Graph Theory II: Shortest Paths, MST

Deadline: <check announcement in your section's slack channel>
Instructions:

1. Prepare a single pdf file containing your works
2. Submit through the google form: <check announcement in your section's slack channel>
3. You need to sign in to your bracu g-suite email account to see the form

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Assume that you are given a function called $dijkstra(G, a)$ which takes a graph G, and a source vertex a as input, then finds the shortest path (distances, $dist$ and parents, par) from a to every other vertex in the graph. Use this function to solve problems #1 and #2. Present your solution idea with a pseudocode, accompanying necessary explanation.

Problem #1: Optimum meeting place

You, and your friend Benji, live in a city which has 50 areas numbered from 1 to 50. Some of the areas are connected with bi-directional roads of various lengths. Your house is in 1 and Benji's is in 50. Both of you want to meet. Now propose an algorithm to choose the meeting point that minimizes the total travel time. In other words, if the meeting area you chose is X, it has the minimum $dist(1, X)\ +\ dist(50, X)$ among all possible area choices.
Expected time complexity: $O(E * log_2(V))$

Problem #2: Meet in the middle!

Continuing on the scenario of Problem #1, but assuming all the roads are one-way (representing a directed graph)...

Benji now has asked you to visit his house in area 50. He likes KFC, so you are thinking about getting a bucket of fried chicken for him. The are KFC outlets available in K different areas in the city. You have to make a stop at one of them. Now propose an algorithm to choose the area from the available K options that minimizes your total travel time. In other words, if the area you chose is X, it has the minimum $dist(1, X)\ +\ dist(X, 50)$ among all possible choices.
Expected time complexity: $O(E * log_2(V))$

Note: If you have to run Dijkstra's algorithm from K different sources, the complexity is $O(K * E * log_2(V))$, which exceeds the expected complexity for this problem.

Problem #3: Dijkstra's algorithm in a 'negative' weight graph!!
Draw a directed graph as described below.

No. of vertices: 7, No. of edges: 9
List of vertices, V: $\{1, 2, 3, 4, 5, 6, 7\}$
List of edges, E [in (u, v, w) format which means there is an outgoing edge from u to v with weight w] :
$\{(1, 2, 8),\ (2, 3, -8),\ (1, 3, 1)\ (3, 4, 4),\ (4, 5, -4),\ (3, 5, 1),\ (5, 6, 2),\ (6, 7, -2),\ (5, 7, 1)\}$

a. Show a simulation of Dijkstra's algorithm to find the *shortest path costs* from vertex 1 to all others. You can use the following pseudocode as reference.

   Note: Dijkstra's algorithm inserts all vertices in a priority-queue/min-heap at the beginning and then repeatedly extracts the vertex with minimum distance. It assumes, whenever a vertex is extracted from the queue, its shortest path cost from the source is finalized.

