PHYS1112 - Electricity and Magnetism Lecture Notes

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January 2005

Contents

Chapter 1

Vector Algebra

1.1 Definitions

A **vector** consists of two components: *magnitude* and *direction* .
(`e.g.` force, velocity, pressure)

A **scalar** consists of *magnitude* only.
(`e.g.` mass, charge, density)

1.2 Vector Algebra

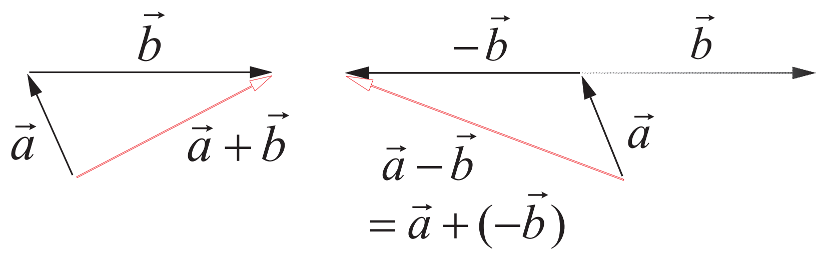


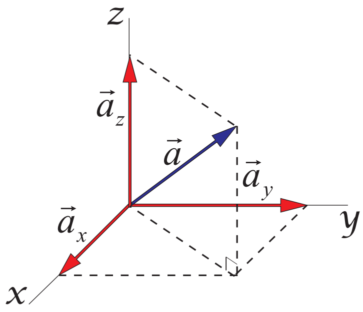
Figure 1.1: Vector algebra

$$\begin{aligned} \vec{a}+\vec{b} &= \vec{b}+\vec{a} \\ \vec{a}+(\vec{c}+\vec{d}) &= (\vec{a}+\vec{c})+\vec{d} \end{aligned}$$

1.3 Components of Vectors

Usually vectors are expressed according to **coordinate system**. Each vector can be expressed in terms of *components*.

The most common coordinate system: **Cartesian**



$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z$$

Magnitude of $\vec{a} = |\vec{a}| = a$,

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

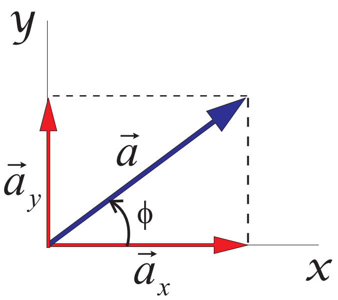


Figure 1.2: ϕ measured anti-clockwise from position x-axis

$$\begin{aligned} \vec{a} &= \vec{a}_x + \vec{a}_y \\ a &= \sqrt{a_x^2 + a_y^2} \\ a_x &= a\,cos\phi;\ a_y = a\,sin\phi \\ tan\phi &= \frac{a_y}{a_x} \end{aligned}$$

Unit vectors have magnitude of 1

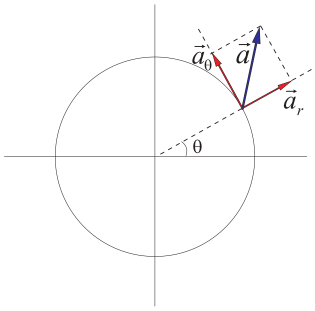
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \text{unit vector along } \vec{a} \text{ direction}$$

$$\begin{array}{cccl} \hat{i} & \hat{j} & \hat{k} & \text{are unit vectors along} \\ \updownarrow & \updownarrow & \updownarrow & \\ x & y & z & \text{directions} \end{array}$$

$$\vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$

Other coordinate systems:

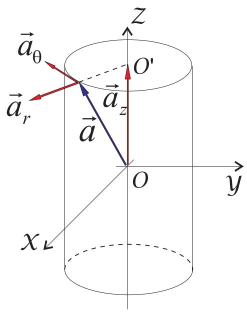
1. Polar Coordinate:



$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta}$$

Figure 1.3: Polar Coordinates

2. Cylindrical Coordinates:

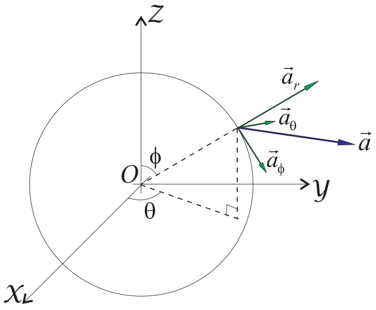


$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_z\,\hat{z}$$

$\hat{r}$ originated from nearest point on z-axis (Point O')

Figure 1.4: Cylindrical Coordinates

3. Spherical Coordinates:



$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_\phi\,\hat{\phi}$$

$\hat{r}$ originated from Origin O

Figure 1.5: Spherical Coordinates

1.4 Multiplication of Vectors

1. Scalar multiplication:

If	$\vec{b}$=m $\vec{a}$	$\vec{b}, \vec{a}$ are vectors; m is a scalar
then	b=m a	(Relation between magnitude)
	b_x=m a_x b_y=m a_y	} Components also follow relation

i.e.

