SSY281 Model Predictive Control
Assignment 2 – Setpoint tracking and disturbance modeling

The purpose of this assignment is to formulate and solve MPC algorithms equipped with setpoint tracking and disturbance models.

Instructions

The assignments comprise an important part of the examination in this course. Hence, it is important to comply with the following rules and instructions:

- The assignment is pursued and reported individually.
- The findings from each assignment are described in a short report, written by each student independently.
- The report should provide clear and concise answers to the questions, including your motivations, explanations, observations from simulations, etc. Conclusions should be supported by relevant results if applicable; e.g., the system is stable since the eigenvalues, $[0.5\ , 0.2 + 0.5j\ , 0.2 - 0.5j]$, are inside the unit circle. Figures included in the report should have legends, should be readable, should have proper scaling to illustrate the relevant information, and axes should be labeled. Try to verify your solutions if possible; e.g., plot the inputs and outputs and see whether they respect the constraints.
- Since the assignments are part of the examination in the course, plagiarism is of course not allowed. If we observe that this happens anyway, it will be reported.
- The report should be uploaded to Canvas *before the deadline*. A report uploaded a second or a day after the deadline are penalized equally. Name the report as A2.pdf.
- A MATLAB code should be uploaded which reproduces all numbers and figures in your report. Make sure that one can run your code and see your results without any error. Name the MATLAB script as A2.m.

Table 1: Points per question

Question:	1	2	Total
Points:	3	12	15

1. **Set-point tracking** Consider the ball and wheel system, described in A1. Assume that we add two magnets, as depicted in Figure 1, and use the coil current in each one to produce a desired force on the ball.

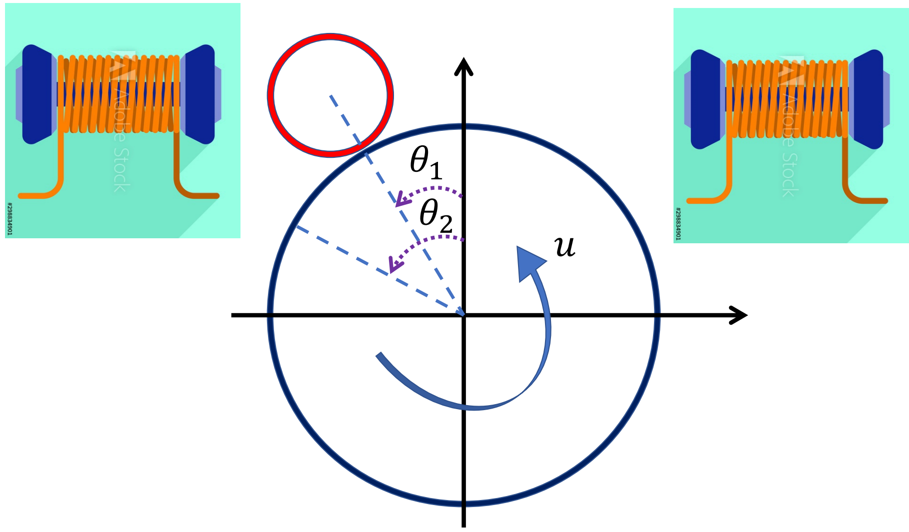

Figure 1: Ball and wheel system with an additional control input.

Assume the system is described by a discrete-time linear dynamics with:

$$A = \begin{bmatrix} 1.0041 & 0.0100 & 0 & 0 \\ 0.8281 & 1.0041 & 0 & -0.0093 \\ 0.0002 & 0.0000 & 1 & 0.0098 \\ 0.0491 & 0.0002 & 0 & 0.9629 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0007 & 0.01 \\ 0.1398 & 1 \\ 0.0028 & 0 \\ 0.5605 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

(a) [1p] Calculate the state and inputs set-points (x_s, u_s) corresponding to the output set-point $y_s = \begin{bmatrix} \frac{\pi}{18} & -\pi \end{bmatrix}^T$. Can the system outputs settle at y_s? Answer this question and motivate your answer in the report.

