General Theory of Natural Equivalences

Samuel Eilenberg Saunders MacLane

15 May 1945

Contents

Introduction

The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual" or "conjugate" space $T(L)$. Let L be a finite-dimensional real vector space, while its conjugate $T(L)$ is, as is customary, the vector space of all real valued linear functions t on L. Since this conjugate $T(L)$ is in its turn a real vector space with the same dimension as L, it is clear that L and $T(L)$ are isomorphic. But such an isomorphism cannot be exhibited until one choose a definite set of basis vectors for L, and furthermore the isomorphism which results will differ for different choices of this basis.

For the iterated conjugate space $T(T(L))$, on the other hand, it is well known that one can exhibit an isomorphism between L and $T(T(L))$ *without* using any special basis in L. This exhibition of the isomorphism $L \cong T(T(L))$ is "natural" in that it is given *simultaneously* for *all* finite-dimensional vector spaces L.

This simultaneity can be further analyzed. Consider two finite-dimensional vector spaces L_1 and L_2 and a linear transformation λ_1 of L_1 into L_2; in symbols

$$\lambda_1 : L_1 \to L_2 \tag{1}$$

This transformation λ_1 induces a corresponding linear transformation of the second conjugate space $T(L_2)$ into the first one, $T(L_1)$. Specifically, since each element t_2 in the conjugate space $T(L_2)$ is itself a mapping, one has two transformations

$$L_1 \xrightarrow{\lambda_1} L_2 \xrightarrow{t_2} \mathbb{R};$$

their product $t_2\lambda_1$ is thus a linear transformation of L_1 into $\mathbb{R}$, hence an element t_1 in the conjugate space $T(L_1)$. We call this correspondence of t_2 to t_1 the mapping $T(\lambda_1)$ *induced* by λ_1; thus $T(\lambda_1)$ is defined by setting $[T(\lambda_1)]t_2 = t_2\lambda_1$, so that

$$T(\lambda_1) : T(L_2) \to T(L_1). \tag{2}$$

In particular, this induced transformation $T(\lambda_1)$ is simply the identity when λ_1 is given as the identity transformation of L_1 into L_1. Furthermore the transformation induced by a product of λ's is the product of the separately induced transformations, for if λ_1 maps L_1 into L_2 while λ_2 maps L_2 into L_3, the definition of $T(\lambda)$ shows that

$$T(\lambda_2\lambda_1) = T(\lambda_1)T(\lambda_2).$$

The process of forming the conjugate space thus actually involves two different operations or functions. The first associates with each space L its conjugate space $T(L)$; the second associates with each linear transformation λ between vector spaces its induced linear transformation $T(\lambda)$.[1]

A discussion of the "simultaneous" or "natural" character of the isomorphism $L \cong T(T(L))$ clearly involves a simultaneous consideration of all spaces L and all transformations λ connecting them; this entails a simultaneous consideration of the conjugate space $T(L)$ and the induced transformations $T(\lambda)$ connecting them. Both functions $T(L)$ and $T(\lambda)$ are thus involved; we regard them as the component parts of what we call a "functor" T. Since the induced mapping $T(\lambda_1)$ of Eq. (2) reversed the direction of the original λ_1 of Eq. (1), this functor T will be called "contravariant".

The simultaneous isomorphisms

$$\tau(L) : L \rightleftarrows T(T(L))$$

compare two *co*variant functors; the first is the identity functor I, composed of the two functions

$$I(L) = L, \qquad I(\lambda) = \lambda;$$

the second is the iterated conjugate functor T^2, with components

$$T^2(L) = T(T(L)), \qquad T^2(\lambda) = T(T(\lambda)).$$

For each L, $\tau(L)$ is constructed as follows. Each vector $x \in L$ and each functional $t \in T(L)$ determine a real number $t(x)$. If in this expression x is fixed while t varies, we obtain a linear transformation of $T(L)$ into $\mathbb{R}$, hence an element y in the double conjugate space $T^2(L)$. This mapping $\tau(L)$ of x to y ma also be defined formally by setting $[[\tau(L)]x]t = t(x)$.

The connections between these isomorphisms $\tau(L)$ and the transformations $\lambda : L_1 \to L_2$ may be displayed thus:

$$\begin{tikzcd}
L_1 \arrow[r, "\tau(L_1)"] \arrow[d, "I(\lambda)"] & T^2(L_1) \arrow[d, "T^2(\lambda)"] \\
L_2 \arrow[r, "\tau(L_2)"] & T^2(L_2)
\end{tikzcd}$$

[1] The two different functions $T(L)$ and $T(\lambda)$ may be safely denoted by the same letter T because their arguments L and λ are always typographically distinct.

The statement that the two possible paths from L_1 to $T^2(L_2)$ in this diagram are in effect identical is what we shall call the "naturality" or "simultaneity" condition for t; explicitly, it reads

$$\tau(L_2)I(\lambda) = T^2(\lambda)\tau(L_1). \tag{3}$$

This equality can be verified from the above definitions of $t(L)$ and $T(\lambda)$ by straightforward substitution. A function t satisfying this "naturality" condition will be called a "natural equivalence" of the functors I and T^2.

On the other hand, the isomorphism of L to its conjugate space $T(L)$ is a comparison of the covariant functor I with the contravariant functor T. Suppose that we are given simultaneous isomorphisms

$$\sigma(L) : L \rightleftarrows T(L)$$

for each L. For each linear transformation $\lambda : L_1 \to L_2$ we than have a diagram

$$\begin{tikzcd}
L_1 \arrow[r, "\sigma(L_1)"] \arrow[d, "I(\lambda)"] & T^2(L_1) \\
L_2 \arrow[r, "\sigma(L_2)"] & T^2(L_2) \arrow[u, "T(\lambda)"']
\end{tikzcd}$$

The only "naturality" condition read from this diagram is $\sigma(L_1){=}T(\lambda)\sigma(L_2)\lambda$. Since $\sigma(L_1)$ is an isomorphism, this condition certainly cannot hold unless λ is an isomorphism of L_1 into L_2. Even in the more restricted case in which $L_2 = L_1 = L$ is a single space there can be no isomorphism $\sigma : L \to T(L)$ which satisfies this naturality condition $\sigma = T(\lambda)\sigma\lambda$ for every non-singular linear transformation λ.[2] Consequently, with our definition of $T(\lambda)$, there is no "natural" isomorphism between the functors I and T, even in a very restricted special case.

Such a consideration of vector spaces and their linear transformations is but one example of many similar mathematical situations; for instance, we may deal with groups and their homomorphisms, with topological spaces and their continuous mappings, with simplicial complexes and their simplicial transformations, with ordered sets and their order preserving transformations. In order to deal in a general way with such situations, we introduce the concept of a *category*. Thus a category **A** will consist of abstract elements of two types: the objects A (for example, vector spaces, groups) and the mappings α (for example, linear transformations, homomorphisms). For some pairs of

[2]For suppose σ has this property. Then $(x, y) = [\sigma(x)]y$ is a non-singular bilinear form (not necessarily symmetric) in the vectors x, y of L, and we would have, for every λ, $(x, y) = [\sigma(x)](y) = [T(\lambda)\sigma\lambda x]y = [\sigma\lambda x]\lambda y = (\lambda x, \lambda y)$, so that the bilinear form is left invariant by every nonsingular linear transformation λ. This is clearly impossible.

mappings in the category there is defined a product (in the examples, the product is the usual composite of two transformations). Certain of these mappings act as identities with respect to this product, and there is a one-to-one correspondence between the objects of the category and these identities. A category is subject to certain simple axioms, so formulated as to include all examples of the character described above.

Some of the mappings α of a category will have a formal inverse mapping in the category; such a mapping α is called and equivalence. In the examples quoted the equivalences turn out to be, respectively, the isomorphisms for vector spaces, the homeomorphisms for topological spaces, the isomorphism for groups and complexes, and so on.

Most of the standard constructions of a new mathematical object from given objects (such as the construction of the direct product of two groups, the homology group of a complex, the Galois group of a field) furnish a function $T(A, B, \dots) = C$ which assigns to given objects $A, B, \dots$ in definite categories $\mathbf{A}, \mathbf{B}, \dots$ a new object C in a category $\mathbf{C}$. As in the special case of the conjugate $T(L)$ of a linear space, where there is a corresponding induced mapping $T(\lambda)$, we usually find that mappings $\alpha, \beta, \dots$ in the categories $\mathbf{A}, \mathbf{B}, \dots$ also induce a definite mapping $T(\alpha, \beta, \dots) = \gamma$ in the category $\mathbf{C}$, properly acting on the object $T(A, B, \dots)$.

These examples suggest the general concept of a functor T on categories $\mathbf{A}, \mathbf{B}, \dots$ to a category $\mathbf{C}$, defined as an appropriate pair of functions $T(A, B, \dots), T(\alpha, \beta, \dots)$. Such a functor may well be covariant in some of its arguments, contravariant in the others. The theory of categories and functors, with a few of the illustrations, constitutes Chapter 1.

The natural isomorphism $L \to T^2(L)$ is but one example of many natural equivalences occurring in mathematics. For instance, the isomorphism of a locally compact abelian group with its twice iterated character group, most of the general isomorphisms in group theory and in the homology theory of complexes and spaces, as well as many equivalences in set theory and in general topology satisfy a naturality condition resembling Eq. (3). In Chapter 2, we provide a general definition of equivalence between functors which includes there cases. A more general notion of a transformation of one functor into another provides a means of comparing functors which may not be equivalent. The general concepts are illustrated by several fairly elementary examples of equivalences and transformations for topological spaces, groups, and Banach spaces.

The third chapter deals especially with groups. In the category of groups the concept of a subgroup establishes a natural partial order for the objects (groups) of the category. For a functor whose values are in the category of groups there is an induced partial order. The formation of a quotient group has as analogue the construction of the quotient functor of a given functor by any normal subfunctor. In the uses of group theory, most groups constructed are obtained as quotient groups of other groups; consequently the operation of building a quotient functor is directly helpful in the representation of such group constructions by functors. The first and second isomorphism theorems of group theory are then formulated for functors; incidentally, this is used to show that these isomorphisms are "natural". The latter part of the chapter establishes the naturality of various known isomorphisms and homomorphisms in group theory.[3]

The fourth chapter starts with a discussion of functors on the category of partially ordered sets, and continues with the discussion of limits of direct and inverse systems of groups, which form the chief topic of this chapter. After suitable categories are introduced, the operations of forming direct and inverse limits of systems of groups are described as functors.

In the fifth chapter we establish the homology and cohomology groups of complexes and spaces as functors and show the naturality of various know isomorphisms of topology, especially those which arise in duality theorems. The treatment of the Čech homology theory utilizes the categories of direct and inverse systems, as discussed in Chapter 4.

The introduction of this study of naturality is justified, in our opinion, both by its technical and by its conceptual advantages.

In the technical sense, it provides the exact hypotheses necessary to apply to both sides of an isomorphism a passage to the limit, in the sense of direct or inverse limits for groups, rings or spaces.[4] Indeed, our naturality condition is part of the standard isomorphism condition for two direct or two inverse systems.[5]

The study of functors also provides a technical background for the intuitive notion of naturality and makes it possible to verify by straightforward computation the naturality of an isomorphism or of an equivalence in all those cases where it has intuitively recognized that the isomorphisms are indeed "natural". In many cases (for example, as in the above isomorphism of L to $T(L)$) we can also assert that certain known isomorphisms are in fact "unnatural", relative to the class of mappings considered.

In metamathematical sense our theory provides general concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards uniform treatment of different mathematical disciplines. In particular, it provides opportunities for the comparison of constructions and of the isomorphisms occurring in different branches of mathematics; in this way it may occasionally suggest new results by analogy.

[3]A brief discussion of this case and of the general theory of functors in the case of groups is given in the authors' note, Eilenberg and MacLane, "Natural isomorphisms in group theory".

[4]Such limiting processes are essential in the transition from the homology theory of complexes to that of spaces. Indeed, the general theory developed here occurred to the authors as a result of the study of the admissibility of such a passage in a relatively involved theorem in homology theory (Eilenberg and MacLane, "Group Extensions and Homology", especially, p. 777 and p. 815).

[5]Freudenthal, "Entwicklungen von Räumen und ihren Gruppen".

The theory also emphasizes that, whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects an an integral part of their definition. The pursuit of this program entails a simultaneous consideration of objects and their mappings (in our terminology, this means the consideration not of individual objects but of categories). This emphasis on the specification of the type of mappings employed gives more insight int the degree of invariance of the various concepts involved. For instance, we show in §3.3, that the concept of the commutator subgroup of a group is in a sense a more invariant one than that of the center, which in its turn is more invariant than the concept of the automorphism group of a group, even though in the classical sense all three concepts are invariant.

The invariant character of a mathematical discipline can be formulated in these terms. Thus, in group theory all basic constructions can be regarded as the definitions of co- or contravariant functors, so we may formulate the dictum: The subject of group theory is essentially the study of those constructions of groups which behave in a covariant or contravariant manner under induced homomorphisms. More precisely, group theory studies functors defined on well specified categories of groups, with values in another such category.

This may be regarded as a continuation of the Klein Erlanger Programm, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings.

Chapter 1

Categories and functors

1.1 Definition of Categories

These investigations will deal with aggregates such as a class of groups together with a class of homomorphisms, each of which maps one of the groups into another one, or such as a class of topological spaces together with their continuous mappings, one into another. Consequently we introduce a notion of "category" which will embody the common formal properties of such aggregates.

From the examples "groups plus homomorphisms" or "spaces plus continuous mappings" we are led to the following definition. A *category* $\mathbf{A} = \{A, \alpha\}$ is aggregate of abstract elements A (for example, groups), called the *objects* of the category, and abstract elements α (for example, homomorphisms), called *mappings* of the category. Certain pair of mappings $\alpha_1, \alpha_2 \in \mathbf{A}$ determine uniquely a product mapping $\alpha = \alpha_2\alpha_1 \in \mathbf{A}$, subject to axioms 1 to 3 below. Corresponding to each object $A \in \mathbf{A}$ there is a unique mapping, denoted by e_A or by $e(A)$, and subject to the axioms 4 and 5. The axioms are:

Axiom 1. *The triple product $\alpha_3(\alpha_2\alpha_1)$ is defined if and only if $(\alpha_3\alpha_2)\alpha_1$ is defined. When either is defined, the associative law*

$$\alpha_3(\alpha_2\alpha_1) = (\alpha_3\alpha_2)\alpha_1$$

holds. This triple product will be written as $\alpha_3\alpha_2\alpha_1$.

Axiom 2. *The triple product $\alpha_3\alpha_2\alpha_1$ is defined whenever both products $\alpha_3\alpha_2$ and $\alpha_2\alpha_1$ are defined.*

Definition. *A mapping $e \in \mathbf{A}$ will be called an* identity *of $\mathbf{A}$ if and only if the existence of any product $e\alpha$ or βe implies that $e\alpha = \alpha$ or $\beta e = \beta$.*

Axiom 3. *For each mapping $\alpha \in \mathbf{A}$ there is at least on identity $e_1 \in \mathbf{A}$ such that αe_1 is defined, and at least on identity $e_2 \in \mathbf{A}$ such that $e_2\alpha$ is defined.*

Axiom 4. *The mapping e_A corresponding to each object A is an identity.*

Axiom 5. *For each identity e of $\mathbf{A}$ there is an unique object A of $\mathbf{A}$ such that $e_A = e$.*

These two axioms assert that the rule $A \to e_A$ provides a one-to-one correspondence between the set of all objects of the category and the set of all its identities.

Lemma 1. *For each mapping $\alpha \in \mathbf{A}$ there is exactly one object A_1 with the product $\alpha e(A_1)$ defined, and exactly one A_2 with $e(A_2)\alpha$ defined.*

The objects A_1, A_2 will be called the *domain* and the *range* of α, respectively. We also say that α acts on A_1 to A_2, and write

$$\alpha : A_1 \to A_2 \text{ in } \mathbf{A}.$$

Proof. Suppose that $\alpha e(A_1)$ and $\alpha e(B_1)$ are both defined. By the properties of an identity, $\alpha e(A_1) = \alpha$, so that axioms 1 and 2 insure that the product $e(A_1)e(B_1)$ is defined. Since both are identities, $e(A_1) = e(A_1)e(B_1) = e(B_1)$, and consequently $A_1 = B_1$. The uniqueness of A_2 is similarly established. □

Lemma 2. *The product $\alpha_2\alpha_1$ is defined if and only if the range of α_1 is the domain of α_2. In other words, $\alpha_2\alpha_1$ is defined if and only if $\alpha_1 : A_1 \to A_2$ and $\alpha_2 : A_2 \to A_3$. In that case $\alpha_1\alpha_2 : A_1 \to A_3$.*

Proof. Let $\alpha_1 : A_1 \to A_2$. The product $e(A_2)\alpha_1$ is then defined and $e(A_2)\alpha_1 = \alpha_1$. Consequently $\alpha_2\alpha_1$ is defined if and only if $\alpha_2 e(A_2)\alpha_1$ is defined. By axioms 1 and 2 this will hold precisely when $\alpha_2 e(A_2)$ is defined. Consequently $\alpha_2\alpha_1$ is defined if and only if A_2 is the domain of α_2 so that $\alpha_2 : A_2 \to A_3$. To prove that $\alpha_2\alpha_1 : A_1 \to A_3$ note that since $\alpha_1 e(A_1)$ and $e(A_3)\alpha_2$ are defined the products $(\alpha_2\alpha_1)e(A_1)$ and $e(A_3)(\alpha_2\alpha_1)$ are defined. □

Lemma 3. *If A is an object, $e_A : A \to A$.*

Proof. If we assume that $e(A) : A_1 \to A_2$ then $e(A)e(A_1)$ and $e(A_2)e(A)$ are defined. Since they are all identities it follows that $e(A) = e(A_1)e(A_2)$ and $A = A_1 = A_2$ □

A "left identity" β is a mapping such that $\beta\alpha = \alpha$ whenever $\beta\alpha$ is defined. Axiom 3 shows that every left identity is an identity. Similarly each right identity is an identity. Furthermore, the product ee^1 of two identities is defined if and only if $e = e^1$.

If $\beta\gamma$ is defined and is an identity, β is called a *left inverse* of γ, γ is a *right inverse* of β. A mapping α is called an *equivalence* of $\mathbf{A}$ if it has in $\mathbf{A}$ at least one left inverse and at least one right inverse.

Lemma 4. *An equivalence α has exactly one left inverse and exactly one right inverse. These inverses are equal, so that the (unique) inverse may be denoted by α^{-1}.*

Proof. It suffices to show that any left inverse β of α equals any right inverse γ. Since $\beta\alpha$ and $\alpha\gamma$ are both defined, $\beta\alpha\gamma$ is defined, by axiom 2. But $\beta\alpha$ and $\alpha\gamma$ are identities, so that $\beta = \beta(\alpha\gamma) = (\beta\alpha)\gamma = \gamma$, as asserted. $\square$

For equivalences α, β one easily proves that α^{-1} and $\alpha\beta$ (if defined) are equivalences, and that

$$(\alpha^{-1})^{-1} = a, \qquad (\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}.$$

Every identity e is an equivalence, with $e^{-1} = e$.

Two objects A_1, A_2 are called *equivalent* if there is an equivalence α such that $\alpha : A_1 \to A_2$. The relation of equivalence between objects is reflexive, symmetric and transitive.

1.2 Examples of Categories

In the construction of examples, it is convenient to use the concept of a subcategory. A subaggregate $\mathbf{A}_0$ of $\mathbf{A}$ will be called a *subcategory* if the following conditions hold:

1. If $\alpha_1, \alpha_2 \in \mathbf{A}_0$ and $\alpha_2\alpha_1$ is defined in $\mathbf{A}$, then $\alpha_2, \alpha_1 \in \mathbf{A}_0$.

2. If $A \in \mathbf{A}_0$, then $e_A \in \mathbf{A}_0$.

3. If $\alpha : A_1 \to A_2$ in $\mathbf{A}$ with $\alpha \in \mathbf{A}_0$, then $A_1, A_2 \in \mathbf{A}_0$

Condition 1 insures that $\mathbf{A}_0$ is "closed" with respect to multiplication in $\mathbf{A}$; from conditions 2 and 3 it then follows that $\mathbf{A}_0$ is itself a category. The intersection of any number of subcategories of $\mathbf{A}$ is again a subcategory of $\mathbf{A}$. Note, however, that an equivalence $\alpha \in \mathbf{A}_0$ of $\mathbf{A}$ need not remain an equivalence in a subcategory $\mathbf{A}_0$, because the inverse α^{-1} may not be in $\mathbf{A}_0$.

For example, if $\mathbf{A}$ is any category, the aggregate $\mathbf{A}_e$ of all the objects and all the equivalences of $\mathbf{A}$ is a subcategory of $\mathbf{A}$. Also if $\mathbf{A}$ is a category and S a subclass of its objects, the aggregate $\mathbf{A}_S$ consisting of all objects of S and all mappings of $\mathbf{A}$ with both range and domain in S is a subcategory. In fact, every subcategory of $\mathbf{A}$ can be obtained in two steps: first, form a subcategory $\mathbf{A}_S$; second, extract from $\mathbf{A}_S$ a subaggregate, consisting of all the objects of $\mathbf{A}_S$ and a set of mappings of $\mathbf{A}_S$ which contains all the identities and is closed under multiplication.

The category $\mathbf{S}$ of all sets has as its objects all sets S.[1] A mapping σ of $\mathbf{S}$ is determined by a pair of sets S_1 and S_2 and a many-one correspondence

[1]This category obviously leads to the paradoxes of set theory. A detailed discussion of this aspect of categories appears in §1.6, below.

between S_1 and a subset of S_2, which assigns to each $x \in S_1$ a corresponding element $\sigma x \in S_2$; we then write $\sigma : S_1 \to S_2$. (Note that any deletion of elements from S_1 or S_2 *changes* the mapping σ.) The product of $\sigma_2 : S_2^1 \to S_3$ and $\sigma_1 : S_1 \to S_2$ is defined if and only if $S_2^1 = S_2$; this product then maps S_1 to S_3 by the usual composite correspondence $(\sigma_2\sigma_1)x = \sigma_2(\sigma_1 x)$, for each $x \in S_1$.[2] The mapping e_S corresponding to the set S is the identity mapping of S onto itself, with $e_S x = x$ for $x \in S$. The axioms 1 to 5 are clearly satisfied. An equivalence $\sigma : S_1 \to S_2$ is simply a one-to-one mapping of S_1 *onto* S_2.

Subcategories of **S** include the category of all finite sets S, with their mappings as before. For any cardinal number m there are two similar categories, consisting of all sets S of power less than m (or, of power less than or equal to m), together with all their mappings. Subcategories of **S** can also be obtained by restricting the mappings; for instance we may require that each σ is a mapping of S_1 *onto* S_2, or that each σ is a one-to-one mapping of S_1 into a subset of S_2.

The category **X** of all topological spaces has as its objects all topological spaces X and as its mappings all continuous transformations $\xi : X_1 \to X_2$ of a space X_1 into a space X_2. The composition $\xi_2\xi_1$ and the identity e_X are both defined as before. An equivalence in **X** is a homeomorphism (=topological equivalence).

