An algorithm for computing the k-numerical range

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SIUE COUGARS

Introduction

The k-numerical range is defined as:

$$W_k(A) = \left\{ \frac{1}{k} \sum_{i=1}^{k} x_i^* A x_i : x_i^* x_j = \delta_{ij} \right\}$$

The k-numerical range:

- is convex
- is compact
- contains averages of k eigenvalues (counting multiplicity)

The real part of the k-numerical range

For $A \in M_n(\mathbb{C})$, Re$(W_k(A)) = W_k(\text{Re}(A))$.

Linear Transform Property

For $z, w \in \mathbb{C}$, $A \in M_n(\mathbb{C})$, $W_k(zA + wI) = zW_k(A) + w$.

Technical Lemma

For $\lambda_1 \geq \cdots \geq \lambda_n$, $0 \leq c_i \leq 1$, $\sum_{i=1}^{n} c_i = k \in \mathbb{N}$. Then $\sum_{i=1}^{n} c_i \lambda_i \leq \sum_{i=1}^{k} \lambda_i$.

Convex Combination Lemma

For $A = A^*$, any $z \in W_k(A)$ has the form $z = \frac{1}{k} \sum_{i=1}^{n} c_i \lambda_i$, where $\lambda_1 \geq \cdots \geq \lambda_n$ are eigenvalues of A, $0 \leq c_i \leq 1$, $\sum_{i=1}^{n} c_i = k$.

Maximum Real Part Theorem

Suppose $A \in M_n(\mathbb{C})$, and $v_1, \cdots, v_n$ are eigenvectors of Re(A) for eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then $\frac{1}{k} \sum_{i=1}^{k} v_i^* A v_i \in W_k(A)$ has maximum real part.

These results are the key tools necessary to generalize an algorithm due to Carl Cowen in the case when $k = 1$.

Explanation of the algorithm

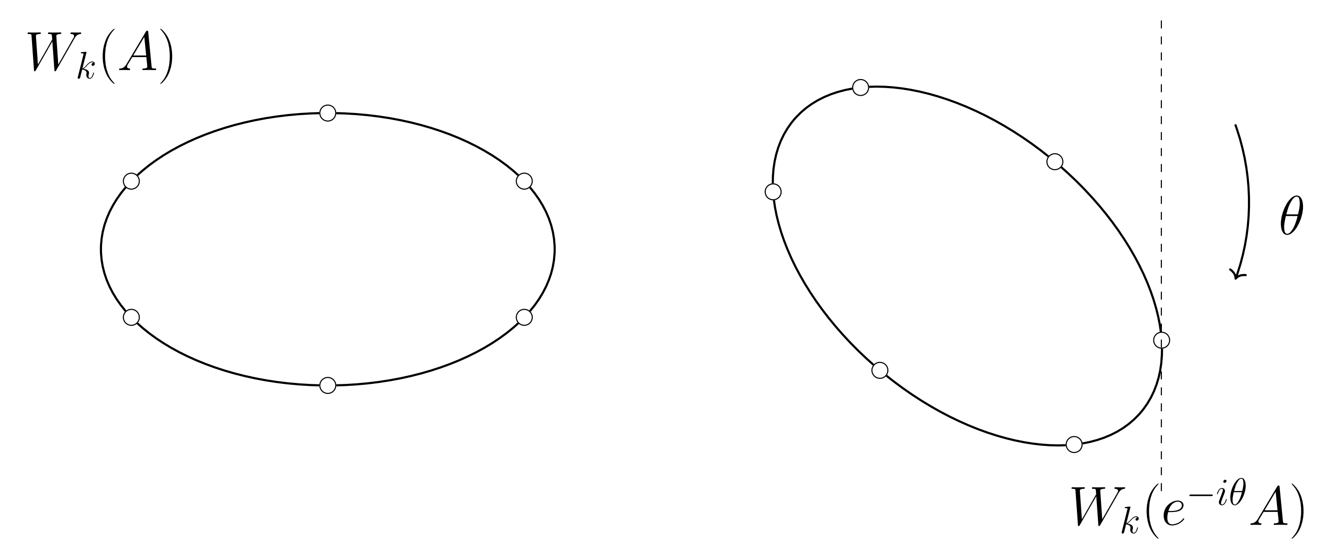

Figure 1: The point on $W_k(e^{-i\theta}A)$ with maximum real part is achieved by the k-largest eigenvectors of $\text{Re}(e^{-i\theta}A)$. So, it suffices to find the eigenvectors corresponding to the k-largest eigenvalues of $\text{Re}(e^{-i\theta}A)$ for various θ.