$$\begin{array}{rcrlrlrl} \vec{a} &=& a_x & \hat{i} &+ a_y & \hat{j} &+ a_z & \hat{k} \\ m\vec{a} &=& ma_x & \hat{i} &+ ma_y & \hat{j} &+ ma_z & \hat{k} \end{array}$$

2. Dot Product (Scalar Product):

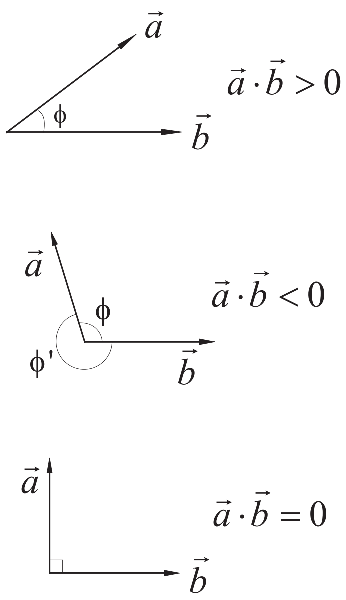


Figure 1.6: Dot Product

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \; cos\phi}$$

Result is **always** a scalar. It can be positive or negative depending on ϕ.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Notice: $\vec{a} \cdot \vec{b} = ab\, cos\phi = ab\, cos\phi'$
i.e. Doesn't matter how you measure angle ϕ between vectors.

$$\begin{aligned} \hat{i} \cdot \hat{i} &= |\hat{i}|\,|\hat{i}| \; cos0^\circ \; = 1 \cdot 1 \cdot 1 = 1 \\ \hat{i} \cdot \hat{j} &= |\hat{i}|\,|\hat{j}|\, cos90^\circ = 1 \cdot 1 \cdot 0 = 0 \end{aligned}$$

$$\boxed{\begin{array}{c} \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{array}}$$

If $\quad \vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$

$\qquad \vec{b} = b_x\,\hat{i} + b_y\,\hat{j} + b_z\,\hat{k}$

then $\quad \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\qquad \vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}|\, cos0^\circ = a \cdot a = a^2$

3. Cross Product (Vector Product):

If $\quad \vec{c} = \vec{a} \times \vec{b}$,
then $\quad c = |\vec{c}| = a\, b\, sin\phi$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \;\; \textbf{!!!}$$

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$$

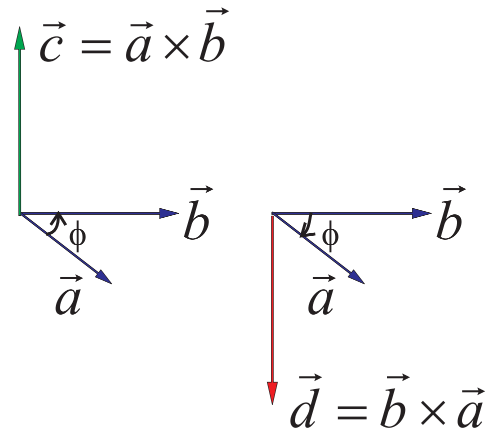


Figure 1.7: Note: How angle ϕ is measured

- Direction of cross product determined from *right hand rule*.
- Also, $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ and $\vec{b}$, `i.e.`

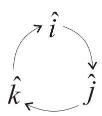
$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$$
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

- IMPORTANT:

$$\boxed{\vec{a} \times \vec{a} = a \cdot a\, sin 0^\circ = 0}$$

$$\begin{aligned} |\hat{i} \times \hat{i}| &= |\hat{i}|\,|\hat{i}|\; sin0^\circ = 1 \cdot 1 \cdot 0 = 0 \\ |\hat{i} \times \hat{j}| &= |\hat{i}|\,|\hat{j}|\, sin90^\circ = 1 \cdot 1 \cdot 1 = 1 \end{aligned}$$

$$\boxed{\begin{array}{c} \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{i} \times \hat{j} = \hat{k};\; \hat{j} \times \hat{k} = \hat{i};\; \hat{k} \times \hat{i} = \hat{j} \end{array}}$$



$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{array}{r} (a_y\, b_z - a_z\, b_y)\, \hat{i} \\ +(a_z\, b_x - a_x\, b_z)\, \hat{j} \\ +(a_x\, b_y - a_y\, b_x)\, \hat{k} \end{array}$$

4. Vector identities:

$$\begin{aligned}
\vec{a}\times(\vec{b}+\vec{c}) &= \vec{a}\times\vec{b}+\vec{a}\times\vec{c} \\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) \\
\vec{a}\times(\vec{b}\times\vec{c}) &= (\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}
\end{aligned}$$

1.5 Vector Field (Physics Point of View)

A **vector field** $\vec{\mathcal{F}}(x, y, z)$ is a mathematical function which has a *vector* output for a *position* input.