Various subcategories of **X** can again be obtained by restricting the type of topological space to be considered, or by restricting the mappings, say to open mappings or to closed mappings.[3]

In particular, **S** can be regarded as a subcategory of **X**, namely, as that subcategory consisting of all spaces with a discrete topology.

The category **G** of all topological groups[4] has as its objects all topological group G and as its mappings γ all those many-one correspondences of a group G_1 into a group G_2 which are homomorphisms.[5]

Subcategories of **G** can be obtained by restricting the groups (discrete, abelian, regular, compact and so on) or by restricting the homomorphisms (open homomorphisms, homomorphisms “onto”, and so on).

The category **B** of all Banach spaces is similar; its objects are the Banach spaces B, its mappings all linear transformations β of norm at most 1 of one

[2]This formal associative law allows us to write $\sigma_2\sigma_1 x$ without fear of ambiguity. In more complicated formulas, parentheses will be inserted to make the components stand out.

[3]A mapping $\xi : x_1 \to x_2$ is *open* (*closed*) if the image under ξ of every open (closed) subset of X is open (closed) in X_2.

[4]A *topological group* G is a group which is also a topological space in which the group composition and the group inverse are continuous functions (no separation axioms are assumed on the space). If, in addition, G is a Hausdorff space, then all the separation axioms up to and including regularity are satisfied, so that we call G a *regular topological group*.

[5]By homomorphism we always understand a continuous homomorphism.

Banach space into another.[6] Its equivalences between two Banach spaces (that is, one-to-one linear transformations which preserve the norm). The assumption above the the mappings of the category **B** all have norm at most 1 is necessary in order to insure that the equivalences in **B** actually preserve the norm. If one admits arbitrary linear transformations as mappings of the category, one obtains a larger category in which the equivalences are isomorphisms (that is, one-to-one linear transformations)[7].

For quick reference, we sometimes describe a category by specifying only the objects involved (for examples, the category of all discrete groups). In such a case, we imply the the mappings of this category are to be all mappings appropriate to the objects in questions (for example, all homomorphisms).

1.3 Functors in two arguments

For simplicity we define only the concept of a functor covariant in one argument and contravariant in another. The generalization to any number of arguments of each type will be immediate.

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three categories. Let $T(A, B)$ be an *object-function* which associates with each pair of objects $A \in \mathbf{A}, B \in \mathbf{B}$ an object $T(A, B) = C \in \mathbf{C}$, and let $T(\alpha, \beta)$ be a *mapping-function* which associates with each pair of mappings $\alpha \in \mathbf{A}, \beta \in \mathbf{B}$ a mapping $T(\alpha, \beta) = \gamma \in \mathbf{C}$. For these functions we formulate certain conditions already indicated in the example in the introduction.

Definition. *The object-function $T(A, B)$ and the mapping-function $T(\alpha, \beta)$ form a* functor T*, covariant in* $\mathbf{A}$ *and contravariant in* $\mathbf{B}$*, with values in* $\mathbf{C}$*, if*

$$T(e_A, e_B) = e_{T(A,B)}, \tag{1.1}$$

if, whenever $\alpha : A_1 \to A_2 \in \mathbf{A}$ *and* $\beta : B_1 \to B_2 \in \mathbf{B}$

$$T(\alpha, \beta) : T(A_1, B_2) \to T(A_2, B_1), \tag{1.2}$$

and if, whenever $\alpha_2\alpha_1 \in \mathbf{A}$ *and* $\beta_2\beta_1 \in \mathbf{B}$*,*

$$T(\alpha_2\alpha_1, \beta_2\beta_1) = T(\alpha_2, \beta_1)T(\alpha_1, \beta_2) \tag{1.3}$$

Condition (1.2) guarantees the existence of the product of mappings appearing on the right of Eq. (1.3).

The formulas (1.2) and (1.3) display the distinction between co- and contravariance. The mapping $T(\alpha, \beta) = \gamma$ induced by α and β from $T(A_1, -)$ to $T(A_2, -)$; that is, in the same direction as does α hence the *co*variance

[6]For each linear transformation β of the Banach space B_1 into B_2, the norm $\|\beta\|$ is defined as the least upper bound $\|\beta b\|$, for all $b \in B_1$ with $\|b\| = 1$.

[7]Banach, *Théorie des opérations linéaires*, p. 180.

of T in the argument $\mathbf{A}$. The induced mapping $T(\alpha, \beta)$ at the same time operates in the direction opposite from that of β; thus it is contravariant in $\mathbf{B}$. Essentially the same shift in direction is indicated by the orders of factors in formula (1.3) (the covariant α's appear in the same order on both sides; the contravariant β's appear in on order on the left and in the opposite order on the right). With this observation, the requisite formulas for functors in more arguments can be set down.

According to this definition, the functor T is composed of an object function and a mapping function. The latter is the more important of the two; in fact, Eq. (1.1) means that it determines the object function and therefore the whole functor, as stated in the following theorem.

Theorem 1. *A function $T(\alpha, \beta)$ which associates to each pair of mappings α and β in the respective categories $\mathbf{A}, \mathbf{B}$ a mapping $T(\alpha, \beta) = \gamma$ in a third category $\mathbf{C}$ is the mapping function of a functor T covariant in $\mathbf{A}$ and contravariant in $\mathbf{B}$ if and only if the following two conditions hold:*

1. *$T(e_A, e_B)$ is an identity mapping in $\mathbf{C}$ for all identities e_A, e_B of $\mathbf{A}$ and $\mathbf{B}$.*

2. *Whenever $\alpha_2\alpha_1 \in \mathbf{A}$ and $\beta_2\beta_1 \in \mathbf{B}$, then $T(\alpha_2, \beta_1)T(\alpha_1, \beta_2)$ is defined and satisfies the equation*

$$T(\alpha_2\alpha_1, \beta_2\beta_1) = T(\alpha_2, \beta_1)T(\alpha_1, \beta_2). \tag{1.4}$$

If $T(\alpha, \beta)$ satisfies Items 1 and 2, the corresponding functor T is uniquely determined, with an object function $T(A, B)$ given by the formula

$$e_{T(A,B)} = T(e_A, e_B) \tag{1.5}$$

Proof. The necessity of Item 1 and Item 2 and the second half of the theorem are obvious.

Conversely, let $T(\alpha, \beta)$ satisfy conditions 1 and 2. Condition 1 means that an object function $T(A, B)$ can be defined by Eq. (1.5). We must show that if $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$, then Eq. (1.2) holds. Since $e(A_2)\alpha$ and $\beta e(B_1)$ are defined, the product $T(\alpha, \beta)T(e(A_1), e(B_2))$ is defined.

In virtue of the definition (1.5), the products

$$e(T(A_2, B_1))T(\alpha, \beta), \qquad T(\alpha, \beta)T(e(A_1), e(B_2))$$

are defined. This implies Eq. (1.2). $\square$

In any functor, the replacement of the arguments A, B by equivalent arguments A', B' will replace the value $T(A, B)$ by an equivalent value $T(A', B')$. This fact may be alternatively stated as follows:

Theorem 2. *If T is a functor on $\mathbf{A}, \mathbf{B}$ to $\mathbf{C}$, and if $\alpha \in \mathbf{A}$ and $\beta \in \mathbf{B}$ are equivalences, then $T(\alpha, \beta)$ is an equivalence in $\mathbf{C}$, with inverse $T(\alpha, \beta)^{-1} = T(\alpha^{-1}, \beta^{-1})$.*

Proof. For the proof we assume that T is covariant in $\mathbf{A}$ and contravariant in $\mathbf{B}$. The products $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are then identities, and the definition of a functor shows that

$$\begin{aligned} T(\alpha, \beta)T(\alpha^{-1}, \beta^{-1}) &= T(\alpha\alpha^{-1}, \beta^{-1}\beta), \\ T(\alpha^{-1}, \beta^{-1})T(\alpha, \beta) &= T(\alpha^{-1}\alpha, \beta\beta^{-1}). \end{aligned}$$

By Eq. (1.1), the terms on the right are both identities, which means that $T(\alpha^{-1}, \beta^{-1})$ is an inverse for $T(\alpha, \beta)$, as asserted. □

1.4 Examples of functors

The same object function may appear in various functors, as is shown by the following example of one covariant and one contravariant functor both with the same object function. In the category $\mathbf{S}$ of all set, the "power" functors P^+ and P^- have the object function

$$P^+(S) = P^-(S) = \text{the set of all subsets of } S.$$

For any many-one correspondence $\sigma : S_1 \to S_2$ the respective mapping functions are defined for any subset $A_1 \subset S_1$ (or $A_2 \subset S_2$) as[8]

$$P^+(\sigma)A_1 = \sigma A_1, \qquad P^-(\sigma)A_2 = \sigma^{-1}A_2.$$

It is immediate that P^+ is a covariant functor and P^- a contravariant one.

The cartesian product $X \times Y$ of two topological spaces is the object function of a functor of two covariant variables X and Y in the category $\mathbf{X}$ of all topological spaces. For a continuous transformations $\xi : X_1 \to X_2$ and $\eta : Y_1 \to Y_2$ the corresponding mapping function $\xi \times \eta$ is defined for any point (x_1, y_1) in the cartesian product $X_1 \times Y_1$ as

$$\xi \times \eta(x_1, y_1) = (\xi x_1, \eta y_1).$$

One verifies that

$$\xi \times \eta : X_1 \times Y_1 \to X_2 \times Y_2,$$

whenever the products $\xi_2\xi_1$ and $\eta_2\eta_1$ are defined. In virtue of these facts, the functions $X \times Y$ and $\xi \times \eta$ constitute a covariant functor of two variables on the category $\mathbf{X}$.

[8]Here σA_1 is the set of all elements of S_2 of the form σx for $x \in A_1$, while $\sigma^{-1}A_2$ consists of all elements $x \in S_1$ with $\sigma x \in A_2$. When σ is an equivalence, with and inverse τ, $\tau A_2 = \sigma^{-1}A_2$, so that no ambiguity as to the meaning of σ^{-1} can arise.

The direct product of two groups is treated in exactly similar fashion; it gives a functor with the set function $G \times H$ and the mapping function $\gamma \times \eta$, defined for $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$ exactly as was $\xi \times \eta$. The same applies to the category **B** of Banach spaces, provided one fixes one of the usual possible definite procedures of norming the cartesian product of two Banach spaces.

For a topological space Y and a locally compact (=locally bicompact) Hausdorff space X one may construct the space Y^X of all continuous mappings f of the whole space X into Y ($fx \in Y$ for $x \in X$). A topology is assigned to Y^X as follows. Let C be any compact subset of X, U any open set in Y. The the set $[C, U]$ of all $f \in Y^X$ with $fC \subset U$ is an open set in Y^X, and the most general open set in Y^X is any union of finite intersections $[C_1, U_1] \cap \cdots \cap [C_n, U_n]$.

This space Y^X may be regarded as the object function of a suitable functor, $\mathrm{Map}(X, Y)$. To construct a suitable mapping function, consider any continuous transformations $\xi : X_1 \to X_2$, $\eta : Y_1 \to Y_2$. For each $f \in Y_1^{X_2}$, one then has mappings acting thus:

$$X_1 \xrightarrow{\xi} X_2 \xrightarrow{f} Y_1 \xrightarrow{\eta} Y_2,$$

so that one may derive a continuous transformation $\eta f \xi$ of $Y_2^{X_1}$. This correspondence $f \to \eta f \xi$, may be shown to be a continuous mapping of $Y_1^{X_2}$ into $Y_2^{X_1}$. Hence we may define object and mapping functions "Map" by setting

$$\mathrm{Map}(X, Y) = Y^X, \qquad [\mathrm{Map}(\xi, \eta)]f = \eta f \xi. \tag{1.6}$$

The construction shows that

$$\mathrm{Map}(\xi, \eta) : \mathrm{Map}(X_2, Y_1) \to \mathrm{Map}(X_1, Y_2),$$

and hence suggests that this functor is contravariant in X and covariant in Y. One observes at once that $\mathrm{Map}(\xi, \eta)$ is an identity when bot ξ and η are identities. Furthermore, if the products $\xi_2\xi_1$ and $\eta_2\eta_1$ are defined, the definition of "Map" gives first,

$$[\mathrm{Map}(\xi_2\xi_1, \eta_2\eta_1)]f = \eta_2\eta_1 f \xi_2\xi_1 = \eta_2(\eta_1 f \xi_2)\xi_1,$$

and second,

$$\mathrm{Map}(\xi_1, \eta_2)\,\mathrm{Map}(\xi_2, \eta_1)f = [\mathrm{Map}(\xi_1, \eta_2)]\eta_1 f \xi_2 = \eta_2(\eta_1 f \xi_2)\xi_1.$$

Consequently

$$\mathrm{Map}(\xi_2\xi_1, \eta_2\eta_1) = \mathrm{Map}(\xi_1, \eta_2)\,\mathrm{Map}(\xi_2, \eta_1),$$

which completes the verification that "Map", defined as in Eq. (1.6), is a functor on $\mathbf{X}_{\mathrm{lc}}, \mathbf{X}$ to $\mathbf{X}$, contravariant in the first variable, covariant in the second, where $\mathbf{X}_{\text{lc}}$ denotes the subcategory of $\mathbf{X}$ defined by the locally compact Hausdorff spaces.

For abelian groups there is a similar functor "Hom". Specifically, let G be a locally compact regular topological group, H a topological abelian group. We construct the set $\mathrm{Hom}(G, H)$ of all (continuous) homomorphisms ϕ of G into H. The sum of two such homomorphisms ϕ_1 and ϕ_2 is defined by setting $(\phi_1 + \phi_2)g = \phi_1 g + \phi_2 g$, for each $g \in G$;[9] this sum is itself a homomorphism because H is abelian.

Under this addition $\mathrm{Hom}(G, H)$ is an abelian group. It is topologized by the family of neighborhoods $[C, U]$ of zero defined as follows. Given C, any compact subset of G, and U, any open set in H containing the zero of H, $[C; U]$ consists of all $\phi \in \mathrm{Hom}(G, H)$ with $\phi C \subset U$. With these definitions, $\mathrm{Hom}(G, H)$ is a topological group. If H has a neighborhood of the identity containing no subgroup but the trivial one, one may prove that $\mathrm{Hom}(G, H)$ is locally compact.

This function of groups is the object function of a functor "Hom". For given $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$ the mapping function is defined by setting

$$[\mathrm{Hom}(\gamma, \eta)]\phi = \eta\phi\gamma \tag{1.7}$$

for each $\phi \in \mathrm{Hom}\, G_2, H_1$. Formally this definition is exactly like Eq. (1.6). One may show that this definition (1.7) does yield a continuous homomorphism

$$\mathrm{Hom}(\gamma, \eta) : \mathrm{Hom}(G_2, H_1) \to \mathrm{Hom}(G_1, H_2).$$

As in the previous case, Hom is a functor with values in the category $\mathbf{G}_{\text{a}}$ of abelian groups, defined for arguments in two appropriate subcategories of $\mathbf{G}$, contravariant in the first argument, G, and covariant in the second, H.

For Banach spaces there is a similar functor. If B and C are two Banach spaces, let $\mathrm{Lin}(B, C)$ denote the Banach space of all linear transformation λ of B into C, with the usual definition of the norm of the transformation. To describe the corresponding mapping function, consider any linear transformations $\beta : B_1 \to B_2$ and $\gamma : C_1 \to C_2$ with $\|\beta\| \leq 1$ and $\|\gamma\| \leq 1$, and set, for each $\lambda \in \mathrm{Lin}(B_2, C_1)$,

$$[\mathrm{Lin}(\beta, \gamma)]\lambda = \gamma\lambda\beta. \tag{1.8}$$

This is in fact a linear transformation

$$\mathrm{Lin}(\beta, \gamma) : \mathrm{Lin}(B_2, C_1) \to \mathrm{Lin}(B_1, C_2)$$

of norm at most 1. As in the previous cases, Lin is a functor on $\mathbf{B}, \mathbf{B}$ to $\mathbf{B}$, contravariant in its first argument and covariant in the second.

[9] The group operation in G,H, and so on, will be written as addition.

In case C is fixed to be the Banach space $\mathbb{R}$ of all real numbers with the absolute value as norm, $\mathrm{Lin}(B, C)$ is just the Banach space conjugate to B, in the usual sense. This leads at once to the functor

$$\mathrm{Conj}(B) = \mathrm{Lin}(b, \mathbb{R}), \qquad \mathrm{Conj}(\beta) = \mathrm{Lin}(\beta, e_{\mathbf{R}}).$$

This is a contravariant functor on $\mathbf{B}$ to $\mathbf{B}$.

Another example of a functor on groups is the tensor product $G \otimes H$ of two abelian groups. This functor has been discussed in more detail in our Proceedings note cited above.

1.5 Slicing of functors

The last example involved the process of holding one of the arguments of a functor constant. This process occurs elsewhere (for example, in the character group theory, §3.6 below), and falls at once under the following theorem.

Theorem 3. *If T is a functor covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$, with values in $\mathbf{C}$, then for each fixed $B \in \mathbf{B}$ the definitions*

$$S(A) = T(A, B), \qquad S(\alpha) = T(\alpha, e_B)$$

yield a functor S on $\mathbf{A}$ to $\mathbf{C}$ with the same variance (in $\mathbf{A}$) as T.

This "slicing" of a functor may be partially inverted, in that the functor T is determined by its object function and its two "sliced" mapping functions, in the following sense.

Theorem 4. *Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be three categories and $T(A, B)$, $T(\alpha, B)$, $T(A, \beta)$ three functions such that for each fixed $B \in \mathbf{B}$ the functions $T(A, B)$, $T(\alpha, B)$ form a covariant functor on $\mathbf{A}$ to $\mathbf{C}$, while for each $A \in \mathbf{A}$ the functions $T(A, B)$ and $T(A, \beta)$ give a contravariant functor on $\mathbf{B}$ to $\mathbf{C}$. If in addition for each $\alpha : A_1 \to A_2$ in $\mathbf{A}$ and $\beta : B_1 \to B_2$ in $\mathbf{B}$ we have*

$$T(A_2, \beta) T(\alpha, B_2) = T(\alpha, B_1) T(A_1, \beta), \tag{1.9}$$

then the functions $T(A, B)$ and

$$T(\alpha, \beta) = T(\alpha, B_1) T(A_1, \beta) \tag{1.10}$$

from a functor covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$, with values in $\mathbf{C}$

Proof. Equation (1.9) merely states the equivalence of the two paths about the following commutative square:

$$\begin{tikzcd}
T(A_1, B_2) \arrow[r, "T(\alpha, B_2)"] \arrow[d, "T(A_1, \beta)"] & T(A_2, B_2) \arrow[d, "T(A_2, \beta)"] \\
T(A_1, B_1) \arrow[r, "T(\alpha, B_1)"] & T(A_2, B_1)
\end{tikzcd}$$

The result of either path is then taken in Eq. (1.10) to define the mapping function, which then certainly satisfies condition (1.1) and (1.2) of the definition of a functor. The proof of the basic product condition (1.3) is best visualized by writing out a 3×3 array of values $T(A_i, B_j)$. □

The significance of this theorem is essentially this: in verifying that a given object and mapping functions do yield a functor, one may replace the verification of the product condition (1.3) in two variables by a separate verification, one variable at a time, provided one *also* proves that the order of application of these one-variable mappings can be interchanged (condition (1.9)).

1.6 Foundations

We remarked in §1.2 that such examples as the "category of *all* sets", the "category of *all* groups" are illegitimate. The difficulties and antinomies here involved are exactly those of ordinary intuitive *Mengenlehre*; no essentially new paradoxes are apparently involved. Any rigorous foundation capable of supporting the ordinary theory of classes would equally well support our theory. Hence we have chosen to adopt the intuitive standpoint, leaving the reader free to insert whatever type of logical foundation (or absence thereof) he may prefer. These ideas will now be illustrated, with particular reference to the category of groups.

It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concept are essentially those of a *functor* and of a natural transformation (the latter is defined in the next chapter). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and range of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as "Hom" is not defined over the category of "all" groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves.

For a more careful treatment, we may regard a group G as a pair, consisting of a set G_0 and a ternary relation $g \cdot h = k$ on this set, subject to the usual axioms of group theory. This makes explicit the usual tacit assumption that a group is not just the set of its elements (two groups can have the *same* elements, yet different operations). If a pair is constructed in the usual manner as a certain class, this means that each subcategory of the category of "all" groups is a class of pairs; each pair being a class of groups with a class of mappings (binary relations). Any given system of foundations will then legitimize those subcategories which are allowable classes in the system in question.

Perhaps the simplest precise device would be to speak not of *the* category of groups, but of *a* category of groups (meaning, any legitimate such category). A functor such as "Hom" is then a functor which can be defined for any two suitable categories of groups, $\mathbf{G}$ and $\mathbf{H}$. This procedure has the advantage of precision, the disadvantage of a multiplicity of categories and functors. This multiplicity would be embarrassing in the study of composite functors (§2.3 below).

One might choose to adopt the (unramified) theory of types as a foundation for the theory of classes. One then can speak of the category $\mathbf{G}_m$ of all abelian groups of type m. The functor "Hom" could then have both arguments in $\mathbf{G}_m$, while its values would be in the same category $\mathbf{G}_{m+k}$ of groups of higher type $m + k$. This procedure affects each functor with the same sort of typical ambiguity adhering to the arithmetical concepts in the Whitehead-Russell development. Isomorphisms between groups of different types would have to be considered, as in the simple isomorphism $\mathrm{Hom}(\mathbf{F}, \mathbf{G}) \cong \mathbf{G}$ (see §2.4); this would somewhat complicate the natural isomorphisms treated below.

One can also choose a set of axioms for classes as in the Fraenkel-von Neumann-Bernays system. A category is then any (legitimate) class in the sense of this axiomatics. Another device would be that of restricting the cardinal number, considering the category of all denumerable groups, of all groups of cardinal at most the cardinal of the continuum, and so on. The subsequent developments may be suitably interpreted under any on of these viewpoints.

Chapter 2

Natural equivalence of functors

2.1 Transformations of functors

Let T and S be two functors on $\mathbf{A}, \mathbf{B}$ to $\mathbf{C}$ which are *concordant*; that is, which have the same variance in $\mathbf{A}$ and the same variance in $\mathbf{B}$. To be specific, assume both T and S covariant in $\mathbf{A}$ and contravariant $\mathbf{B}$. Let τ be a function which associates to each pair of objects $A \in \mathbf{A}, B \in \mathbf{B}$ a mapping $\tau(A, B) = \gamma$ in $\mathbf{C}$.