To compute a boundary point of $W_k(A)$ in the direction θ, we rotate the matrix: $e^{-i\theta}A$. The maximum part theorem shows that the point in $W_k(e^{-i\theta}A)$ with maximum real part is achieved by the k eigenvectors corresponding to the k-largest eigenvalues of $\text{Re}(e^{-i\theta}A)$.

By evaluating this over many angles θ, we trace the boundary of $W_k(A)$.

> **How it works:** Rotate A $\rightarrow$ Take real part Re$(e^{-i\theta}A)$ $\rightarrow$ Use k eigenvectors with largest eigenvalues $\rightarrow$ Average $\rightarrow$ Plot the result.

To approximate the k-numerical range of $A_0 = \text{diag}(i, -i, 1, \ldots, 1) \in M_7(\mathbb{C})$, we compute x^*Ax for 1000 random unit vectors x. The vertices of the range lie at i, $-i$, and 1. Since most diagonal entries are 1, the values cluster near 1 on the real axis. The resulting approximation for $k = 1$ is shown below.

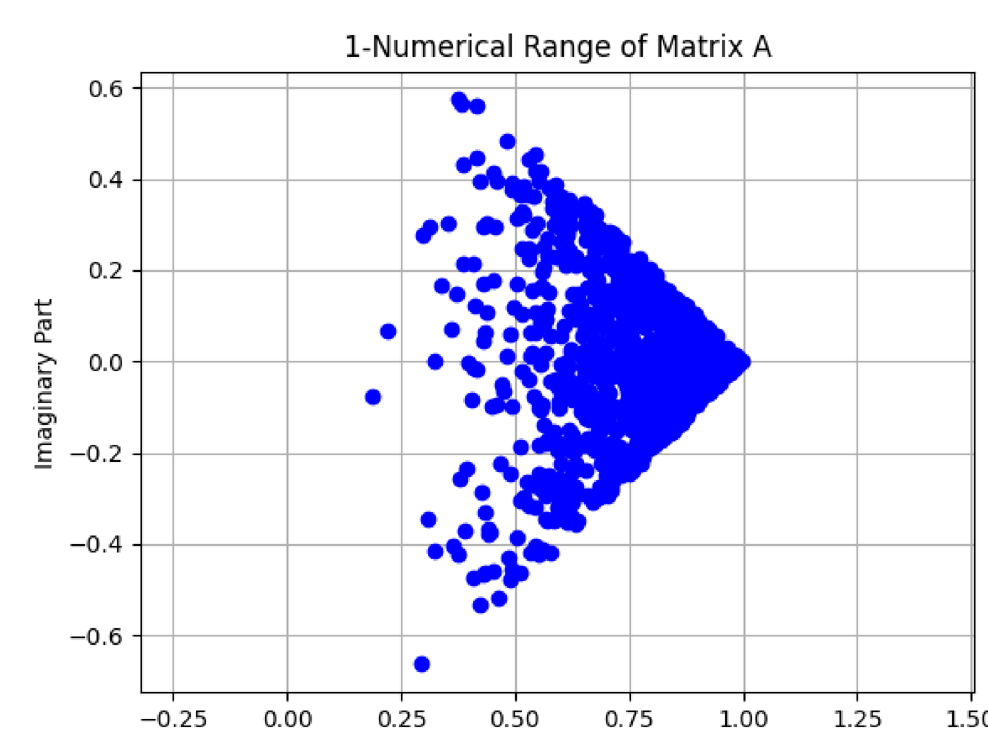

Figure 2: Approximation of the numerical range of $\text{diag}(i, -i, 1, \ldots, 1) \in M_7(\mathbb{C})$; computed x^*Ax for 1000 randomly chosen unit vectors x

