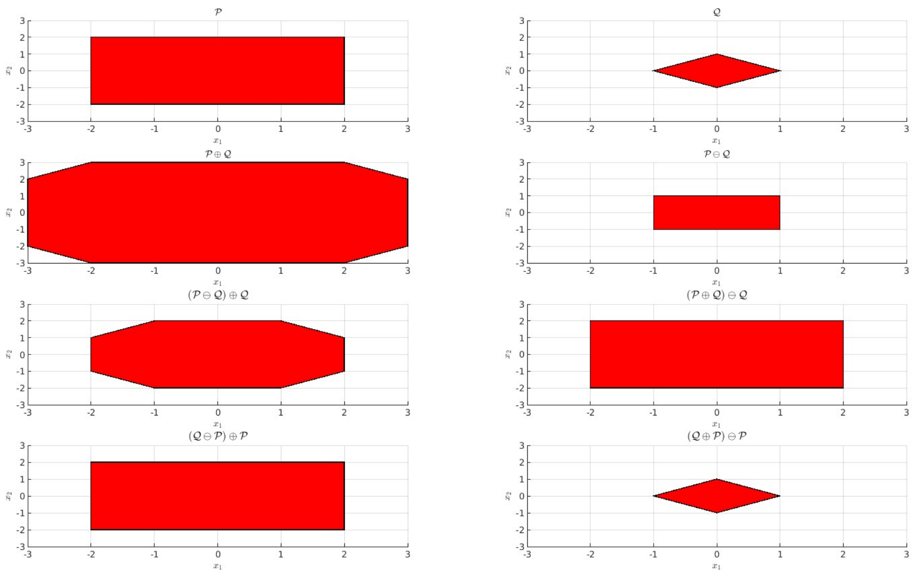

Figure 2: Polyhedron for different stes

Figure 2 shows the result of different sets operations. The *mpt3* toolbox incorrectly calculates $(\mathcal{Q} \ominus \mathcal{P}) \oplus \mathcal{P} = \mathcal{P}$. Whereas the result should be $\{\}$ since $\mathcal{Q} \ominus \mathcal{P} = \{\}$.

2 Forward and backward reachability

a)

The system

$$x(k+1) = Ax(k) \tag{2}$$

is autonomous with $A = \begin{bmatrix} 0.8 & 0.4 \\ -0.4 & 0.8 \end{bmatrix}$

And the set

$$\mathcal{S} := \{x : A_{in}x \leq b_{in}, x \in \mathcal{R}^2\} \tag{3}$$

$$A_{in} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} b_{in} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix} \tag{4}$$

From the eigenvalues of the matrix A we conclude that the system is asymptotically stable. Since there is no bound on the states $\Rightarrow$ any subset $\mathcal{S} \in \mathcal{R}^2$ such that $x_0 \in \mathcal{S}$ will be positively invariant for the system for all future time k. Figure **??** shows the set $\mathcal{S}$ in blue and the one-step reachable set $reach(\mathcal{S})$.
Where the one-step reachable set has been calculated according to:

$$reach(\mathcal{S}) = \{x(k+1) : A_{in}A^{-1}x(k+1) \leq b_{in}, x \in \mathcal{R}^2\}$$