Definition. *The function τ is a "natural" transformation of the functor T, covariant in $\mathbf{A}$ and contravariant in $\mathbf{B}$, into the concordant functor S provided that, for each pair of objects $A \in \mathbf{A}, B \in \mathbf{B}$,*

$$\tau(A, B) : T(A, B) \to S(A, B) \text{ in } \mathbf{C}, \tag{2.1}$$

and provided, whenever $\alpha : A_1 \to A_2$ in $\mathbf{A}$ and $\beta : B_1 \to B_2$ in $\mathbf{B}$, that

$$\tau(A_2, B_1)T(\alpha, \beta) = S(\alpha, \beta)\tau(A_1, B_2). \tag{2.2}$$

When these conditions hold, we write

$$\tau : T \to S.$$

If in addition each $\tau(A, B)$ is an equivalence mapping of the category $\mathbf{C}$, we call τ a *natural equivalence* of T to S (notation: $\tau : T \rightleftarrows S$) and say that the functors T and S are *naturally equivalent*. In this case condition (2.2) can be rewritten as

$$\tau(A_2, B_1)T(\alpha, \beta)[\tau(A_1, B_2)]^{-1} = S(\alpha, \beta). \tag{2.3}$$

Condition (2.1) of this definition is equivalent to the requirement that both product in Eq. (2.2) are always defined. Condition (2.2) is illustrated by the equivalence of the two paths indicated in the following diagram:

$$\begin{tikzcd}
T(A_1, B_2) \arrow[r, "T(\alpha,\beta)"] \arrow[d, "\tau(A_1,B_2)"] & T(A_2, B_1) \arrow[d, "\tau(A_2,B_1)"] \\
S(A_1, B_2) \arrow[r, "S(\alpha,\beta)"] & S(A_2, B_1)
\end{tikzcd}$$

Given three concordant functors T, S, and R on $\mathbf{A}, \mathbf{B}$ to $\mathbf{C}$, with natural transformations $\tau : T \to S$ and $\sigma : S \to R$, the product

$$\rho(A, B) = \sigma(A, B)\tau(A, B)$$

is defined as a mapping in $\mathbf{C}$, and yields a natural transformation $\rho : T \to R$. If τ and σ are natural equivalences, so is $\rho = \sigma\tau$.

Observe also that if $\tau : T \to S$ is a natural equivalence, then the function τ^{-1} is defined by $\tau^{-1}(A, B) = [\tau(A, B)]^{-1}$ is a natural equivalence $\tau^{-1} : S \to T$. Given any functor T on $\mathbf{A}, \mathbf{B}$ to $\mathbf{C}$, the function

$$\tau_0(A, B) = e_{T(A,B)}$$

is a natural equivalence $\tau_0 : T \rightleftarrows T$. These remarks imply that the concept of natural equivalence of functors is reflexive, symmetric and transitive.

In demonstrating that a given mapping $\tau(A, B)$ is actually a natural transformation, it suffices to prove the rule (2.2) only in these cases in which all except one of the mappings $\alpha, \beta, \ldots$ is an identity. To state this result it is convenient to introduce a simplified notation for the mapping function when one argument is an identity, by setting

$$T(\alpha, \beta) = T(\alpha, e_B), \qquad T(A, \beta) = T(e_A, \beta).$$

Theorem 5. *Let T and S be functors covariant in $\mathbf{A}$ and contravariant in $\mathbf{B}$, with values in $\mathbf{C}$, and let τ be a function which associates to each pair of objects $A \in \mathbf{A}$, $B \in \mathbf{B}$ a mapping with Eq. (2.1). A necessary and sufficient condition that τ be a natural transformation $\tau : T \to S$ is that for each mapping $\alpha : A_1 \to A_2$ and each object $B \in \mathbf{B}$ one has*

$$\tau(A_2, B)T(\alpha, B) = S(\alpha, B)\tau(A_1, B), \tag{2.4}$$

and that, for each $A \in \mathbf{A}$ and each $\beta : B_1 \to B_2$ one has

$$\tau(A, B_1)T(A, \beta) = S(A, \beta)\tau(A, B_2). \tag{2.5}$$

Proof. The necessity of these conditions is obvious, since they are simply the special case of Eq. (2.2) in which $\beta = e_B$ and $\alpha = e_A$, respectively. The sufficiency can best be illustrated by the following diagram, applying to any mappings $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$:

$$\begin{tikzcd}
T(A_1, B_2) \arrow[r, "\tau(A_1,B_2)"] \arrow[d, "T(\alpha,B_2)"] & S(A_1, B_2) \arrow[d, "S(\alpha,B_2)"] \\
T(A_2, B_2) \arrow[r, "\tau(A_2,B_2)"] \arrow[d, "T(A_2,\beta)"] & S(A_2, B_2) \arrow[d, "S(A_2,\beta)"] \\
T(A_2, B_1) \arrow[r, "\tau(A_2,B_1)"] & S(A_2, B_1)
\end{tikzcd}$$

Equation (2.4) states the equivalence of the results found by following either path around the upper small rectangle, and Eq. (2.5) makes a similar assertion for the bottom rectangle. Combining these successive equivalences, we have the equivalence of the two paths around the edges of the whole rectangle; this is the requirement (2.2). This argument can be easily set down formally. □

2.2 Categories of functors

The functors may be made the objects of a category in which the mapping are natural transformation. Specifically, given three fixed categories **A**, **B** and **C**, form the category **F** for which the objects are the functors T covariant in **A** and contravariant **B**, with values in **C**, and for which the mappings are the natural transformation $\tau : T \to S$. This requires some caution, because we may have $\tau : T \to S$ and $\tau : T' \to S'$ for the same function τ with different functors T, T' (which would have the same object function but different mapping functions). To circumvent this difficulty we define a mapping in the category T to be a triple $[\tau, T, S]$ with $\tau : T \to S$. The product of mappings $[\tau, T, S]$ and $[\sigma, S', R]$ is defined if and only if $S = S'$; in this case it is

$$[\sigma, S, R] \circ [\tau, T, S] = [\sigma\tau, T, R].$$

We verify that the axioms 1 to 3 of §1.1 are satisfied. Furthermore we define, for each functor T

$$e_T = [\tau_T, T, T], \text{ with } \tau_T(A, B) = e_{T(A,B)},$$

and verify the remaining axioms 4 and 5. Consequently **F** is a category. In this category it can be proved easily that $[\tau, T, S]$ is an equivalence mapping if and only if $\tau : T \rightleftarrows S$; consequently the concept of the natural equivalence of functors agrees with the concept of equivalence of objects in the category **T** of functors.

This category **T** is useful chiefly in simplifying the statements and proofs of various facts about functors, as will appear subsequently.

2.3 Composition of functors

This process arises by the familiar "function of a function" procedure, in which for the argument of a functor we substitute the value of another functor. For example, let T be a functor on $\mathbf{A}, \mathbf{B}$ to $\mathbf{C}$, R a functor on $\mathbf{C}, \mathbf{D}$ to $\mathbf{E}$. Then $S = R \circ (T, I)$, defined by setting

$$S(A, B, D) = R(T(A, B), D), \qquad S(\alpha, \beta, \gamma) = R(T(\alpha, \beta), \gamma),$$

for objects $A \in \mathbf{A}$, $B \in \mathbf{B}$, $D \in \mathbf{D}$ and mappings $\alpha \in \mathbf{A}$, $\beta \in \mathbf{B}$, $\gamma \in \mathbf{D}$, is a functor on $\mathbf{A}, \mathbf{B}, \mathbf{D}$ to $\mathbf{E}$. In the argument $\mathbf{D}$, the variance of S is just the the variance of R. The variance of R in $\mathbf{A}$ (or $\mathbf{B}$) may be determined by the rule of signs (with $+$ for covariance, $-$ for contravariance): (variance of $S \in \mathbf{A}$) = (variance of $R \in \mathbf{C}$) $\times$ (variance of $T \in \mathbf{A}$).

Composition can also be applied to natural transformations. To simplify the notation, assume that R is a functor in *one* variable, contravariant on $\mathbf{C}$ to $\mathbf{E}$, and that T is covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$ with values in $\mathbf{C}$. The composite $R \circ T$ is then contravariant in $\mathbf{A}$, covariant in $\mathbf{B}$. Any pair of natural transformations

$$\rho : R \to R', \qquad \tau = T \to T'$$

give rise to a natural transformation

$$\rho \circ \tau : R \circ T' \to R' \circ T$$

defined by setting

$$\rho \circ \tau(A, B) = \rho(T(A, B))R(\tau(A, B)).$$

Because ρ is natural, $\rho \circ \tau$ could equally well be defined as

$$\rho \circ \tau(A, B) = R'(\tau(A, B))\rho(T'(A, B)).$$

This alternative means that the passage from $R \circ T'(A, B)$ to $R' \circ T(A, B)$ can be made either through $R \circ T(A, B)$ or through $R' \circ T'(A, B)$, without altering the final result. The resulting *composite transformation* $\rho \circ \tau$ has all the usual formal properties appropriate to the mapping function of the "functor" $R \circ T(A, B)$; specifically,

$$(\rho_2 \rho_1) \circ (\tau_1 \tau_2) = (\rho_2 \circ \tau_2)(\rho_1 \circ \tau_1),$$

as may be verified by a suitable 3×3 diagram.

These properties show that the functions $R \circ T$ and $\rho \circ \tau$ determine a functor C, defined on the categories $\mathbf{R}$ and $\mathbf{T}$ of functors, with values in a category $\mathbf{S}$ of functors, covariant in $\mathbf{R}$ and contravariant in $\mathbf{T}$ (because of the contravariance of R). Here $\mathbf{R}$ is the category of all contravariant functors R on $\mathbf{C}$ to $\mathbf{E}$, while $\mathbf{S}$ and $\mathbf{T}$ are the categories of all functors S and T, of appropriate variances, respectively. In each case, the mappings of the category of functors are natural transformations, as described in the previous section. To be more explicit, the mapping function $C(\rho, \tau)$ of this functor is not the simple composite $\rho \circ \tau$, but the triple $[\rho \circ \tau, R \circ T', R' \circ T]$.

Since $\rho \circ \tau$ is essentially the mapping function of a functor, we know by Theorem 2 that if ρ and τ are natural equivalences, then $\rho \circ \tau$ is an equivalence. Consequently, if he pairs R and R', T and T' are naturally equivalent, so is the pair of composite $R \circ T$ and $R' \circ T'$.

It is easy to verify that the composition of functors and natural transformations is associative, so that symbols like $R \circ T \circ S$ may be written without parentheses.

If in the definition of $\rho \circ \tau$ above it occurs that $T = T'$ and that τ is the identity transformation $T \to T$ we shall write $\rho \circ T$ instead of $\rho \circ \tau$. Similarly we shall write $R \circ \tau$ in the case when $R = R'$ and ρ is the identity transformation on $R \to R$.

2.4 Examples of transformations

The associative and commutative laws for the direct and cartesian products are isomorphisms which can be regarded as equivalences between functors. For example, let X, Y and Z be three topological spaces, and let the homeomorphism

$$(X \times Y) \times Z \cong X \times (Y \times Z) \tag{2.6}$$

be established by the usual correspondence $\tau = \tau(X, Y, Z)$, defined for any point $((x, y), z)$ in the iterated cartesian product $(X \times Y) \times Z$ by

$$\tau(X, Y, Z)((x, y), z) = (x, (y, z)).$$

Each $\tau(X, Y, Z)$ is then an equivalence mapping in the category $\mathbf{X}$ of spaces. Furthermore each side of Eq. (2.6) may be considered as the object function of a covariant functor obtained by composition of the cartesian product functor with itself. The corresponding mapping function are obtained by parallel composition as $(\xi \times \eta) \times \zeta$ and $\xi \times (\eta \times \zeta)$. To show that $\tau(X, Y, Z)$ is indeed a natural equivalence, we consider three mappings $\xi : X_1 \to X_2$, $\eta : Y_1 \to Y_2$ and $\zeta : Z_1 \to Z_2$, and show that

$$\tau(X_2, Y_2, Z_2)[(\xi \times \eta) \times \zeta] = [\xi \times (\eta \times \zeta)]\tau(X_1, Y_1, Z_1).$$

This identity may be verified by applying each side to an arbitrary point $((x_1, y_1), z_1)$ in the space $(X_1 \times Y_1) \times Z_1$; each transforms it into the point $(\xi x_1, (\eta y_1, \zeta z_1))$ in $X_2 \times (Y_2 \times Z_2)$.

In similar fashion the homeomorphism $X \times Y \cong Y \times X$ may be interpreted as a natural equivalence, defined as $\tau(X, Y)(x, y) = (y, x)$. In particular, if X, Y and Z are discrete spaces (that is, are simply sets), there remarks show that the associative and commutative laws for the (cardinal) product of two sets are natural equivalences between functors.

For similar reasons, the associative and commutative laws for the direct product of groups are natural equivalences (or *natural isomorphisms*) between functors of groups. The same laws for Banach spaces, with a fixed convention as to the construction of the norm in the cartesian product of two such spaces, are natural equivalences between functors.

If $\mathbb{Z}$ is the (fixed) additive group of integers, H any topological abelian group, there is an isomorphism

$$\mathrm{Hom}(\mathbb{Z}, H) \cong H \tag{2.7}$$

in which both sides may be regarded as covariant functors of a single argument H. This isomorphism $\tau = \tau(H)$ is defined for any homomorphism $\phi \in \mathrm{Hom}(\mathbb{Z}, H)$ by setting $\tau(H)\phi = \phi(1) \in H$. One observes that $\tau(H)$ is indeed a (bicontinuous) isomorphism, that is, an equivalence in the category of topological abelian groups. That $\tau(H)$ actually is a natural equivalence between functors is shown by proving, for any $\eta : H_1 \to H_2$, that

$$\tau(H_2)\,\mathrm{Hom}(e_J, \eta) = \eta\tau(H_1).$$

There is also a second natural equivalence between the functors indicated in Eq. (2.7), obtained by setting $\tau'(H)\phi = \phi(-1)$.

With the fixed Banach space $\mathbb{R}$ of real numbers there is a similar formula

$$\mathrm{Lin}(\mathbb{R}, B) \cong B \tag{2.8}$$

for any Banach space B. This gives a natural equivalence $\tau = \tau(B)$ between two covariant functors of one argument in the category $\mathbf{B}$ of all Banach spaces. Here $\tau(B)$ is defined by setting $\tau(B)l = l(1)$ for each linear transformation $l \in \mathrm{Lin}(\mathbb{R}, B)$; another choice of τ would set $\tau(B)l = l(-1)$.

For topological spaces there is a distributive law for the functors “Map” and the direct product functor, which may be written as a natural equivalence

$$\mathrm{Map}(Z, X) \times \mathrm{Map}(Z, Y) \cong \mathrm{Map}(Z, X \times Y) \tag{2.9}$$

between two composite functors, each contravariant in the first argument Z and covariant in the other two arguments X and Y. To define this natural equivalence

$$\tau(X, Y, Z) : \mathrm{Map}(Z, X) \times \mathrm{Map}(Z, Y) \rightleftarrows \mathrm{Map}(Z, X \times Y)$$

consider any pair of mappings $f \in \mathrm{Map}(Z, X)$ and $g \in \mathrm{Map}(Z, Y)$ and set, for each $z \in Z$,

$$[\tau(f, g)](z) = (f(z), g(z)).$$

It can be shown that this definition does indeed give the homeomorphism (2.9). It is furthermore natural, which means that, for mappings $\xi : X_1 \to X_2$, $\eta : Y_1 \to Y_2$ and $\zeta : Z_1 \to Z_2$,

$$\tau(X_2, Y_2, Z_1)[\mathrm{Map}(\zeta, \xi) \times \mathrm{Map}(\zeta, \eta)] = \mathrm{Map}(\zeta, \xi \times \eta)\tau(X_1, Y_1, Z_2).$$

The proof of this statement is a straightforward application of the various definitions involved. Both sides are mappings carrying $\mathrm{Map}(Z_2, X_1) \times \mathrm{Map}(Z_2, Y_1)$ into $\mathrm{Map}(Z_1, X_2 \times Y_2)$. They will be equal if they give identical results when applied to an arbitrary element (f_2, g_2) in the first space. These applications give, by the definition of the mapping functions of the functors "Map" and "$\times$", the respective elements

$$\tau(X_2, Y_2, Z_1)(\xi f_2 \zeta, \eta g_2 \zeta), \qquad (\xi \times \eta)\tau(X_1, Y_1, Z_2)(f_2, g_2)\zeta.$$

Both are in $\mathrm{Map}(Z_1, X_2 \times Y_2)$. Applied to an arbitrary $z \in Z_1$, we obtain in both cases, by the definition of τ, the same element $(\xi f_2 \zeta(z), \eta g_2 \zeta(z)) \in X_2 \times Y_2$.

For groups and Banach spaces there are analogous natural equivalences

$$\mathrm{Hom}(G, H) \times \mathrm{Hom}(G, K) \cong \mathrm{Hom}(G, H \times K), \tag{2.10}$$

$$\mathrm{Lin}(B, C) \times \mathrm{Lin}(B, D) \cong \mathrm{Lin}(B, C \times D). \tag{2.11}$$

In each case the equivalence is given by a transformation defined exactly as before. In the formula for Banach spaces we assume that the direct product is normed by the maximum formula. In the case of any other formula for the norm in a direct product, we can assert only that τ is a one-to-one linear transformation of norm one, but not necessarily a transformation preserving the norm. In such a case τ then gives merely a natural transformation of the functor on the left to the functor on the right.

For groups there is another type of distributive law, which is an equivalence transformation,

$$\mathrm{Hom}(G, K) \times \mathrm{Hom}(H, K) \cong \mathrm{Hom}(G \times H, K),$$

The transformation $\tau(G, H, K)$ is defined for each pair $(\phi, \psi) \in \mathrm{Hom}(G, K) \times \mathrm{Hom}(H, K)$ by setting

$$[\tau(G, H, K)(\phi, \psi)](g, h) = \phi g + \psi h$$

for every element (g, h) in the direct product $G \times H$. The properties of τ are proved as before.

It is well known that a function $g(x, y)$ of two variables x and y may be regarded as a function τg of the first variable x for which the values are in turn functions of the second variable y. In other words, τg is defined by

$$[[\tau g](x)](y) = g(x, y).$$

It may be shown that the correspondence $g \to \tau g$ does establish a homeomorphism between the spaces

$$Z^{X \times Y} \cong \left(Z^Y\right)^X,$$

where Z is any topological space and X and Y are locally compact Hausdorff spaces. This is a "natural" homeomorphism, because the correspondence $\tau = \tau(X, Y, Z)$ defined above is actually a natural equivalence

$$\tau(X, Y, Z) : \mathrm{Map}(X \times Y, Z) \rightleftarrows \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

between the two composite functors whose object functions are displayed here.

To prove that τ is natural, we consider any mappings $\xi : X_1 \to X_2$, $\eta : Y_1 \to Y_2$, $\zeta : Z_1 \to Z_2$, and show that

$$\tau(X_1, Y_1, Z_2)\,\mathrm{Map}(\xi \times \eta, \zeta) = \mathrm{Map}(\xi, \mathrm{Map}(\eta, \zeta))\tau(X_2, Y_2, Z_1). \qquad (2.12)$$

Each side of this equation is a mapping which applies to any element $g_2 \in \mathrm{Map}(X_2 \times Y_2, Z_1)$ to give an element of $\mathrm{Map}(X_1, \mathrm{Map}(Y_1, Z_2))$. The resulting elements may be applied to an $x_1 \in X_1$ to give an element of $\mathrm{Map}(Y_1, Z_2)$, which in turn may be applied to any $y_1 \in Y_1$. If each side of Eq. (2.12) is applied in this fashion, and simplified by the definitions of τ and of the mapping functions of the functors involved, one obtains in both cases the same element $\zeta g_2(\xi x_1, \eta y_1) \in Z_2$. Hence Eq. (2.12) holds, and τ is natural.

Incidentally, the analogous formula for groups uses the tensor product $G \otimes H$ of two groups, and gives an equivalence transformation

$$\mathrm{Hom}(G \otimes H, k) \cong \mathrm{Hom}(G, \mathrm{Hom}(H, K)).$$

The proof appears in our Proceedings note[1] quoted in the introduction.

Let D be a fixed Banach space, while B and C are two (variable) Banach spaces. To each pair of linear transformations λ and μ, with $\|\lambda\| \leq 1$ and $\|\mu\| \leq 1$, and with

$$B \xrightarrow{\lambda} C \xrightarrow{\mu} D,$$

there is associated a composite linear transformation $\mu\lambda$, with $\mu\lambda : B \to D$. Thus there is a correspondence $\tau = \tau(B, C)$ which associates to each $\lambda \in \mathrm{Lin}(B, C)$ a linear transformation $\tau\lambda$ with

$$[\tau\lambda](\mu) = \mu\lambda \in \mathrm{Lin}(B, D).$$

[1]Eilenberg and MacLane, "Natural isomorphisms in group theory".

Each $\tau\lambda$ is a linear transformation of $\mathrm{Lin}(C, D)$ into $\mathrm{Lin}(B, D)$ with norm at most one; consequently τ establishes a correspondence

$$\tau(B, C) : \mathrm{Lin}(B, C) \to \mathrm{Lin}(\mathrm{Lin}(C, D), \mathrm{Lin}(B, D)). \tag{2.13}$$

It can be readily shown that τ itself is a linear transformation, and that $\|\tau(\lambda)\| = \|\lambda\|$, so that τ is an isometric mapping. This mapping τ actually gives a transformation between the functors in Eq. (2.12). If the space D is kept fixed[2], the functions $\mathrm{Lin}(B, C)$ and $\mathrm{Lin}(\mathrm{Lin}(C, D), \mathrm{Lin}(B, D))$ are object functions of functors contravariant in B and covariant in C, with values in the category $\mathbf{B}$ of Banach spaces. Each $\tau = \tau(B, C)$ is a mapping of this category; thus τ is a natural transformation of the first functor in the second provided that, whenever $\beta : B_1 \to B_2$ and $\gamma : C_1 \to C_2$,

$$\tau(B_1, C_2)\,\mathrm{Lin}(\beta, \gamma) = \mathrm{Lin}(\mathrm{Lin}(\gamma, e), \mathrm{Lin}(\beta, e))\tau(B_2, C_1), \tag{2.14}$$

where $e = e_D$ is the identity mapping of D into itself. Each side of Eq. (2.14) is a mapping of $\mathrm{Lin}(B_2, C_1)$ into $\mathrm{Lin}(\mathrm{Lin}(C_2, D), \mathrm{Lin}(B_1, D))$. Apply each side to any $\lambda \in \mathrm{Lin}(B_2, C_1)$, and let the result act on any $\mu \in \mathrm{Lin}(C_2, D)$. On the left side, the result of these applications simplifies as follows (in each step the definition used is cited at the right):

$$\begin{aligned}
&\{[\tau(B_1, C_2)]\,\mathrm{Lin}(\beta, \gamma)\lambda\}\mu && \\
&\quad = \{[\tau(B_1, C_2)](\gamma\lambda\beta)\}\mu && \text{(Definition of } \mathrm{Lin}(\beta, \gamma)) \\
&\quad = \mu\gamma\lambda\beta && \text{(Definition of } \tau(B_1, C_2)).
\end{aligned}$$

The right side similarly becomes

$$\begin{aligned}
&\{\,\mathrm{Lin}(\mathrm{Lin}(\gamma, e), \mathrm{Lin}(\beta, e))[\tau(B_2, C_1)\lambda]\}\tau && \\
&\quad = \{\mathrm{Lin}(\beta, e)[\tau(B_2, C_1)\lambda]\,\mathrm{Lin}(\gamma, e)\}\tau && \text{(Definition of } \mathrm{Lin}(-, -)) \\
&\quad = \mathrm{Lin}(\beta, e)\{[\tau(B_2, C_1)\lambda](\mu\gamma)\} && \text{(Definition of } \mathrm{Lin}(\gamma, e)) \\
&\quad = \mathrm{Lin}(\beta, e)(\mu\gamma\lambda) && \text{(Definition of } \tau(B_2, C_1)) \\
&\quad = \mu\gamma\lambda\beta && \text{(Definition of } \mathrm{Lin}(\beta, e)).
\end{aligned}$$

The identity of these two results shows that τ is indeed a natural transformation of functors.