(**Scalar field** $\vec{\mathcal{U}}(x, y, z)$)

1.6 Other Topics

Tangential Vector

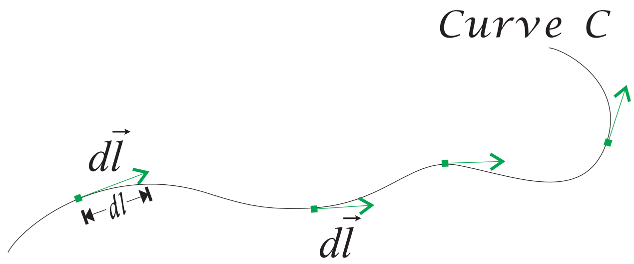


Figure 1.8: $d\vec{l}$ is a vector that is always tangential to the curve C with infinitesimal length dl

Surface Vector

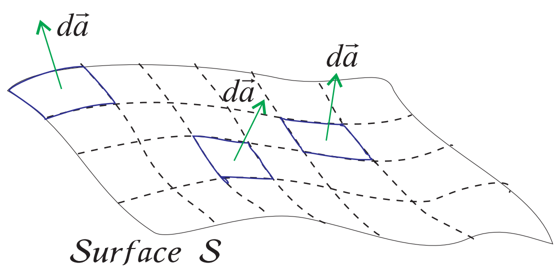


Figure 1.9: $d\vec{a}$ is a vector that is always perpendicular to the surface S with infinitesimal area da

Some uncertainty! $(d\vec{a} \quad versus \quad -d\vec{a})$

Two conventions:

- Area formed from a closed curve

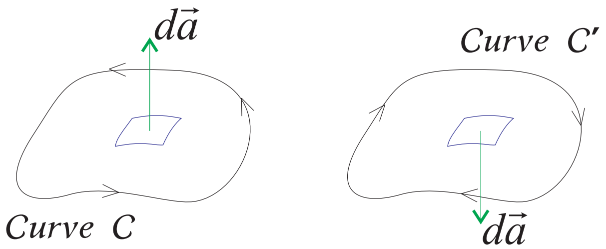


Figure 1.10: Direction of $d\vec{a}$ determined from right-hand rule

- Closed surface enclosing a volume

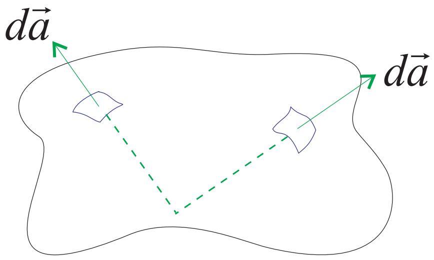


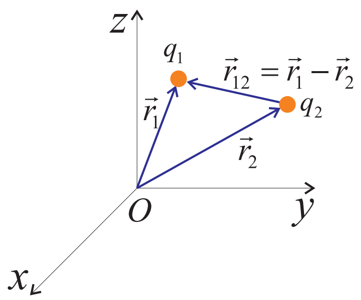
Figure 1.11: Direction of $d\vec{a}$ going from inside to outside

Chapter 2

Electric Force & Electric Field

2.1 Electric Force

The electric force between two **charges** $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ can be described by **Coulomb's Law**.



$\vec{F}_{12} = Force$ on $\boldsymbol{q_1}$ exerted by $\boldsymbol{q_2}$

$$\boxed{\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r_{12}^2} \cdot \hat{r}_{12}}$$

where $\hat{r}_{12} = \dfrac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is the *unit vector* which locates particle 1 relative to particle 2.

$$\texttt{i.e.} \quad \boxed{\vec{r}_{12} = \vec{r}_1 - \vec{r}_2}$$

- q_1, q_2 are electrical charges in units of *Coulomb*(C)
- Charge is *quantized*
  Recall 1 electron carries $1.602 \times 10^{-19} C$
- ϵ_0 = Permittivity of free space $= 8.85 \times 10^{-12} C^2/Nm^2$

COULOMB'S LAW:

(1) q_1, q_2 can be either positive or negative.

(2) If q_1, q_2 *are of same sign*, then the force experienced by q_1 is in direction *away from* q_2, that is, *repulsive.*

(3) Force on q_2 exerted by q_1:

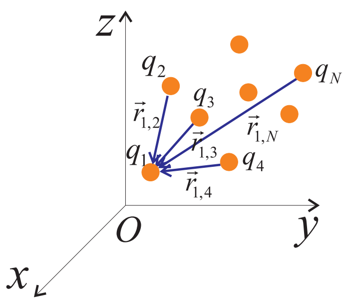
$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_2 q_1}{r_{21}^2} \cdot \hat{r}_{21}$$

BUT:

$$r_{12} = r_{21} = \text{distance between } q_1, q_2$$
$$\hat{r}_{21} = \frac{\vec{r}_{21}}{r_{21}} = \frac{\vec{r}_2 - \vec{r}_1}{r_{21}} = \frac{-\vec{r}_{12}}{r_{12}} = -\hat{r}_{12}$$

$$\therefore \boxed{\vec{F}_{21} = -\vec{F}_{12} \text{ \textit{Newton's 3rd Law}}}$$

<u>**SYSTEM WITH MANY CHARGES**</u>:



The total force experienced by charge q_1 is the *vector sum* of the forces on q_1 exerted by other charges.

