Chalmers University of Technology

SSY281 - Model predictive control

Assignment 3

Author:
Hasan Sharineh

February 22, 2024

1 Constrained optimization

a)

A function $\mathbf{f}(\mathbf{x})$ is convex on $[\mathbf{a}\,,\ \mathbf{b}]$ if for any two points $\mathbf{x_1}, \mathbf{x_2} \in [\mathbf{a}\,,\ \mathbf{b}]$ and any $\lambda \in [\mathbf{0}\,,\ \mathbf{1}]$ such that:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \tag{1}$$

And strictly convex if the following is satisfied

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2) \tag{2}$$

The graphic interpretation, is that the function is convex if any two points lying on the graph i.e. $x_1, x_2 \in \mathcal{C}$, the line $\mathcal{L}$ between x_1, x_2 lies above or on the graph i.e. $\mathcal{L} \geq \mathcal{C}$

b)

A set $\mathbf{S}$ is convex if $\forall\ \mathbf{x}, \mathbf{y} \in \mathbf{S}$ and $\lambda \in [\mathbf{0}\,,\ \mathbf{1}]$

$$(1-\lambda)x + \lambda y\ \in \mathbf{S} \tag{3}$$

Graphically this means, the line connecting any two points in the set $\mathbf{s}$ is also contained in $\mathbf{S}$.

c)

$$\begin{gathered}\min_x f(x) \qquad (4)\\ s.t\\ g(x) \leq 0\\ h(x) = 0\end{gathered}$$

The optimization problem (4) is convex if and only if $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$ are convex and $\mathbf{h}(\mathbf{x})$ is affine.
That is if $\mathbf{f}(\mathbf{x})$ is convex and $\mathbf{S}$ is a polyhedron.

$$S = \{x \mid gi(x) \leq 0, hi(x) = 0, \forall i\} \tag{5}$$

Then (4) is a convex optimization problem.

2 Convexity

a)

Consider the set

$$S_1 = \{x \in R^n \,|\, \alpha \leq a^\top x \leq \beta\} \tag{6}$$

Consider the point $z = (1-\lambda)x_1 + \lambda x_2$ where $x_1, x_2 \in S_1$ and $0 \leq \lambda \leq 1$

$$\Rightarrow a^\top z = (1-\lambda)a^\top x_1 + \lambda a^\top x_2 \tag{7}$$

$$Since\ x_1, x_2 \in S_1\ \ and\ \ \lambda \in [0,1]$$

$$\Rightarrow\ \ \lambda\alpha \leq \lambda a^\top x_2 \leq \lambda\beta \tag{8}$$

$$and$$

$$(1-\lambda)\alpha \leq (1-\lambda)a^\top x_1 \leq (1-\lambda)\beta \tag{9}$$

By summing (8)and (9) we obtain:

$$\alpha \leq (1-\lambda)a^\top x_1 + \lambda a^T x_2 \leq \beta$$

$$\alpha \leq a^\top z \leq \beta$$

Hence $z \in S_1$ and the set S_1 is convex.

b)

Consider the set

$$S_2 = \{x \,|\, \|x - y\| \leq f(y)\ \forall\, y \in S\}$$

where $S \subseteq R^n$, $f(y) \geq 0$ and $||.||$ is an arbitrary norm.
Let $x_1, x_2 \in S_2$ and $\lambda \in [0,1]$ Consider the point :

$$x = (1-\lambda)x_1 + \lambda x_2 \tag{10}$$

$$\Rightarrow \|x - y\| = \|(1-\lambda)x_1 + \lambda x_2 - y\|$$

$$\Rightarrow \|(1-\lambda)x_1 + \lambda x_2 - y\| = \|(1-\lambda)x_1 + \lambda x_2 - ((1-\lambda)y + \lambda y)\|$$

$$\|(1-\lambda)x_1 + \lambda x_2 - ((1-\lambda)y + \lambda y) \leq (1-\lambda)\|x_1 - y\| + \lambda\|x_2 - y\|$$

$$Since\ \ \|x_1 - y\| \leq f(y)\ \ and\ \ \|x_2 - y\| \leq f(y)$$

$$\Rightarrow \|x - y\| \leq \lambda f(y) + (1-\lambda)f(y) = f(y)$$

Hence the set S_2 is convex

c)

Consider the set

$$S_3 = \{(x, y) \mid y \leq 2^x,\ \forall\ (x, y) \in \mathbf{R^2}\}. \tag{11}$$

Consider the two points $x_1(0, 1)$ and $x_2(1, 1)$ the middle point of x_1 and x_2 wich lies on the line between the two points is $z_m(\frac{1}{2}, 1)$

$$y_{zm} = 1\ and\ x_{zm} = \frac{1}{2}$$

Which implies according to (11)

$$1 \leq 2^{\frac{1}{2}} \tag{12}$$

$$\Rightarrow z_m \notin S_3 \tag{13}$$

Hence the set S_3 is not convex.

