Assignment-5 Solution

#1 (a) From the given equations, we find:

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 3 & -2 \\ 3 & -1 & -1 \end{pmatrix}; \quad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and } b = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

(b) Here: $\det A = 2(-3-2) - (-2)(-1+6) + 1(-1-9)$
$= -\cancel{10} + \cancel{10} - 10 = -10 \neq 0$

Since $\det A \neq 0$, this system has a unique solution.

(c) Here the Augmented matrix of A is

$$Aug(A) = \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right)$$

Now, $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{2} = 0.5$ and $m_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2} = 1.5$

So, the first row operation ~~gives~~ are:

$r_2 \to r_2' = r_2 - m_{21} r_1 = (1 \quad 3 \quad -2 \quad 1) - 0.5(2 \quad -2 \quad 1 \quad -3)$
$= (0 \quad 4 \quad -2.5 \quad 2.5)$

and $r_3 \to r_3' = r_3 - m_{31} r_1 = (3 \quad -1 \quad -1 \quad 2) - 1.5(2 \quad -2 \quad 1 \quad -3)$
$= (0 \quad 2 \quad -2.5 \quad 6.5)$

So, $Aug(A') = \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -2.5 & 2.5 \\ 0 & 2 & -2.5 & 6.5 \end{array}\right)$

Now we apply the 2nd row operation. The only row multiplier is:

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{2}{4} = 0.5$$

So, the second row operation is:

$$r_3 \to r_3' = r_3 - m_{32} r_2 = (0 \quad 2 \quad -2.5 \quad 6.5) - 0.5(0 \quad 4 \quad -2.5 \quad 2.5)$$
$$= (0 \quad 0 \quad -1.25 \quad 5.25)$$

Therefore, $$\mathrm{Aug}(A'') = \left(\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -2.5 & 2.5 \\ 0 & 0 & -1.25 & 5.25 \end{array}\right)$$

Finally, the upper triangular matrix U is given by

$$\boxed{U = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 4 & -2.5 \\ 0 & 0 & -1.25 \end{pmatrix}}$$

(d) From the previous part, we write:

$$\begin{pmatrix} 2 & -2 & 1 \\ 0 & 4 & -2.5 \\ 0 & 0 & -1.25 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2.5 \\ 5.25 \end{pmatrix}$$

Therefore: $-1.25z = 5.25 \Rightarrow \boxed{z = -4.2}$

Also, $4y - 2.5z = 2.5 \Rightarrow y = \frac{2.5 + 2.5(-4.2)}{4} \Rightarrow \boxed{y = -2}$

and $2x - 2y + z = -3$

$$\cancel{\Rightarrow x = \frac{-3 + 2x + 2y}{2} = -3 - 2(}$$

$$\Rightarrow x = \frac{-3 + 2y - z}{2} = \frac{-3 + 2(-2) - (-4.2)}{2}$$

$$\Rightarrow \boxed{x = -1.4}$$

#2 From the given equations, we identify:

(a) $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$; $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Since all ~~dig~~ diagonal elements of A are non-zero, we can calculate all row multipliers. Hence, there is no Pivoting problem. ✓

(b) Since A is a 3×3 matrix, there will be only two Frobenius matrices, $F^{(1)}$ and $F^{(2)}$ due to 1st and 2nd row operations respectively. ✓

For 1st row multiplication, the row multipliers are:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2 \quad \text{and} \quad m_{31} = \frac{a_{31}}{a_{11}} = \frac{-1}{1} = -1 \checkmark$$

Hence:

$$r_2 \to r_2' = r_2 - m_{21} r_1 = (2 \;\; -1 \;\; 1) - 2(1 \;\; 2 \;\; -1)$$
$$= (0 \;\; -5 \;\; 3)$$
$$r_3 \to r_3' = r_3 - m_{31} r_1 = (-1 \;\; 1 \;\; 2) - (-1)(1 \;\; 2 \;\; -1)$$
$$= (0 \;\; 3 \;\; 1)$$

Therefore, the 1st Frobenius matrix $F^{(1)}$ is

$$F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} \Rightarrow \boxed{F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \checkmark$$

We now have:

$$A^{(2)} = F^{(1)} A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

We now apply 2nd row operation on $A^{(2)}$. The row multiplier is

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{3}{-5} = -\frac{3}{5}$$

and hence the 2nd row operation gives.

$$r_3 \to r_3' = r_3 - m_{32} r_2 = \begin{pmatrix} 0 & 3 & 1 \end{pmatrix} - \left(-\tfrac{3}{5}\right)\begin{pmatrix} 0 & -5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 14/5 \end{pmatrix}$$

Therefore, we obtain the 2nd Frobenius matrix $F^{(2)}$ as

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix} \Rightarrow \boxed{F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/5 & 0 \end{pmatrix}}$$

(c) Using the results of the previous part, we obtain.

$$L = (F^{(1)})^{-1}(F^{(2)})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} \Rightarrow \boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/5 & 1 \end{pmatrix}}$$

And $U \equiv A^{(3)} = F^{(2)} A^{(2)}$

$$\Rightarrow \boxed{U = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 14/5 \end{pmatrix}}$$

(d) From $Ly = b$, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/5 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Solving for y's we get: $y_1 = 0$; $y_2 = 1$ and $y_3 = 13/5$

Now using: $Ux = y \Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 14/5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 13/5 \end{pmatrix}$, we calculate,

the solution, and obtain.

$$\boxed{z = \frac{13}{14}}, \quad \boxed{y = \frac{5}{14}} \quad \text{and} \quad \boxed{x = \frac{3}{14}}$$