Visual examples of the k-numerical range

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$

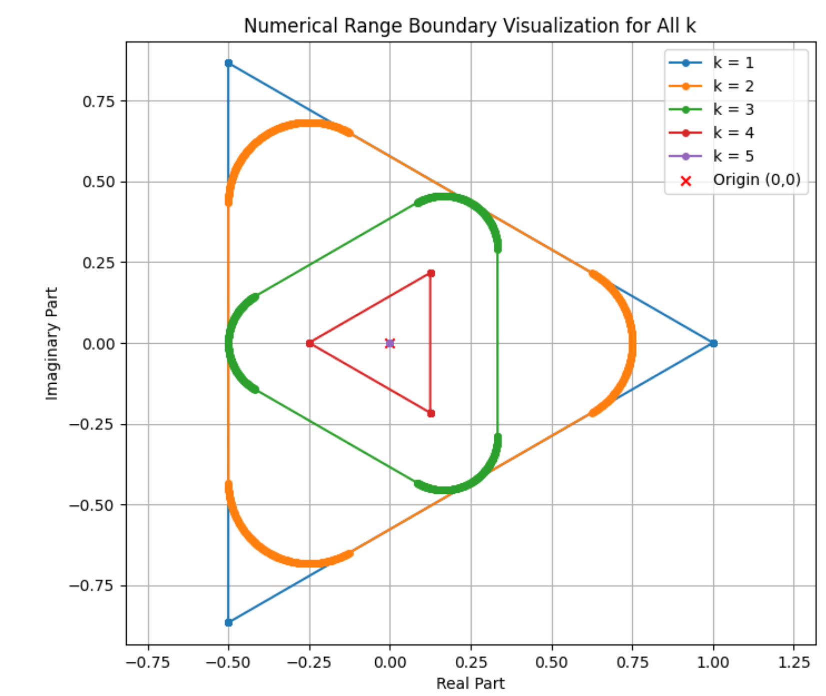

Figure 3: k-numerical range of A_1

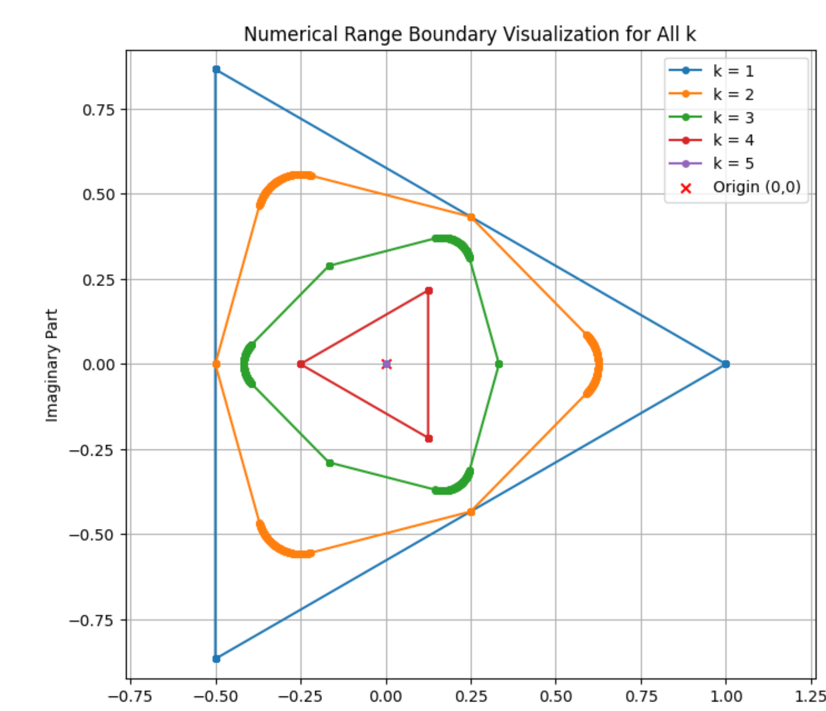

Figure 4: k-numerical range of A_2

References

[1] Carl C. Cowen, *An effective algorithm for computing the numerical range*, Aug 1995, Unpublished Manuscript.

[2] Panayiotis J. Psarrakos and Michael J. Tsatsomeros, *Numerical range: (in) a matrix nutshell*, https://www.math.wsu.edu/math/faculty/ tsat/files/short.pdf, Aug 2002.