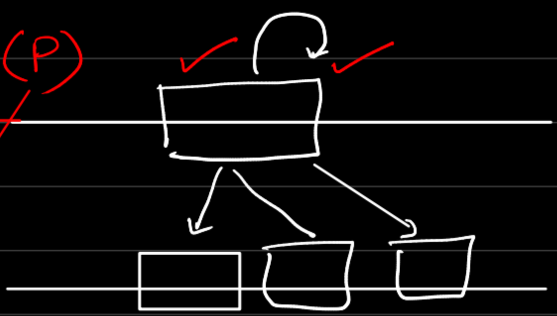
Pumping lemma

⇒ Every regular language has a special property.

↓

L is a Regular Language.
Every long string in L can be pumped.

$|W| \geq$ pumping length (P)



$\in L$

L ✓

$W_1 \; W_2 \; W_3 \dots$

THE PUMPING LEMMA FOR REGULAR LANGUAGES

Our technique for proving nonregularity stems from a theorem about regular languages, traditionally called the ***pumping lemma***. This theorem states that all regular languages have a special property. If we can show that a language does not have this property, we are guaranteed that it is not regular. The property states that all strings in the language can be "pumped" if they are at least as long as a certain special value, called the ***pumping length***. That means each such string contains a section that can be repeated any number of times with the resulting string remaining in the language.

THEOREM 1.70 ....

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Recall the notation where $|s|$ represents the length of string s, y^i means that i copies of y are concatenated together, and y^0 equals ε.

When s is divided into xyz, either x or z may be ε, but condition 2 says that $y \neq \varepsilon$. Observe that without condition 2 the theorem would be trivially true. Condition 3 states that the pieces x and y together have length at most p. It is an extra technical condition that we occasionally find useful when proving certain languages to be nonregular. See Example 1.74 for an application of condition 3.

