1. **Find the rank of the matrix**

a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

b) $B = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

Solution:

a) Subtract row 1 multiplied by 2 from row 2: $R_2 = R_2 - 2R_1$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 3 & 5 & 7 \end{bmatrix}$$

Subtract row 1 multiplied by 3 from row 3: $R_3 = R_3 - 3R_1$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

Multiply row 2 by -1 : $R_2 = -R_2$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

Subtract row 2 multiplied by 2 from row 1 : $R_1 = R_1 - 2R_2$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

Add row 2 to row 3: $R_3 = R_3 + R_2$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the rank is 2.

b) Swap the rows 1 and 2 :

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$

Subtract row 1 multiplied by 3 from row 3: $R_3 = R_3 - 3R_1$.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$$

Subtract row 2 multiplied by 2 from row 1 : $R_1 = R_1 - 2R_2$.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$$

Add row 2 multiplied by 5 to row 3: $R_3 = R_3 + 5R_2$.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Divide row 3 by 2: $R_3 = \frac{R_3}{2}$.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Add row 3 to row 1: $R_1 = R_1 + R_3$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Subtract row 3 multiplied by 2 from row 2: $R_2 = R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

So, the rank is 3.

2. **Let V be a subset of $\mathbf{R}^4$ consisting of vectors that are perpendicular to vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{and}\ \boldsymbol{c}$ where $\boldsymbol{a} = <1, 0, 1, 0>, \boldsymbol{b} = <1, 1, 0, 0>, \boldsymbol{c} = <0, 1, -1, 0>$,**

   Namely, $V = \{x \in R^4 | a^T x = 0, b^T x = 0, and\ C^T x = 0\}$

   a. **Prove that V is a subspace of R^4**

   b. **Find a basis for V**

   c. **Determine the Dimension of V**

   Solution:

(a) Prove that V is a subspace of $\mathbb{R}^4$.

Observe that the conditions

$$\mathbf{a}^{\mathrm{T}}\mathbf{x} = 0, \mathbf{b}^{\mathrm{T}}\mathbf{x} = 0, \text{ and } \mathbf{c}^{\mathrm{T}}\mathbf{x} = 0$$

can be combined into the following matrix equation, $A\mathbf{x} = \mathbf{0}$

Where, $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

and $\mathbf{0}$ is the three-dimensional zero vector.

Note that the rows of the matrix A are $\mathbf{a}^{\mathrm{T}}, \mathbf{b}^{\mathrm{T}}$, and $\mathbf{c}^{\mathrm{T}}$.

It follows that the subset V is the null space $\mathcal{N}(A)$ of the matrix A.

Being the null space, $V = \mathcal{N}(A)$ is a subspace of $\mathbb{R}^4$.

(b) Find a basis of V.

In the proof of Part (a), we saw that $V = \mathcal{N}(A)$.

To find a basis, we determine the solutions of $A\mathbf{x} = \mathbf{0}$. Applying elementary row operations to the augmented matrix, we see that

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the general solution is given by, $x_1 = -x_3, x_2 = x_3$.

The vector form solution is, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Hence, we have,

$$
\begin{aligned}
V &= \mathcal{N}(A) \\
&= \left\{ \mathbf{x} \in \mathbb{R}^4 \;\middle|\; \mathbf{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right. \\
&= \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\end{aligned}
$$

Let $B := \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Then we just showed that B is a spanning set for V.

It is straightforward to see that B is linearly independent.

Hence B is a basis for V.

(c) Determine the dimension of V.

As the basis B for V that we obtained in Part (b) consists of two vectors, the dimension of the subspace V is $\dim(V) = 2$.

