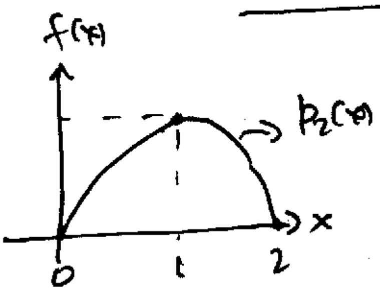
Solution: Assignment #2

#1

(a)



The curve is facing down, shifted right and also shifted upward. Writing $p_2(x)$ as $a(x-b)^2+c$, the parameters $a<0$, $b=1$ and $c=1$.

(b) $x_0=0,\ x_1=1,\ x_2=2$ and $f(x_0)=0,\ f(x_1)=1$ and $f(x_2)=0$

$$\therefore l_0(x) = \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2} = \frac{x-1}{0-1}\cdot\frac{x-2}{-2} = \frac{1}{2}(x-1)(x-2).$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2} = \frac{x-0}{1-0}\cdot\frac{x-2}{1-2} = -x(x-2).$$

and
$$l_2(x) = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1} = \frac{x-0}{2-0}\cdot\frac{x-1}{2-1} = \frac{1}{2}x(x-1).$$

(c)
$$p_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$
$$= \cancel{\left[\tfrac{1}{2}(x-1)(x-2)\right]\times 0}^{\,0} + (-x)(x-2)\cdot 1 + \cancel{\tfrac{1}{2}(x)(x-1)\times 0}^{\,0}$$
$$= -x^2+2x \quad \text{(Natural basis).}$$
$$= -(x^2-2x+1)+1 = -(x-1)^2+1$$

Comparing to $a(x-b)^2+c$, we find: $\underline{a=-1<0}$, $\underline{b=1}$ & $\underline{c=1}$ as expected

the properties found in Part-(a).

Yes!!! The parameters agree with the properties found in Part-(a).

#2 Here: $f(x)=\cos x$; $x_0=0,\ x_1=\pi/2,\ x_2=\pi$ and added $x_3=-\pi/2$.

$\Rightarrow f(x_0)=1;\ f(x_1)=0;\ f(x_2)=-1$ & $f(x_3)=0$

(a) $x_0=0: f[x_0]=1=a_0$

$$f[x_0,x_1]=a_1 = \frac{0-1}{\pi/2-0} = -\frac{2}{\pi}$$

$x_1=\pi/2: f[x_1]=0$

$$f[x_0,x_1,x_2]=a_2 = \frac{-\frac{2}{\pi}-\left(-\frac{2}{\pi}\right)}{\pi-0} = \frac{0}{\pi} = 0$$

$$f[x_1,x_2] = \frac{-1-0}{\pi-\pi/2} = -\frac{2}{\pi}$$

$x_2=\pi: f[x_2]=-1$

Therefore,
$$p_2(x) = a_0 + a_1(x-x_0) + \cancel{a_2(x-x_0)(x-x_1)}^{\,0}$$
$$= 1+\left(-\tfrac{2}{\pi}\right)(x-0)+0 = 1-\tfrac{2}{\pi}x$$
$$\Rightarrow \boxed{p_2(x) = -\frac{2}{\pi}x+1}$$

(b) $p_2(\pi/4) = -\frac{2}{\pi}\left(\frac{\pi}{4}\right)+1 = -\frac{1}{2}+1 = \frac{1}{2} = 0.5$ & $f(\pi/4) = \cos\pi/4 = 0.707$

$\therefore$ Relative Error $= \left|\frac{f(\pi/4) - p_2(\pi/4)}{f(\pi/4)}\right| \times 100\% = \frac{0.707 - 0.5}{0.707} \times 100\%$

$= 29.3\%$. ✓

(c) Now we add the Node $x_3 = -\pi/2$. From Part (a), we copy the calculation of $a_0, a_1 \& a_2$ as below and add x_3:

$x_0 = 0 : f[x_0] = a_0 = 1$

$x_1 = \pi/2 : f[x_1] = 0$ $\quad a_1 = -\frac{2}{\pi}$

$x_2 = \pi : f[x_2] = -1$ $\quad = -\frac{2}{\pi}$ $\quad = a_2 = f[x_0,x_1,x_2] = 0$

$x_3 = -\pi/2 : f[x_3] = 0$ $\quad = f[x_2,x_3] = \frac{0-(-1)}{-\pi/2-\pi} = -\frac{2}{3\pi}$ $\quad = f[x_1,x_2,x_3] = \frac{-\frac{2}{3\pi}+\frac{2}{\pi}}{-\frac{\pi}{2}-\frac{\pi}{2}} = -\frac{4}{3\pi^2}$

$f[x_0,x_1,x_2,x_3] = a_3 = \frac{-\frac{4}{3\pi^2}-0}{-\pi/2-0} = \frac{8}{3\pi^3}$ ✓

So, the interpolation polynomial after adding x_3-node is

$$p_3(x) = p_2(x) + a_3(x-x_0)(x-x_1)(x-x_2)$$
$$= -\frac{2}{\pi}x+1 + \left(\frac{8}{3\pi^3}\right)(x-0)(x-\pi/2)(x-\pi)$$

$$\Rightarrow \boxed{p_3(x) = -\frac{2}{\pi}x+1+\frac{8}{3\pi^3}x(x-\pi/2)(x-\pi)}$$ ✓

(d) Here: $I = [-\pi/2, \pi] = [-1.571, 3.141]$ & $\xi, x \in I$, Also, $n = 3$.

So Upperbound $\leq \frac{1}{4!}|f^{(4)}(\xi)|_{max}|\omega_4(x)|_{max}$

Now, $|f^{(4)}(\xi)| = \left|\frac{d^4}{d\xi^4}\cos\xi\right|_{max} = |\cos\xi|_{max} = 1.$

and $W_4(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3) = x(x-\pi/2)(x-\pi)(x+\pi/2)$

$= x^4 - \pi x^3 - \frac{\pi^2}{4}x^2 + \frac{\pi^3}{4}x.$

$\therefore W_4'(x) = 0 \Rightarrow 4x^3 - \frac{\pi^2}{2}x - 3\pi x^2 + \frac{\pi^3}{4} = 0 \Rightarrow x = \begin{cases}\pi/4 \\ \frac{\pi}{4}+\frac{\sqrt{5}}{4}\pi \\ \frac{\pi}{4}-\frac{\sqrt{5}}{4}\pi\end{cases}$ (Using MAPLE)

$= 0.785, 2.5416, -0.9708$

But: $|W(0.785)| = \cancel{1.556}$ 3.425 ; $|W(2.5416)| = \cancel{23.94}$ 6.088 ;

and $|W(-0.9708)| = \cancel{4.546}$ 6.088 . $= |W_4(x)|_{max}$

$\therefore \boxed{\text{Upperbound} \leq \frac{1}{4!}(1)(23.94)\ 6.088 = \cancel{0.9975}\ 0.2537}$ ✓