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L, → P

w ∈ L

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x | y | z

x | y | z

w = x | y | z

$xy^iz \in L$, $i \geq 0$

$y \neq \varepsilon$

xy^iz, $i \geq 0$

$xy^0z = xz$

x | y | z

$y \neq \varepsilon$

xy^iz $\quad$ xy^0z

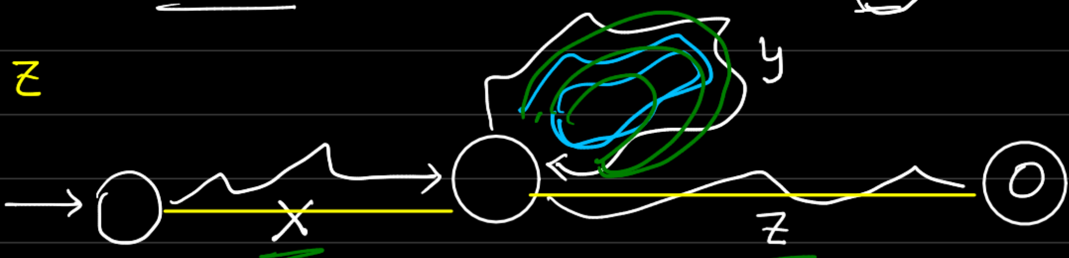
$i \geq 0$

$xyz \to xz$

↳ xyz

↳ $xyyz$

↳ $xyyyz$



xy^2z $\quad$ xy^3z

xy^0z

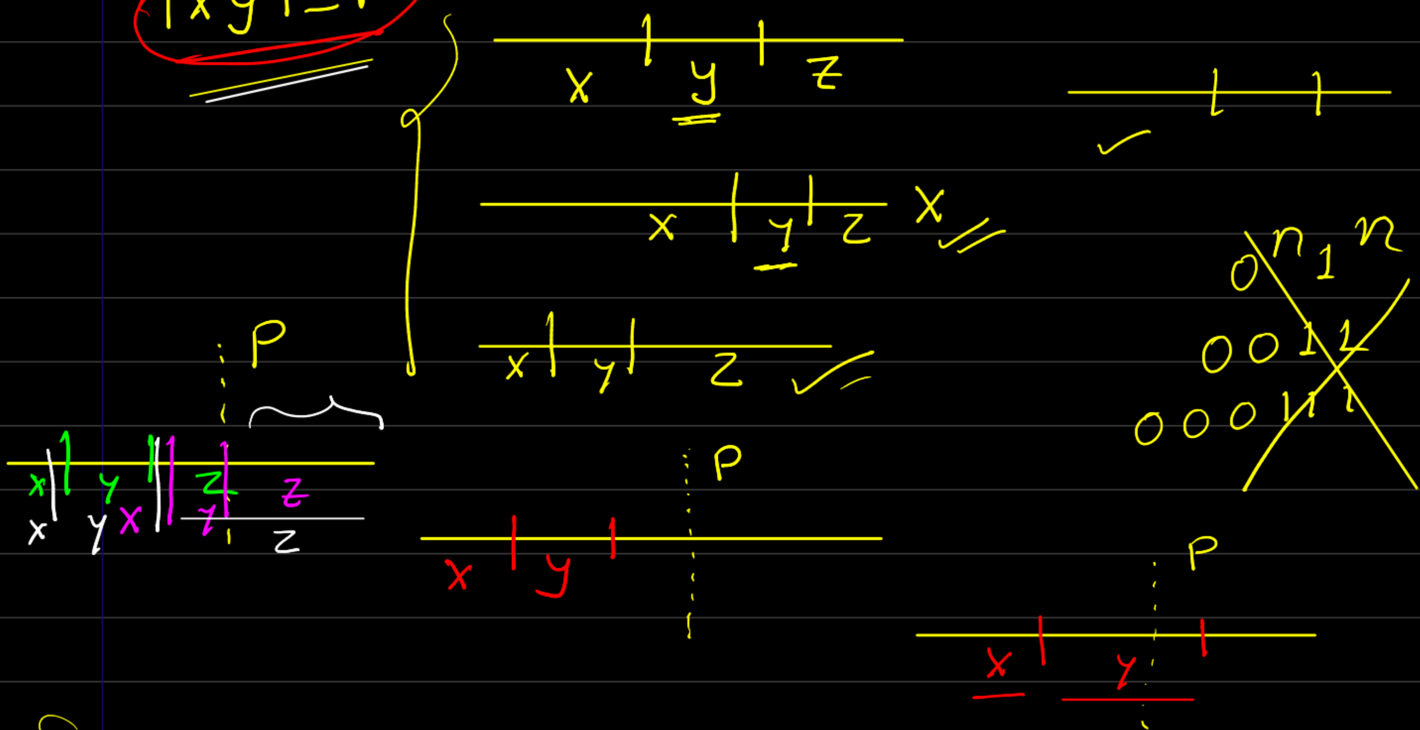
xz

Regular language:

For $\forall$ long strings w, $\exists\ xyz$ (at least one) with [ ] such that $\forall i,\ xy^iz \in L,\ i \geq 0$

$y \neq \varepsilon$

$|xy| \leq P$



Non Regular Language:

$\exists$ long string w, $\forall\ xyz$ with $y \neq \varepsilon$, $|xy| \leq P$ such that $\exists i,\ xy^iz \notin L,\ i \geq 0$

$i = ?\ \notin L$

$i = ?\ \in L$

$L = \{ ww \mid w \in \{0,1\}^* \}$

Let's assume, L is a regular Language.
Then let p be the pumping length for L.