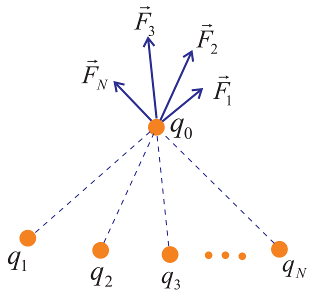
$$\begin{aligned} \vec{F}_1 &= \text{Force experienced by } q_1 \\ &= \vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} + \cdots + \vec{F}_{1,N} \end{aligned}$$

PRINCIPLE OF SUPERPOSITION:

$$\boxed{\vec{F}_1 = \sum_{j=2}^{N} \vec{F}_{1,j}}$$

2.2 The Electric Field

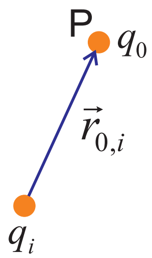
While we need two charges to quantify the **electric force**, we define the **electric field** for any single charge distribution to describe its effect on other charges.



Total force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$
The **electric field** is defined as

$$\boxed{\lim_{q_0 \to 0} \frac{\vec{F}}{q_0} = \vec{E}}$$

(a) E-field due to a single charge q_i:



From the definitions of **Coulomb's Law**, the force experienced at location of q_0 (point P)

$$\vec{F}_{0,i} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_0 q_i}{r_{0,i}^2} \cdot \hat{r}_{0,i}$$

where $\hat{r}_{0,i}$ is the unit vector along the direction *from charge* q_i *to* q_0,

$$\begin{aligned} \hat{r}_{0,i} &= \text{Unit vector from charge } q_i \text{ to point P} \\ &= \hat{r}_i \text{ (radical unit vector from } q_i) \end{aligned}$$

Recall $\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$
$\therefore$ E-field due to q_i at point P:

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_i}{r_i^2} \cdot \hat{r}_i$$

where $\vec{r}_i$ = Vector pointing from q_i to point P,
thus $\hat{r}_i$ = Unit vector pointing from q_i to point P
Note:

(1) E-field is a **vector**.

(2) Direction of E-field depends on **both** position of P and sign of q_i.

(b) E-field due to system of charges:

Principle of Superposition:
In a system with N charges, the **total** E-field due to all charges is the **vector sum** of E-field due to individual charges.

$$\texttt{i.e.} \quad \boxed{\vec{E} = \sum_i \vec{E}_i = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{r}_i}$$

(c) Electric Dipole

System of *equal and opposite* charges separated by a distance d.

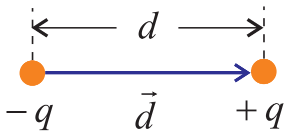
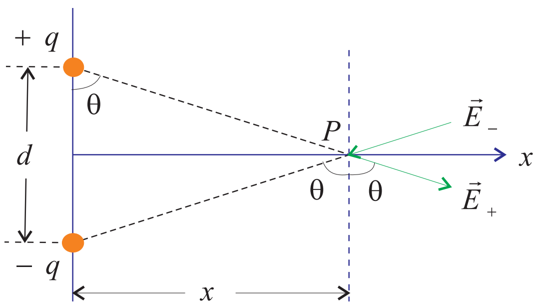


Figure 2.1: An electric dipole. (Direction of $\vec{d}$ from negative to positive charge)

$$\boxed{\begin{array}{c}\text{Electric Dipole Moment}\\ \vec{p} = q\vec{d} = qd\hat{d}\end{array}}$$

$$\boxed{p = qd}$$

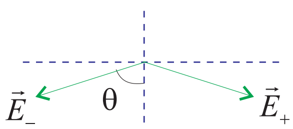
<u>Example</u>: $\vec{E}$ due to dipole along x-axis



Consider point P at distance x along the perpendicular axis of the dipole $\vec{p}$:

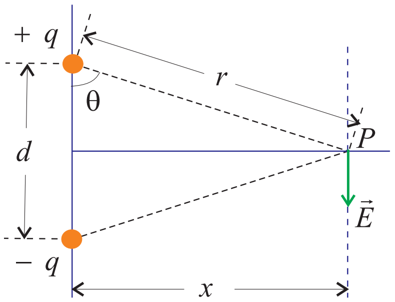
$$\begin{array}{ccccc} \vec{E} & = & \vec{E}_+ & + & \vec{E}_- \\ & & \uparrow & & \uparrow \\ & & \text{(E-field} & & \text{(E-field} \\ & & \text{due to } +q) & & \text{due to } -q) \end{array}$$

Notice: Horizontal E-field components of $\vec{E}_+$ and $\vec{E}_-$ cancel out.