3. **Determine which of the following is a subspace of R^3.**

a) $x + 2y - 3z = 4$

b) $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{4}$

c) $x + y + z = 0$ and $x - y + z = 1$

d) $x = -z$ and $x = z$

e) $x^2 + y^2 = z$

f) $\frac{x}{2} = \frac{y-3}{5}$

4. **Suppose $rref(R_0) = A$ where $R_0 = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 2 & 6 & 10 & 1 & 16 \\ 3 & 9 & 15 & 1 & 23 \end{bmatrix}$. Show that –**

   a) **The row space has dimension 2, matching the rank**
   b) **The column space of R_0 has also dimension $r = 2$**
   c) **The null space of R_0 has dimension 3**
   d) **The null space of R_0^T, which can also be called the left null space of R_0; has dimension 1.**

Solution:

Subtract row 1 multiplied by 2 from row 2: $R_2 = R_2 - 2R_1$.

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 3 & 9 & 15 & 1 & 23 \end{bmatrix}$$

Subtract row 1 multiplied by 3 from row 3: $R_3 = R_3 - 3R_1$.

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Since the element at row 2 and column 2 (pivot element) equals 0 , we need to swap the rows.
Find the first nonzero element in column 2 under the pivot entry.
As can be seen, there are no such entries. Move to the next column.
Since the element at row 2 and column 3 (pivot element) equals 0 , we need to swap the rows.
Find the first nonzero element in column 3 under the pivot entry.
As can be seen, there are no such entries. Move to the next column.
Subtract row 2 from row 3: $R_3 = R_3 - R_2$.

$$rref(R_0) = A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) The row space is a space spanned by the nonzero rows of the reduced matrix.
   Thus, the row space is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.
   Therefore, the dimension of row space is 2.

b) The column space is a space spanned by the columns of the initial matrix that correspond to the pivot columns of the reduced matrix.

   Thus, the column space is $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Therefore, the dimension of column space is 2.

c) To find a basis for the null space, we form an augmented matrix by appending a column of zeros to the right, and then put this matrix in reduced row-echelon form.
   We begin with the matrix:

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is already in reduced row-echelon form.

Convert the matrix equation back to an equivalent system:

$$\begin{aligned} x_1 + 3x_2 + 5x_3 + 7x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}$$

Collect terms into vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -7x_5 \\ 0 \\ 0 \\ -2x_5 \\ x_5 \end{bmatrix}$$

Factor out variables on the right side:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Thus, a basis for the null space is:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

So. The dimension of the null space is 3.

d) The left null space of A is basically the null space of A^T. So, we will calculate the null space of A^T

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 0 \end{bmatrix}$$

We begin with the matrix: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 2 & 0 & 0 \end{bmatrix}$

Add -3 times row 1 to row 2 : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 2 & 0 & 0 \end{bmatrix}$

Add -5 times row 1 to row 3 : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 2 & 0 & 0 \end{bmatrix}$

Add -7 times row 1 to row 5 : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$

Add -2 times row 2 to row 5 : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Convert the matrix equation back to an equivalent system:

$x_1 = 0$

$x_2 = 0$

Add an equation for each free variable:

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 \qquad &= 0 \\ x_3 &= x_3 \end{aligned}$$

Solve for each variable in terms of the free variables:

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 \qquad &= 0 \\ x_3 &= x_3 \end{aligned}$$

Collect terms into vectors: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$

Factor out variables on the right side: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus, a basis for the null space is: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

So, the dimension of the left null space is 1.

5. Find a basis for each of the four fundamental subspaces associated with the matrix.

$$\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution:

Subtract row 1 from row 3: $R_3 = R_3 - R_1$. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Subtract row 2 multiplied by 2 from row 1: $R_1 = R_1 - 2R_2$. $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So $rref\ (A) = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Colum Space: The column space is a space spanned by the columns of the initial matrix that correspond to the pivot columns of the reduced matrix.

Thus, the column space is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Row Space: The row is a space spanned by the nonzero rows of the reduced matrix.

