Solution: Assignment - 3

#1 $h_k'(x) = \frac{d}{dx}\left(h_k(x)\right) = -2l_k'(x_k)\cdot l_k^2(x) + \left(1-2(x-x_k)\,l_k'(x_k)\right)2l_k(x)\,l_k'(x)$

Putting $x = x_k$ and using $l_k(x_k) = 1 \Rightarrow l_k^2(x_k) = 1$, we get

$$h_k'(x_k) = -2l_k'(x_k)\cdot \underbrace{l_k^2(x_k)}_{=1} + \Big(1-\underbrace{2(x_k-x_k)\,l_k'(x_k)}_{=0}\Big)\,2\underbrace{l_k(x_k)}_{=1}\,l_k'(x_k)$$

$$= -2l_k'(x_k) + 2l_k'(x_k) = 0.$$

Now Putting $x = x_j$ $(\forall j \neq k)$, and using $l_k(x_j) = 0$ if $j \neq k$, we get

$$h_k'(x_j) = -2l_k'(x_k)\cdot\underbrace{l_k^2(x_j)}_{=0} + \left(1-2(x_j-x_k)\,l_k'(x_k)\right)2\underbrace{l_k(x_j)}_{=0}\,l_k'(x_j)$$

$$= 0$$

Therefore $\boxed{h_k'(x_j) = 0 \quad \forall j,k \quad (\text{when } j=k \text{ and } j \neq k)}$ IK

#2 $f(x) = xe^{-3x} + x^2$; $x_0 = 2$ and $h = 0.1$.

(a) $$D_h(x_0) = D_{0.1}(x_0) = \frac{(x_0+h)e^{-3(x_0+h)} + (x_0+h)^2 - (x_0-h)e^{-3(x_0-h)} - (x_0-h)^2}{2h}$$

$$= \frac{2.1\,e^{-3(2.1)} + (2.1)^2 - (1.9)e^{-3(1.9)} - (1.9)^2}{2(0.1)}$$

$$\Rightarrow \boxed{D_{0.1}(2) = 3.9875}$$ IK

(b) Here $f'(x_0) = \frac{d}{dx}\left(xe^{-3x}+x^2\right)\Big|_{x_0} = e^{-3x_0} - 3x_0e^{-3x_0} + 2x_0$

$\therefore f'(2) = e^{-3(2)} - 3(2)e^{-3(2)} + 2(2) = 3.9876.$

$\therefore$ Truncation Error $= |f'(x_0) - D_h| = |3.9876 - 3.9875| = \underline{0.0001}$. IK

(c) $D_{0.1}^{(1)}(x_0) = \dfrac{4D_{0.05} - D_{0.1}}{3}$ and $h/2 = 0.05$

and $$D_{0.5}(2) = \frac{(x_0+h/2)\,e^{-3(x_0+h/2)} + (x_0+h/2)^2 - (x_0-h/2)e^{-3(x_0-h/2)} - (x_0-h/2)^2}{2(h/2)}$$

$$= \frac{2.05\,e^{-3(2.05)} + (2.05)^2 - (1.95)e^{-3(1.95)} - (1.95)^2}{2(0.05)} = 3.9876$$

$\therefore D_{0.1}^{(1)}(2) = \dfrac{4(3.9876) - 3.9875}{3} \Rightarrow \boxed{D_{0.1}^{(1)}(2) = 3.9876}$ LK

$\therefore \boxed{\text{Truncation Error} = |f'(x_0) - D_{0.1}^{(1)}(2)| = 0}$ LK (within 5 decimals)

#3. From the lecture during class, we already have.

(a) $D_h^{(1)}(x_0) = f'(x_0) - \dfrac{h^4}{4 \cdot 5!} f^{(5)}(x_0) - \dfrac{5}{16 \cdot 7!} h^6 f^{(7)}(x_0) + \cdots$

Now $D_{h/2}^{(1)}(x_0) = f'(x_0) - \dfrac{h^4 f^{(5)}(x_0)}{2^4 \cdot 4 \cdot 5!} - \dfrac{5}{16 \cdot 7!} \dfrac{h^6}{2^6} f^{(7)}(x_0) + \mathcal{O}(h^8)$

$\therefore 2^4 D_{h/2}^{(1)}(x_0) = 2^4 f'(x_0) - \dfrac{h^4 f^{(5)}(x_0)}{4 \cdot 5!} - \dfrac{5}{16 \cdot 7!} \dfrac{h^6}{2^2} f^{(7)}(x_0) + \mathcal{O}(h^8)$

So $2^4 D_{h/2}^{(1)}(x_0) - D_h^{(1)}(x_0) = (2^4 - 1) f'(x_0) - \left(\dfrac{1}{2^2} - 1\right) \dfrac{5}{16 \cdot 7!} h^6 f^{(7)}(x_0) + \mathcal{O}(h^8)$

Hence, the 2nd order Richardson extrapolation is

$$D_h^{(2)}(x_0) \equiv \frac{2^4 D_{h/2}^{(1)}(x_0) - D_h^{(1)}(x_0)}{2^4 - 1} = \frac{2^4 D_{h/2}^{(1)}(x_0) - D_h^{(1)}(x_0)}{15}$$

$$= \frac{1}{15}\left[15 f'(x_0) + \frac{3}{4} \cdot \frac{5}{16 \cdot 7!} h^6 f^{(7)}(x_0) + \mathcal{O}(h^8)\right]$$

$\Rightarrow \boxed{D_h^{(2)}(x_0) = f'(x_0) + \dfrac{h^6}{2^6 \cdot 7!} f^{(7)}(x_0) + \mathcal{O}(h^8)}$ LK

(b) The upperbound of Error for $D_h^{(1)} = \left|\dfrac{h^4}{480} f^{(5)}(\xi)\right|_{\max,\ \xi \in [x_0 - h, x_0 + h]}$

Here, $x_0 = 1$ and $h = 0.1 \Rightarrow \xi \in [0.9, 1.1]$ and $f(\xi) = \xi^2 \ln \xi$.

So $f'(\xi) = \xi + 2\xi \ln \xi \Rightarrow f''(\xi) = 1 + 2\ln\xi + 2\xi\left(\frac{1}{\xi}\right) = 3 + 2\ln\xi$

$f'''(\xi) = 0 + \dfrac{2}{\xi} = \dfrac{2}{\xi} \Rightarrow f^{(4)}(\xi) = -\dfrac{2}{\xi^2}$ and $f^{(5)}(\xi) = \dfrac{4}{\xi^3}$

$\therefore$ Upper bound $= \left|\dfrac{h^4}{480} \cdot \dfrac{4}{\xi^3}\right|_{\max;\ \xi \in [0.9, 1.1]}$

$$= \frac{(0.1)^4}{480} \cdot \frac{4}{(0.9)^3}$$

$\Rightarrow \boxed{\text{Upperbound} = 1.14312 \times 10^{-6}}$ LK