A. Dimension, Rank, Basis, Four Fundamental Subspace

B. Orthogonality, Projection, Component, Eigenvectors, and Eigenvalues

1. Find the rank of the matrix

   a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

   b) $\mathrm{B} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

2. Let V be a subset of R^4 consisting of vectors that are perpendicular to vectors $a, b, and\ c$ where $a\ = < 1, 0, 1, 0 >, b = < 1, 1, 0, 0 >, c =< 0, 1, -1, 0 >,$

   Namely, $V = \{x \in R^4 | a^T x = 0, b^T x = 0, and\ C^T x = 0\}$

   a. Prove that V is a subspace of R^4

   b. Find a basis for V

   c. Determine the Dimension of V

   Solution Hint:

   a) Observe that the conditions $a^T x = 0, b^T x = 0, and\ c^T x = 0$, combining $Ax = 0$

   Where, $\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$. Note that the rows of the matrix **A** are $a^T, b^T, and\ c^T$. It follows that the subset V is in the null space $N(A)$ of the matrix A. Being the null space $V = N(A)$, is a subspace of R^4.

   b) To find a basis, we determine the solutions of $Ax = 0$

   Applying elementary row operations to the augmented matrix, we see that,

   $$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} (R_2 - R_3) \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Then, Determine the general solution and determine the basis and you will have it.

3. Determine which of the following is a subspace of R^3.

   a) $x + 2y - 3z = 4$
   b) $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{4}$
   c) $x + y + z = 0$ and $x - y + z = 1$
   d) $x = -z$ and $x = z$
   e) $x^2 + y^2 = z$
   f) $\frac{x}{2} = \frac{y-3}{5}$

4. Suppose $rref(R_0) = A\ where\ R_0 = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 2 & 6 & 10 & 1 & 16 \\ 3 & 9 & 15 & 1 & 23 \end{bmatrix}$. Show that –

   a) The row space has dimension 2, matching the rank
   b) The column space of R_0 has also dimension $r = 2$
   c) The null space of R_0 has dimension 3
   d) The null space of R_0^T, which can also be called the left null space of R_0; has dimension 1.

5. Find a basis for each of the four fundamental subspaces associated with the matrix.

   A= $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

6. Let A be a real 7x3 matrix such that the null space is spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\ and\ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Find the rank of the matrix A.

7. Let V be a subset of the vector space R^n consisting only of the zero vector of R^n, Namely $V = \{0\}$. Then prove that V is a subspace of R^n.

8. Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and consider the following subset V of the 2-dimensional vector space R^2, Namely $V = \{x \in R^2 | Ax = 5x\}$

a) Prove that the subset V is a subspace of R^2

b) Find a basis for V and determine the dimension of V

9. The smallest subspace of R^3 containing the vectors (2, -3, -3) and (0, 3, 2) is the plane whose equation is $ax + by + 6z = 0$. Determine the value of a, b.

10. Determine The matrix representation of the orthogonal projection operator taking R^3 onto the plane $x + y + z \ = \ 0$.

11. Let $u \ = \ (8, \sqrt{3}, \sqrt{7}, -1, 1)$ and $u \ = \ (1, -1, 0, 2, \sqrt{3})$. If the orthogonal projection of u onto v is $\frac{a}{b}v$, then determine a and b.

12. Find the point q in R^3 on the ray connecting the origin to the point (2, 4, 8) which is closest to the point (1, 1, 1).

13. Find the eigenvalues and eigenvectors of the following matrix A.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Show that these eigenvectors are perpendicular. [Hint: It will always be perpendicular when A is symmetric]

14. Suppose you want a vector to rotate about 90 Degree anti-clockwise. Determine the transformation matrix that should operate on that vector to produce such result? Determine for 180, and 270 degrees too.

15. Find the rank and the four eigenvalues of A, where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

16. [Page 201, Worked Example 4.1A, Introduction to Linear Algebra (4th Edition), Gilbert Strang]

    Suppose S is a six-dimensional subspace of nine-dimensional space R^9.

    a. What are the possible dimensions of subspaces orthogonal to S?
    b. What are the possible dimensions of the orthogonal complement $S^\wedge$ of S?
    c. What is the smallest possible size of a matrix A that has row space S?
    d. What is the shape of its null space matrix N?

17. (Bonus Problem)

Find all eigenvalues and eigenvectors of the matrix A,

$$where\ A = \begin{bmatrix} 10001 & 3 & 5 & 7 & 9 & 11 \\ 1 & 10003 & 5 & 7 & 9 & 11 \\ 1 & 3 & 10005 & 7 & 9 & 11 \\ 1 & 3 & 5 & 10007 & 9 & 11 \\ 1 & 3 & 5 & 7 & 10009 & 11 \\ 1 & 3 & 5 & 7 & 9 & 10011 \end{bmatrix}$$

Solution Hint: Let $B = A - 10000I$, where I is the $6 * 6$ identity matrix. That is, we have,

$B = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}$, since all row are same, B is singular and hence $\lambda = 0$ is an eigenvalue of B. With elementary row operation, we find $B \longrightarrow \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

By inspection, we see that $Bv = 36v$, where $v =< 1, 1, 1, 1, 1, 1 >$. Thus it yields that $\lambda = 36$ is the eigenvalue of $B\ and\ v$ is the corresponding eigenvector.