Now, find one long string
s, $s \in L$ and $|s| \geq p$
**

$0^p 1 0^p 1$

$0^p 1^p$

Now, we take a string

$s = 0^p 0^p \in L$ ✗ $0^p 1^p 0^p 1^p$

$|s| = 2p \geq p$

So, s can be split into xyz such that $|y| > 0$, $|xy| \leq p$ and $xy^i z \in L$, $i \geq 0$

$$\underbrace{000}_{x} \ldots \underbrace{\ldots 000}_{y} \overset{p}{\vdots} \underbrace{000 \ldots 000}_{z}$$

$0^p 0^p$

$i = 2$, xy^2z

$xy^2z = xyyz$

$= 0^{p+|y|} 0^p$

$s = xyz = 0^p 0^p$

$xyyz = 0^{p+|y|} 0^p$

$p + |y| \neq p$

~~p = 4~~ [This is not how we prove ... $0^4 0^4$

$\underbrace{00}_{x}|\underbrace{0}_{y}|\underbrace{0\,0000}_{z}$

$xyyz \Rightarrow 00\ 00\ 00000$

$xyyyz$

$\underline{0}\ \underbrace{00000}_{w}0 \mid 0\ \underbrace{00000}_{w}$

xz $\quad \underbrace{000}_{w} \mid \underbrace{000}_{w}$

$p = 4 \qquad 0^4 0^4$

$\underbrace{0}_{x} \mid \underbrace{\cancel{00}0}_{y} \mid 0 \ \vdots\ p\ \underbrace{0000}_{z}$

$xyyz \quad 0\ 0000 \mid 00000$

$\forall xyz$

$\underbrace{\quad}_{x} \mid \underbrace{\quad}_{y} \mid \underbrace{\quad}_{z}$

$\exists i$

$xyyz \in L$

$xyyyz \notin L$ ✓

$|xy| \le p$

✓ $S = 0^p 1 0^p 1 \rightarrow$ $|S| \ge p$, $|2p+2| \ge p$, $S \in L$

$\underbrace{000}_{x} \ldots \underbrace{\ldots 000}_{y} \mid \overset{p}{\underbrace{1\ 000 \ldots\ldots 1}_{z}}$

$x \qquad y \qquad z$

✓ first p characters are 0, $\underline{y \text{ consist}}$ of only 0s

Then for $\underline{i = 2}$, $\underline{xy^2z}$ will be

$x\underline{y}z = 0^p 1 0^p 1 = \underbrace{0^p}_{w} \underbrace{1 0^p 1}_{w}$

$x\underline{y}yz =$

$xy^2z = xyyz = \underbrace{0^{p+|y|}1}_{w}\ \underbrace{0^p 1}_{w}$

Since $p + |y| \neq p$, so we can't split the string $\underline{\text{in the ww form}}$

$xyyz \notin L$

So, L is not regular language.

$p = 4$ ×

$0^4 1 0^4 1$

$\underbrace{0000}_{y}|\overset{p}{1}\ 00001$

$xyyz$ $\underbrace{0\,00000\,1}_{w}\ \underbrace{00001}_{w}$

$$L = \{ 1^{n^2} \mid n \geq 0 \}$$

$1^{0^2} = 1^0 = \varepsilon$

$1^{1^2} = 1^1 = 1$

$1^{2^2} = 1^4 = 1111$

$1^{3^2} = 1^9 = 111\ldots11$

1^m $\quad 1^m \to$ length of string

how many 1s are there

1^{n^2}

Solution:

Let's assume L is a regular language. Then let p be the pumping length for L.

Let's,

$$W = 1^{p^2} \in L \quad p^2 \to \text{length}$$

$$|W| = p^2 \geq p$$

$i = 2$, $xyyz$

$$1^{p^2} = xyz$$

$$xy^2z = xyyz$$

$$= 1^{p^2 + |y|} \in L, \text{ if}$$

$\to (p^2 + |y|)$ is perfect square

1^{p^2} → perfect square

$1^{\underbrace{p^2 + |y|}} \in L$

$$p^2 + |y| \geq (p+1)^2$$

$$\Rightarrow |y| \geq (p+1)^2 - p^2$$

$$\Rightarrow |y| \geq 2p + 1 \quad \text{---} (1)$$

$2^2 + |y|$

3^2

4^2

p^2

$(p+1)^2$

however, $|xy| \leq p$

$\Rightarrow |y| \leq p$ --- (2)