$\therefore$ Net E-field points along the axis opposite to the dipole moment vector.

Magnitude of E-field $= 2E_+ \cos\theta$



E_+ or E_- magnitude!

$$\therefore E = 2\Big(\overbrace{\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}}\Big)\cos\theta$$

$$\text{But}\quad r = \sqrt{\Big(\frac{d}{2}\Big)^2 + x^2}$$

$$\cos\theta = \frac{d/2}{r}$$

$$\therefore E = \frac{1}{4\pi\epsilon_0}\cdot\frac{p}{[x^2 + (\frac{d}{2})^2]^{\frac{3}{2}}}$$

$$(p = qd)$$

<u>Special case:</u> When $x \gg d$

$$[x^2 + (\frac{d}{2})^2]^{\frac{3}{2}} = x^3[1 + (\frac{d}{2x})^2]^{\frac{3}{2}}$$

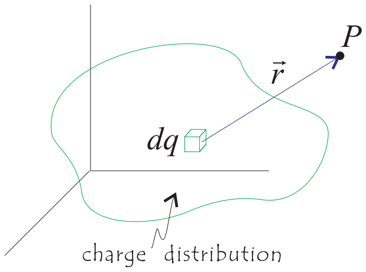
- Binomial Approximation:

$$(1+y)^n \approx 1 + ny \qquad \text{if } y \ll 1$$

$$\boxed{\text{E-field of dipole} \doteqdot \frac{1}{4\pi\epsilon_0}\cdot\frac{p}{x^3} \sim \frac{1}{x^3}}$$

- Compare with $\frac{1}{r^2}$ E-field for single charge
- Result also valid for point P along any axis with respect to dipole

2.3 Continuous Charge Distribution



E-field at point P due to dq:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r^2}\cdot\hat{r}$$

$\therefore$ E-field due to charge distribution:

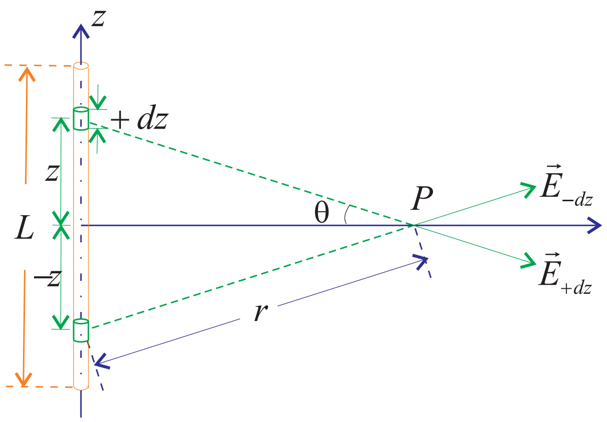
$$\vec{\boldsymbol{E}} = \int_{Volume} d\vec{\boldsymbol{E}} = \int_{Volume} \frac{\mathbf{1}}{\mathbf{4}\boldsymbol{\pi\epsilon_0}} \cdot \frac{\boldsymbol{dq}}{\boldsymbol{r^2}} \cdot \hat{\boldsymbol{r}}$$

(1) In many cases, we can take advantage of the *symmetry* of the system to simplify the integral.

(2) To write down the small charge element dq:

1-D	$dq = \lambda\ ds$	$\lambda =$ linear charge density	$ds =$ small length element
2-D	$dq = \sigma\ dA$	$\sigma =$ surface charge density	$dA =$ small area element
3-D	$dq = \rho\ dV$	$\rho =$ volume charge density	$dV =$ small volume element

Example 1: Uniform line of charge

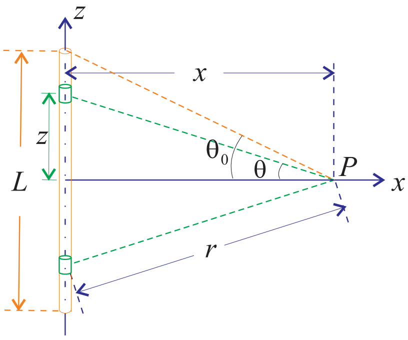


charge per unit length $= \lambda$

(1) Symmetry considered: The E-field from $+z$ and $-z$ directions *cancel along z-direction*, $\therefore$ Only horizontal E-field components need to be considered.