In the special case when D is the space of real numbers, $\mathrm{Lin}(C, D)$ is simply the conjugate space $\mathrm{Conj}(C)$. Thus we have the natural transformation

$$\tau(B, C) : \mathrm{Lin}(B, C) \to \mathrm{Lin}(\mathrm{Conj}(C), \mathrm{Conj}(B)). \tag{2.15}$$

A similar argument for locally compact abelian groups G and H yields a natural transformation

$$\tau(G, H) : \mathrm{Hom}(G, M) \to \mathrm{Hom}(\mathrm{Ch}(H), \mathrm{Ch}(G)). \tag{2.16}$$

[2]We keep the space D fixed because in one of these functors it appears twice, once as a covariant argument and once as a contravariant one.

In the theory of character groups it is shown that each $\tau(G, H)$ is an isomorphism, so Eq. (2.16) is actually a natural isomorphism. The well known isomorphism between a locally compact abelian group G and its twice iterated character group is also a natural isomorphism

$$\tau(G) : G \rightleftarrows \mathrm{Ch}(\mathrm{Ch}(G))$$

between functors.[3] The analogous natural transformation

$$\tau(B) : B \to \mathrm{Conj}(\mathrm{Conj}(B))$$

for Banach spaces is an equivalence only when B is restricted to the category of reflexive Banach spaces.

2.5 Groups as categories

Any group G may be regarded as a category $\mathbf{G}_G$ in which there is only one object. This object may either be the set G or, if G is a transformation group, the space on which G acts. The mappings of the category are to be the elements γ of the group G, and the product of two elements in the group is to be their product as mappings in the category. In this category every mapping is an equivalence, and there is only one identity mapping (the unit element of G). A covariant functor T with one argument in $\mathbf{G}_G$ and with values in (the category of) the group H is just a homomorphic mapping $\eta = T(\gamma)$ of G into H. A natural transformation τ of one such functor T_1 into another one, T_2, is defined by a single element $\tau(G) = \eta_0 \in H$. Since η_0 has an inverse, every natural transformation is automatically an equivalence. The naturality condition (2.3) for τ becomes simply $\eta_0 T_1(\gamma)\eta_0^{-1} = T_2(\gamma)$. Thus the functors T_1 and T_2 are naturally equivalent if and only if T_1 and T_2, considered as homomorphisms, are conjugate.

Similarly, a contravariant functor T on a group G, considered as a category, is simply a “dual” or “counter” homomorphism $(T(\gamma_2\gamma_1) = T(\gamma_1)T(\gamma_2))$.

A ring R with unity also gives a category, in which the mappings are the elements of R, under the operation of multiplication in R. The unity of the ring is the only identity of the category, and the units of the ring are the equivalences of the category.

2.6 Construction of functors as transformations

Under suitable conditions a mapping-function $\tau(A, B)$ acting on a given functor $T(A, B)$ can be used to construct a new functor S such that $\tau : T \to S$. The case in which each τ is an equivalence mapping is the simplest, so will be stated first.

[3] The proof of naturality appears in the note quoted in Footnote 3.

Theorem 6. *Let T be a functor covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$, with values in $\mathbf{C}$. Let S and τ be functions which determine for each pair of objects $A \in \mathbf{A}, B \in \mathbf{B}$ an object $S(A, B) \in \mathbf{C}$ and an equivalence mapping*

$$\tau(A, B) : T(A, B) \to S(A, B) \text{ in } \mathbf{C}.$$

Then S is the object function of a uniquely determined functor S, concordant with T and such that τ is a natural equivalence $\tau : T \rightleftarrows S$.

Proof. One may readily show that the mapping function appropriate to S is uniquely determined for each $\alpha : A_1 \to A_2$ in $\mathbf{A}$ and $\beta : B_1 \to B_2$ in $\mathbf{B}$ by the formula

$$S(\alpha, \beta) = \tau(A_2, B_1)T(\alpha, \beta)[\tau(A_1, B_2)]^{-1}.$$

□

The companion theorem for the case of a transformation which is not necessarily an equivalence is somewhat more complicated. We first define mappings cancellable from the right. A mapping $\alpha \in \mathbf{A}$ will be called cancellable from the right if $\beta\alpha = \gamma\alpha$ always implies $\beta = \gamma$. To illustrate, if each "formal" mapping is an actual many-to-one mapping of one set into another, and if the composition of formal mappings is the usual composition of correspondences, it can be shown that every mapping α of one set *onto* another is cancellable from the right.

Theorem 7. *Let T be a functor covariant in $\mathbf{A}$ and contravariant in $\mathbf{B}$, with values in $\mathbf{C}$. Let $S(A, B)$ and $S(\alpha, \beta)$ be two functions on the objects (and mappings) of $\mathbf{A}$ and $\mathbf{B}$, for which it is assumed only, when $\alpha : A_1 \to A_2 \in \mathbf{A}$ and $\beta : B_1 \to B_2 \in \mathbf{B}$, that*

$$S(\alpha, \beta) : S(A_1, B_2) \to S(A_2, B_1) \in \mathbf{C}.$$

If a function τ on the objects of $\mathbf{A}, \mathbf{B}$ to the mappings of $\mathbf{C}$ satisfies the usual conditions for a natural transformation $\tau : T \to S$; namely that

$$\tau(A, B) : T(A, B) \to S(A, B) \in \mathbf{C}, \tag{2.17}$$

$$\tau(A_2, B_1)T(\alpha, \beta) = S(\alpha, \beta)\tau(A_1, B_2), \tag{2.18}$$

and if in addition each $\tau(A, B)$ is cancellable from the right, then the functions $S(\alpha, \beta)$ and $S(A, B)$ form a functor S, concordant with T, and τ is a transformation $\tau : T \to S$.

Proof. We need to show that

$$S(e_A, e_B) = e_{S(A,B)}, \tag{2.19}$$

$$S(\alpha_2\alpha_1, \beta_2\beta_1) = S(\alpha_2, \beta_1)S(\alpha_1, \beta_2). \tag{2.20}$$

Since T is a functor, $T(e_A, e_B)$ is an identity, so that the condition (2.18) with $A_1 = A_2$, $B_1 = B_2$ becomes

$$\tau(A, B) = S(e_A, e_B)\tau(A, B).$$

Because $\tau(A, B)$ is cancellable from the right, it follows that $S(e_A, e_B)$ must be the identity mapping of $S(A, B)$, as desired.

To consider the second condition, let $\alpha : A_1 \to A_2$, $\alpha_2 : A_2 \to A_3$, $\beta_1 : B_1 \to B_2$ and $\beta_2 : B_2 \to B_3$, so that $\alpha_2\alpha_1$ and $\beta_2\beta_1$ are defined. By condition (2.18) and the properties of the functor T

$$\begin{aligned} S(\alpha_2\alpha_1, \beta_2\beta_1)\tau(A_1, B_3) &= \tau(A_3, B_1)T(\alpha_2\alpha_1, \beta_2\beta_1) \\ &= \tau(A_3, B_1)T(\alpha_2, \beta_1)T(\alpha_1, \beta_2) \\ &= S(\alpha_2, \beta_1)\tau(A_2, B_2)T(\alpha_1, \beta_2) \\ &= S(\alpha_2, \beta_1)S(\alpha_1, \beta_2)\tau(A_1, B_3) \end{aligned}$$

Again because $\tau(A_1, B_3)$ may be cancelled on the right, Eq. (2.20) follows. □

2.7 Combination of the arguments of functors

For n given categories $\mathbf{A}_1, \ldots, \mathbf{A}_n$, the cartesian product category

$$\mathbf{A} = \prod_i \mathbf{A}_i = \mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n \tag{2.21}$$

is defined as a category in which the objects are the n-tuples of objects $[A_1, \ldots, A_n]$, with $A_i \in \mathbf{A}_i$, the mappings are the n-tuples $[\alpha_1, \ldots, \alpha_n]$ of mappings $\alpha_i \in \mathbf{A}_i$. The product

$$[\alpha_1, \ldots, \alpha_n][\beta_1, \ldots, \beta_n] = [\alpha_1\beta_1, \ldots, \alpha_n\beta_n]$$

is defined if and only if each individual product $\alpha_i\beta_i$ is defined in $\mathbf{A}_i$, for $i \in \{1, \ldots, n\}$. The identity corresponding to the object $[A_1, \ldots, A_n]$ in the product category is to be the mapping $[e(A_1), \ldots, e(A_n)]$. The axioms which assert that the product $\mathbf{A}$ is a category follow at once. The natural correspondence

$$P(A_1, \ldots, A_n) = [A_1, \ldots, A_n] \tag{2.22}$$

$$P(\alpha_1, \ldots, \alpha_n) = [\alpha_1, \ldots, \alpha_n] \tag{2.23}$$

is a covariant functor on the n categories $\mathbf{A}_1, \ldots, \mathbf{A}_n$ to the product category. Conversely, the correspondences given by "projection" into the ith coordinate,

$$Q_i([A_1, \ldots, A_n]) = A_i, \qquad Q_i([\alpha_1, \ldots, \alpha_n]) = \alpha_i, \tag{2.24}$$

is a covariant functor in one argument, on $\mathbf{A}$ to $\mathbf{A}_i$.

It is now possible to represent a functor covariant in any number of arguments as a functor in one argument. Let T be a functor on the categories $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{B}$, with the same variance in $\mathbf{A}_i$, as in $\mathbf{A}_1$; define a new functor T^* by setting

$$T^*([A_1, \dots, A_n], B) = T(A_1, \dots, A_n, B),$$
$$T^*([\alpha_1, \dots, \alpha_n], \beta) = T(\alpha_1, \dots, \alpha_n, \beta).$$

This is a functor, since it is a composite of T and the projections Q_i of Eq. (2.24); its variance in the first argument is that of T in any A_i. Conversely, each functor S with arguments in $\mathbf{A}_i \times \cdots \times \mathbf{A}_n$ and $\mathbf{B}$ can be represented as $S = T^*$, for a T with $n+1$ arguments in $\mathbf{A}_i, \dots, \mathbf{A}_n, \mathbf{B}$, defined by

$$T(A_1, \dots, A_n, B) = S([A_1, \dots, A_n], B) = S(P(A_1, \dots, A_n), B),$$
$$T(\alpha_1, \dots, \alpha_n, \beta) = S([\alpha_1, \dots, \alpha_n], \beta) = S(P(\alpha_1, \dots, \alpha_n), \beta).$$

Again T is a composite functor. These reduction arguments combine to give the following theorem.

Theorem 8. *For given categories $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{B}_1, \dots, \mathbf{B}_m, \mathbf{C}$, there is a one-to-one correspondence between the functors T covariant in $\mathbf{A}_1, \dots, \mathbf{A}_n$, contravariant in $\mathbf{B}_1, \dots, \mathbf{B}_m$, with values in $\mathbf{C}$, and the functors S in two arguments, covariant in $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ and $\mathbf{B}_1 \times \cdots \times \mathbf{B}_m$, with values in the same category $\mathbf{C}$. Under this correspondence, equivalent functors T correspond to equivalent functors S, and a natural transformation $\tau : T_1 \to T_2$ gives rise to a natural transformation $\sigma : S_1 \to S_2$ between the functors S_1 and S_2 corresponding to T_1 and T_2 respectively.*

By this theorem, all functors can be reduced to functors in two arguments. To carry this reduction further, we introduce the concept of "dual" category.

Given a category $\mathbf{A}$, the dual category $\mathbf{A}^{\mathrm{op}}$ is defined as follows. The objects of $\mathbf{A}^{\mathrm{op}}$ are those of $\mathbf{A}$; the mappings α^{op} of $\mathbf{A}^{\mathrm{op}}$ are in one-to-one correspondence $\alpha \rightleftarrows \alpha^{\mathrm{op}}$ with the mappings of $\mathbf{A}$. If $\alpha : A_1 \to A_2$ in $\mathbf{A}$, then $\alpha^{\mathrm{op}} : A_2 \to A_1$ in $\mathbf{A}^{\mathrm{op}}$. The composition law is defined by the equation

$$\alpha_2^{\mathrm{op}} \alpha_1^{\mathrm{op}} = (\alpha_1 \alpha_2)^{\mathrm{op}},$$

if $\alpha_1 \alpha_2$ is defined in $\mathbf{A}$. We verify that $\mathbf{A}^{\mathrm{op}}$ is a category and that there are equivalences

$$(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}} \cong \mathbf{A}, \qquad \prod_i \mathbf{A}_i^{\mathrm{op}} \cong \left(\prod \mathbf{A}_i\right)^{\mathrm{op}}.$$

The mapping

$$D(A) = A, \qquad D(\alpha) = \alpha^{\mathrm{op}}$$

is a contravariant functor on $\mathbf{A}$ to $\mathbf{A}^{\mathrm{op}}$, while D^{-1} is contravariant on $\mathbf{A}^{\mathrm{op}}$ to $\mathbf{A}$.

Any contravariant functor T on $\mathbf{A}$ to $\mathbf{C}$ can be regarded as a covariant functor T^{op} on $\mathbf{A}^{\mathrm{op}}$ to $\mathbf{C}$, and vice versa. Explicitly, T^{op} is defined as a composite

$$T^{\mathrm{op}}(A) = T(D^{-1}(A)), \qquad T^{\mathrm{op}}(\alpha^{\mathrm{op}}) = T(D^{-1}(\alpha^{\mathrm{op}})).$$

Theorem 9. *Every functor T covariant on $\mathbf{A}_1, \ldots, \mathbf{A}_n$ and contravariant on $\mathbf{B}_1, \ldots, \mathbf{B}_m$ with values in $\mathbf{C}$ may be regarded as a covariant functor T' on*

$$\left(\prod_i \mathbf{A}_i\right) \times \left(\prod_j \mathbf{B}_j^{op}\right)$$

with values in $\mathbf{C}$, and vice versa. Each natural transformation (or equivalence) $\tau : T_1 \to T_2$ yields a corresponding transformation (or equivalence) $\tau' : T_1' \to T_2'$.

Chapter 3

Functors and groups

3.1 Subfunctors

This chapter will develop the fashion in which various particular properties of groups are reflected by properties of functors with values in a category of groups. The simplest such case is the fact that subgroups can give rise to "subfunctors". The concept of subfunctor thus developed applies with equal force to functors whose values are in the category of rings, spaces, and so on.

In the category $\mathbf{G}$ of all topological groups we say that a mapping $\gamma' : G_1' \to G_2'$ is a *submapping* of a mapping $\gamma : G_1 \to G_2$ (notation: $\gamma' \subset \gamma$) whenever $G_1' \subset G_1$, $G_2' \subset G_2$ and $\gamma'(g_1) = \gamma(g_1)$ for each $g_1 \in G_1'$. Here $G_1' \subset G_1$ means of course that G_1' is a subgroup (not just a subset) of G_1.

Given two concordant functors T' and T on $\mathbf{A}$ and $\mathbf{B}$ to $\mathbf{C}$, we say that T' is a subfunctor of T (notation: $T' \subset T$) provided $T'(A, B) \subset T(A, B)$ for each pair of objects $A \in \mathbf{A}$, $B \in \mathbf{B}$ and $T'(\alpha, \beta) \subset T(\alpha, \beta)$ for each pair of mapping $\alpha \in \mathbf{A}$, $\beta \in \mathbf{B}$. Clearly $T' \subset T$ and $T \subset T'$ imply $T = T'$; furthermore this inclusion satisfies the transitive law. If T' and T'' are both subfunctors of the same functor T, then in order to prove that $T' \subset T''$ it is sufficient to verify that $T'(A, B) \subset T''(A, B)$ for all A and B.

A subfunctor can be completely determined by giving its object function alone. The requisite properties for this object function may be specified as:

Theorem 10. *Let the functor T covariant in $\mathbf{A}$ and contravariant in $\mathbf{B}$ have values in the category $\mathbf{G}$ of groups, while T' is a function which assigns to each pair of objects $A \in \mathbf{A}$ and $B \in \mathbf{B}$ a subgroup $T'(A, B)$ of $T(A, B)$. Then T' is the object function of a subfunctor of T if and only if for each $\alpha : A_1 \to A_2 \in \mathbf{A}$ and each $\beta : B_1 \to B_2 \in \mathbf{B}$ the mapping $T(\alpha, \beta)$ carries the subgroup $T'(A_1, B_2)$ into part of $T'(A_2, B_1)$. If T' satisfies this condition, the corresponding mapping function is uniquely determined.*

Proof. The necessity of this condition is immediate. Conversely, to prove the sufficiency, we define for each α and β a homomorphism $T'(\alpha, \beta)$ of $T'(A_1, B_2)$

into $T'(A_2, B_1)$ by setting $T'(\alpha, \beta)g = T(\alpha, \beta)g$, for each $g \in T'(A_1, B_2)$. The fact that T' satisfies the requisite conditions for the mapping function of a functor is then immediate, since T' is obtained by "cutting down" T. $\square$

The concept of a subtransformation may also be defined. If T, S, T', S' are concordant functors on $\mathbf{A}, \mathbf{B}$ to $\mathbf{C}$, and if $\tau : T \to S$ and $\tau' : T' \to S'$ are natural transformations, we say that τ' is a *subtransformation* of τ (notation: $\tau' \subset \tau$) if $T' \subset T$, $S' \subset S$ and if, for each pair of arguments $A, B, \tau'(A, B)$ is a submapping of $\tau(A, B)$. Any such subtransformation of τ may be obtained by suitably restricting both the domain and the range of τ. Explicitly, let $\tau : T \to S$, let $T' \subset T$ and $S' \subset S$ be such that for each $A, B, \tau(A, B)$ maps the subgroup $T'(A, B)$ of $T(A, B)$ into the subgroup $S'(A, B)$ of $S(A, B)$. If then $\tau'(A, B)$ is defined as the homomorphism $\tau(A, B)$ with its domain restricted to the subgroup $T'(A, B)$ and its range to the subgroup $S'(A, B)$, it follows readily that τ' is indeed a natural transformation $\tau' : T' \to S'$.

Let τ be a natural transformation $\tau : T \to S$ of concordant functors T and S on $\mathbf{A}$ and $\mathbf{B}$ to the category $\mathbf{G}$ of groups. If T' is a subfunctor of T, then the map of each $T'(A, B)$ under $\tau(A, B)$ is a subgroup of $S(A, B)$, so that we may define an object function

$$S'(A, B) = \tau(A, B)[T'(A, B)], \qquad A \in \mathbf{A}, B \in \mathbf{B}.$$

The naturality condition on τ shows that the function S' satisfies the condition of Theorem 10; hence $S' = \tau T'$ gives a subfunctor of S, called the τ-transformation of T'. Furthermore there is a natural transformation $\tau' : T' \to S'$ obtained by restricting τ. In particular, if τ is a natural equivalence, so is τ'.

Conversely, for a given $\tau : T \to S$ let S'' be a subfunctor of S. The inverse image of each subgroup $S''(A, B)$ under the homomorphism $\tau(A, B)$ is then a subgroup of $T(A, B)$, hence gives an object function

$$T''(A, B) = \tau(A, B)^{-1}[S''(A, B)], \qquad A \in \mathbf{A}, B \in \mathbf{B}.$$

As before this is the object function of a subfunctor $T'' \subset T$ which may be called the inverse transform $\tau^{-1}S'' = T''$ of S''. Again, τ ma be restricted to give a natural transformation $\tau'' : T'' \to S''$. In case each $\tau(A, B)$ is homomorphism of $T(A, B)$ *onto* $S(A, B)$, we may assert that $\tau(\tau^{-1})S'' = S''$.

Lattice operations on subgroups can be applied to functors. If T' and T'' are two subfunctors of a functor T with values in G, we define their meet $T' \wedge T''$ and their join $T' \vee T''$ by giving object functions,

$$[T' \wedge T''](A, B) = T'(A, B) \wedge T''(A, B),$$
$$[T' \vee T''](A, B) = T'(A, B) \vee T''(A, B).$$

We verify that the condition of Theorem 10 is satisfied here, so that these object functions do uniquely determine corresponding subfunctors of T. Any lattice identity for groups may then be written directly as an identity for the subfunctors of a fixed functor T with values in **G**.

3.2 Quotient functors

The operation of forming a quotient group leads to an analogous operation of taking the "quotient functor" of a functor T by a "normal" subfunctor T'. If T is a functor covariant in **A** and contravariant in **B**, with values in **G**, a *normal functor* T' will mean a subfunctor $T' \subset T$ such that each $T'(A, B)$ is a normal subgroup of $T(A, B)$, while a *closed* subfunctor T' will be one in which each $T'(A, B)$ is a closed subgroup of the topological group $T(A, B)$. If T' is a normal subfunctor of T, the quotient functor $Q = T/T'$ has an object function given as the factor group,

$$Q(A, B) = T(A, B)/T'(A, B).$$

For homomorphisms $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ the corresponding mapping function $Q(\alpha, \beta)$ is defined for each coset[1] $x + T'(A_1, B_2)$ as

$$Q(\alpha, \beta)[x + T'(A_1, B_2)] = [T(\alpha, \beta)x] + T'(A_2, B_1).$$

Before we prove that Q is actually a functor, we introduce for each $A \in \mathbf{A}$ and $B \in \mathbf{B}$ the homomorphism

$$\nu(A, B) : T(A, B) \to Q(A, B)$$

defined for each $x \in T(A, B)$ by the formula

$$\nu(A, B)(x) = x + T'(A, B).$$

When $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ we now show that

$$Q(\alpha, \beta)\nu(A_1, B_2) = \nu(A_2, B_1)T(\alpha, \beta).$$

For, given any $x \in T(A_1, B_2)$, the definitions of ν and Q give at once

$$\begin{aligned} Q(\alpha, \beta)[\nu(A_1, B_2)(x)] &= Q(\alpha, \beta)[x + T'(A_1, B_2)] \\ &= [T(\alpha, \beta)(x)] + T'(A_2, B_1) \\ &= \nu(A_2, B_1)[T(\alpha, \beta)(x)]. \end{aligned}$$

Notice also that $\nu(A, B)$ maps $T(A, B)$ *onto* the factor group $Q(A, B)$, hence is cancellable from the right. Therefore, Theorem 10 shows that $Q = T/T'$ is a functor, and that ν is a natural transformation of T onto T/T'.

[1] For convenience of notation we write the group operations (commutative or not) with a plus sign.

In particular, if the functor T has its values in the category of regular topological groups, while T' is a *closed* normal subfunctor of T, the quotient functor T/T' has its values in the same category of groups, since a quotient of a regular topological group by a *closed* subgroup is again regular.