Thus, the row space is $\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Null Space: Convert the matrix equation back to an equivalent system:

$$\begin{aligned} x_1 + -2x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Add an equation for each free variable:

$$\begin{aligned} x_1 + -2x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Collect terms into vectors: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$

Factor out variables on the right side: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Thus, a basis for the null space is: $\left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$

Left Null Space: The left null space of a matrix A is basically the null space of A^T

So, we begin with determining A^T and arrange in the gauss elimination where the result is zero.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\-2&1&0&0\\1&0&0&0\end{bmatrix}$$

Add 2 times row 1 to row 3 : $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$

Add -1 times row 1 to row 4 : $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}$

Add -1 times row 2 to row 3 : $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

Convert the matrix equation back to an equivalent system: $\begin{aligned}x_1 &= 0\\ x_2 &= 0\end{aligned}$

Add an equation for each free variable: $\begin{aligned}x_1 &\quad= 0\\ x_2 &\quad= 0\\ = x_3 &\quad= x_3\end{aligned}$

Solve for each variable in terms of the free variables: $\begin{aligned}x_1 &= 0\\ x_2 &= 0\\ x_3 &= x_3\end{aligned}$

Collect terms into vectors: $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\x_3\end{bmatrix}=x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$

Thus, a basis for the null space is $\begin{bmatrix}0\\0\\1\end{bmatrix}$

6. **Let A be a real 7x3 matrix such that the null space is spanned by the vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$ *and* $\begin{bmatrix}1\\-1\\0\end{bmatrix}$. Find the rank of the matrix A.**

Solution: We first determine the nullity of A and deduce the rank of A by the rank-nullity theorem.

The null space $\mathcal{N}(A)$ of the matrix A is spanned by

$$\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \text{ and } \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

Let us find a basis of the null space $\mathcal{N}(A)$ among these vectors.

We use the "leading 1 method".

Form the matrix whose column vectors are these three vectors and we apply elementary row operations as follows.

$$\begin{bmatrix}1 & 2 & 1\\2 & 1 & -1\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix}1 & 2 & 1\\0 & -3 & -3\\0 & 0 & 0\end{bmatrix} \xrightarrow{-\frac{1}{3}R_2}$$
$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}.$$

The last matrix is in reduced row echelon form and the first and second column contain the leading 1 's.

Therefore, the first two vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$

form a basis of the null space $\mathcal{N}(A)$.

Hence the nullity, which is the dimension of $\mathcal{N}(A)$, is 2 .

Since the size of the matrix A is 7×3, the rank-nullity theorem gives

$$3 = \text{ nullity of } A + \text{rank } \text{ of } A.$$

Thus, the rank of A is 1 .

7. Let V be a subset of the vector space R^n consisting only of the zero vector of R^n, Namely $V = \{0\}$. Then prove that V is a subspace of R^n.

Solution: To prove that $V = \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, we check the following subspace criteria.

Subspace Criteria

(a) The zero vector $\mathbf{0} \in \mathbb{R}^n$ is in V.

(b) If $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} + \mathbf{y} \in V$.

(c) If $\mathbf{x} \in V$ and $c \in \mathbb{R}$, then $c\mathbf{x} \in V$.

Condition (a) is clear since V consists of the zero vector $\mathbf{0}$.

To check condition (b), note that the only element in $V = \{\mathbf{0}\}$ is $\mathbf{0}$. Thus if $\mathbf{x}, \mathbf{y} \in V$, then both $\mathbf{x}, \mathbf{y}$ are $\mathbf{0}$. Hence

$$\mathbf{x} + \mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} \in V$$

and condition (b) is met.

To confirm condition (c), let $\mathbf{x} \in V$ and $c \in \mathbb{R}$. Then $\mathbf{x} = \mathbf{0}$.

We have

$$c\mathbf{x} = c\mathbf{0} = \mathbf{0} \in V$$

and condition (c) is satisfied.

Hence we have checked all the subspace criteria, and hence the subset $V = \{\mathbf{0}\}$ consisting only of the zero vector is a subspace of $\mathbb{R}^n$.

What's the dimension of the zero-vector space?

What's the dimension of the subspace $V = \{\mathbf{0}\}$?