(2) For each element of length dz, charge $dq = \lambda dz$
$\therefore$ Horizontal E-field at point P due to element $dz =$

$$|d\vec{E}|\cos\theta = \underbrace{\frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dz}{r^2}}_{dE_{dz}} \cos\theta$$



$\therefore$ E-field due to entire line charge at point P

$$E = \int_{-L/2}^{L/2} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dz}{r^2} \cos\theta$$

$$= 2\int_{0}^{L/2} \frac{\lambda}{4\pi\epsilon_0} \cdot \frac{dz}{r^2} \cos\theta$$

To calculate this integral:

- First, notice that x is fixed, but $z,\ r,\ \theta$ all varies.
- Change of variable (from z to θ)

  (1) $\begin{array}{ll} z = x\tan\theta & \therefore dz = x\sec^2\theta\ d\theta \\ x = r\cos\theta & \therefore r^2 = x^2\sec^2\theta \end{array}$

  (2) When $\begin{array}{lll} z = 0\ , & \theta = 0^\circ & \\ z = L/2 & \theta = \theta_0 & \text{where } \tan\theta_0 = \dfrac{L/2}{x} \end{array}$

$$\begin{aligned} E &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{x\sec^2\theta\ d\theta}{x^2\sec^2\theta}\cdot\cos\theta \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{1}{x}\cdot\cos\theta\ d\theta \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot(\sin\theta)\Big|_0^{\theta_0} \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\sin\theta_0 \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\frac{L/2}{\sqrt{x^2+(\frac{L}{2})^2}} \end{aligned}$$

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda L}{x\sqrt{x^2+(\frac{L}{2})^2}}} \qquad \text{along } x\text{-direction}$$

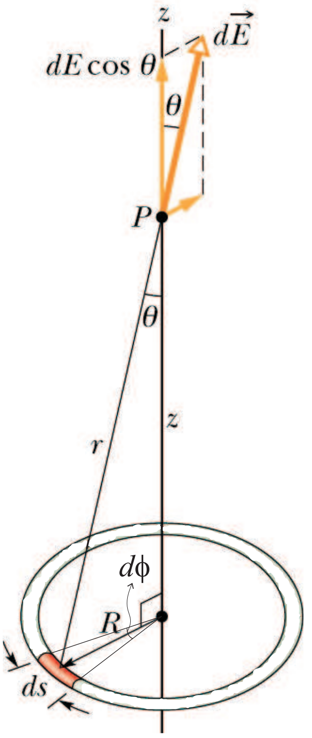
Important limiting cases:

1. $x \gg L: \quad E \doteqdot \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda L}{x^2}$
   But $\lambda L =$ Total charge on rod
   $\therefore$ System behave like a point charge
2. $L \gg x: \quad E \doteqdot \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda L}{x\cdot\frac{L}{2}}$

$$\boxed{E_x = \frac{\lambda}{2\pi\epsilon_0 x}}$$

ELECTRIC FIELD DUE TO INFINITELY LONG LINE OF CHARGE

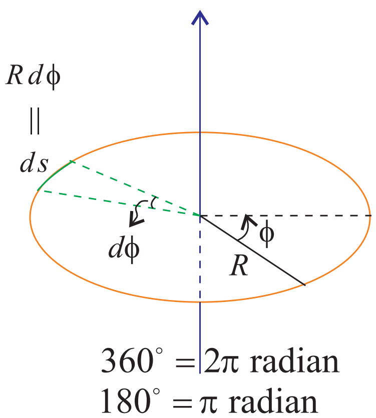
Example 2: Ring of Charge



E-field at a height z above a ring of charge of radius R

(1) Symmetry considered: For every charge element dq considered, there exists $dq\prime$ where the horizontal $\vec{E}$ field components cancel.
$\Rightarrow$ Overall E-field lies along z-direction.

(2) For each element of length dz, charge



$$dq = \underset{\substack{\uparrow \\ \text{Linear} \\ \text{charge density}}}{\lambda} \cdot \underset{\substack{\uparrow \\ \text{Circular} \\ \text{length element}}}{ds}$$

$dq = \lambda \cdot R\, d\phi$, where ϕ is the angle measured on the ring plane

$\therefore$ Net E-field along z-axis due to dq:

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \cos\theta$$

$$\begin{aligned}\text{Total E-field} &= \int dE \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\phi}{r^2} \cdot \cos\theta \qquad (\cos\theta = \frac{z}{r})\end{aligned}$$

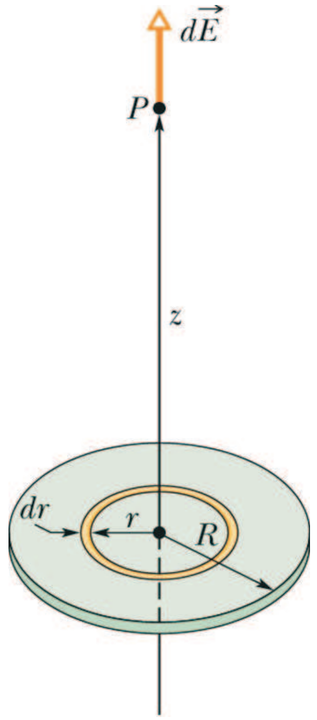
Note: Here in this case, θ, R and r are *fixed* as ϕ varies! BUT we want to convert r, θ to R, z.