To consider the behavior of quotient functors under natural transformations we first recall some properties of homomorphisms. Let $\alpha : G \to H$ be a homomorphism of the group G into H, while $\alpha' : G' \to H'$ is a submapping of α, with G' and H' normal subgroups of G and H, respectively, and ν and μ are the natural homomorphisms $\nu : G \to G/G'$, $\mu : H \to H/H'$. Then we may define a homomorphism $\beta : G/G' \to H/H'$ be setting $\beta(x + G') = \alpha x + H'$ for each $x \in G$. This homomorphism is the only mapping of G/G' into H/H' with the property that $\beta\nu = \mu\alpha$, as indicated in the figure

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\nu"] & H \arrow[d, "\mu"] \\ G/G' \arrow[r, "\beta"] & H/H' \end{tikzcd}$$

We may write $\beta = \alpha/\alpha'$. The corresponding statement for functors is as follows:

Theorem 11. *Let $\tau : T \to S$ be a natural transformation between functors with values in $\mathbf{G}$; and let $\tau' : T' \to S'$ be a subtransformation of τ such that T' and S' are normal subfunctors of T and S respectively. Then the definition of $\rho(A, B) = \tau(A, B)/\tau'(A, B)$ gives a natural transformation $\rho = \tau/\tau'$,*

$$\rho : T/T' \to S/S'.$$

Furthermore, $\rho\nu = \mu\tau$, where ν is the natural transformation $\nu : T \to T/T'$ and μ is the natural transformation $\mu : S \to S/S'$.

Proof. This requires only the verification of the naturality condition for ρ, which follows at once from the relevant definitions. □

The "kernel" of a transformation appears as a special case of this theorem. Let $\tau : T \to S$ be given, and take S' be the identity-element subfunctor of S; that is, let each $S'(A, B)$ be the subgroup consisting only of the identity (zero) element of $S(A, B)$. Then the inverse transformation $T' = \tau^{-1}S'$ is by §3.1 a (normal) subfunctor of T, and τ may be restricted to give the natural transformation $\tau' : T' \to S'$. We may call T' the kernel functor of the transformation τ. Theorem 11 applied in this case shows that there is then a natural transformation $\rho : T/T' \to S$ such that $\rho = \tau\nu$. Furthermore each $\rho(A, B)$ is a one-to-one mapping of the quotient group $T(A, B)/T'(A, B)$ into $S(A, B)$. If in addition we assume that each $\tau(A, B)$ is an open mapping of $T(A, B)$ *onto* $S(A, B)$, we may conclude, exactly as in group theory, that ρ is a natural equivalence.

3.3 Examples of subfunctors

Many characteristic subgroups of a group may be written as subfunctors of the identity functor. The (covariant) identity functor I on **G** to **G** is defined by setting

$$I(G) = G, \qquad I(\gamma) = \gamma.$$

Any subfunctor of I, is by Theorem 10, determined by an object function

$$T(G) \subset G$$

such that whenever γ maps G_1 homomorphically into G_2, then $\gamma[T(G_1)] \subset T(G_2)$. Furthermore, if each $T(G)$ is a normal subgroup of G, we can form the quotient functor I/T.

For example, the commutator subgroup $C(G)$ of the group G determines in this fashion a normal subfunctor of I. The corresponding quotient functor $(I/C)(G)$ is the functor determining for each G the factor commutator group of G (the group G made abelian).

The center $Z(G)$ does no determine in this fashion a subfunctor of I, because a homomorphism of G_1 *into* G_2 ma carry central elements of G_1 into non-central elements of G_2. However, we may choose to restrict the category **G** by using as mappings only homomorphisms of one group *onto* another. For *this* category, Z is a subfunctor of I, and we may form a quotient functor I/Z.

Thus various types of subgroups of G may be classified in terms of the degree of invariance of the "subfunctors" of the identity which they generate. This classification is similar to, but not identical with, the known distinction between normal subgroups, characteristic subgroups, and strictly characteristic subgroups of a single group.[2] The present distinction by functors refers not to the subgroups of an individual group, but to a definition yielding a subgroup for each of the groups in a suitable category. It includes the standard distinction, in the sense that one may consider functors on the category with only one object (a single group G) and with mappings which are the inner automorphisms of G (the subfunctors of I = normal subgroups), the automorphisms of G (subfunctors = characteristic subgroups), or the endomorphisms of G (subfunctors = strictly characteristic subgroups).

Still another example of the degree of invariance is given by the automorphism group $A(G)$ of a group G. This is a functor A defined on the category **G** of groups with the mappings restricted to the isomorphisms $\gamma : G_1 \to G_2$ of one group onto another. The mapping function $A(\gamma)$ for any automorphism σ_1 of G_1 is then defined by setting

$$[A(\gamma)\sigma_1]g_2 = \gamma\sigma_1\gamma^{-1}g_2, \qquad g_2 \in G_2.$$

[2]A subgroup S of G is characteristic if $\sigma_1(S) \subset S$ for every automorphism σ_1 of G, and strictly (or "strongly") characteristic if $\sigma_2(S) \subset S$ for every endomorphism σ_2 of G.

The types of invariance for functors on **G** may thus be indicated by a table, showing how the mappings of the category must be restricted in order to make the indicated set function a functor:

Functor	*Mappings* $\gamma : G_1 \to G_2$
$C(G)$	Homomorphisms into,
$Z(G)$	Homomorphisms onto,
$A(G)$	Isomorphisms onto.

For the subcategory of **G** consisting of all (additive) abelian groups there are similar subfunctors

1. G_0, the set of all elements of finite order in G;
2. G_m, the set of all elements of G of order dividing the integer m;
3. mG, the set of all elements of the form $mg \in G$.

The corresponding quotient functors will have object functions G/G_0 (the "Betti group" of G), G/G_m and G/mG (the group G reduced modulo m).

3.4 The isomorphism theorems

The isomorphism theorems of group theory can be formulated for functors; from this it will follow that these isomorphisms between groups are "natural".

The "first isomorphism theorem" asserts that if G has two normal subgroups G_1 and G_2 with $G_2 \subset G_1$, then G_1/G_2 is a normal subgroup of G/G_2, and there is an isomorphisms $\tau : (G/G_2)/(G_1/G_2) \to G/G_1$. The elements of the first group (in additive notation) are coset of coset, of the form $(x + G_2) + G_1/G_2$, and the isomorphism τ is defined as

$$\tau[(x + G_2) + G_1/G_2] = x + G_1. \tag{3.1}$$

This may be stated in terms of functors as follows.

Theorem 12. *Let T_1 and T_2 be two normal subfunctors of a functor T with values in the category of groups* **G**. *If $T_2 \subset T_1$, then T_1/T_2 is a normal subfunctor of T/T_2 and the functors*

$$T/T_1 \quad \textit{and} \quad (T/T_2)/(T_1/T_2) \tag{3.2}$$

are naturally equivalent.

Proof. We assume that the given functor T depends on the usual typical arguments A and B. Since $(T_1/T_2)(A, B)$ is clearly a normal subgroup of $(T/T_2)(A, B)$, a proof that T_1/T_2 is a normal subfunctor of T/T_2 requires only a proof that each $(T_1/T_2)(\alpha, \beta)$, is a submapping of the corresponding $(T/T_2)(\alpha, \beta)$ for any $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$. To show this, apply

$(T_1/T_2)(\alpha,\beta)$ to a typical coset $x+T_2(A_1,B_2)$. Applying the definitions, one has

$$\begin{aligned}(T_1/T_2)(\alpha,\beta)[x+T_2(A_1,B_2)&=(T_1)(\alpha,\beta)(x)+T_2(A_2,B_1)\\&=(T)(\alpha,\beta)(x)+T_2(A_2,B_1)\\&=(T/T_2)(\alpha,\beta)[x+T_2(A_2,B_1)],\end{aligned}$$

for $T_1(\alpha,\beta)$ was assumed to be a submapping of $T(\alpha,\beta)$. The asserted equivalence (3.2) is established by setting, as in Eq. (3.1),

$$\tau(A,B)\{[x+T_2(A,B)]+(T_1/T_2)(A,B)\}=x+T_1(A,B).$$

The naturality proof then requires that, for any mappings $\alpha:A_1\to A_2$ and $\beta:B_1\to B_2$

$$\tau(A_2,B_1)S(\alpha,\beta)=(T/T_1)(\alpha,\beta)\tau(A_1,B_2),$$

where $S=(T/T_2)/(T_1/T_2)$. This equality may be verified mechanically by applying each side to a general element $[x+T_2(A_1,B_2)]+(T_1/T_2)(A_1,B_2)$ in the group $S(A,B)$. □

The theorem may also be stated and proved in the following equivalent form:

Theorem 13. *Let T' and T'' be two normal subfunctors of a functor T with values in the category* **G** *of groups. Then $T'\wedge T''$ is a normal subfunctor of T' and of T, $T'/T'\wedge T''$ is a normal subfunctor of $T/T'\wedge T''$, and the functors*

$$T/T' \qquad \text{and} \qquad (T/T'\wedge T'')/(T'/T'\wedge T'') \tag{3.3}$$

are naturally equivalent.

Proof. Set $T_1=T'$, $T_2=T'\wedge T''$. □

The second isomorphism theorem for groups is fundamental in the proof of the Jordan-Hölder Theorem. It states that if G has normal subgroups G_1 and G_2, then $G_1\wedge G_2$ is a normal subgroup of G_1, G_2 is a normal subgroup of $G_1\vee G_2$, and there is an isomorphism μ of $G_1/(G_1\wedge G_2)$ to $(G_1\vee G_2)/G_2$. (Because G_1 and G_2 are normal subgroups, the join $G_1\vee G_2$ consists of all "sums" g_1+g_2, for $g_i\in G_i$, so is often written as $G_1\vee G_2=G_1+G_2$.) For any $x\in G_1$, this isomorphism is defined as

$$\mu[x+(G_1\wedge G_2)]=x+G_2. \tag{3.4}$$

The corresponding theorem for functors reads:

Theorem 14. *If T_1, T_2 are normal subfunctors of a functor T with values in* $\mathbf{G}$*, then $T_1 \wedge T_2$ is a normal subfunctor of T_1, and T_2 is a normal subfunctor of $T_1 \vee T_2$, and the quotient functors*

$$T_1/(T_1 \wedge T_2) \qquad \text{and} \qquad (T_1 \vee T_2)/T_2 \tag{3.5}$$

are naturally equivalent.

Proof. It is clear that both quotients in Eq. (3.5) are functors. The requisite equivalence $\mu(A, B)$ is given, as in Eq. (3.4), by the definition

$$\mu(A, B)[x + (T_1(A, B) \wedge T_2(A, B))] = x + T_2(A, B),$$

for any $x \in T_1(A, B)$. The naturality may be verified as before. □

From these theorems we may deduce that the first and second isomorphism theorems yield natural isomorphisms between groups in another and more specific way. To this end we introduce an appropriate category $\mathbf{G}^*$. An object of $\mathbf{G}^*$ is to be a triple $G^* = [G, G', G'']$ consisting of a group G and two of its normal subgroups. A mapping $\gamma : [G_1, G_1', G_1''] \to [G_2, G_2', G_2''] \in \mathbf{G}^*$ is to be a homomorphism $\gamma : G_1 \to G_2$ with the special properties that $\gamma(G_1') \triangleleft G_2'$ and $\gamma(G_1'') \triangleleft G_2''$. It is clear that these definitions do yield a category $\mathbf{G}^*$. on this category $\mathbf{G}^*$ we may defined three (covariant) functors with values in the category $\mathbf{G}$ of groups. The first is a "projection" functor

$$P([G, G', G'']) = G \qquad P(\gamma) = \gamma;$$

the others are two normal subfunctors of P, which may be specified by their object functions as

$$P'([G, G', G'']) = G', \qquad P''([G, G', G'']) = G''.$$

Consider now the first isomorphism theorem, in the second form,

$$G/G' \cong (G/(G' \wedge G''))/(G'/(G' \wedge G'')). \tag{3.6}$$

If we set $G^* = [G, G', G'']$, the left side here is a value of the object function of the functor, P/P', and the right side is similarly a value of $(P/(P' \wedge P''))/(P'/(P' \wedge P''))$. Theorem 12 asserts that these two functors are indeed naturally equivalent. Therefore, the isomorphism (3.6) is itself natural, in that it can be regarded as natural isomorphism between the object functions of suitable functors on the category $\mathbf{G}^*$.

The second isomorphism theorem

$$(G' \vee G'')/G'' \cong G'/(G' \wedge G'')$$

is natural in a similar sense, for both sides can be regarded as object functions of suitable (covariant) functors on $\mathbf{G}^*$.

It is clear that this technique of constructing a suitable category $\mathbf{G}^*$ could be used to establish the naturality of even more complicated "isomorphism" theorems.

3.5 Direct product of functors

We recall that there are essentially two different ways of defining the direct product of two groups G and H. The "external" direct product $G \times H$ is the group of all pairs (g, h) with $g \in G, h \in H$, with the usual multiplication. This product $G \times H$ contains a subgroup G', of all pairs $(g, 0)$, which is isomorphic to G, and a subgroup H' isomorphic to H. Alternatively, a group L with subgroups G and H is said to be the "internal" direct product $L = G \boxtimes H$ of its subgroup G and H if $gh = hg$ for every $g \in G, h \in H$ and if every element in L can be written uniquely as a product gh with $g \in G, h \in H$. The intimate connection between the two types of direct products is provided by the isomorphisms $G \times H \cong G \boxtimes H$ and by the equality $G \times H = G' \boxtimes H'$, where $G' \cong G, H' \cong H$.

As in §1.4, the *external* direct product can be regarded as a covariant functor on $\mathbf{G}$, and $\mathbf{G}$ to $\mathbf{G}$, with object function $G \times H$, and mapping function $\gamma \times \eta$, defined as in §1.4.

Direct products of functors may also be defined, with the same distinction between "external" and "internal" products. We consider throughout functors covariant in a category $\mathbf{A}$, contravariant in $\mathbf{B}$, with values in the category $\mathbf{G}_0$ of discrete groups. If T_1 and T_2 are two such functors, the external direct product is a functor $T_1 \times T_2$ for which the object and mapping functions are respectively

$$(T_1 \times T_2)(A, B) = T_1(A, B) \times T_2(A, B), \tag{3.7}$$

$$(T_1 \times T_2)(\alpha, \beta) = T_1(\alpha, \beta) \times T_2(\alpha, \beta), \tag{3.8}$$

If $T_1'(A, B)$ denotes the set of all pairs $(g, 0)$ in the direct product $T_1(A, B) \times T_2(A, B)$, T_1' is a subfunctor of $T_1 \times T_2$, and the correspondence $g \to (g, 0)$ provides a natural isomorphism of T_1 to T_1'. Similarly T_2 is naturally isomorphic to a subfunctor T_2' of $T_1 \times T_2$.

On the other hand, let S be a functor on $\mathbf{A}, \mathbf{B}$ to $\mathbf{G}_0$ with subfunctors S_1 and S_2. We call S the *internal* direct product $S_1 \boxtimes S_2$ if, for each $A \in \mathbf{A}$ and $B \in \mathbf{B}$, $S(A, B)$ is the internal direct product $S_1(A, B) \boxtimes S_2(A, B)$. From this definition it follows that, whenever $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ are given mappings and $g_i \in S_i(A_1, B_2)$ are given elements (for $i \in \{1, 2\}$), then, since $S_i(\alpha, \beta) \subset S(\alpha, \beta)$

$$S(\alpha, \beta) g_1 g_2 = [S_1(\alpha, \beta) g_1][S_2(\alpha, \beta) g_2].$$

This means that the correspondence τ defined by setting $[\tau(A_1, B_2)](g_1 g_2) = g_2$ is a natural transformation $\tau : S \to S_2$. Furthermore this transformation is idempotent, for $\tau(A_1, B_2)\tau(A_1, B_2) = \tau(A_1, B_2)$.

The connection between the two definitions is immediate; there is a natural isomorphism of the internal direct product $S_1 \boxtimes S_2$ to the external direct product $S_1 \times S_2$; furthermore any external product $T_1 \times T_2$ is the internal product $T_1' \boxtimes T_2'$ of its subfunctors $T_1' \cong T_1, T_2' \cong T_2$.

There are in group theory various theorems giving direct product decompositions. These decompositions can now be classified as to "naturality". Consider for example the theorem that every finite abelian group G can be represented as the (internal) direct product of its Sylow subgroups. This decomposition is "natural"; specifically, we may regard the Sylow subgroup $S_p(G)$ (the subgroup consisting of all elements in G of order some power of the prime p) as the object function of a subfunctor S_p of the identity. The theorem in question then asserts in effect that the identity functor I is the internal direct product of (a finite number of) the functors S_p. This representation of the direct factors by functors is the underlying reason for the possibility of extending the decomposition theorem in question to infinite groups in which every element has finite order.

On the other hand consider the theorem which asserts that every finite abelian group is the direct product of cyclic subgroups. It is clear here that the subgroups cannot be given as the values of functors, and we observe that in this case the theorem does not extend to infinite abelian groups.

As another example of non-naturality, consider the theorem which asserts that any abelian group G with a finite number of generators can be represented as a direct product of a free abelian group by the subgroup $T(G)$ of all elements of finite order in G. Let us consider the category $\mathbf{G}_{\mathrm{af}}$ of all discrete abelian groups with a finite number of generators. In this category the "torsion" subgroup $T(G)$ does determine the object function of a subfunctor $T \subset I$. However, there is no such functor giving the complementary direct factor of G.

Theorem 15. *In the category $\mathbf{G}_{af}$ there is no subfunctor $F \subset I$ such that $I = F \boxtimes T$, that is such that, for all G*

$$G = F(G) \boxtimes T(G). \tag{3.9}$$

Proof. It suffices to consider just one group, such as the group G which is the (external) direct product of the additive group if integers and the additive group of integers mod m, for $m \neq 0$. Then no matter which free subgroup $F(G)$ may be chosen so that Eq. (3.9) holds for this G, there clearly is an isomorphism of G to G which does not carry F into itself. Hence F cannot be a functor. □

This result could also be formulated in the statement that, for any G with $G \neq T(G) \neq \{0\}$, there is no decomposition (3.9) with $F(G)$ a (strongly) characteristic subgroup of G. In order to have a situation which cannot be reformulated in this way, consider the closely related (and weaker) group theoretic theorem which asserts that for each $G \in \mathbf{G}_{\mathrm{af}}$ there is an isomorphism of $G/T(G)$ into G. This isomorphism cannot be natural.

Theorem 16. *For the category $\mathbf{G}_{af}$ there is no natural transformation, $\tau : I/T \to I$, which gives for each G an isomorphism $\tau(G)$ of $G/T(G)$ into a subgroup of G.*

Proof. This proof will require consideration of an infinite class of groups, such as the groups $G_m = \mathbb{Z} \times \mathbb{Z}_m$ where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Z}_m$ the additive group of integers, modulo m. Suppose that $\tau(G) : G/T(G) \to G$ existed. If $\mu(G) : G \to G/T(G)$ is the natural transformation of G into $G/T(G)$ the product $\sigma(G) = \tau(G)\mu(G)$ would be a natural transformation of G into G with kernel $T(G)$. For each of the groups G_m with elements (a, b_m) for $a \in \mathbb{Z}$ and $b_m \in \mathbb{Z}_m$, this transformation $\sigma_m = \sigma(G_m)$ must be a homomorphism with kernel $\mathbb{Z}_m$, hence must have the form

$$\sigma_m(a, b_m) = (r_m a, (s_m a)_m),$$

where r_m and s_m are integers. Now consider the homomorphism $\gamma : G_m \to G_m$ defined by setting $\gamma(a, b_m) = (0, b_m)$. Since σ_m is natural, we must have $\sigma_m\gamma = \gamma\sigma_m$. Applying this equality to an arbitrary element we conclude that $s_m \equiv 0 \pmod{m}$. Next consider $\delta(a, b_m) = (0, a_m)$. the condition $\sigma_m\delta = \delta\sigma_m$ here gives $r_m \equiv 0 \pmod{m}$, so that we can write $r_m = mt_m$. Therefore for each m

$$\sigma_m(a, b_m) = (mt_m a, 0).$$

Now consider two groups G_m, G_n with a homomorphism $\beta : G_m \to G_n$ defined by setting $\beta(a, b_m) = (a, 0_n)$. The naturality condition $\sigma_n\beta = \beta\sigma_m$ now gives $mt_m = nt_n$. If we hold m fixed and allow n to increase indefinitely, this contradicts the fact that mt_m is a finite integer. □

It may be observed that the use of an infinite number of distinct groups is essential to the proof of this theorem. For any subcategory of $\mathbf{G}_{\text{af}}$ containing only a finite number of groups, Theorem 16 would be false, for it would be possible to define a natural transformation $\tau(G)$ bye setting $[\tau(G)]g = kg$ for every g, where the integer k is chosen as any multiple of the order of all the subgroups $T(G)$ for G in the given category.

The examples of "non-natural" direct products adduced here are all examples which mathematicians would usually recognize as not in fact natural. What we have done is merely to show that our definition of naturality does indeed properly apply to cases of intuitively clear non-naturality.

3.6 Characters[3]

The character group of a group may be regarded as a contravariant functor on the category $\mathbf{G}_{\text{lca}}$ of locally compact regular abelian groups, with values

[3]General references: Weil, *L'integration Dans Les Groupes Topologiques Et Ses Applications*, chap.1; Lefschetz, *Algebraic Topology*, chap.2

in the same category. Specifically, this functor "Char" may be defined by "slicing" (see §1.5 the functor Hom of §1.4 as follows. Let P be the (fixed) topological group of real numbers modulo 1, define "Char" by setting

$$\operatorname{Char} G = \operatorname{Hom}(G, P), \qquad \operatorname{Char} \gamma = \operatorname{Hom}(\gamma, e_P). \tag{3.10}$$

Given $g \in G$ and $\chi \in \operatorname{Char} G$ it will be convenient to denote the element $\chi(g)$ of P by (χ, g). Using this terminology and the definition of Hom we obtain for $\gamma : G_1 \to G_2, \chi \in \operatorname{Char} G_2$ and $g \in G_1$

$$(\operatorname{Char}(\gamma)\chi, g) = (\chi, \gamma g). \tag{3.11}$$

As mentioned before (§2.4) the familiar isomorphism $\operatorname{Char}(\operatorname{Char} G) \cong G$ is a natural equivalence.

The functor "Char" can be compounded with other functors. Let T be any functor covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$, with values in $\mathbf{G}_{\mathrm{lca}}$. The composite functor $\operatorname{Char} T$ is then defined on the same categories $\mathbf{A}$ and $\mathbf{B}$ but is contravariant in $\mathbf{A}$ and covariant in $\mathbf{B}$. Let S be any closed subfunctor of T. Then for each pair of objects $A \in \mathbf{A}, B \in \mathbf{B}$, the closed subgroup $S(A,B) \subset T(A,B)$ determines a corresponding subgroup $\operatorname{Annih} S(A,B)$ in $\operatorname{Char} T(A,B)$; this annihilator is defined as the set of all those characters $\chi \in \operatorname{Char} T(A,B)$ with $(\chi, g) = 0$ for each $g \in S(A,B)$. This leads to a closed subfunctor $\operatorname{Annih}(S;T)$ of the functor $\operatorname{Char} T$ determined by the object function

$$[\operatorname{Annih}(S;T)](A,B) = \operatorname{Annih} S(A,B) \text{ in } \operatorname{Char} T(A,B).$$

It is well known that

$$\operatorname{Char}[T(A,B)/S(A,B)] = \operatorname{Annih} S(A,B),$$
$$\operatorname{Char} S(A,B) = \operatorname{Char} T(A,B)/\operatorname{Annih} S(A,B).$$

These isomorphisms in fact yield natural equivalences

$$\sigma : \operatorname{Annih}(S;T) \rightleftarrows \operatorname{Char}(T/S), \tag{3.12}$$
$$\tau : \operatorname{Char} T/\operatorname{Annih}(S;T) \rightleftarrows \operatorname{Char} S. \tag{3.13}$$

For example, to prove Eq. (3.13) one observes that each $\chi \in \operatorname{Char} T(A,B)$ may be restricted to give a character $\tau_0(A,B)\chi$ of $S(A,B)$ by setting

$$(\tau_0(A,B)\chi, h) = (\chi, h), \qquad h \in S(A,B). \tag{3.14}$$

This gives a homomorphism

$$\tau_0(A,B) : \operatorname{Char} T(A,B) \to \operatorname{Char} S(A,B)$$

with kernel Annih $S(A,B)$. This homomorphism τ_0 will yield the required isomorphism τ of Eq. (3.13); by Theorem 11 a proof that τ_0 is natural will imply that τ is natural.