The dimension of a subspace is the number of vectors in a basis. So let us first find a basis of V.

Note that a basis of V consists of vectors in V that are linearly independent spanning set. Since 0 is the only vector in V, the set $S = \{\mathbf{0}\}$ is the only possible set for a basis.

However, S is not a linearly independent set since, for example, we have a nontrivial linear combination $\mathbf{1} \cdot \mathbf{0} = \mathbf{0}$.

Therefore, the subspace $V = \{\mathbf{0}\}$ does not have a basis.

Hence the dimension of V is zero.

8. **Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and consider the following subset V of the 2-dimensional vector space R^2, Namely $V = \{x \in R^2 | Ax = 5x\}$**

   a) **Prove that the subset V is a subspace of R^2**

   b) **Find a basis for V and determine the dimension of V**

Solution: To prove that V is a subspace of R^2, we need to show that it satisfies the three conditions of a subspace:

1. V contains the zero vector:
   Since A is a 2x2 matrix, we can compute A.0 = 0, where 0 is the 2-dimensional zero vector. So, 0 is an element of V, and V is not empty.
2. V is closed under vector addition:
   Let x, y be vectors in V, so we have Ax = 5x and Ay = 5y. We want to show that x + y is also in V, i.e., $A(x + y) = 5(x + y)$.
   Using matrix multiplication, we have:
   $$A(x + y) = Ax + Ay = 5x + 5y = 5(x + y)$$
   Therefore, x + y is also in V, and V is closed under vector addition.
3. V is closed under scalar multiplication:
   Let x be a vector in V, and let c be a scalar. We want to show that cx is also in V, i.e., $Acx = 5(cx)$.
   Using matrix multiplication, we have:
   $$Acx = c(Ax) = c(5x) = 5(cx)$$
   Therefore, cx is also in V, and V is closed under scalar multiplication.
   Since V satisfies all three conditions of a subspace, V is a subspace of R^2.

To find a basis for V, we need to find a set of linearly independent vectors that span V. Since V is defined as the set of all vectors x such that $Ax = 5x$, we can rewrite this as $(A - 5I)x = 0$, where I is the 2x2 identity matrix.
So, we want to find the null space of the matrix A - 5I. Using row reduction, we have:

$$A - 5I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$$

$R_2 = R_2 + 3R_1$ => $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

R1 = -R1 => $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

So, the null space of A - 5I is spanned by the vector (1, 1), since any scalar multiple of this vector satisfies $(A - 5I)x = 0$. Therefore, a basis for V is {(1, 1)}, and the dimension of V is 1.

9. **The smallest subspace of R^3 containing the vectors (2, -3, -3) and (0, 3, 2) is the plane whose equation is $ax + by + 6z = 0$. Determine the value of a, b.**

Solution: Since the given subspace contains the vectors (2,-3,-3) and (0,3,2), it must contain all linear combinations of these vectors. So, any vector of the form c(2,-3,-3) + d(0,3,2) is also in the subspace, where c and d are scalars.

We want to find the equation of the plane that contains this subspace. We know that any vector in this subspace can be written as $(x, y, z) = c(2, -3, -3) + d(0,3,2)$, so we can write:

$$x = 2c$$

$$y = -3c + 3d$$

$$z = -3c + 2d$$

We want to find the equation of the plane that contains all such vectors. This means that we need to find values of a, b, and c such that the equation $ax + by + 6z = 0$ is satisfied for all such vectors.