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R z}{r^3} \int_0^{2\pi} d\phi$$

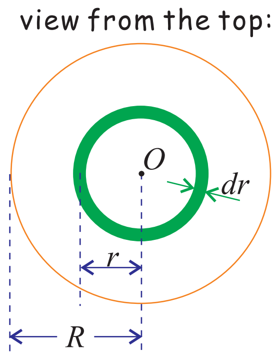
$$\boxed{E = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda(2\pi R)z}{(z^2+R^2)^{3/2}}} \qquad \text{along } z\text{-axis}$$

BUT: $\lambda(2\pi R)$ = total charge on the ring

Example 3: E-field from a disk of surface charge density σ



We find the E-field of a disk by integrating concentric rings of charges.



Total charge of ring

$$dq = \sigma \cdot (\underbrace{2\pi r\ dr}_{\text{Area of the ring}})$$

Recall from Example 2:

$$\text{E-field from ring: } dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq\ z}{(z^2+r^2)^{3/2}}$$

$$\begin{aligned} \therefore \quad E &= \frac{1}{4\pi\epsilon_0} \int_0^R \frac{2\pi\sigma r\ dr \cdot z}{(z^2+r^2)^{3/2}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^R 2\pi\sigma z \frac{r\ dr}{(z^2+r^2)^{3/2}} \end{aligned}$$

- Change of variable:

$$\begin{aligned} & u = z^2 + r^2 && \Rightarrow && (z^2+r^2)^{3/2} = u^{3/2} \\ \Rightarrow\ & du = 2r\ dr && \Rightarrow && r\ dr = \tfrac{1}{2} du \end{aligned}$$

- Change of integration limit:

$$\begin{cases} r = 0 & , \quad u = z^2 \\ r = R & , \quad u = z^2 + R^2 \end{cases}$$

$$\therefore\ E = \frac{1}{4\pi\epsilon_0} \cdot 2\pi\sigma z \int_{z^2}^{z^2+R^2} \frac{1}{2}\, u^{-3/2} du$$

BUT: $$\int u^{-3/2} du = \frac{u^{-1/2}}{-1/2} = -2u^{-1/2}$$

$$\begin{aligned} \therefore\ E &= \frac{1}{2\epsilon_0}\, \sigma z\, (-u^{-1/2})\Big|_{z^2}^{z^2+R^2} \\ &= \frac{1}{2\epsilon_0}\, \sigma z \left(\frac{-1}{\sqrt{z^2+R^2}} + \frac{1}{z} \right) \end{aligned}$$

$$\boxed{E = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{z^2+R^2}} \right]}$$

VERY IMPORTANT LIMITING CASE:

If $R \gg z$, that is if we have an infinite sheet of charge with charge density σ:

$$
\begin{aligned}
E &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\
&\simeq \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{R}\right]
\end{aligned}
$$

$$\boxed{E \approx \frac{\sigma}{2\epsilon_0}}$$

E-field is normal to the charged surface

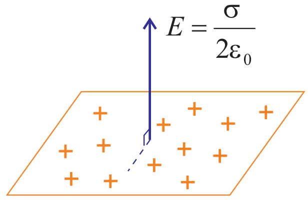
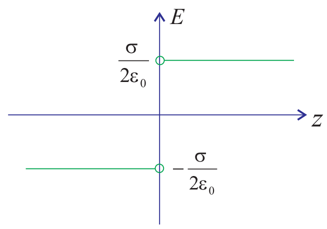


Figure 2.2: E-field due to an infinite sheet of charge, charge density $= \sigma$

Q: What's the E-field belows the charged sheet?