To show that τ_0 is natural, consider any mappings $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ in the argument categories of T. Then $\gamma = T(\alpha, \beta)$ maps $T(A_1, B_2)$ into $T(A_2, B_1)$, while $\delta = S(\alpha, \beta)$ is a submapping of γ. The naturality requirements for τ_0 is

$$(\operatorname{Char}\gamma)\tau_0(A_2, B_1) = \tau_0(A_1, B_2)\operatorname{Char}\gamma. \tag{3.15}$$

Each side is a homomorphism of $\operatorname{Char} T(A_2, B_1)$ into $\operatorname{Char} S(A_1, B_2)$. If the left-hand side is applied to an element $\chi \in \operatorname{Char} T(A_2, B_1)$, and the resulting character of $S(A_1, B_2)$ is then applied to an element h in the latter group, we obtain

$$(\operatorname{Char}\gamma(\tau_0(A_2, B_1)\chi), h) = (\tau(A_2, B_1)\chi, \delta h) = (\chi, \delta h)$$

by using the definition (3.11) of $\operatorname{Char}\delta$ and the definition (3.14) of τ_0. If the right-hand side of Eq. (3.15) be similarly applied to χ and then to h, the result is

$$(\tau_0(A_1, B_2)((\operatorname{Char}\gamma)\chi), h) = ((\operatorname{Char}\gamma)\chi, h) = (\chi, \gamma h).$$

Since $\delta \subset \gamma$, these two results are equal, and both τ_0 and τ are therefore natural.

The proof of naturality for Eq. (3.12) is analogous.

if R is a closed subfunctor of S which is in turn a closed subfunctor of T, both of these natural isomorphisms may be combined to give a single natural isomorphism

$$\rho : \operatorname{Char}(S/R) \rightleftarrows \operatorname{Annih}(S;T)/\operatorname{Annih}(R;T). \tag{3.16}$$

Chapter 4

Partially ordered sets and projective limits

4.1 Quasi-ordered sets

The notions of functors and their natural equivalences apply to partially ordered sets, to lattices, and to related mathematical systems. The category **P** of all quasi-ordered sets[1] has as its objects the quasi-ordered sets P and its mappings $\pi : P_1 \to P_2$ the order preserving transformations of one quasi-ordered sets, P, into another. An equivalence in this category is thus an isomorphism in the sense of order.

An important subcategory of **P** is the category $\mathbf{P}_{\mathrm{d}}$ of all directed sets.[2] One may also consider subcategories which are obtained by restricting both the quasi-ordered sets and their mappings. For example, the category of lattices has as objects all those partially ordered sets which are lattices and as mappings those correspondences which preserve both joins and meets. Alternatively, by using these mappings which preserve only joins, or those which preserve only meets, we obtain two other categories of lattices.

The category **S** of sets may be regarded as a subcategory of **P**, if each set S is considered as a (trivially) quasi-ordered set in which $p_1 \leq p_2$ in S means that $p_1 = p_2$. The category **W** of well-ordered sets in another subcategory of **P**. These categories provide a basis for applying the study of functors to cardinal and ordinal arithmetic. Specifically, the general theory of arithmetic of partially ordered sets, as developed recently by Birkhoff,[3] can be viewed as the construction of a large number of functors (cardinal power, ordinal power,

[1] A *quasi-ordered sets* P is a set of elements $p_1, p_2 \ldots$ with a reflexive and transitive binary relation $p_1 \leq p_2$ between the elements. If, in addition, the antisymmetric law ($p_1 \leq p_2$ and $p_2 \leq p_1 \implies p_1 = p_2$) holds, P is a *partially ordered set.*

[2] A quasi-ordered set P is *directed* if for each pair of elements $p_1, p_2 \in P$ there exist a $p_3 \in P$ with $p_1 \leq p_2 \leq p_3$.

[3] Birkhoff, “Generalized arithmetic”.

and so no) defined on suitable subcategories of $\mathbf{P}$, together with a collection of natural equivalences and transformations between these functors.[4]

The construction of the category $\mathbf{P}$ of all quasi-ordered sets is not the only such interpretation of a partial order. It is also possible to regard the elements of a *single* quasi-ordered set P as the objects of a category; with this device, one can represent an inverse or a direct system of groups (or of spaces) as a functor on P.

If a quasi-ordered set P be regarded as a category $\mathbf{C}_P$, the objects of the category are all the elements $p \in P$ and the mappings are the pairs $\pi = (p_2, p_1)$ of elements $p_i \in P$ such that $p_1 \leq p_2$. To each object p we assign the pair $e_p = (p, p)$ as the corresponding identity mapping, while the product $(p_3, p_2')(p_2, p_1)$ of two mappings of $\mathbf{C}_P$ is defined if and only if $p_2' = p_2$ and is in this case the mapping (p_3, p_1). The axioms 1 to 5 for a category are readily verified, and it develops that the only identities are the pairs (p, p), that the equivalence mappings of $\mathbf{C}_P$ are the pairs (p_2, p_1) with $p_1 \leq p_2$ and $p_2 \leq p_1$ and that any pair (p_2, p_1) with $p_1 \leq p_2$ is a mapping $(p_2, p_1) : p_1 \to p_2$. It further follows that any two mappings $\pi_1 : p_1 \to p_2$ and $\pi_2 : p_1 \to p_2$ of this category which have the same range and the same domain are necessarily equal. Conversely any given category $\mathbf{C}$ which has the property that any two mappings π_1 and π_2 of $\mathbf{C}$ with the same range and the same domain are equal is isomorphic to the category $\mathbf{C}_P$ for a suitable quasi-ordered set P. In fact, P can be defined to be the set of all objects $C \in \mathbf{C}$ with $C_1 \leq C_2$ if and only if there is in $\mathbf{C}$ a mapping $\gamma : C_1 \to C_2$.

Consider now two quasi-ordered sets P and Q, with their corresponding categories $\mathbf{C}_P$ and $\mathbf{C}_Q$. A covariant (contravariant) functor T on $\mathbf{C}_P$ with values in $\mathbf{C}_Q$ is determined uniquely by and order preserving (reversing) mapping $t : P \to Q$. Specifically, each such correspondence t is the object function $t(p) = q$ of a functor T, for which the corresponding mapping function is defined as $T(p_2, p_1) = (tp_2, tp_1)$ (or, in case t is order-reversing, as (tp_1, tp_2)). Each functor T of one variable can be obtained in this way.

4.2 Direct systems as functors

Let D be a directed set. If for every $d \in D$ a discrete group G_d is defined and for every pair $d_1 \leq d_2$ in D a homomorphism

$$\phi_{d_2,d_1} : G_{d_1} \to G_{d_2} \tag{4.1}$$

is given such that $\phi_{d,d}$ is the identity and that

$$\phi_{d_3,d_1} = \phi_{d_3,d_2}\phi_{d_2,d_1} \text{ for } d_1 \leq d_2 \leq d_3 \tag{4.2}$$

[4]Note, however, that the ordinal cardinal sum of two sets A and B does not give rise to a functor, because the definition applies only when the sets A and B are disjoint.

then we say that the groups $\{G_d\}$ and the homomorphisms $\{\phi_{d_2,d_1}\}$ constitute a direct systems of groups indexed by D.

Let us now regard the directed set D as a category. For every object $d \in D$ define

$$T(d) = G_d.$$

For every mapping $\delta = (d_2, d_1)$ in D define

$$T(\delta) = T(d_2, d_1) = \phi_{d_2,d_1}.$$

Equations (4.1) and (4.2) imply that T is a contravariant functor on D with values in the category $\mathbf{G}_0$ of discrete groups. Conversely any such functor give rise to a unique direct system. Consequently the terms “direct system of groups indexed by directed set D” and “covariant functor on D to $\mathbf{G}_0$” may be regarded as synonyms.

With each direct system groups T there is associated a discrete limit group $G = \mathrm{Lim}_{\rightarrow} T$ defined as follows. The elements of the limit group G are pairs (g, d) for $g \in T(d)$; two elements (g_1, d_1) and (g_2, d_2) are considered equal if and only if there is an index d_3 with $d_1 \leq d_3, d_2 \leq d_3$ and with $T(d_3, d_1)g_1 = T(d_3, d_2)g_2$. The sum is defined by setting $(g_1, d) + (g_2, d) = (g_1 + g_2, d)$; since the set D is directed, this provides for the addition of any two pairs in G. For a fixed $d \in D$ one may also consider the homomorphisms, called projections, $\lambda(d) : T(d) \to G$ defined by setting

$$\lambda(d)g = (g, d) \tag{4.3}$$

for $g \in T(d)$. Clearly

$$\lambda(d_1) = \lambda(d_2)T(d_2, d_1) \text{ for } d_1 \leq d_2. \tag{4.4}$$

To treat this limit group, we enlarge the given directed set D by adjoining one new element $\top$, ordered by the specification that $d \leq \top, \forall d \in D$. This enlarge directed set $\overline{D}$ also determines a category containing D as a subcategory, with new mappings $(\top, d)$ for each $d \in D$. Let now T be any covariant functor on D to $\mathbf{G}_0$ (that is, any direct system of groups indexed by D). We define an extension $\overline{T}$ of the object function T by setting

$$\overline{T}(\top) = \mathrm{Lim}_{\rightarrow} T = G, \tag{4.5}$$

the limit group of the given directed system T, and we similarly extend the mapping function of T by letting $\overline{T}$, for a new mapping $(\top, d)$, be the corresponding projection of $T(d)$ into the limit group

$$\overline{T}(\top, d) = \lambda(d). \tag{4.6}$$

Equation (4.5) implies that $\overline{T}$ is indeed a covariant function on $\overline{D}$ with values in $\mathbf{G}_0$. The properties of the limit group may be described in terms of this extended functor $\overline{T}$.

Theorem 17. *Let D be a directed set and T a covariant functor on D (regarded as a category) to $\mathbf{G}_0$. Then the limit group G of the direct system T and the projections of each group $T(d)$ into this limit determine as in Eqs.* (4.5) *and* (4.6) *an extension of T to a covariant functor $\overline{T}$ on $\overline{D}$ to $\mathbf{G}_0$. If $\overline{S}$ is any other extension of T to a covariant functor on $\overline{D}$ to $\mathbf{G}_0$, there is a unique natural transformation $\sigma : \overline{T} \to \overline{S}$ such that each $\sigma(d)$ with $d \neq \top$ is the identity.*

Proof. We have already seen that $\overline{T}$ is a covariant functor on $\overline{D}$ to $\mathbf{G}_0$, extending T. Let now $\overline{S}$ be any other functor extending T. Since $S(d_2, d_1) = T(d_2, d_1)$ for $d_2 \leq d_1$ in D, it follows from the functor condition on $\overline{S}$ that

$$\overline{S}(\top, d_2)T(d_2, d_1) = \overline{S}(\top, d_1). \tag{4.7}$$

We define a homomorphism

$$\sigma(\top) : \overline{T}(\top) \to \overline{S}(\top)$$

by setting $\sigma(\top)(g, d) = \overline{S}(\top, d)g$ for every element $(g, d) \in \overline{T}(\top) = \mathrm{Lim}_{\to} T$. Equation (4.7) implies that $\sigma(\top)$ is single-valued. If we now set $\sigma(d)$ to be the identity mapping $\overline{T}(d) \to \overline{S}(d)$ for $d \neq \top$, we have the desired transformation $\sigma : \overline{T} \to \overline{S}$. □

The extension $\overline{T}$ and hence the limit group $G = \overline{T}(\top)$ of the given direct system is completely determined by the property given in the last sentence of the theorem. In fact if $\hat{T}$ is any other extension of T with the same property as $\overline{T}$, there will exist transformations $\sigma : \overline{T} \to \hat{T}$ and $\tau : \hat{T} \to \overline{T}$. Then $\rho = \tau\sigma : \overline{T} \to \overline{T}$ with $\rho(d)$ the identity whenever $d \neq \top$. It follows that

$$\rho(\top)(g, d) = \rho(\top)\lambda(d)g = \lambda(d)g = (g, d).$$

Hence $\rho(\top)$ is the identity and σ a natural equivalence $\sigma : \overline{T} \to \hat{T}$. In this way a limit group of a direct system of groups can be defined up to an isomorphism by means of such extensions of functors. This indicates that the concept (but not necessarily the existence) of direct "limits" could be set up not only for groups, but also for objects of any category.

Theorem 18. *If T_1 and T_2 are two covariant functors on the directed category D with values in $\mathbf{G}_0$ and τ is a natural transformation $\tau : T_1 \to T_2$, there is only one extension $\overline{\tau}$ of τ which is a natural transformation $\overline{\tau} : \overline{T_1} \to \overline{T_2}$ between the extended functors on $\overline{D}$. When τ is a natural equivalence so is $\overline{\tau}$.*

Proof. The naturality condition for τ, when applied to any mapping (d_2, d_1) with $d_1 \leq d_2$ in the directed set D reads

$$\tau(d_2)T_1(d_2, d_1) = T_2(d_2, d_1)\tau(d_1). \tag{4.8}$$

Given any element (g_1, d) of the limit group $\overline{T_1}(\top) = \mathrm{Lim}_{\to} T_1$ we define

$$\omega(g_1, d) = (\tau(d)g_1, d) \in \mathrm{Lim}_{\to} T_2 = \overline{T_2}(\top). \tag{4.9}$$

Equation (4.8) implies that this definition of ω gives a result independent of the special representation (g_1, d) chosen for the limit element. Hence we get a homomorphism

$$\omega : \overline{T_1} \to \overline{T_2}.$$

In virtue of Eqs. (4.3) and (4.6), the definition (4.9) becomes

$$\omega\overline{T_1}(\top, d) = \overline{(T_2)}(\top, d)\tau(d). \tag{4.10}$$

This means simply that by setting $\overline{\tau}(d) = \tau(d)$, $\overline{\tau}(\top) = \omega$ we get an extension of τ which is still natural and which gives a transformation $\overline{\tau} : \overline{T_1} \to \overline{T_2}$. Since the naturality condition (4.10) is equivalent with Eq. (4.9) which completely determines the value of $\overline{\tau}(\top)$, the requisite uniqueness follows. □

In particular, if τ is an equivalence, each $\tau(d)$ is an isomorphism "onto", hence it follows that $\omega = \overline{\tau}(\top)$ is also an isomorphism onto, and is an equivalence. This is just a restatement of the known theorem that "isomorphic" direct systems determine isomorphic limit groups.

Theorem 19. *If T is a direct system of groups indexed by a directed set D, while H is a fixed discrete group, regarded as a (constant) covariant functor on D to $\mathbf{G}_0$, then for each natural transformation $\tau : T \to H$ there is a unique homomorphism τ_0 of the limit group $\mathrm{Lim}_{\to} T$ into H with the property that $\tau(d) = \tau_0\lambda(d)$ for each $d \in D$, where $\lambda(d)$ is the projection of $T(d)$ into $\mathrm{Lim}_{\to} T$.*

Proof. This follows from the preceding theorem and from the remark that $\overline{H}$ is also a constant functor from $\overline{D}$ to $\mathbf{G}_0$. □

4.3 Inverse limits as functors

Let D be a directed set. If for every $d \in D$ a topological group G_d is defined and for every pair $d_1 \leq d_2$ in D a homomorphism

$$\phi(d_2, d_1) : G_{d_2} \to G_{d_1} \tag{4.11}$$

is given such that $\phi(d, d)$ is the identity and that

$$\phi(d_3, d_3) = \phi(d_2, d_1)\phi(d_3, d_2) \quad \text{for } d_1 \leq d_2 \leq d_3 \tag{4.12}$$

then we say that the groups $\{G_d\}$ and the homomorphisms $\{\phi(d_2, d_1)\}$ constitute an inverse systems of groups indexed by D.

If we now regard D as a category, and define as before

$$T(d) = G_d \tag{4.13}$$

for every object $d \in D$, and

$$T(\delta) = T(d_2, d_1) = \phi(d_2, d_1) \tag{4.14}$$

for every mapping $\delta = (d_2, d_1) \in D$, it is clear that T is a contravariant functor on D with values in the category $\mathbf{G}$ of topological groups. Conversely any such functor may be regarded as an inverse system of groups.

With each inverse system of groups T there is associated a limit group $G = \mathrm{Lim}_{\leftarrow} T$ defined as follows. An element of G is a function $g(d)$ which assigns to each element $d \in D$ an element $g(d) \in T(d)$, in such way that these elements "match" under mappings; that is, such that $T(d_2, d_1)g(d_2) = g(d_1)$ whenever $d_1 \leq d_2$. The sum of $g_1 + g_2$ is defined as $(g_1 + g_2)(d) = g_1(d) + g_2(d)$. This limit group G is assigned a topology, in known fashion, by treating G as a subgroup of the direct product of the groups $T(d)$, with the usual direct product topology. For fixed d, the (continuous) projection $\mu(d)$ of the limit group G into $T(d)$ is defined by setting $[\mu(d)]g = g(d)$, for $g \in G$.

Again we may consider the extended category $\overline{D}$ and define the extension $\overline{T}$ of T by setting

$$\overline{T}(\top) = G \qquad \overline{T}(\top, d) = \mu(d). \tag{4.15}$$

As before the following theorem can be established:

Theorem 20. *Let D be a directed set and T a contravariant functor on D (regarded as a category) to $\mathbf{G}$. Then the limit group G of the inverse system T and the projections of this limit group into each group $T(d)$ determine as in Eq. (4.15) an extension of T to a contravariant functor $\overline{T}$ on $\overline{D}$ to $\mathbf{G}$. If $\overline{S}$ is any other extension of T to a contravariant functor on $\overline{D}$ to $\mathbf{G}$, there is an unique natural transformation $\sigma : \overline{S} \to \overline{T}$ such that each $\sigma(d)$ with $d \neq \top$ is the identity.*

As before we can also verify that the second half of the theorem determines the extended functor $\overline{T}$ to within a natural equivalence, and therefore it determines the limit group to within an isomorphism. The following two theorems may also be proved as in the preceding section.

Theorem 21. *If T_1 and T_2 are two contravariant functors on the directed category D with values in $\mathbf{G}$, and τ is a natural transformation $\tau : T_1 \to T_2$, there is only one extension $\overline{\tau}$ of τ which is a natural transformation $\overline{\tau} : \overline{T_1} \to \overline{T_2}$, between the the extended functors on $\overline{D}$. When τ is a natural equivalence so is $\overline{\tau}$.*

Theorem 22. *If T is an inverse system of groups indexed by the directed set D, while K is a fixed topological group regarded as a (constant) contravariant functor on D to $\mathbf{G}$, then for each natural transformation $\tau : T \to K$ there is a unique homomorphism $\tau_0 : \mathrm{Lim}_{\leftarrow} T \to K$ such that $\tau_0 = \tau(d)\lambda(d), \forall d \in D$.*

The preceding discussion carries over to inverse systems of spaces, by a mere replacement of the category of topological groups **G** by the category of topological spaces **X**.

4.4 The categories "Dir" and "Inv"

The process of forming a direct or inverse limit of a system of groups can be treated as a functor "$\mathrm{Lim}_{\rightarrow}$" or "$\mathrm{Lim}_{\leftarrow}$" which operates on an appropriately defined category. Thus, the functor '$\mathrm{Lim}_{\rightarrow}$" will operate on any direct system T defined on any directed set D. Consequently we define a category "**Dir**" of directed systems whose objects are such pairs (D, T). Here we may regard D itself as a category and T as a covariant functor on D to $\mathbf{G}_0$. To introduce the mappings of this category, observe first that each order preserving transformation R of a directed set D_1 into another such set D_2 will give for each direct system T_2 of groups indexed by D_2 an induced direct system indexed by D_1. Specifically, the induced direct system is just the composite $T_2 \circ R$ of the (covariant) functor R on D_1 to D_2 and the (covariant) functor T_2 on D_2 to $\mathbf{G}_0$. Given two objects (D_1, T_1) and (D_2, T_2) of **Dir**, a mapping

$$(R, \rho) : (D_1, T_1) \to (D_2, T_2)$$

of the category **Dir** is a pair (R, ρ) composed of a covariant functor R on D_1 to D_2 and a natural transformation

$$\rho : T_1 \to T_2 \circ R$$

of T_1 into the composite functor $T_2 \circ R$.

To form the product of two such mappings

$$(R_1, \rho_1) : (D_1, T_1) \to (D_2, T_2), \qquad (R_2, \rho_2) : (D_2, T_2) \to (D_3, T_3) \quad (4.16)$$

observe first that the functors T_2 and $T_3 \circ R_2$ on D_2 to $\mathbf{G}_0$ can be compounded with the functor R_1 on D_1 to D_2, and hence that the given transformation $\rho_2 : T_2 \to T_3 \circ R_2$ can be compounded with the identity transformation of R_1 into itself, just as in §2.3.

The result is a composite transformation

$$\rho_2 \circ R_1 : T_2 \circ R_1 \to T_3 \circ R_2 \circ R_1 \quad (4.17)$$

which assigns to each object $d_1 \in D_1$ the mapping $[\rho_2 \circ R_1](d_1) = \rho_2(R_1 d)$ of $T_2(R_1 d)$ into $T_3 \circ R_2(R_1 d_1)$. The transformation (4.17) and $\rho_1 : T_1 \to T_2 \circ R_1$ yield as in §2.3 a composite transformation $\rho_2 \circ R_1 \circ \rho_1 : T_1 \to (T_3 \circ R_2 \circ R_1)$. We may now define the product of two mappings given by Eq. (4.16) to be

$$(R_2, \rho_2)(R_1, \rho_1) = (R_2 \circ R_1, \rho_2 \circ R_1 \circ \rho_1).$$

With these conventions, we verify that **Dir** is a category. Its identities are the pairs (R, ρ) in which both R and ρ are identities; its equivalences are the pairs (R, ρ) in which R is an isomorphism and ρ a natural equivalence.

The effect of fixing the directed set D in the objects (D, T) of the category **Dir** is to restrict **Dir** to the subcategory which consist of all direct systems of groups indexed by D (that is, the category of all covariant functors on D to $\mathbf{G}_0$, as defined in §2.2).