3 Norm problems as linear programs

a)

$$\min_{x,\epsilon} \epsilon \tag{14}$$

$$s.t. \quad -\epsilon \leq (Ax - b)_i \leq \epsilon,\ \forall i \in \{1, ..., n\},$$

$$\min_x \|Ax - b\|_\infty \tag{15}$$

(14) and (15) yield the same result since

$$\|Ax - b\| = Ax - b\ \ if\ Ax - b > 0$$

$$and$$

$$\|Ax - b\| = -(Ax - b)\ \ if\ Ax - b < 0$$

Which essentially means choosing the smallest ϵ that is bigger or equal to $(Ax - b)$ or $-(Ax - b)$ Therefore solving (14) is equivalent to solving (15) and since (15) is not linear, it's preferable to solve the linear program instead.

b)

Assuming $z^\top = [x^\top \;\; \epsilon] \Rightarrow z = \begin{bmatrix} x^\top \\ \epsilon \end{bmatrix}$

$$\Rightarrow c^\top = \begin{bmatrix} 0 & I \end{bmatrix}$$
$$F = \begin{bmatrix} A^\top & -I \\ -A^\top & -I \end{bmatrix}$$
$$g = \begin{bmatrix} b^\top \\ -b^\top \end{bmatrix}$$

c)

Consider

$$A = \begin{bmatrix} 0.4889 & 0.2939 \\ 1.0347 & -0.7873 \\ 0.7269 & 0.8884 \\ -0.3034 & -1.1471 \end{bmatrix}$$
$$b = \begin{bmatrix} -1.0689 \\ -0.8095 \\ -2.9443 \\ 1.4384 \end{bmatrix}$$

The problem to minimize is

$$\min_x \|Ax - b\|_\infty \tag{16}$$

Which can be rewritten as :

$$\min_{x,\epsilon} \; \epsilon \tag{17}$$
$$s.t. \;\; (Ax - b)_i \leq \epsilon$$
$$(-Ax + b)_i \leq \epsilon$$

Rewriting the minimization problem in matrix form gives the following:

Where the optimization variable is $z = \begin{bmatrix} x_1 \\ x_2 \\ \epsilon \end{bmatrix}$

$$A_{inq} = \begin{bmatrix} A & -1_{4\times1} \\ -A & -1_{4\times1} \end{bmatrix} \tag{18}$$
$$b_{inq} = \begin{bmatrix} b \\ -b \end{bmatrix} \tag{19}$$
$$c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{20}$$

Using the Matlab function **linprog()** the following solution was found :

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \epsilon^* \end{bmatrix} = \begin{bmatrix} -2.0674 \\ -1.1067 \\ 0.4583 \end{bmatrix} \tag{21}$$

d)

Consider the problem:

$$\begin{aligned} \min_{z} \; & c^\top z \\ s.t. \; & Fz \le g \end{aligned} \tag{22}$$

The Lagrangian can be formulated as follow:

$$\begin{aligned} \mathcal{L}(z, \mu) &= c^\top z + \mu^\top (Fz - g) \\ &= (c + F^\top \mu)^\top z - \mu^\top g \\ \implies q(\mu) = \inf_z \mathcal{L}(z,\mu) \implies q(\mu) &= \begin{cases} -\mu^\top g & \text{if } c + F^\top \mu = 0 \\ -\infty & Otherwise \end{cases} \end{aligned}$$

The dual problem of the primal probelm (22) can be written as :

$$\begin{aligned} \max \quad & -\mu^\top g \\ s.t. \quad & F^\top \mu + c = 0 \\ & \mu \ge 0 \end{aligned} \tag{23}$$

e)

The dual problem of (17) with the matrices A and b as in sub-task c can be written in the following form:

$$\begin{aligned} \max_{\mu} & -f^\top \mu \\ \text{s.t.} \;\; & \mu \ge 0 \\ \begin{bmatrix} -A^\top & A^\top \\ -1_{1\times 4} & -1_{1\times 4} \end{bmatrix} \mu &= - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{24}$$