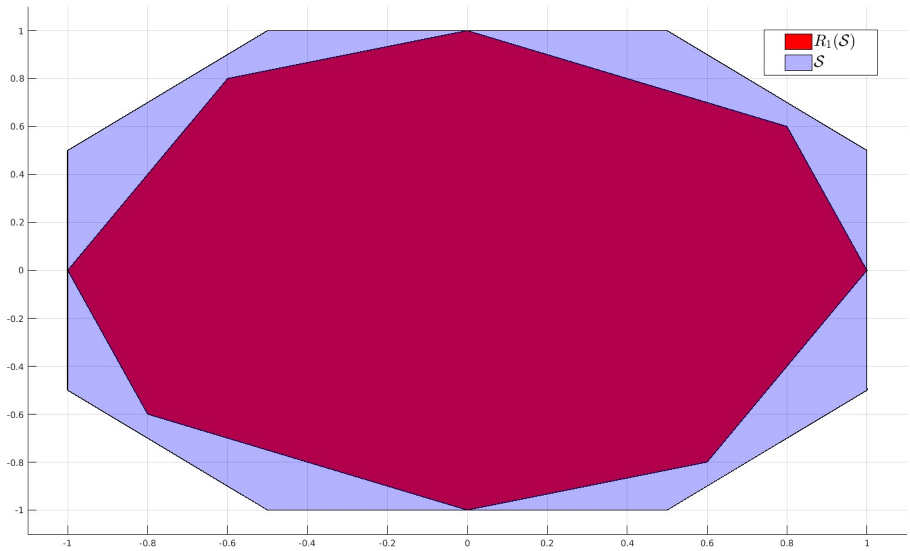

Figure 3: The set $\mathcal{S}$ and the one-step reachable set

From figure 3, $reach(\mathcal{S}) \subset \mathcal{S}$, hence the set $\mathcal{S}$ is positively invariant.

b)

Consider the system

$$x^+ = Ax + Bu \tag{5}$$

Where A and B :

$$A = \begin{bmatrix} 0.8 & 0.4 \\ -0.4 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And

$$-1 \leq u \leq 1 \tag{6}$$

The one-step reachable set from $\mathcal{S}$ in (3) is calculated as follows:

$$reach(\mathcal{S}) = (A \circ \mathcal{S}) \oplus (B \circ \mathbb{U}) \tag{7}$$

Where $\mathbb{U}$ is the set formed from the inequality constraints (6) on the control input.

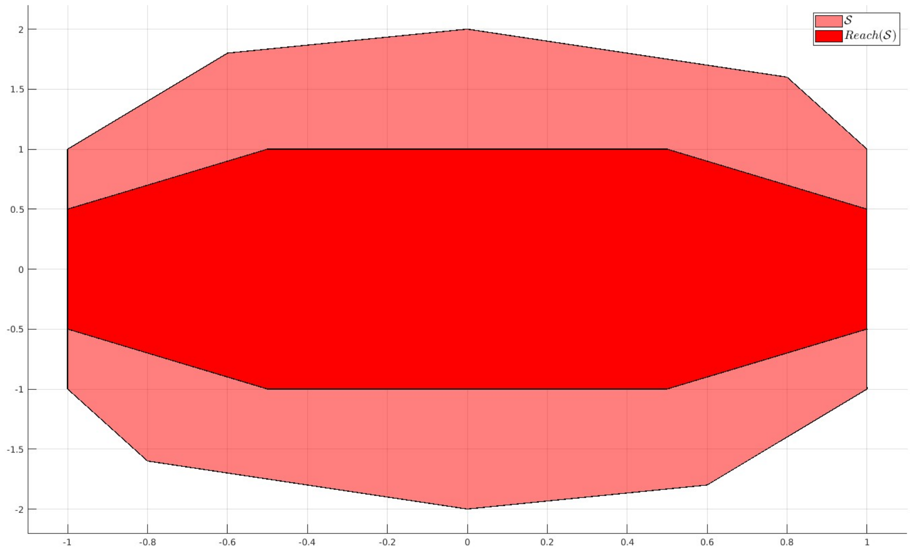

Figure 4: One-step reachable set

c)

The one-step backward reachable set has been calculated acording to the following:

$$pre(\mathcal{S}) = \left\{ x(k) \in \mathbb{R}^n | \exists u(k) \in \mathbb{R}^m, \overbrace{\begin{bmatrix} GA & GB \\ 0 & G_u \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq \begin{bmatrix} h \\ h_u \end{bmatrix}}^{\mathcal{T}} \right\} = proj_x(\mathcal{T}) \tag{8}$$

Where

$$G = A_{in},\ G_u = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ h = b_{in}\ h_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A_{in} and b_{in} as in (4)
After constructing the set $\mathcal{T}$ the result was projected onto the x_1x_2 plan to get the set $pre(\mathcal{S})$