Substituting the expressions for x, y, and z, we get:

$$a(2c) + b(-3c + 3d) + 6(-3c + 2d) = 0$$

Simplifying this equation, we get:

$$(2a - 3b - 18c) + (3b + 12d) = 0$$

Since this equation must hold for all values of c and d, we get the following system of equations:

$$2a - 3b - 18c = 0\,3b + 12d = 0$$

We want to solve for a and b in terms of constants. To do this, we can eliminate c and d from the system of equations. Solving the second equation for b, we get: $b = -4d$

Substituting this into the first equation, we get: $2a + 12d = 18c$

Simplifying this equation, we get: $a + 6d = 9c$

We can solve for a and b in terms of d and c by expressing d and c in terms of a and b: $d = -\frac{b}{4}\ c = \frac{a + 6d}{9} = \frac{a - \left(\frac{3}{2}\right)b}{9}$

Substituting these expressions into the original equation for the plane, we get: $ax + by + 6z = a(2c) + b(-3c + 3d) + 6(-3c + 2d) = 0$

Substituting the expressions for c and d, we get: $ax + by + 6z = a(\frac{a - \left(\frac{3}{2}\right)b}{9}) - b(\frac{a - \left(\frac{3}{2}\right)b}{3}) - 18(\frac{a - \left(\frac{3}{2}\right)b}{9})$

Simplifying this equation, we get: $ax + by + 6z = (2a - 3b)x + 6z = 0$

Therefore, the equation of the plane is $2a - 3b + 6z = 0$. Comparing this to the given equation $ax + by + 6z = 0$, we see that a = 2 and b = -3. So, the equation of the plane is $2x - 3y + 6z = 0$.

10. Determine The matrix representation of the orthogonal projection operator taking R^3 onto the plane $x + y + z = 0$.

Solution: Let P be the orthogonal projection operator taking R^3 onto the plane $x + y + z = 0$. We can find the matrix representation of P by finding its action on the standard basis vectors $e_1 = (1,0,0), e_2 = (0,1,0), and\ e_3 = (0,0,1)\ of\ R^3$.

To project a vector v onto the plane $x + y + z = 0$, we need to find its projection onto a vector n=(1,1,1) that is normal to the plane. We can normalize this vector to get $\hat{n} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Then, the projection of v onto n̂ is given by (v·n̂)n̂, where · denotes the dot product. We can write this as:

$$proj_{\hat{n}}(v) = (\frac{v \cdot \hat{n}}{(\| \hat{n} \|)^2})\hat{n} = (\frac{v \cdot \hat{n}}{3})(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

The orthogonal projection of v onto the plane is obtained by subtracting this projection from v:

$$proj_P(v) = v - proj_{\hat{n}}(v) = v - (\frac{v \cdot \hat{n}}{3})(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

Now, we can find the matrix representation of P by expressing $proj_P(e_1), proj_P(e_2), and\ proj_P(e_3)$ as column vectors and putting them together as the columns of a matrix. We get:

$$proj_P(e_1) = (1,0,0) - \left(\frac{1}{3}\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

$$proj_P(e_2) = (0,1,0) - \left(\frac{1}{3}\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

$$proj_P(e_3) = (0,0,1) - (\frac{1}{3})(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$$

Therefore, the matrix representation of P with respect to the standard basis of R^3 is:

$$P = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

11. Let $\boldsymbol{u} = (\mathbf{8}, \sqrt{\mathbf{3}}, \sqrt{\mathbf{7}}, -\mathbf{1}, \mathbf{1})$ and $\boldsymbol{v} = (\mathbf{1}, -\mathbf{1}, \mathbf{0}, \mathbf{2}, \sqrt{\mathbf{3}})$. If the orthogonal projection of u onto v is $\frac{\boldsymbol{a}}{\boldsymbol{b}}\boldsymbol{v}$, then determine a and b.

Solution: Let u = (8, √3, √7, -1, 1) and v = (1, -1, 0, 2, √3). We want to find the orthogonal projection of u onto v, which is given by:

$$proj_v(u) = (\frac{u \cdot v}{(\| v \|)^2})\, v$$

where · denotes the dot product.

First, we need to calculate the dot product u·v:

$$u \cdot v = (8)(1) + (\sqrt{3})(-1) + (\sqrt{7})(0) + (-1)(2) + (1)(\sqrt{3}) = 6 + \sqrt{3}$$

Next, we need to calculate the norm ‖v‖ of v:

$$\| v \| = \sqrt{1^2 + (-1)^2} + \sqrt{0^2 + 2^2} + \left(\sqrt{3}\right)^2 = \sqrt{7}$$

Putting these together, we get:

$$proj\_v(u) = (\frac{6 + \sqrt{3}}{7})v$$

Therefore, $a = 6 + \sqrt{3}\ and\ b = 7$.