$p^2 + 2p + 1 - p^2$

$= 2p + 1$

from (1) & (2) $\rightarrow$

$$p \geq |y| \geq 2p+1$$

So, we have a contradiction. So, $p^2 + |y|$ is not a perfect square

$$W = 1^{p^2}$$

$$xyz = 1^{p^2}$$

$$xyyz = 1^{p^2+|y|}$$

$|xy| \leq p$

$|y| \leq p$

p^2

$(p+1)^2$

$$\Rightarrow p^2 < p^2 + |y| \leq p^2 + p < p^2 + 2p + 1 < (p+1)^2$$

$$\Rightarrow p^2 < p^2 + |y| < (p+1)^2$$

$|y| \leq p$

$$\underbrace{p^2 < p^2 + |y|}_{y \neq \varepsilon} \leq p^2 + \underline{p} < p^2 + \underbrace{2p+1}$$

$$< (p+1)^2$$

$$p^2 < \underbrace{p^2 + |y|} < (p+1)^2$$

$L = \{ 0^{3^n}, \underline{n \geq 0} \}$ $\quad n = 0$

$L \rightarrow$ regular Language
pumping length $\underline{p}$

$0^{3^0} = 0^1 = 0$

$0^{3^1} = 0^3$
$= 000$

$0^{3^2} = 0^9$
$= 000\ldots$

$W = 0^{3^p} \in L$

$|W| = \underline{3}^p \geq p$

$xyz = 0^{3^p}$

$xy^2z = 0^{3^p + |y|} \in L$

$3^p + |y|$ is a power of 3 $\quad 3^p$

$3^p + |y| \geq 3^{p+1}$

$|y| \geq 2 . 3^p$ — (1)

$3^{p+1} - 3^p$

$3^p . 3 - 3^p$

$3^p (3 - 1)$

$$|xy| \leq P$$

$$\therefore \; |y| \leq P \quad \text{—— (2)}$$

$$\underbrace{P \geq |y| \geq 2.3^{P}}$$

Another sol:

$$\underbrace{3^{P} < \qquad\qquad < 3^{P+1}}$$

$$L = \{ w w^{R} \mid w \in \{0,1\}^{*} \}$$

ww^R → even Palindrome

w^R reverse of w

$W = 10100$
$W^R = 00101$

1010000101

$W = 100$
$W^R = 001$

100001

$$S = \underbrace{0^{P}}_{w} 1 \, \underbrace{1^{P} 0}_{w^R} \, x \notin L$$

$$1^{P} = 0 \, 1^{P} 1^{P} 0 \in L \quad |w| \geq P$$

$\underbrace{011}_{y} \mid \underbrace{\ldots\ldots}_{z} \mid 1$

$W = 0^P 1 1 0^P$

P

$00 \ldots\ldots 00 \, 1100 \ldots\ldots 0$

$\underbrace{\quad}_{x} \; \underbrace{\quad}_{y} \; \underbrace{\qquad\qquad}_{z}$

y consist of only 0s, $|xy| \leq P$

$xyyz = 0^{P+|y|} 1 1 0^P$

$|y| \neq \varepsilon$ $\quad 10^P$ is not reverse of $0^{P+|y|} 1$

P

w contains more 0s than w^R

$0000 \ldots \mid 11 \ldots 1$

$n = m$

$n > m$

$L = \{ 0^n 1^m , \; n != m \}$

$n < m$

$\underbrace{0^{P} \, 1^{P+1}}$ | H.W $|xy| \leq P$

00

000

$0^{P+1} \; 1^P$ $\qquad 000 \ldots \mid 011 \ldots$

$P + |y| > P+1$

$P + |y| \geq P+1$