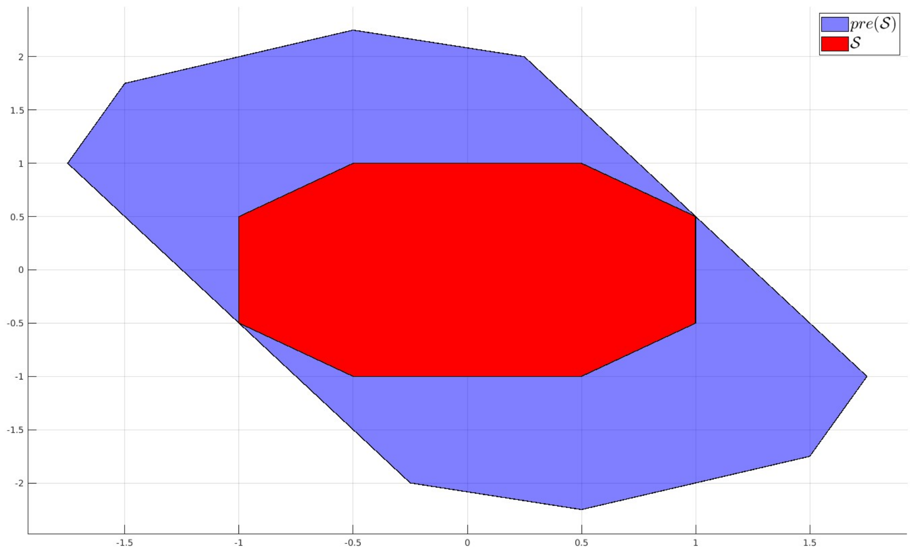

Figure 5: The one-step backward reachable set $pre(\mathcal{S})$ and the set $\mathcal{S}$

3 Persistent feasibility

3.1 a)

Consider the linear system:

$$A = \begin{bmatrix} 0.9 & 0.4 \\ -0.4 & 0.9 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9}$$

With $P_f = \mathbf{0}$, $Q = I_{2\times 2}$ and, $R = 1$. $x(0) = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top$ is the initial condition.
Such that

$$|x_i(k)| \leq 3,\ |u(k)| \leq 0.1\ \forall k \in \{0, 1, 2, ...\},\ \forall i \in \{1, 2\} \tag{10}$$

And $\mathbb{X}_f = [0\ \ 0]^\top$
Using the ***mpt3*** toolbox an RHC was designed taking into consideration the above parameters. The command ***evaluate*** has been utilized to test if the RHC is feasible for different value of the parameter N. It was found that the shortest prediction horizon that makes the controller feasible is