12. Find the point q in R^3 on the ray connecting the origin to the point (2, 4, 8) which is closest to the point (1, 1, 1).

Solution: The ray connecting the origin to the point (2, 4, 8) can be parameterized as:

$$r(t) = t(2,4,8)$$

where t is a scalar parameter. We want to find the point q on this ray that is closest to the point (1, 1, 1).

The distance between a point (x1, y1, z1) and a point (x2, y2, z2) is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

We want to minimize this distance for points of the form r(t), subject to the constraint that t is a nonnegative scalar. Therefore, we can minimize the square of the distance:

$$f(t) = (2t - 1)^2 + (4t - 1)^2 + (8t - 1)^2$$

This is a quadratic function of t, which can be expanded and simplified:

$$f(t) = 84t^2 - 52t + 7$$

The minimum value of this function occurs at the vertex of the parabola, which is at:

$$t = -\frac{-52}{2 * 84} = \frac{13}{21}$$

Therefore, the point q on the ray connecting the origin to (2, 4, 8) that is closest to (1, 1, 1) is:

$$q = r\left(\frac{13}{21}\right) = \left(\frac{13}{21}\right)(2,4,8) = \left(\frac{26}{21}, \frac{52}{21}, \frac{104}{21}\right)$$

13. Find the eigenvalues and eigenvectors of the following matrix A.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Show that these eigenvectors are perpendicular. [Hint: It will always be perpendicular when A is symmetric]

Solution:

Start from forming a new matrix by subtracting λ from the diagonal entries of the given matrix:

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

The determinant of the obtained matrix is $-\lambda(\lambda-3)(\lambda-1)$

Solve the equation $-\lambda(\lambda-3)(\lambda-1)=0$.

The roots are $\lambda_1=3, \lambda_2=1, \lambda_3=0$

Next, find the eigenvectors.

- $\lambda = 3$

$$A-\lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

We begin with the matrix: $\begin{bmatrix} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$

Multiply row 1 by $-1/2$: $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$

Add 1 times row 1 to row 2 : $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$

Multiply row 2 by -2 : $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$

Add 1 times row 2 to row 3 : $\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Add $-1/2$ times row 2 to row 1 : $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Convert the matrix equation back to an equivalent system: $\begin{matrix} x_1 & +-x_3 & =0 \\ x_2+2x_3 & & =0 \end{matrix}$

Collect terms into vectors: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \right\}$

The null space of this matrix is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

This is the eigenvector.

- $\lambda = 1$

$$A-\lambda I = \begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

We begin with the matrix: $\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

Swap rows 1 and 2 : $\begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

Multiply row 1 by -1: $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

Multiply row 2 by -1 : $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

Add 1 times row 2 to row 3 : $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Add 1 times row 2 to row 1 : $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Convert the matrix equation back to an equivalent system: $\begin{aligned} x_1 + x_1 &= 0 \\ x_2 &= 0 \end{aligned}$

Add an equation for each free variable: $\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \\ x_3 &= x_3 \end{aligned}$

Solve for each variable in terms of the free variables: $\begin{aligned} x_1 &= -x_3 \\ x_2 &= 0 \\ x_{13} &= x_3 \end{aligned}$

Collect terms into vectors: $\begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} -x_{11} \\ 0 \\ x_3 \end{bmatrix}$

Factor out variables on the right side: $\begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Thus, a basis for the null space is: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

The null space of this matrix is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ (for steps, see null space calculator).

This is the eigenvector.

