Chalmers University of Technology

SSY281 - Model predictive control

Assignment 2

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1 Set-point tracking

1.1 a)

Given the following system of linear equations:

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_s \end{bmatrix} \tag{1}$$

The state x_s and control input u_s set point can be calculated by solving (1) for the given system matrices.

$$x_s = \begin{bmatrix} 0.1745 \\ 0.0722 \\ -3.1416 \\ 0.0008 \end{bmatrix} \tag{2}$$

$$u_s = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix} \tag{3}$$

Since there exist a unique solution to (1), the system output can settle at the given point $y_s = [\frac{\pi}{18} - \pi]$.

1.2 b)

Considering only the second control input the system can settle at the given point $y_s = [\frac{\pi}{18} - \pi]$. The steady-state that minimizes the 2-norm of the output error is :

$$x_{ss} = \begin{bmatrix} 0 \\ 0 \\ -3.1416 \\ 0 \end{bmatrix}$$

$$u_s = 7.4578 \times 10^{-14} \approx 0$$

The procedure that was used to obtain the steady state, is first to build the quadratic function that has to be minimized in order to minimize the 2-norm of the output error

$$\min_{x_s, u_s} \; [Cx_s - y_s]_2^2 \tag{4}$$

$$s.\,t$$

$$[I - A \;\; - B] = 0$$

Where

$$x_s = Ax_s + Bu_s \text{ in steady state}$$
$$y_s \text{ is the settle point}$$

The objective function was minimized using the Matlab function **quadprog()**.

1.3 c)

Considering both control inputs are available and the output matrix c is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

The state and input set-points corresponding to the output set point y_s

$$x_{sp} = \begin{bmatrix} 0.1745 \\ 0.0722 \\ 0 \\ 0.0008 \end{bmatrix} \tag{5}$$

$$u_{sp} = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix} \tag{6}$$

The system can settle at the given set point $y_s = [\frac{\pi}{18}]$ since there exists a uniqe solution to (1).
The steady-state that minimizes the input while the output error is zero is the same as in (5) and (6). The procedure in this sub-task was similar to the one in sub-question b. The difference is in the objective function, and with an additional constrain that the output error is zero

$$\min_{x_s, u_s} \quad ||u_s||_2^2 \tag{7}$$

$$s.\,t$$

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} = 0 \tag{8}$$

Again Matlab function **quadprog()** was used to solve the quadratic problem.

2 Control of a ball and wheel system

2.1 a)

The augmented system becomes after adding the disturbance to the state vector

$$\begin{aligned} \xi(k+1) &= A_a \xi(k) + B_a u(k) \\ y(k) &= C_a \xi(k) \end{aligned} \tag{9}$$

Where

$$\xi = \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \tag{10}$$

And

$$A_a = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \tag{11}$$
$$B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$C_a = \begin{bmatrix} C & C_d \end{bmatrix}$$

The detectability of each new augmented system was checked according to the formula (12)

$$rank \begin{bmatrix} I_A & B_d \\ C & C_d \end{bmatrix} = n + n_d \tag{12}$$

Applying the formula (12) we get that :
system 1 corresponding to $n_d = 1$,$B_d = 0_{4\times1}$ and $C_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is detectable.
System 2 corresponding to $n_d = 2$,$B_d = 0_{4\times2}$ and $C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not detectable.
System 3 corresponding to $n_d = 2$,$B_d = [0_{4\times1} \; B_p]$ and $C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is detectable

2.2 b)

Since only the first and the third systems are detectable. Kalman filters has been designed for only detectable systems.
System 1 Kalman gain is:

$$L_1 = \begin{bmatrix} 1.0912 & -0.4025 \\ 8.7809 & -3.1649 \\ 0.4246 & 0.2332 \\ 0.3672 & -0.0835 \\ -0.3508 & 0.4763 \end{bmatrix} \tag{13}$$

And system 3 Kalman gain is:

$$L_3 = \begin{bmatrix} 1.2496 & 0.0099 \\ 10.4751 & 0.2567 \\ 0.0111 & 0.2279 \\ 0.2721 & 1.8818 \\ -0.4815 & -0.0090 \\ -0.0102 & 0.5109 \end{bmatrix} \tag{14}$$

Where the Kalman filter gains L_1 and L_3 has been computed according to the following formulas for the stationary Kalman filter:

$$L = PC^T[CPC^T + R]^{-1} \tag{15}$$
$$P = APA^T - APC^T[CPC^T + R]^{-1}CPA^T + Q \tag{16}$$

2.3 c)