$N = 26$

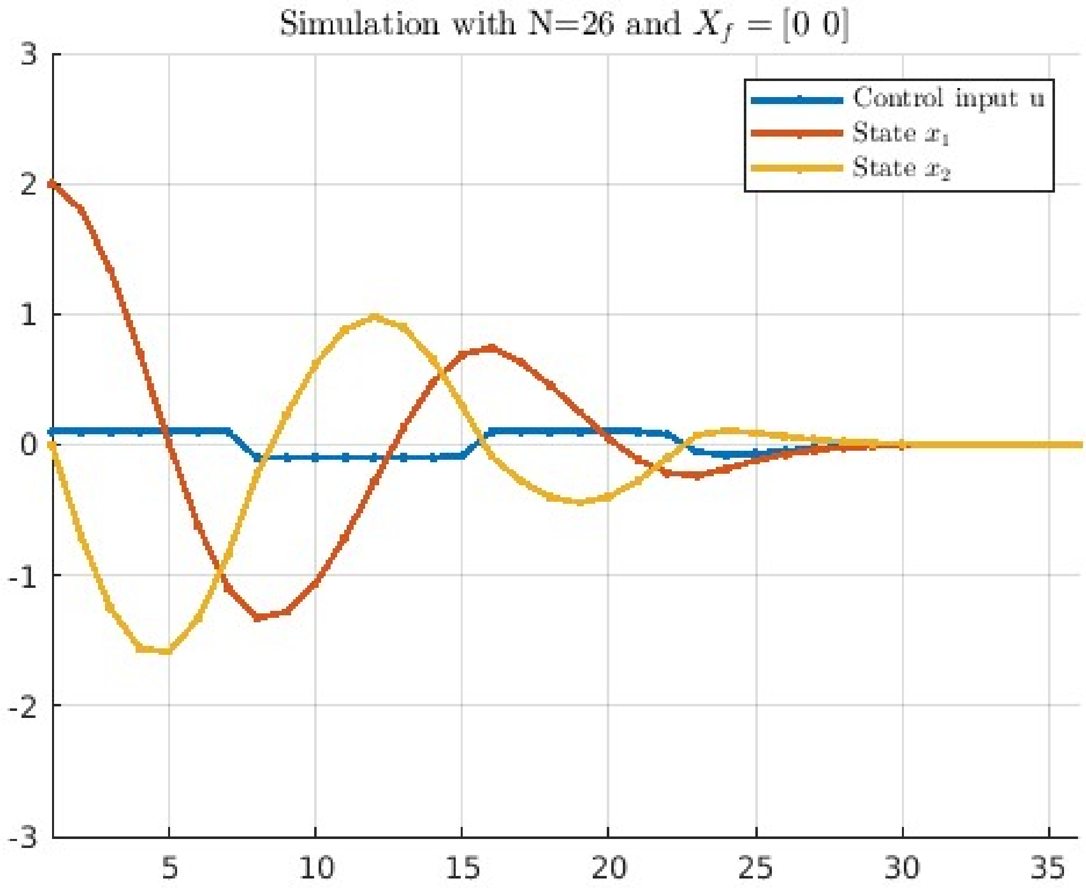

Figure 6

Figure 6 shows the simulation of the system with RHC with horizon $N = 26$. It is clear that the system reaches the terminal set $\mathbb{X}_f = [0\ \ 0]^\top$ and never leaves after that.

3.2 b)

For this task a new controller has been designed according to the parameters in (a) with the following modifications:

- Setting the horizon to $N = 2$
- changing the terminal set to $\mathbb{X}_f = \mathcal{C}_\infty$ maximal control invariant set

Where the ***mpt3*** toolbox has also been utilized in this task to design the RHC, and the command ***invariantSet*** to find the maximum control invariant-set. After the construction of the RHC the system has been simulated.