- $\lambda = 0$

$$A - \lambda I = \begin{bmatrix} 1-0 & -1 & 0 \\ -1 & 2-0 & -1 \\ 0 & -1 & 1-0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

We begin with the matrix: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

Add 1 times row 1 to row 2 : $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

Add 1 times row 2 to row 3 : $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Add 1 times row 2 to row 1 : $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Convert the matrix equation back to an equivalent system: $\begin{aligned} x_1 + -x_3 &= 0 \\ x_2 + -x_3 &= 0 \end{aligned}$

Add an equation for each free variable: $\begin{aligned} x_1 + -x_3 &= 0 \\ x_2 + -x_3 &= 0 \\ x_3 &= x_3 \end{aligned}$

Solve for each variable in terms of the free variables: $\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \\ x_3 &= x_3 \end{aligned}$

Collect terms into vectors: $\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_3 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_3 \end{bmatrix}$

Factor out variables on the right side: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The null space of this matrix is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

14. Suppose you want a vector to rotate about 90 Degree anti-clockwise. Determine the transformation matrix that should operate on that vector to produce such result? Determine for 180, and 270 degrees too.

Solution: To rotate a vector counterclockwise about the origin, we can use a transformation matrix. The general form of the 2D transformation matrix for rotation about the origin is: $\begin{bmatrix} cos\ \theta & -sin\ \theta \\ \sin\theta & \cos\theta \end{bmatrix}$

For a 90-degree counterclockwise rotation, **θ** would be **π/2** radians.

To rotate a vector 180 degrees counterclockwise, **θ** would be **π** radians.

To rotate a vector 270 degrees counterclockwise, **θ** would be **3π/2** radians.

15. Find the rank and the four eigenvalues of A, where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Solution: First we will determine rref(A) which is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Start from forming a new matrix by subtracting λ from the diagonal entries of the given matrix:

$$\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix}$$

The determinant of the obtained matrix is $\lambda^2(\lambda-2)^2$

Solve the equation $\lambda^2(\lambda-2)^2 = 0$

The roots are $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = 0$

These are the eigenvalues.

Next, find the eigenvectors.

- $\lambda = 2$

$$\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Multiply row 1 by -1: $R_1 = -R_1$. $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Subtract row 1 from row 3: $R_3 = R_3 - R_1$. $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Multiply row 2 by -1 : $R_2 = -R_2$. $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Subtract row 2 from row 4: $R_4 = R_4 - R_2$. $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The reduced row echelon form of the matrix is $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

To find the null space, solve the matrix equation $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

If we take $x_3 = x_3, x_4 = x_4$, then $x_1 = x_3, x_2 = x_4$.

Thus, $\vec{\mathbf{x}} = \begin{bmatrix} t \\ s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_4$

This is the null space.

The nullity of a matrix is the dimension of the basis for the null space.

Thus, the nullity of the matrix is 2 .

The null space of this matrix is $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$

These are the eigenvectors.

- $\lambda = 0$

$$\begin{bmatrix}1-\lambda & 0 & 1 & 0\\ 0 & 1-\lambda & 0 & 1\\ 1 & 0 & 1-\lambda & 0\\ 0 & 1 & 0 & 1-\lambda\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$$

The reduced row echelon form of the matrix is $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

Subtract row 1 from row 3: $R_3 = R_3 - R_1$. $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$

Subtract row 2 from row 4: $R_4 = R_4 - R_2$. $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

To find the null space, solve the matrix equation $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

If we take $x_3 = x_3, x_4 = x_4$, then $x_1 = -x\_3$, $x_2 = -x_4$.

Thus, $\vec{\mathbf{x}} = \begin{bmatrix}-x_3\\-x_4\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}x_3 + \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}x_4$.

This is the null space.

The nullity of a matrix is the dimension of the basis for the null space.

Thus, the nullity of the matrix is 2 .