We shall now define $\mathrm{Lim}_{\to}$ as a covariant functor on **Dir** with values in $\mathbf{G}_0$. For each object (D, T) of **Dir** we define $\mathrm{Lim}_{\to}(D, T)$ to be the group obtained as the direct limit of the direct system of groups T indexed by the directed set D. Given a mapping

$$(R, \rho) : (D_1, T_1) \to (D_2, T_2) \quad \text{in } \mathbf{Dir} \tag{4.18}$$

we define the mapping function of $\mathrm{Lim}_{\to}$,

$$\mathrm{Lim}_{\to}(R, \rho) : \mathrm{Lim}_{\to}(D_1, T_1) \to \mathrm{Lim}_{\to}(D_2, T_2), \tag{4.19}$$

as follows. An element in the limit group $\mathrm{Lim}_{\to}(D_1, T_1)$ is a pair (g_1, d_1) with $d_1 \in D_1, g_1 \in T_1(d_1)$. For each such element define $\phi(g_1, d_1)$ to be the pair $(\rho(d_1)g_1, Rd_1)$. Since $\rho(d_1)$ maps $T_1(d_1)$ into $T_2(Rd_1)$ we have $\rho(d_1)g_1 \in T_2(Rd_1)$, so that the resulting pair is indeed in the limit group $\mathrm{Lim}_{\to}(D_2, T_2)$. The mapping ρ carries equal pairs into equal pairs, and yields the requisite homomorphism (4.19). We verify that $\mathrm{Lim}_{\to}$, defined in this manner, is a covariant functor on **Dir** to $\mathbf{G}_0$.

Alternatively, the mapping function of this functor "$\mathrm{Lim}_{\to}$" can be obtained by extensions of mappings to the directed sets $\overline{D_1}, \overline{D_2}$ (with $\top$ added), defined as in §4.2. Given the mapping (R, ρ) of Eq. (4.18), first extend the given object of **Dir** to obtain new objects $(\overline{D_1}, \overline{T_1})$ and $(\overline{D_2}, \overline{T_2})$. The given functor R on D_1 to D_2 can also be extended by setting $\overline{R}(\top) = \top$; this gives a functor $\overline{R}$ on $\overline{D_1}$ to $\overline{D_2}$. Furthermore, Theorem 17 asserts that the transformation $\rho : T_1 \to T_2 \circ R$ has then a unique extension $\overline{\rho} : \overline{T_1} \to \overline{T_2} \circ \overline{R}$. All told, we have a new mapping

$$(\overline{R}, \overline{\rho}) : (\overline{D_1}, \overline{T_1}) \to (\overline{D_2}, \overline{T_2}) \quad \text{in } \mathbf{Dir}$$

In particular, when $\overline{\rho}$ is applied to the new element $\top \in \overline{D_1}$, it yields a homomorphism of the limit group of T_1 into the limit group of $T_2 \circ R$. On the other hand, R determines a homomorphism $\hat{R}$ of the limit group of $T_2 \circ R$ into the limit group of T_2; explicitly, for (g_1, d_1) in the first limit group, the image $\hat{R}(g_1, d_1)$ is the element (g_1, Rd_1) in the second limit group. The requisite mapping function of the functor "$\mathrm{Lim}_{\to}$" is now defined by setting

$$\mathrm{Lim}_{\to}(R, \rho) = \hat{R}(\overline{\rho}(\top)).$$

In a similar way we define the category **Inv**. The objects of **Inv** are pairs (D, T) where D is a directed set and T is an inverse system of topological groups indexed by D (that is, T is a contravariant functor on D to $\mathbf{G}$). The mappings in **Inv** are pairs (R,ρ)

$$(R,\rho):(D_1,T_1)\to(D_2,T_2)$$

where R is a covariant functor on D_2 to D_1 (that is, an order preserving transformation of D_2 into D_1) and ρ is a natural transformation of the functors

$$\rho:T_1\circ R\to T_2$$

both contravariant on D_2 to $\mathbf{G}$. The product of two mappings

$$(R_1,\rho_1):(D_1,T_1)\to(D_2,T_2),\qquad(R_2,\rho_2):(D_2,T_2)\to(D_3,T_3)$$

is defined as

$$(R_2,\rho_2)(R_1,\rho_1)=(R_1\circ R_2,\rho_2\circ\rho_1\circ R_2)$$

where $\rho_1\circ R_2$ is the transformation

$$\rho_1\circ R_2:T_1\circ R_1\circ R_2\to T_2\circ R_2$$

induced (as in §2.3) by

$$\rho_1:T_1\circ R_1\to T_2.$$

With these conventions, we verify that **Inv** is a category.

We shall now define $\mathrm{Lim}_{\leftarrow}$ as a covariant functor on **Inv** with values in $\mathbf{G}$. For each object (D,T) in **Inv** we define $\mathrm{Lim}_{\leftarrow}(D,T)$ to be the inverse limit of the inverse system of groups T indexed by the directed set D. Given a mapping

$$(R,\rho):(D_1,T_1)\to(D_2,T_2)\in\mathbf{Inv}\tag{4.20}$$

we define the mapping function of $\mathrm{Lim}_{\leftarrow}$

$$\mathrm{Lim}_{\leftarrow}(R,\rho):\mathrm{Lim}_{\leftarrow}(D_1,T_1)\to\mathrm{Lim}_{\leftarrow}(D_2,T_2)\tag{4.21}$$

as follows. Each element of $\mathrm{Lim}_{\leftarrow}(D_1,T_1)$ is a function $g(d_1)$ with values $g(d_1)\in T_1(d_1)$, for $d_1\in D_1$, which match properly under the projections in T_1. Now we define a new function h, with

$$h(d_2)=\rho(_d2)g(Rd_2),\quad d_2\in D_2;$$

it is easy to verify that h is an element of the limit group $\mathrm{Lim}_{\leftarrow}(D_2,T_2)$. The correspondence $g\to h$ is the homomorphism (4.21) required for the definition of the mapping function of $\mathrm{Lim}_{\leftarrow}$. One may verify that this definition does yield a covariant functor $\mathrm{Lim}_{\leftarrow}$ on the category **Inv** to $\mathbf{G}$.

The mapping function of $\mathrm{Lim}_{\leftarrow}$ may again be obtained by first extending the given mapping (4.20) to

$$(\overline{R},\overline{\rho}):(\overline{D_1},\overline{T_1})\to(\overline{D_2},\overline{T_2})\in\mathbf{Inv}.$$

In particular, when the extended transformation $\overline{\rho}$ is applied to the element $\top$ of $\overline{D_1}$, we obtain a homomorphism of the limit group of $T_1 \circ R$ into the limit group T_2. On the other hand, the covariant functor $R : D_2 \to D_1$ determines a morphism $\breve{R}$ of the limit group (D_1, T_1) into the limit group of $(D_2, T_1 \circ R)$; explicitly, for each function $g(d_1)$ in the first limit group, the image $h = \breve{R}g$ in the second limit group is defined by setting $h(d_2) = g(Rd_2)$, $\forall d_2 \in D_2$. The mapping function of the functor "$\mathrm{Lim}_{\leftarrow}$" is now $\mathrm{Lim}_{\leftarrow}(R, \rho) = \overline{\rho}(\top)\breve{R}$.

4.5 The lifting principle

Let Q be a functor whose arguments and values are groups, while T is any direct or inverse system of groups. If the object function of Q is applied to each group $T(d)$ of the given system, while the mapping function of Q is applied to each projection $T(d_1, d_2)$ of the given system, we obtain a new system of groups, which may be called $Q \circ T$. If Q is covariant, T and $Q \circ T$ are both direct or inverse, while if Q is contravariant, $Q \circ T$ is inverse when T is direct, and vice versa.

Actually this new system $Q \circ T$ is simply the composite of the functor T with the functor Q (see §2.3). We may regard this composition as a process which "lifts" a functor Q whose arguments and values are groups to a functor Q_L whose arguments and values are direct (or inverse) systems of groups. We may then regard the lifted functor as one acting on the categories **Dir** and **Inv**, as the case may be. In every case, the lifted functor has its object and mapping functions given formally by the equations (in the "$\circ$-notation" for composites)

$$Q_L(D, T) = (D, Q \circ T) \tag{4.22}$$

$$Q_L(R, \rho) = (R, Q \circ \rho). \tag{4.23}$$

These formulas includes the following four cases:

1. Q covariant on $\mathbf{G}_0$ to $\mathbf{G}_0$; Q_L covariant on **Dir** to **Dir**.
2. Q contravariant on $\mathbf{G}_0$ to $\mathbf{G}$; Q_L contravariant on **Dir** to **Inv**.
3. Q covariant on $\mathbf{G}$ to $\mathbf{G}$; Q_L covariant on **Inv** to **Inv**.
4. Q contravariant on $\mathbf{G}$ to $\mathbf{G}_0$; Q_L contravariant on **Inv** to **Dir**.

For illustration, we discuss case 2, in which Q is given contravariant on $\mathbf{G}_0$ to $\mathbf{G}$. The object function of Q_L, as defined in Eq. (4.22), assigns to each object (D, T) of the category **Dir** a pair $(D, Q \circ T)$. Since T is covariant on D to $\mathbf{G}_0$ and Q contravariant on $\mathbf{G}_0$ to $\mathbf{G}$, the composite $Q \circ T$ is contravariant on D to $\mathbf{G}$, so that $Q \circ T$ is an inverse system of groups, and the pair $(D, Q \circ T)$ is an object of **Inv**. On the other hand, given a mapping

$$(R, \rho) : (D_1, T_1) \to (D_2, T_2) \in \mathbf{Dir},$$

with $\rho : T_1 \to (T_2 \circ R)$, the composite transformation $Q \circ \rho$ is obtained by applying the mapping function of Q to each homomorphism $\rho(d_1) : T_1(d_1) \to (T_2 \circ R(d_1))$, and this gives a transformation $Q \circ \rho : (Q \circ T_2 \circ R) \to (Q \circ T_1)$. Thus the mapping function of Q_L, as defined in Eq. (4.23), does give a mapping $(R, Q \circ \rho) : (D_2, Q \circ T_2) \to (D_1, Q \circ T_1)$ in the category **Inv**. We verify that Q_L is a contravariant functor on **Dir** to **Inv**.

Any natural transformation $k_1 : Q \to P$ induces a transformation on the lifted functors, $k_L : Q_L \to P_L$, obtained by composition of the transformation k with the identity transformation of each T, as

$$k_L(D, T) = (D, k \circ T).$$

If k is an equivalence, so is this "lifted" transformation.

Just as in the case of composition, the operation of "lifting" can itself be regarded as a functor "Lift", defined on a suitable category of functors Q. In all four cases 1 to 4, this functor "Lift" is covariant.

In all these cases the functor Q may originally contain any number of additional variables. The lifted functor Q_L will then involve the same extra variables with the same variance. With proper caution the lifting process may also be applied simultaneously to a functor Q with two variables, both of which are groups.

4.6 Functors which commute with limits

Certain operations, such as the formation of the character groups of discrete or compact groups, are known to "commute" with the passage to a limit. Using the lifting operation, this can be formulated exactly.

To illustrate, let Q be a covariant functor on $\mathbf{G}_0$ to $\mathbf{G}_0$, and Q_L the corresponding covariant lifted functor on **Dir** to **Dir**, as in case 1 of §4.5. Since $\mathrm{Lim}_{\to}$ is a covariant functor on **Dir** to $\mathbf{G}_0$, we have two composite functors

$$\mathrm{Lim}_{\to} \circ Q_L \text{ and } Q \circ \mathrm{Lim}_{\to},$$

both covariant on **Dir** to $\mathbf{G}_0$. There is also an explicit natural transformation

$$\omega_1 : (\mathrm{Lim}_{\to} \circ Q_L) \to (Q \circ \mathrm{Lim}_{\to}), \tag{4.24}$$

defined as follows. Let the pair (D, T) be a direct system of groups in the category **Dir**, and let $\lambda(d)$ be the projection

$$\lambda(d) : T(d) \to \mathrm{Lim}_{\to} T,\ d \in D.$$

Then, on applying the mapping function of Q to λ, we obtain the natural transformation

$$Q\lambda(d) : QT(d) \to Q[\mathrm{Lim}_{\to} T].$$

Theorem 19 now gives a homomorphism

$$\omega_1(T) : \mathrm{Lim}_{\to}[Q \circ T] \to Q[\mathrm{Lim}_{\to} T],$$

or, exhibiting D explicitly, a homomorphism

$$\omega_1(D, T) : \mathrm{Lim}_{\to} Q_L(D, T) \to Q[\mathrm{Lim}_{\to}(D, T)].$$

We verify that ω_1, so defined, satisfies the naturality condition.

Similarly, to treat case 2, consider a contravariant functor Q on $\mathbf{G}_0$ to $\mathbf{G}$ and the lifted functor Q_L on **Dir** to **Inv**. We then construct an explicit natural transformation

$$\omega_2 : (Q \circ \mathrm{Lim}_{\to}) \to (\mathrm{Lim}_{\leftarrow} \circ Q_L) \tag{4.25}$$

(note the order!), defined as follows. Let the pair (D, T) be in **Dir**, and let $\lambda(d)$ be the projection

$$\lambda(d) : T(d) \to \mathrm{Lim}_{\to} T,\ d \in D.$$

On applying Q, we get

$$Q\lambda(d) : Q[\mathrm{Lim}_{\to} T] \to QT(d).$$

Theorem 22 for inverse systems now gives a homomorphism

$$\omega_2(D, T) : Q[\mathrm{Lim}_{\to}(D, T)] \to Q[\mathrm{Lim}_{\leftarrow} Q_L(D, T)].$$

In the remaining cases 3 and 4 similar arguments give natural transformation

$$\omega_3 : (Q \circ \mathrm{Lim}_{\leftarrow}) \to (\mathrm{Lim}_{\leftarrow} \circ Q_L), \tag{4.26}$$

$$\omega_4 : (\mathrm{Lim}_{\to} \circ Q_L) \to (Q \circ \mathrm{Lim}_{\leftarrow}). \tag{4.27}$$

Definition. *The functor Q defined on groups to groups is said to commute (more precisely to ω-commute) with* $\mathrm{Lim}_{\leftrightarrows}$ *if the appropriate one of the four natural transformations ω above is an equivalence.*

In other words, the proof that a functor Q commutes with $\mathrm{Lim}_{\to}$ requires only the verification that the homomorphisms defined above are isomorphisms. The naturality condition holds in general!

To illustrate these concepts, consider the functor C which assigns to each discrete group G its commutator subgroup $C(G)$, and consider a direct system T of groups, indexed by D. Then the lifted functor Q (case 1 of §4.5) applied to the pair (D, T) in **Dir** gives a new direct system of groups, still indexed by D, with the groups $T(d)$ of the original system replaced by their commutator subgroups $CT(d)$, and with the projections correspondingly cut down. It may be readily verified that this functor does commute with $\mathrm{Lim}_{\to}$.

Another functor Q is the subfunctor of the identity which assigns to each discrete abelian group G the subgroup $Q(G)$ consisting of those elements $g \in G$ such that there is for each integer m an $x \in G$ with $mx = g$ (that is, of those elements of G which are divisible by every integer), Q is a covariant functor with arguments and values in the subcategory $\mathbf{G}_{0a}$ of discrete abelian groups. This functor Q clearly does not commute with $\mathrm{Lim}_{\rightarrow}$, since one may represent the additive group of rational numbers as a direct limit cyclic group $\mathbb{Z}$ for which each subgroup $Q(\mathbb{Z})$ is the group consisting of zero alone.

The formation of character groups gives further examples. If we consider the functor Char as a contravariant functor on the category $\mathbf{G}_{0a}$ of compact abelian groups, the lifted functor Char_L will be covariant on the appropriate subcategory of **Dir** to **Inv** as in case 2 of §4.5. This lifted functor Char_L applied to any direct system (D, T) of discrete abelian groups will yield an inverse system of compact abelian group, indexed by the same set D. Each group of the inverse system is the character group of the corresponding group of the direct system, and the projections of the inverse system are the induced mappings.

On the other hand, there is a contravariant functor Char on $\mathbf{G}_{ca}$ to $\mathbf{G}_{0a}$. In this case the lifted functor Char_L will be contravariant on a suitable subcategory of **Inv** with values in **Dir**, just as in case 3 of §4.5. Both these functors Char commute with Lim.

Chapter 5

Applications to topology[1]

5.1 Complexes

An abstract complex K (in the sense of W. Mayer) is a collection

$$\{C^q(K)\}, \quad q \in \mathbb{Z},$$

of free abelian discrete groups, together with a collection of homomorphisms

$$\partial^q : C^q(K) \to C^{q-1}(K)$$

called *boundary homomorphism*, such that

$$\partial^q \partial^{q+1} = 0.$$

By selecting for each of the free groups C^q a fixed basis $\{\sigma_i^q\}$ we obtain a complex which is substantially an abstract complex in the sense of A. W. Tucker. The σ_i^q will be called q-dimensional cells. The boundary operator ∂ can be written as a finite sum

$$\partial \sigma^q = \sum_{\sigma^{q-1}} [\sigma^q : \sigma^{q-1}] \sigma^{q-1}.$$

The integers $[\sigma^q : \sigma^{q-1}]$ are called incidence numbers, and satisfy the following conditions:

1. Given $\sigma^q, [\sigma^q : \sigma^{q-1}] \neq 0$ only for a finite number of $(q-1)$-cells σ^{q-1}.
2. Given σ^{q+1} and $\sigma^{q-1}, \sum_{\sigma^q} [\sigma^{q+1} : \sigma^q][\sigma^q : \sigma^{q-1}] = 0$.

Condition 1 indicates that we are confronted with an abstract complex of the closure finite type. Consequently we shall define (in §5.2) homology based on finite chains and cohomologies based on infinite cochains.

[1]General reference: Lefschetz, *Algebraic Topology*.

Our preferences for complexes à la W. Mayer is due to the fact that they seem to be best adapted for the exposition of the homology theory in terms of functors.

Given two abstract complexes K_1 and K_2, a chain transformation

$$k : K_1 \to K_2$$

will mean a collection $k = \{k^q\}$ of homomorphisms,

$$k^q : C^q(K_1) \to C^q(K_2),$$

such that

$$k^{q-1}\partial^q = \partial^q k^q.$$

In this way we are led to the category $\mathbf{K}$ whose objects are the abstract complexes (in the sense of W. Mayer) and whose mappings are the chain transformations with obvious definition of the composition of chain transformations.

The consideration of simplicial complexes and of simplicial transformations leads to a category $\mathbf{K}_s$. As is well known, every simplicial complex uniquely determines an abstract complex, and every simplicial transformation a chain transformation. This leads to a covariant functor on $\mathbf{K}_s$ to $\mathbf{K}$.

5.2 Homology and cohomology groups

For every complex K in the category $\mathbf{K}$ and every group G in the category $\mathbf{G}_{0a}$ of discrete abelian groups we define the groups $C^q(K, G)$ of the q-dimensional chains of K over G as the tensor product

$$C^q(K, G) = G \otimes C^q(K),$$

that is, $C^q(K, G)$ is the group with the symbols

$$gc^q, \text{ for } g \in G, c^q \in C^q(K)$$

as generators, and

$$(g_1 + g_2)c^q = g_1c^q + g_2c^q, \qquad g(c_1^q + c_2^q) = g_1c^q + g_2c^q$$

as relations.

For every chain transformation $k : K_1 \to K_2$ and for every homomorphism $\gamma : G_1 \to G_2$ we define a homomorphism

$$C^q(k, \gamma) : C^q(K_1, G_1) \to C^q(K_2, G_2)$$

by setting

$$C^q(k, \gamma)(g_1c_1^q) = \gamma(g_1)k^q(c_1^q)$$

for each generator $g_1 c_1^q \in C^q(K_1, G_1)$.

These definitions of $C^q(K, G)$ and of $C^q(k, \gamma)$ yield a functor C^q covariant in $\mathbf{K}$ and in $\mathbf{G}_{0a}$ with values in $\mathbf{G}_{0a}$. This functor will be called the q-chain functor.

We define an homomorphism

$$\partial^q(K, G) : C^q(K, G) \to C^{q-1}(K, G)$$

by setting

$$\partial^q(K, G)(gc^q) = g\partial c^q$$

for each generator gc^q of $C^q(K, G)$. Thus the boundary operator becomes a natural transformation of the functor C^q into the functor C^{q-1}

$$\partial^q : C^q \to C^{q-1}.$$

The kernel of this transformation will be denoted by Z^q and will be called the q-cycle functor. Its object function is the group $Z^q(K, G)$ of the q-dimensional cycles of the complex K over G.

The image of C^q under the transformation ∂^q is a subfunctor $B^{q-1} = \partial^q(C^q)$ of C^{q-1}. Its object function is the group $B^{q-1}(K, G)$ of the of the $(q-1)$-dimensional boundaries in K over G.

The fact that $\partial^q \partial^{q+1} = 0$ implies that $B^q(K, G)$ is a subgroup of $Z^q(K, G)$. Consequently B^q is a subfunctor of Z^q. The quotient functor

$$H^q = Z^q / B^q$$

is called the qth homology functor. Its object function associates with each complex K and with each discrete abelian coefficient group G the qth homology group $H^q(K, G)$ of K over G. The functor H^q is covariant in $\mathbf{K}$ and $\mathbf{G}_{0a}$ and has values in $\mathbf{G}_{0a}$.

In order to define the cohomology groups as functors we consider the category $\mathbf{K}$ as before and the category $\mathbf{G}_a$ of topological abelian groups. Given a complex $K \in \mathbf{K}$ and a group $G \in \mathbf{G}$ we define the group $C_q(K, G)$ of the q-dimensional cochains of K over G as

$$C_q(K, G) = \operatorname{Hom}(C^q(K), G).$$

Given a chain transformation $k : K_1 \to K_2$ and a homomorphism $\gamma : G_1 \to G_2$ we define a homomorphism

$$C_q(k, \gamma) : C_q(K_2, G_1) \to C_q(K_1, G_2)$$

by associating with each homomorphism $f \in C_q(K_2, G_1)$ the homomorphism $\bar{f} = C_q(k, \gamma) f$, defined as follows:

$$\bar{f}(c_1^q) = \gamma[f(k^q c_1^q)], \qquad c_1^q \in C^q(K_1).$$

By comparing this definition with the definition of the functor Hom, we observe that $C_q(k,\gamma)$ is in fact just $\mathrm{Hom}(k^q,\gamma)$.

The definition of $C_q(K,G)$ and $C_q(k,\gamma)$ yield a functor C_q contravariant in $\mathbf{K}$, covariant in $\mathbf{G}_a$, and with values in $\mathbf{G}_a$. This functor will be called the qth cochain functor.

The coboundary homomorphism

$$\delta_q(K,G) : C_q(K,G) \to C_{q+1}(K,G)$$

is defined by setting, for each cochain $f \in C_q(K,G)$,

$$(\delta_q f)(c^{q+1}) = f(\partial^{q+1}c^{q+1}).$$

This leads to a natural transformation of functors

$$\delta_q : C_q \to C_{q+1}.$$

We may observe that in terms of the functor "Hom" we have $\delta_q(K,G) = \mathrm{Hom}(\partial^{q+1}, e_G)$.

The kernel of the transformation δ_q is denoted by Z_q and is called the q-cocycle functor. The image functor of δ_q is denoted by B_{q+1} and is called the $(q+1)$-coboundary functor. Since $\partial^q\partial^{q+1} = 0$, we may easily deduce that B_q is a subfunctor of Z_q. The quotient-functor

$$H_q = Z_q/B_q$$

is, by definition, the qth cohomology functor. H_q is contravariant in $\mathbf{K}$, covariant in $\mathbf{G}_a$, and has values in $\mathbf{G}_a$. Its object function associates with each complex K and each topological abelian group G the (topological abelian) qth cohomology group $H_q(K,G)$.