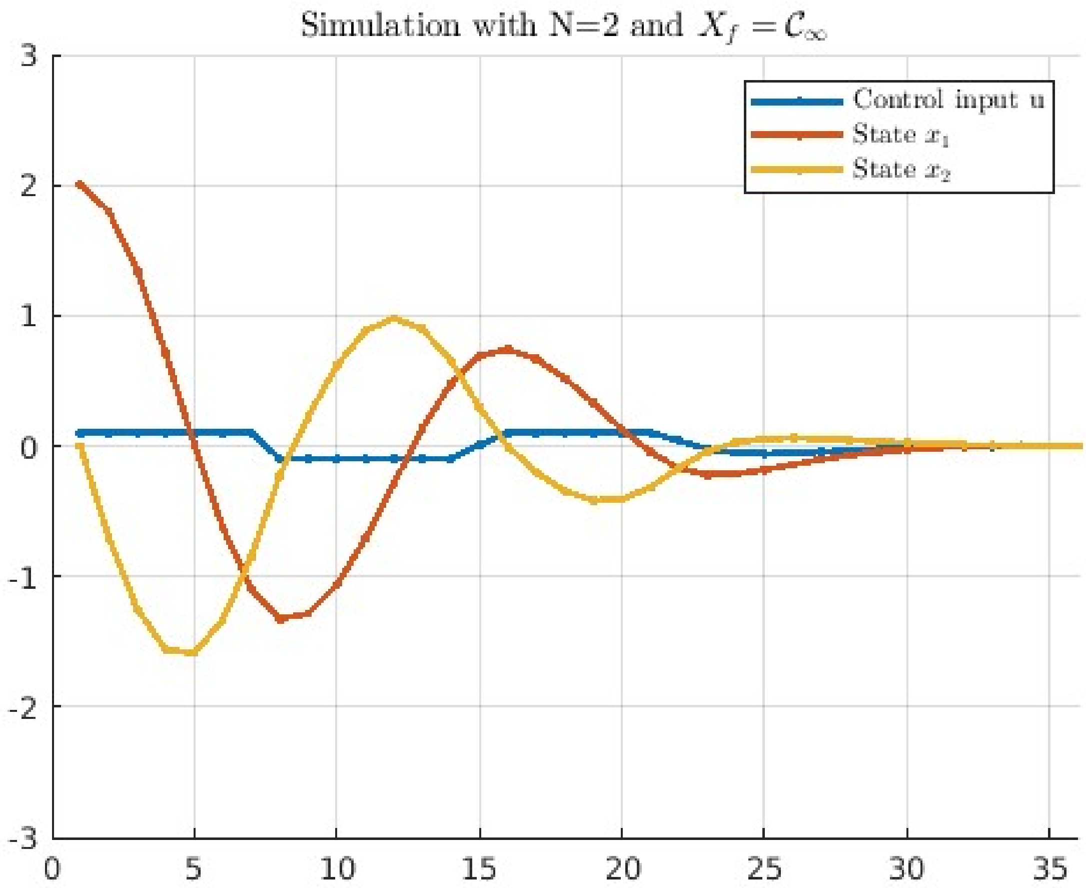

Figure 7

Figure 7 shows the simulation of the system with $N = 2$ and $\mathbb{X}_f = \mathcal{C}_\infty$. It is clear that the RHC is feasible until convergence to the origin.

3.3 c)

The command ***reachableSet*** has been used to calculate the set of feasible inital states for both controllers.