(b) [1p] Assume that only the second control input is available for control, i.e.,

$$B = \begin{bmatrix} 0.01 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Can the system outputs settle at $y_s = \begin{bmatrix} \frac{\pi}{18} & -\pi \end{bmatrix}^T$? Find the steady state that minimizes the two-norm of the output error and motivate the procedure in the report.
Hint: The Matlab command `quadprog` can be used.

(c) [1p] Assume both control inputs are available and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Calculate the state and inputs set-points (x_s, u_s) corresponding to the output set-point $y_s = \frac{\pi}{18}$. Can the system output settle at y_s? Find the steady state that minimizes two-norm of the input while the output error is zero. Motivate the procedure in the report.
Hint: The Matlab command `quadprog` can be used.

2. **Control of a ball and wheel system**

In this task, you will apply MPC to the ball and wheel system, described by

$$x(k+1) = Ax(k) + Bu(k) + B_p p(k), \tag{1a}$$
$$y(k) = Cx(k) + C_p p(k) \tag{1b}$$

with

$$A = \begin{bmatrix} 1.0041 & 0.0100 & 0 & 0 \\ 0.8281 & 1.0041 & 0 & -0.0093 \\ 0.0002 & 0.0000 & 1 & 0.0098 \\ 0.0491 & 0.0002 & 0 & 0.9629 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0007 \\ 0.1398 \\ 0.0028 \\ 0.5605 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_p = B, \quad C_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $p(k)$ is a disturbance (unknown to the control designer).

An offset-free RH controller can be designed with an augmented model to account for the unknown disturbance. Consider the three following disturbance models

1. $n_d = 1,\ B_d = 0_{4\times 1},\ C_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,
2. $n_d = 2,\ B_d = 0_{4\times 2},\ C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
3. $n_d = 2,\ B_d = \begin{bmatrix} 0_{4\times 1} & B_p \end{bmatrix},\ C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In these disturbance models, it is assumed that beside the disturbance $p(k)$, matrices B_p and C_p are also unknown to the control designer. Hence, the designer is assuming

$$x(k+1) = Ax(k) + Bu(k) + B_d d(k), \tag{2a}$$
$$y(k) = Cx(k) + C_d d(k). \tag{2b}$$

(a) [3p] Construct an augmented model (assuming $d(k)$ is unknown but constant) with the matrices B_d and C_d for each disturbance model given above. In which case the augmented system is detectable? Answer and motivate your answer in the report.

(b) [2p] Design a Kalman filter for each detectable augmented model to estimate its states (use identity as noise covariance matrices).

(c) [2p] Assume that we are only interested to regulate the second output. Therefore, for the steady state, consider $y(k) = HCx(k) + HC_d d(k)$ with

$$H = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In case the output set-point, i.e. y_{sp}, is zero, one can find the steady state target using the estimation of the disturbance, look at Section **5.2** of the lecture notes. Find matrix M_{ss}, defined as follows, for each detectable augmented system.

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = M_{ss}\, \hat{d}(k)$$

(d) [5p] Design an RHC for each detectable augmented system and simulate it with the following parameters, where N and M are the prediction and control horizons, respectively. Consider zero as the initial condition for your observer.

$$N = 50, \quad M = 40, \quad R = 0.1, \ t_f = 1000,$$

$$Q = P_f = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} \frac{\pi}{36} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To simulate the controlled system, use the disturbance $p(k)$ defined as:

$$p(k) = \begin{cases} 0 & \text{if } k \leq 50 \\ 0.2 & \text{if } k > 50 \end{cases} \tag{3}$$

In which case the controller is able to remove the off-set? Motivate your answer in the report.

Hint: While one should use (1) to simulate the system, the controller and the observer are designed based on (2). Make sure you understand this before simulating the closed-loop system.
Hint 2: Keep in mind that having the control horizon shorter than the prediction horizon means that you should be using a constant input (the last one you compute at the last control horizon time step) for the last time samples of the prediction horizon. In this case, the prediction horizon is 10 time steps longer than the control horizon, which means that during the 10 last time steps of the prediction horizon you will not be computing new control inputs, but rather you will use the last control input computed at the 40-th time step of the control horizon.
Hint 3: Keep also in mind that you can (and should) use the stationary Kalman filters previously computed in question 2b.