The null space of this matrix is $\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$

16. **[Page 201, Worked Example 4.1A, Introduction to Linear Algebra (4th Edition), Gilbert Strang]**
Suppose S is a six-dimensional subspace of nine-dimensional space R^9.
   a. **What are the possible dimensions of subspaces orthogonal to S?**
   b. **What are the possible dimensions of the orthogonal complement $S^\perp$ of S?**
   c. **What is the smallest possible size of a matrix A that has row space S?**
   d. **What is the shape of its null space matrix N?**

Solution:

(a) If S is six-dimensional in $\mathbf{R}^9$, subspaces orthogonal to S can have dimensions 0,1,2,3.

(b) The complement $S^\perp$ is the largest orthogonal subspace, with dimension 3 .

(c) The smallest matrix A is 6 by 9 (its six rows are a basis for S).

(d) Its nullspace matrix N is 9 by 3 . The columns of N contain a basis for $S^\perp$.

If a new row 7 of B is a combination of the six rows of A, then B has the same row space as A. It also has the same nullspace matrix N. The special solutions s_1, s_2, s_3 will be the same. Elimination will change row 7 of B to all zeros.

17. (Bonus Problem)

Find all eigenvalues and eigenvectors of the matrix A,

$$\boldsymbol{where\ A} = \begin{bmatrix} \mathbf{10001} & \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{9} & \mathbf{11} \\ \mathbf{1} & \mathbf{10003} & \mathbf{5} & \mathbf{7} & \mathbf{9} & \mathbf{11} \\ \mathbf{1} & \mathbf{3} & \mathbf{10005} & \mathbf{7} & \mathbf{9} & \mathbf{11} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{10007} & \mathbf{9} & \mathbf{11} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{10009} & \mathbf{11} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{9} & \mathbf{10011} \end{bmatrix}$$

Solution: Let $B = A - 10000I$

where I is the 6×6 identity matrix. That is, we have

$$\mathrm{B} = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}$$

Since all the rows are the same, the matrix B is singular and hence $\lambda = 0$ is an eigenvalue of B.

Let us determine the geometric multiplicity of $\lambda = 0$ (namely, the dimension of the null space of B).

We apply elementary row operations to B and obtain

$$\mathrm{B} \xrightarrow{\text{elementary row operations}} \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, if $\mathrm{B}\mathbf{x} = \mathbf{0}$, then we have

$$x_1 = -3x_2 - 5x_3 - 7x_4 - 9x_5 - 11x_6.$$

It follows from this that basis vectors of the eigenspace $\mathrm{E}_0 = \mathcal{N}(\mathrm{B})$ are

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and hence the geometric multiplicity corresponding to $\lambda = 0$ is 5.

By inspection, we see that, $\mathrm{B}\mathbf{v} = 36\mathbf{v}$

Where, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Thus, it yields that $\lambda = 36$ is an eigenvalue of B and $\mathbf{v}$ is a corresponding eigenvector.

Recall that the algebraic multiplicity of an eigenvalue is greater than or equal to the geometric multiplicity.

Also the sum of algebraic multiplicities of all eigenvalues of B is equal to 6 since B is a 6×6 matrix.

It follows from this observation that we determine that the algebraic multiplicity of $\lambda = 0$ is 5 and the algebraic and geometric multiplicities of $\lambda = 36$ are both 1 .

Hence the vector $\mathbf{v}$ form a basis of the eigenspace E_{36}.

Now that we have determined eigenvalues and eigenvectors of B, we can deduce those of A as follows.

In general, if $A = B + cI$, then the eigenvalues of A are $\lambda + c$, where λ are eigenvalues of B. The eigenvectors for A corresponding to $\lambda + c$ are exactly the eigenvectors for B corresponding λ.

(See the post "Eigenvalues and algebraic/geometric

multiplicities of matrix $A + cI$ " for a proof.)

In the current problem, we have $A = B + 10000I$, and thus $c = 10000$.

Therefore, the eigenvalues of A are 10000,10036 .

Eigenvectors corresponding to 10000 are

$$x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $(x_2, x_3, x_4, x_5, x_6) \neq (0,0,0,0,0,0)$.

The eigenvector corresponding to 10036 is

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where a is any nonzero scalar.