Where $f = \begin{bmatrix} b \\ -b \end{bmatrix}$. And the optimal solution to (24) is:

$$\mu^* = \begin{bmatrix} 0 \\ 0 \\ 0.4095 \\ 0.4284 \\ 0 \\ 0.1621 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

f)

Since strong duality holds i.e. $c^\top z = -f^\top \mu$.From the complementary slackness theorem $A_i x^* = b_i$ if and only if $\mu_i^* > 0$.
Utilizing the dual solution (25). The indexes of the active constraints are $i \in \{3, 4, 6\}$. Solving a system of linear equations where A and b as in (18), (19) respectively, the following primal solution has been obtained:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \epsilon^* \end{bmatrix} = \begin{bmatrix} -2.0674 \\ -1.1067 \\ 0.4583 \end{bmatrix} \tag{26}$$

Comparing (21) and (26), the solutions are identical.

4 Quadratic programming

a)

The solution for the optimization problem is found using **quadprog()**. Using the Matlab formulation

$$\min_x \ \frac{1}{2} x^\top H x + f^{\text{ such that}} \begin{cases} Ax \leq 0 \\ A_{eq} x = b_{eq} \\ lb \leq x \leq ub \end{cases} \tag{27}$$

Where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ u_0 \\ u_1 \end{bmatrix} H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, lb = \begin{bmatrix} 2.5000 \\ -0.5000 \\ -2.0000 \\ -2.0000 \end{bmatrix} \text{ and } ub = \begin{bmatrix} 5.0000 \\ 0.5000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$$
$$A_{eq} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -A & 1 & 0 & -B \end{bmatrix}, b_{eq} = \begin{bmatrix} Ax_0 \\ 0 \end{bmatrix}$$

The solution for the optimaization problem is

$$x^* = \begin{bmatrix} 2.5000 \\ 0.5000 \\ 1.9000 \\ -0.5000 \end{bmatrix} \tag{28}$$

b)

The KKT conditions are :

$$\nabla f(x^*) + \lambda \nabla h(x^*) + \mu \nabla g(x^*) = 0 \tag{29}$$
$$g(x^*) \leq 0 \tag{30}$$
$$h(x^*) = 0 \tag{31}$$
$$\mu_i g_i(x^*) = 0 \tag{32}$$
$$\mu \geq 0 \tag{33}$$

the result is :

$$\nabla f(x^*) + \lambda \nabla h(x^*) + \mu \nabla g(x^*) = 10^{-10} \times \begin{bmatrix} -0.1095 \\ 0.2739 \\ 0.0702 \\ -0.1095 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \text{(29) is satisfied}$$

$$g(x^*) = \begin{bmatrix} -2.5000 \\ -0.0000 \\ -0.1000 \\ -2.5000 \\ -0.0000 \\ -1.0000 \\ -3.9000 \\ -1.5000 \end{bmatrix} \implies \text{(30) is satisfied}$$

$$h(x^*) = 10^{-15} \times \begin{bmatrix} 0 \\ -0.1110 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \text{(31) is satisfied}$$

$$\mu_i * g_i(x^*) = -2.4003 \times 10^{-09} \approx 0 \implies \text{(32) is satisfied}$$

$$\mu = \begin{bmatrix} 0 \\ 0.0001 \\ 0.0000 \\ 0 \\ 4.6000 \\ 0 \\ 0 \\ 0.0000 \end{bmatrix} \implies \text{(33) is satisfied}$$

From the above result, the KKT conditions for (28) are satisfied.

c)

By removing the lower bound on x_1 the following result was obtained

$$\begin{bmatrix} 0.2885 \\ 0.0577 \\ -0.3115 \\ -0.0577 \end{bmatrix} \tag{34}$$

Which result in $V(x^*) = 0.0935$ compared to $V(x^*) = 5.1800$ when x_1 is bounded from below. This result is reasonable since the constraint corresponding to the lower bound was active, meaning preventing from reaching lower objective value. Removing the constraint allows for better objective function value.

Removing the upper bound will still give the same result as in (28). This is also reasonable since the problem is to minimize a quadratic cost function with positive definite coefficient matrix $H \succ 0$. That is lower optimal solution will result in lower objective function value. So increasing the upper bound changes nothing.