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DIJKSTRA(G, s)
for each v ∈ V[G]
      do d[v] ← ∞
         π[v] ← NIL
d[s] ← 0
S ← ∅
Q ← V[G]
while Q ≠ ∅
      do u ← EXTRACT-MIN(Q)
         S ← S ∪ {u}
         for each vertex v ∈ Adj[u]
               do if d[v] > d[u] + w(u, v)
                     then d[v] ← d[u] + w(u, v)
                          π[v] ← u
```

b. Explain why the algorithm could not find the correct shortest path costs for some of the vertices.
c. Imagine you have a modified version of Dijkstra's algorithm with the following changes:
   i. At the beginning, it inserts <u>only the source vertex</u> in the priority-queue.
   ii. Whenever the distance of a vertex is updated, it inserts that vertex into the priority-queue.
   iii. It <u>does not</u> assume that extracting a vertex from the queue means that its shortest path cost is finalized.

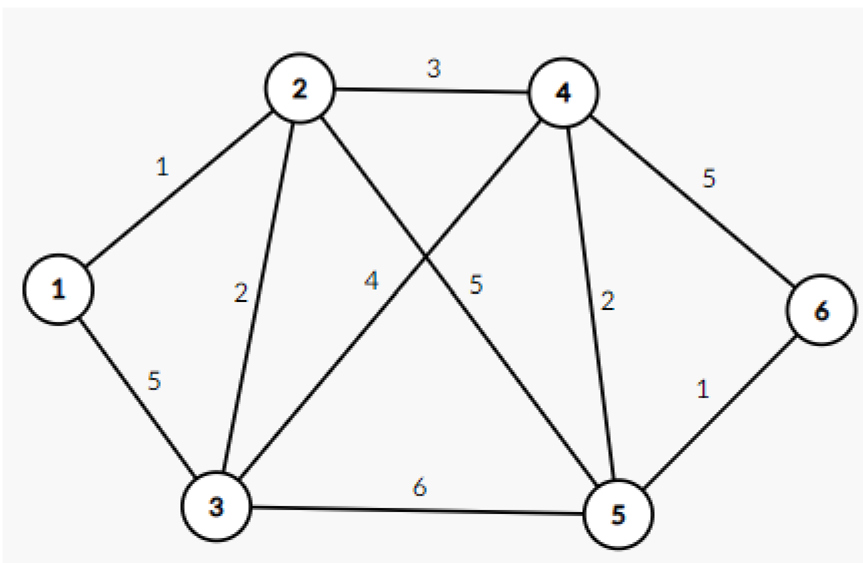
   Will this modified Dijkstra's algorithm be able to find the correct shortest path costs in the graph given above? Explain with reasoning or a simulation.
d. Analyze the time-complexity of this modified algorithm.

Problem #4: A divide-and-conquer MST algorithm?!

Your friend Mriaslun loves the divide and conquer strategy! He recently invented a divide-and-conquer algorithm to find the Minimum Spanning Tree of a given undirected weighted graph. The algorithm is quite simple, just having 4 steps in it:

1. Divide the set of vertices into two equal halves (one side can have an extra, like we do in merge sort). Assume that this divides the graph in three parts:
    a. G_{left} which contains all the vertices in the left half and the edges between them
    b. G_{right} containing the vertices in the right half, and the corresponding edges
    c. $G_{remains}$ containing the edges where one vertex is from the left half, and another one is from the right half
2. Use a recursive approach (repeating steps 1 to 4) to find the MST of G_{left}, let's call it MST_{left}
3. Do the same to find MST_{right}
4. Now you only need to select one edge from $G_{remains}$ to connect MST_{left} to MST_{right}. So you pick the one with the lowest weight value and you have the MST of the original graph!

Let's see an example.



In G_{left}, we have three vertices $\{1, 2, 3\}$ and three edges $\{(1, 2),\ (1, 3),\ (2, 3)\}$
In G_{right}, we have vertices $\{4, 5, 6\}$ and edges $\{(4, 5),\ (4, 6),\ (5, 6)\}$
And $G_{remains}$ is defined by the edges $\{(2, 4),\ (2, 5),\ (3, 4),\ (3, 5)\}$

If we can find MST_{left} and MST_{right} separately, then we can connect them with the minimum weight edge from $G_{remains}$, which is $(2, 4)$. And we can use the same divide and conquer approach recursively to know that MST_{left} consists of the edges $\{(1, 2), (2, 3)\}$ and MST_{right} contains the edges $\{(4, 5), (5, 6)\}$.
So, the MST of the original graph is $\{(1, 2), (2, 3), (2, 4), (4, 5), (5, 6)\}$ which is the correct answer!

Now your job is to prove that this algorithm will not always give a correct answer. Provide necessary reasonings to do that. You can also create an input graph and show that the algorithm fails or provides a wrong answer for this input.

Problem #5: Coin Change

There are a few types of banknotes available in your country's currency system. Any amount of taka that you want to give someone has to be constructed using these notes. Suppose you are given an infinite supply of these banknotes. You have to find the minimum number of notes needed to construct the amount.

Now, propose an algorithm applying your knowledge on graph theory which will take the currency notes and the amount to make, K as input. Then as output, return the minimum number of notes and the value of the notes.

Examples:

Input	Output
Notes: 1, 2, 5, 10 Amount to make: 21	3 notes needed: 10, 10, 1
Notes: 3, 5, 8 Amount to make: 20	4 notes needed: 5, 5, 5, 5
Notes: 3, 5, 8 Amount to make: 7	Impossible

The following questions might help:

1. Which graph theory algorithm might be relevant here?
2. How do you convert this into a graph? What are the vertices? What are the edges?

END!