Since the number of inputs are the same as the number of outputs. Given the relation

$$\begin{bmatrix} xs \\ us \end{bmatrix} = M_{ss}\hat{d}(k)$$

The following formula has been used to find M_{ss}

$$M_{ss} = \begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} & B_d \\ \underbrace{y_{sp}}_{0} & -C_d \end{bmatrix} \hat{d}(k) \tag{17}$$

Where M_{ss1} for the first system is

$$M_{ss1} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

And M_{ss2} for the second system is

$$M_{ss2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{19}$$

2.4 d)

Ball and wheel system:

$$x(k+1) = Ax(k) + Bu(k) + B_p p(k) \tag{20}$$
$$y(k) = Cx(k) + C_p p(k) \tag{21}$$

2.4.1 Model 1

Figure 1 shows the simulation of model 1 with the following parameters:

$$n_d = 1$$
$$B_d = 0_{4\times 1}$$
$$C_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From the figure 1, the controller is able to remove the off-set and drive state x_1 to the desired set point. Nevertheless the controller fails to remove the off-set and drive state x_3 to the desired set point. Which means that the wheel angle drifts away. In other words, the controller is able to balance the ball on top of the wheel, but the wheel drifts from the desired set-point.
This can be explained by the choice of B_d, which in this models assumes that there is only measurement noise and no process disturbances. One can see from equation (20) and (21) in the problem description that the disturbances act on both the process and measurement.

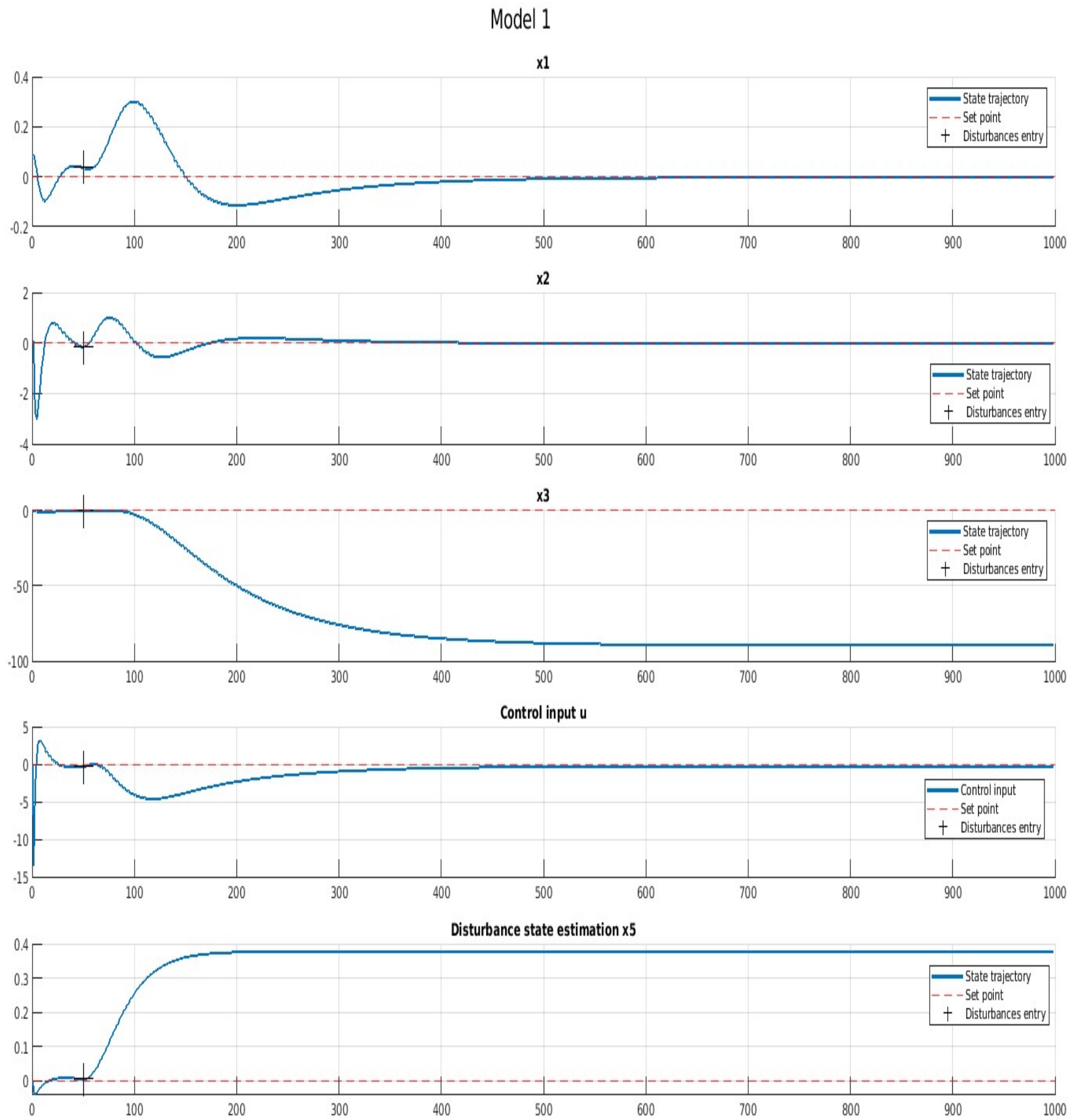

Figure 1: Simulation of model 1

2.4.2 Model 3

Figure 2 shows the simulation of model 1 with the following parameters:

$$n_d = 2$$
$$B_d = [0_{4\times1} \;\; B_p]$$
$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In comparison to model 1, this controller is able to remove the off-set and drive both state x_1 and x_3 to the desired set-point. In this model the controller design accounts for both process disturbances and measurement noise. Which is the correct choice taking into account equations (20) and (21).

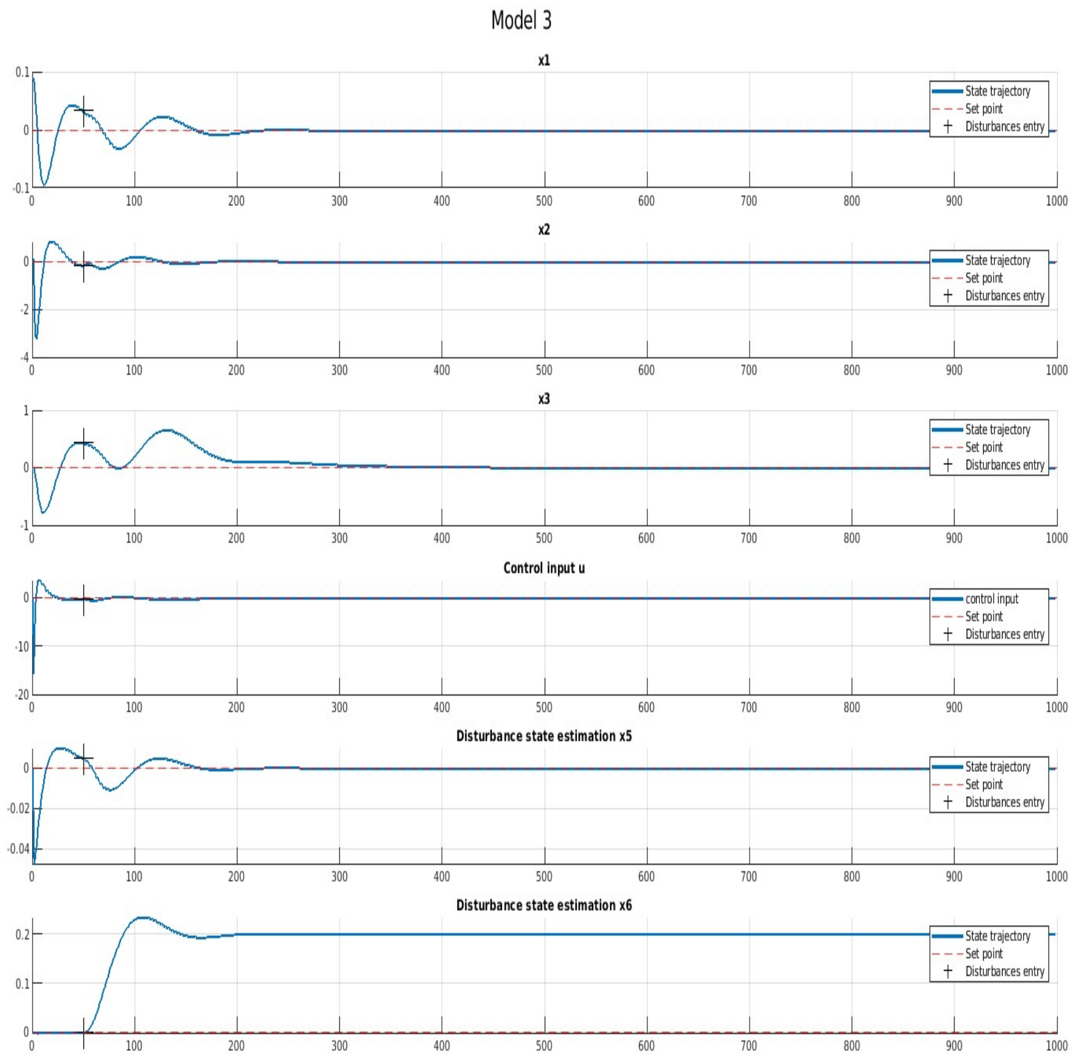

Figure 2: Simulation of model 3