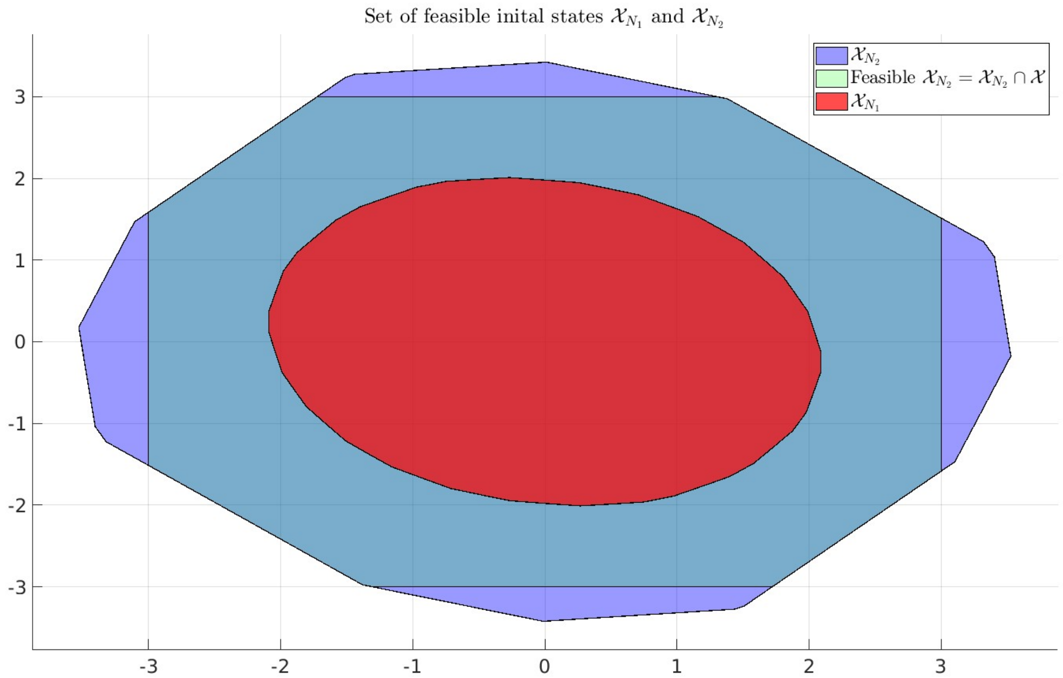

Figure 8

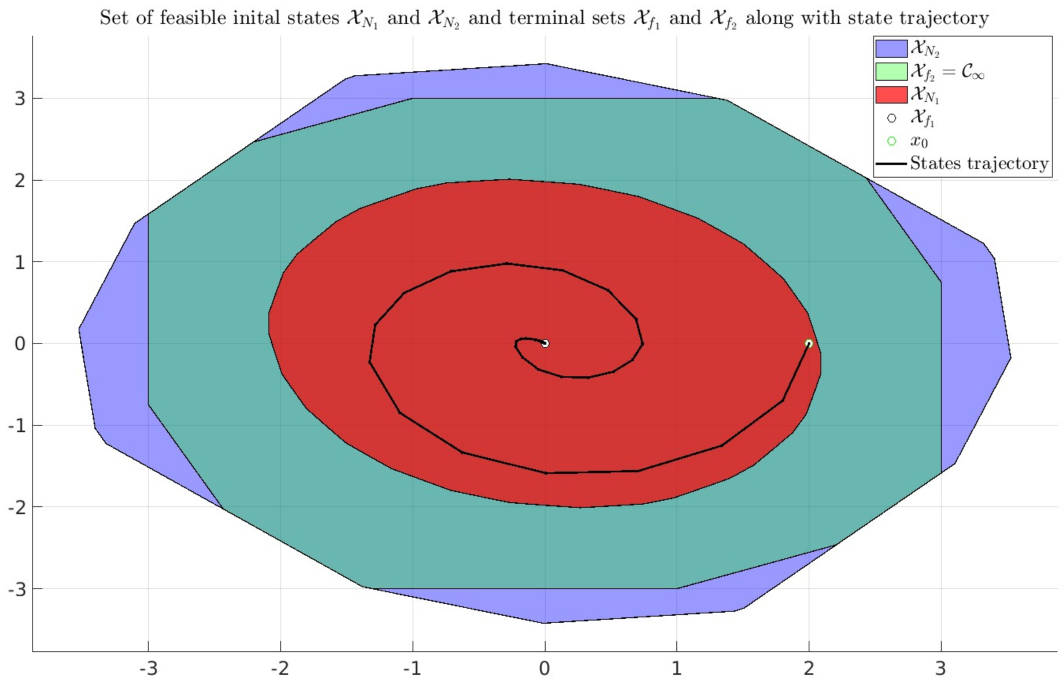

Figure 9

Figure 8 shows the set of feasible initial states for the both controllers.

From 8, $\mathcal{X}_{N_2}$ for the second RHC is bigger than for the first RHC. This is expected since the first controller has a single point as terminal set $\mathbb{X}_{f_1} = [0 \ \ 0]^\top$, whereas the second controller has $\mathbb{X}_{f_2} = \mathcal{C}_\infty$ as the terminal set. Figure 9 shows the set $\mathbb{X}_f$ for the different controllers along with the set of feasible initial states and the states trajectory.

Even though the first controller has much longer horizon $N = 26$, which increases the size of $\mathcal{X}_{N_1}$. The second controller with $N = 2$ still has a larger $\mathcal{X}_{N_2}$ and that is because the terminal set makes up for the shorter horizon.

The amount of optimization variables for both controllers are the same, i.e. 3 variables (2 states x_1, x_2 and 1 control signal u)

The first RHC with N=26 has $26 \times 6 = 156$ inequality constraints (two for each variable) and $2 \times 26 = 52$ equality constraints (2×2 system). Whereas the second RHC with N=2 has $2 \times 6 = 12$ inequality constraints and $2 \times 2 = 4$ equality constraints. The set of feasible initial states for the first RHC is defined by 52 inequality constraints meanwhile only 16 for the second RHC.
It means that it is much more computationally efficient to use the second RHC with N=2 and $\mathbb{X}_f = \mathcal{C}_\infty$