The fact the the homology groups are discrete and have discrete coefficient groups, while the cohomology groups are topologized and have topological coefficient groups, is due to the circumstance that the complexes considered are closure finite. In a star finite complex the relation would be reversed.

For "finite" complexes both homology and cohomology groups may be topological. Let $\mathbf{K}_f$ denote the subcategory of $\mathbf{K}$ determined by all those complexes K such that all the groups $C^q(K)$ have finite rank. If $K \in \mathbf{K}_f$ and G is a topological group, then the group $C^q(K,G) = G \otimes C^q(K)$ can be topologized in a natural fashion and consequently $H^q(K,G)$ will be topological. Hence both H^q and H_q may be regarded as functors on $\mathbf{K}_f$ and $\mathbf{G}_a$ with values in $\mathbf{G}_a$. The first one is covariant in both $\mathbf{K}_f$ and $\mathbf{G}_a$, while the second one is contravariant in $\mathbf{K}_f$ and covariant in $\mathbf{G}_a$.

5.3 Duality

Let G be a discrete abelian group and $\mathrm{Char}(G)$ be its (compact) character group (see §3.6).

Given a chain

$$c^q \in C^q(K, G)$$

where

$$c^q = \sum_i g_i c_i^q, \quad g_i \in G, c_i^q \in C^q(K),$$

and given a cochain

$$f \in C_q(K, \operatorname{Char} G),$$

we may define the Kronecker index

$$\operatorname{Ki}(f, c^q) = \sum_i (f(c_i^q), g_i). \tag{5.1}$$

Since (c_i^q) is an element of $\operatorname{Char} G$, its application to g_i gives an element of the group P of reals reduced $\bmod 1$. The continuity of $\operatorname{Ki}(f, c^q)$ as a function of f follows from the definition of the topology in $\operatorname{Char} G$ and in $C_q(K, \operatorname{Char} G)$.

As a preliminary to the duality theorem, we define an isomorphism

$$\tau^q(K, G) : C_q(K, \operatorname{Char} G) \rightleftarrows \operatorname{Char} C^q(K, G), \tag{5.2}$$

by defining for each cochain $f \in C_q(K, \operatorname{Char} G)$ a character

$$\tau^q(K, G)f : C^q(K, G) \to P$$

as follows:

$$(\tau^q f, c^q) = \operatorname{Ki}(f, c^q).$$

The fact that $\tau^q(K, G)$ is an isomorphism is a direct consequence of the character theory. In Eq. (5.2) both sides should be interpreted as object functions of functors (contravariant in both K and G), suitably compounded from the functors C^q, C_q and Char. In order to prove that Eq. (5.2) is natural, consider

$$k : K_1 \to K_2 \in \mathbf{K}, \qquad \gamma : G_1 \to G_2 \in \mathbf{G}_{0a}.$$

We must prove that

$$\tau^q(K_1, G_1) C_q(k, \operatorname{Char} \gamma) = [\operatorname{Char} C^q(k, \gamma)] \tau^q(K_2, G_2). \tag{5.3}$$

If now

$$f \in C_q(K,_2 G_2), \qquad c^q \in C^q(K_1, G_1),$$

then the definition of τ^q shows that Eq. (5.3) is equivalent to the identity

$$\operatorname{Ki}(C_q(k, \operatorname{Char} \gamma) f, c^q) = \operatorname{Ki}(f, C^q(k, \gamma) c^q). \tag{5.4}$$

It will be sufficient to establish Eq. (5.4) in the case when c^q is a generator of $C^q(K_1, G_1)$,

$$c^q = g_1 c_1^q, \qquad g_1 \in G_1, c_1^q \in C^q(K_2).$$

Proof. Using the definitions of the terms involved in Eq. (5.4) we have on the one hand

$$\begin{aligned}\mathrm{Ki}(C_q(k, \mathrm{Char}\,\gamma)f, g_1c_1^q) &= ([C_q(k, \mathrm{Char}\,\gamma)f]c_1^q, g_1) \\ &= (\mathrm{Char}\,\gamma[f(kc_1^q)]g_1 \\ &= (f(kc_1^q, \gamma g_1),\end{aligned}$$

and on the other hand

$$\mathrm{Ki}(f, C^q(k,\gamma)g_1c_1^q) = \mathrm{Ki}(f, (\gamma g_1)(kc_1^q)) = (f(kc_1^q, \gamma g_1).$$

This completes the proof of the naturality of Eq. (5.4). □

Using the well known property of the Kronecker index

$$\mathrm{Ki}(f, \partial^{q+1}c^{q+1}) = \mathrm{Ki}(\delta_q f, c^{q+1}),$$

one shows easily that under the isomorphisms τ^q of Eq. (5.2)

$$\begin{aligned}\tau^q[Z_q(K, \mathrm{Char}\,G)] &= \mathrm{Annih}\,B^q(K,G), \\ \tau^q[B_q(K, \mathrm{Char}\,G)] &= \mathrm{Annih}\,Z^q(K,G)\end{aligned}$$

with "Annih" defined as in §3.6. Both $\mathrm{Annih}(B^q; C^q)$ and $\mathrm{Annih}(Z^q; C^q)$ are functors covariant in K and G; the latter is a subfunctor of the former, so that τ^q induces a natural isomorphism

$$\sigma^q : Z_q(K, \mathrm{Char}\,G)/B_q(K, \mathrm{Char}\,G) \rightleftarrows \mathrm{Annih}\,B^q(K,G)/\,\mathrm{Annih}\,Z^q(K,G).$$

The group on the left is $H_q(K, \mathrm{Char}\,G)$. The group on the right is, according to Eq. (3.16), naturally isomorphic to $\mathrm{Char}\,Z^q(K,G)/B^q(K,G)$. All told we have a natural isomorphism:

$$\rho^q : H_q(K, \mathrm{Char}\,G) \rightleftarrows \mathrm{Char}\,H^q(K,G).$$

This is the customary Pontrjagin-type duality between homology and cohomology. Thus we have established the naturality of this duality.

5.4 Universal coefficient theorems

The theorems of this name express the cohomology groups of a complex, for an arbitrary coefficient group, in terms of the integral homology groups and the coefficient group itself. A quite general form of such theorems can be stated in terms of certain groups of group extensions;[2] hence we first show that the basic constructions of group extensions may be regarded as functors.

[2]Eilenberg and MacLane, "Group Extensions and Homology", pp. 757–831.

Let G be a topological abelian group and H a discrete abelian group. A factor set of H in G is a function $f(h,k)$ which assigns to each pair (h,k) of elements in H an element $f(h,k) \in G$ in such wise that

$$f(h,k) = f(k,h), \qquad f(h,k) + f(h+k,l) = f(h,k+l) + f(k,l)$$

for all h,k, and l in H. With the natural addition and topology, the set of all factor set f of H in G constitute a topological abelian group $\mathrm{Fact}(G,H)$. If $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$ are homomorphisms, we can defined a corresponding mapping

$$\mathrm{Fact}(\gamma,\eta) : \mathrm{Fact}(G_1,H_2) \to \mathrm{Fact}(G_2,H_1)$$

by setting

$$[\mathrm{Fact}(\gamma,\eta)f](h_1,k_1) = \gamma f(\eta h_1, \eta k_1)$$

for each factor set $f \in \mathrm{Fact}(G_1,H_2)$. Thus it appears that Fact is a functor, covariant on the category $\mathbf{G}_a$ of topological abelian groups and contravariant in the category $\mathbf{G}_{0a}$ of discrete abelian groups.

Given any function $g(h)$ with values in G, the combination

$$f(h,k) = g(h) + g(k) - g(h+k)$$

is always a factor set; the factor sets of this special form are said to be transformation sets, and the set of all transformation sets is a subgroup $\mathrm{Trans}(G,H)$ of the group $\mathrm{Fact}(G,H)$. Furthermore, this subgroup is the object function of a subfunctor. The corresponding quotient functor

$$\mathrm{Ext} = \mathrm{Fact}\,/\,\mathrm{Trans}$$

is thus covariant in $\mathbf{G}_a$, contravariant in $\mathbf{G}_{0a}$, and has values in $\mathbf{G}_a$. Its object function assigns to the groups G and H the group $\mathrm{Ext}(G,H)$ the so-called abelian group extension of G by H.

Since $C_q(K,G) = \mathrm{Hom}(C^q(K),G)$ and since $C^q(K,\mathbb{Z}) = \mathbb{Z} \otimes C^q(K) = C^q(K)$ where $\mathbb{Z}$ is the additive group of integers, we have

$$C_q(K,G) = \mathrm{Hom}(C^q(K,\mathbb{Z}),G).$$

We, therefore, may define a subgroup

$$A_q(K,G) = \mathrm{Annih}\, Z^q(K,\mathbb{Z})$$

of $C_q(K,G)$ consisting of all homomorphisms f such that $f(z^q) = 0$ for $z^q \in Z^q(K,\mathbb{Z})$. Thus we get a subfunctor A_q of C_q, and one may show that the coboundary functor B_q is a subfunctor of A_q which, in turn, is a subfunctor of the cocycle functor Z_q. Consequently, the quotient functor

$$Q_q = A_q/B_q$$

is a subfunctor of the cohomology functor H_q, and we may consider the quotient functor H_q/Q_q. The functors Q_q and H_q/Q_q have the following object functions

$$
\begin{aligned}
Q_q(K,G) &= A_q(K,G)/B_q(K,G),\\
(H_q/Q_q)(K,G) &= H_q(K,G)/Q_q(K,G) \cong Z_q(K,G)/A_q(K,G).
\end{aligned}
$$

The universal coefficient theorem now consists of these three assertions:[3]

$$Q_q(K,G) \text{ is a direct factor of } H_q(K,G). \tag{5.5}$$

$$Q_q(K,G) \cong \operatorname{Ext}(G, H^{q+1}(K,\mathbb{Z})). \tag{5.6}$$

$$H_q(K,G)/Q_q(K,G) \cong \operatorname{Hom}(H^q(K,\mathbb{Z}),G). \tag{5.7}$$

Both the isomorphisms (5.6) and (5.7) can be interpreted as equivalences of functors. The naturality of these equivalences with respect to K has been explicitly verified,[4] while the naturality with respect to G ca be verified without difficulty. We have not been able to prove and we doubt that the functor Q_q is a direct factor of the functor H_q (see §3.5).

5.5 Čech homology groups

We shall present now a treatment of the Čech homology in terms of functors.

By a covering U of a topological space X we shall understand a finite collection:

$$U = \{A_1, \ldots, A_n\}$$

of open sets whose union is X. The sets A_i may appear with repetitions, and some of them may be empty. If U_1 and U_2 are two such coverings, we write $U_1 \prec U_2$ whenever U_2 is a refinement of U_1, that is, whenever each set of the covering U_2 is contained in some set of the covering U_1. With this definition the coverings U of X form a directed set which we denote by $C(X)$.

Let $\xi : X_1 \to X_2$ be a continuous mapping of the space X_1 into the space X_2. Given a covering

$$U = \{A_1, \ldots, A_n\} \in C(X_2),$$

we define

$$C(\xi)U = \{\xi^{-1}(A_1), \ldots, \xi^{-1}(A_n)\} \in C(X_1)$$

and we obtain an order preserving mapping

$$C(\xi) : C(X_2) \to C(X_1).$$

[3] Eilenberg and MacLane, "Group Extensions and Homology", p. 808.

[4] Ibid., p. 815.

We verify that the functions $C(X), C(\xi)$ define a contravariant functor C on the category $\mathbf{X}$ of topological spaces to the category $\mathbf{D}$ of directed sets.

Given a covering U of X we define, in the usual fashion, the nerve $N(U)$ of U. $N(U)$ is a finite simplicial complex; it will be treated, however, as an object of the category K_f of §5.2.

If two coverings $U_1 \prec U_2$ of X are given, then we select for each set of the covering U_2 a set of the covering U_1 containing it. This leads to a simplicial mapping of the complex $N(U_2)$ into the complex $N(U_1)$ and therefore gives a chain transformation

$$k : N(U_2) \to N(U_1).$$

This transformation k will be called a projection. The projection k is not defined uniquely by U_1 and U_2, but it is known that any two projections k_1 and k_2 are chain homotopic and consequently the induced homomorphisms

$$H^q(k, e_G) : H^q(N(U_2), G) \to H^q(N(U_1), G), \tag{5.8}$$

$$H_q(k, e_G) : H_q(N(U_1), G) \to H_q(N(U_2), G) \tag{5.9}$$

of the homology and cohomology groups do not depend upon the particular choice of the projection k.

Given a topological group G we consider the collection of the homology groups $H^q(N(U), G)$ for $U \in C(X)$. These groups together with the mappings Eq. (5.8) form an inverse system of groups defined on the directed set $C(X)$. We denote the inverse system by $\overline{C}^q(X, G)$ and treat it as an object of the category $\mathbf{Inv}$ (§4.4).

Similarly, for a discrete G the cohomology groups $H_q(N(U), G)$ together with the mappings (5.9) form a direct system of groups $\overline{C}_q(X, G)$ likewise defined on the directed set $C(X)$. The system $\overline{C}_q(X, G)$ will be treated as an object of the category $\mathbf{Dir}$.

The functions $\overline{C}^q(X, G)$ and $\overline{C}_q(X, G)$ will be object functions of functors $\overline{C}^q$ and $\overline{C}_q$. In order to complete the definition we shall define the mapping functions $\overline{C}^q(\xi, \gamma)$ and $\overline{C}_q(\xi, \gamma)$ for given mappings

$$\xi : X_1 \to X_2, \qquad \gamma : G_1 \to G_2.$$

We have the order preserving mapping

$$C(\xi) : C(X_2) \to C(X_1) \tag{5.10}$$

which with each covering

$$U = \{A_1, \ldots, A_n\} \in C(X_2)$$

associates the covering

$$V = C(\xi) = \{\xi^{-1}A_1, \ldots, \xi^{-1}A_n\} \in C(X_1).$$

Thus to each set of the covering V corresponds uniquely a set of covering U; this yields a simplicial mapping

$$k : N(V) \to N(U),$$

which leads to the homomorphisms

$$H^q(k,\gamma) : H^q(N(V), G_1) \to H^q(N(U), G_2), \tag{5.11}$$
$$H_q(k,\gamma) : H_q(N(U), G_1) \to H_q(N(V), G_2) \tag{5.12}$$

Equations (5.10) to (5.12) define the transformations

$$\overline{C}^q(\xi,\gamma) : \overline{C}^q(X_1, G_1) \to \overline{C}^q(X_2, G_2) \in \mathbf{Inv},$$
$$\overline{C}_q(\xi,\gamma) : \overline{C}_q(X_2, G_1) \to \overline{C}_q(X_1, G_2) \in \mathbf{Dir}.$$

Hence we see that $\overline{C}^q$ is a functor covariant in $\mathbf{X}$ and in $\mathbf{G}_a$ with values in $\mathbf{Inv}$ while $\overline{C}_q$ is contravariant in $\mathbf{X}$, covariant in $\mathbf{G}_{0a}$ with values in $\mathbf{Dir}$.

The Čech homology and cohomology functors are now defined as

$$\overline{H}^q = \mathrm{Lim}_{\leftarrow} \overline{C}^q, \qquad \overline{H}_q = \mathrm{Lim}_{\rightarrow} \overline{C}_q.$$

$\overline{H}^q$ is covariant in $\mathbf{X}$ and in $\mathbf{G}_a$ with values in $\mathbf{G}_a$, while $\overline{H}_q$ is contravariant in $\mathbf{X}$, covariant in $\mathbf{G}_{0a}$, and has values in $\mathbf{G}_{0a}$. The object functions $\overline{H}^q(X,G)$ and $\overline{H}_q(X,G)$ are the Čech homology and cohomology groups of the space X with the group G as coefficients.

5.6 Miscellaneous remarks

The process of setting up the various topological invariants as functors will require the construction of many categories. For instance, if we wish to discuss the so-called relative homology theory, we shall need the category $\mathbf{X}_S$ whose objects are the pairs (X, A), where X is a topological space and A is a subset of X. A mapping

$$\xi : (X, A) \to (Y, B) \in \mathbf{X}_S$$

is a continuous mapping $\xi : X \to Y$ such that $\xi(A) \subset B$. The category $\mathbf{X}$ may be regarded as the subcategory of $\mathbf{X}_S$ in the category $\mathbf{X}_b$ defined by the pairs (X, A) in which the sets A consists of a single point, called the base point. This category $\mathbf{X}_b$ would be used in a functorial treatment of the fundamental group and of the homotopy groups.

Appendix

Representations of Categories

The purpose of this appendix is to show that every category is isomorphic with a suitable subcategory of the category of sets **S**.

Let **A** be any category. A covariant functor T on **A** with values in **S** will be called a representation of **A** in **S**. A representation T will be called *faithful* if for every two mappings, $\alpha_1, \alpha_2 \in \mathbf{A}$, we have $T(\alpha_1) = T(\alpha_2)$ only if $\alpha_1 = \alpha_2$. This implies a similar proposition for the objects of **A**. It is clear that a faithful representation is nothing but an isomorphic mappings of **A** onto some subcategory of **S**.

If the functor T on **A** to **S** is contravariant, we shall say that T is a dual representation. T is then obviously a representation of the dual category $\mathbf{A}^{\mathrm{op}}$, as defined in §2.7.

Given a mapping $\alpha : A_1 \to A_2$ in **A**, we shall denote the domain A_1 of α by $d(\alpha)$ and the range A_2 of α by $r(\alpha)$. In this fashion we have

$$\alpha : d(a) \to r(a).$$

Given an object $A \in \mathbf{A}$ we shall denote by $R(A)$ the set of $\alpha \in \mathbf{A}$, such that $A = r(\alpha)$. In symbols

$$R(A) = \{\alpha | \alpha \in \mathbf{A}, r(\alpha) = A\}. \tag{5.13}$$

For every mapping $\alpha \in \mathbf{A}$ we defined a mapping

$$R(\alpha) : R(d(\alpha)) \to R(r(\alpha)) \tag{5.14}$$

in the category **S** by setting

$$[R(\alpha)]\xi = \alpha\xi, \qquad \forall \xi \in R(d(\alpha)). \tag{5.15}$$

This mapping is well defined because if $\xi \in R(d(\alpha))$, then $r(\xi) = d(\alpha)$, so that $\alpha\xi$ is defined and $r(\alpha\xi) = r(\alpha)$ which implies $\alpha\xi \in R(r(\alpha))$.

Theorem 23. *For every category* **A** *the pair of functions* $R(A), R(\alpha)$, *defined above, establishes a faithful representation* R *of* **A** *in* **S**.

Proof. We first verify that R is a functor. If $\alpha = e_A$ is an identity then the definition (5.15) implies that $[R(\alpha)]\xi = \xi$, so that $R(\alpha)$ is the identity mapping of $R(A)$ into itself. Thus R satisfies Eq. (1.1). Equation (1.2) has already been verified. In order to verify Eq. (1.3) let us consider the mappings

$$\alpha_1 : A_1 \to A_2 \qquad \alpha_2 : A_2 \to A_3.$$

We have for every $\xi \in R(A_1)$,

$$[R(\alpha_2\alpha_1)]\xi = \alpha_2\alpha_1\xi = [R(\alpha_2)]\alpha_1\xi = [R(\alpha_2)R(\alpha_1)]\xi,$$

so that $R(\alpha_2\alpha_1) = R(\alpha_2)R(\alpha_1)$. This concludes the proof that R is a representation.

In order to show that R is faithful, let us consider two mappings $\alpha_1, \alpha_2 \in \mathbf{A}$ and let assume that $R(\alpha_1) = R(\alpha_2)$. It follows from Eq. (5.14) that $R(d(\alpha_1)) = R(d(\alpha_2))$, and, therefore, according to Eq. (5.13), $d(\alpha_1) = d(\alpha_2)$. Consider the identity mapping $e = e_{d(\alpha_1)} = e_{d(\alpha_2)}$. Following Eq. (5.15) we have

$$\alpha_1 = \alpha_1 e = [R(\alpha_1)]e = [R(\alpha_2)]e = \alpha_2 e = \alpha_2,$$

so that $\alpha_1 = \alpha_2$. This concludes the proof of the theorem. $\square$

In a similar fashion we could define a faithful dual representation D of $\mathbf{A}$ by setting

$$D(A) = \{\alpha | \alpha \in \mathbf{A}, d(\alpha) = A\}.$$

and

$$[D(\alpha)]\xi = \xi\alpha$$

for every $\xi \in D(r(\alpha))$.

The representations R and D are the analogues of the left and right regular representations in group theory.

We shall conclude with some remarks concerning partial order in categories. Most of the categories which we have considered have an intrinsic partial order. For instance, in the categories $\mathbf{S}, \mathbf{X}$, and $\mathbf{G}$ the concept of subset,subspace and subgroup furnish a partial order. In view of Eq. (5.13), $A_1 \neq A_2$ implies that $R(A_1)$ and $R(A_2)$ are disjoint, so that the representation R destroys this order completely. The problem of getting "order preserving representations" would require probably a suitable formalization of the concept of a partially ordered category.

As an illustration of the type of arguments which may be involved, let us consider the category $\mathbf{G}_0$ of discrete groups. With each group G we can associate the set $R_1(G)$ which is the set of elements constituting the group G. With the obvious mappings function, R_1 becomes a covariant functor on $\mathbf{G}_0$ to $\mathbf{S}$, that is, R_1 is a representation of $\mathbf{G}_0$ in $\mathbf{S}$. This representation is not faithful, since the same set may carry two different group structures. The group structure of G is entirely described by means of a ternary relation $g_1 g_2 = g$.

This ternary relation is nothing by a subset $R_2(G) \subset R_1(G) \times R_1(G) \times R_1(G)$. All of the axioms of group theory can be formulated in terms of the subset $R_2(G)$. Moreover a homomorphism $\gamma : G_1 \to G_2$ includes a mapping $R_2(\gamma) : R_2(G_1) \to R_2(G_2)$. Consequently R_2 is a subfunctor of a suitably defined functor $R_1 \times R_1 \times R_1$. The two functors R_1 and R_2 together give a complete description of $\mathbf{G}_0$, preserving the partial order.

Bibliography

Banach, Stefan. *Théorie des opérations linéaires*. Monografje Matematyczne. Hafner Publishing Company, 1932.

Birkhoff, Garrett. “Generalized arithmetic”. In: *Duke Mathematical Journal* 9.2 (1942), pp. 283–302.

Eilenberg, Samuel and Saunders MacLane. “Group Extensions and Homology”. In: *Annals of Mathematics* 43.4 (1942), pp. 757–831.

Eilenberg, Samuel and Saunders MacLane. “Natural isomorphisms in group theory”. In: *Proceedings of the National Academy of Sciences* 28.12 (1942), pp. 537–543.

Freudenthal, Hans. “Entwicklungen von Räumen und ihren Gruppen”. In: *Compositio Mathematica* 4 (1937), pp. 145–234.

Lefschetz, Solomon. *Algebraic Topology*. Vol. 27. American Mathematical Society Colloquium Publications. American Mathematical Society, 1942.

Weil, André. *L'integration Dans Les Groupes Topologiques Et Ses Applications*. Vol. 4. Publications de l'Institut de mathématique de l'Université de Strasbourg. Hermann, 1938.