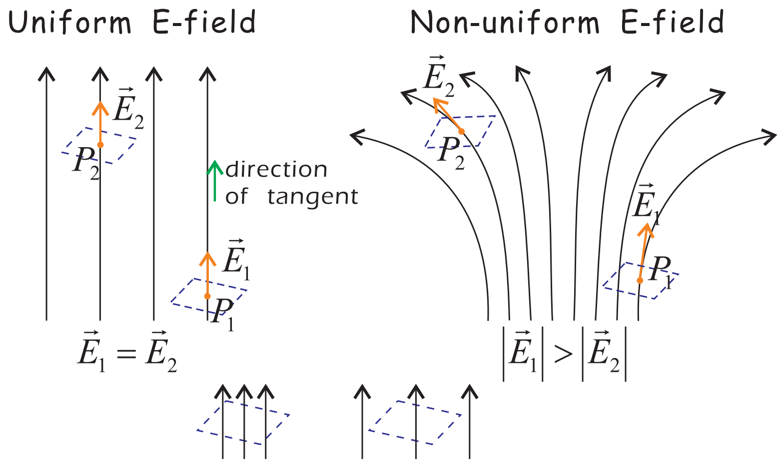


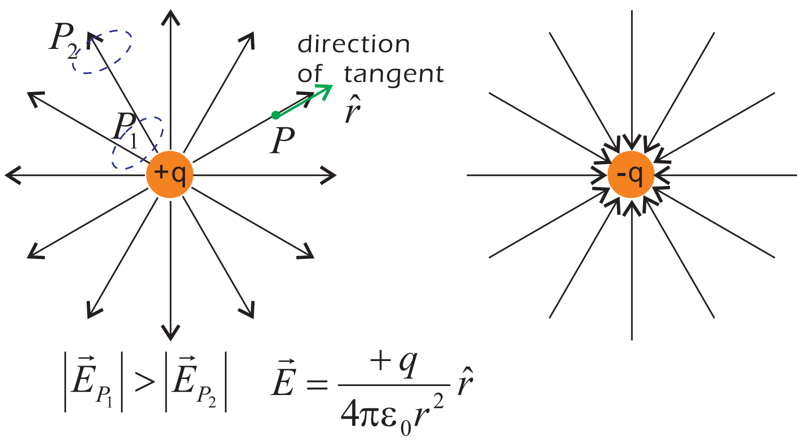
2.4 Electric Field Lines

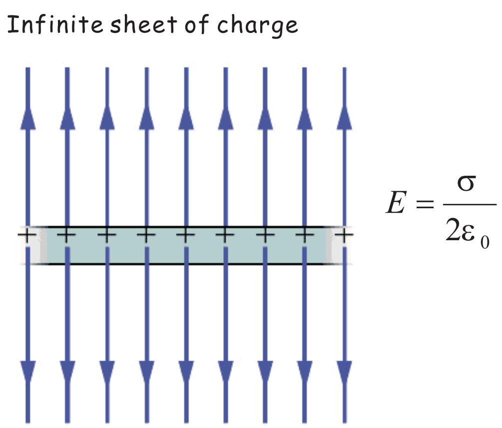
To visualize the electric field, we can use a graphical tool called the **electric field lines**.

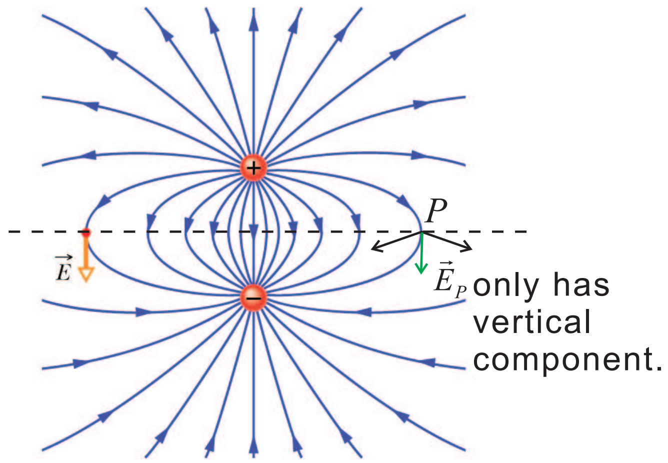
Conventions:

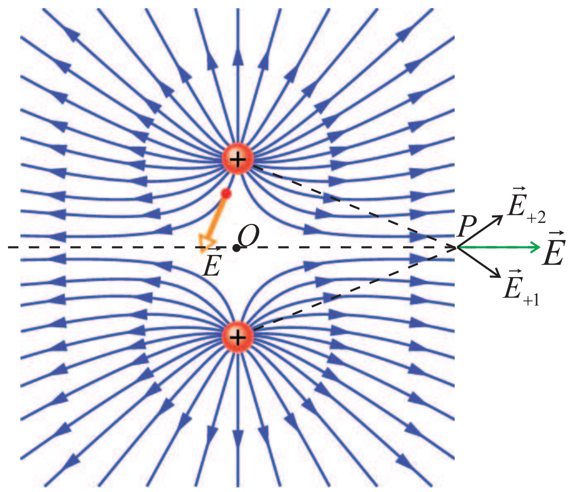
1. The start on position charges and end on negative charges.
2. *Direction* of E-field at any point is given by *tangent* of E-field line.
3. *Magnitude* of E-field at any point is proportional to *number of E-field lines per unit area perpendicular to the lines.*


Uniform E-field
Non-uniform E-field
direction
of tangent
$\vec{E}_1 = \vec{E}_2$
$|\vec{E}_1| > |\vec{E}_2|$


direction
of tangent
+q
-q
$|\vec{E}_{P_1}| > |\vec{E}_{P_2}|$
$\vec{E} = \frac{+q}{4\pi\varepsilon_0 r^2}\hat{r}$


Infinite sheet of charge
$E = \frac{\sigma}{2\varepsilon_0}$





$$\vec{E}_{\text{at point } O} = 0$$

2.5 Point Charge in E-field

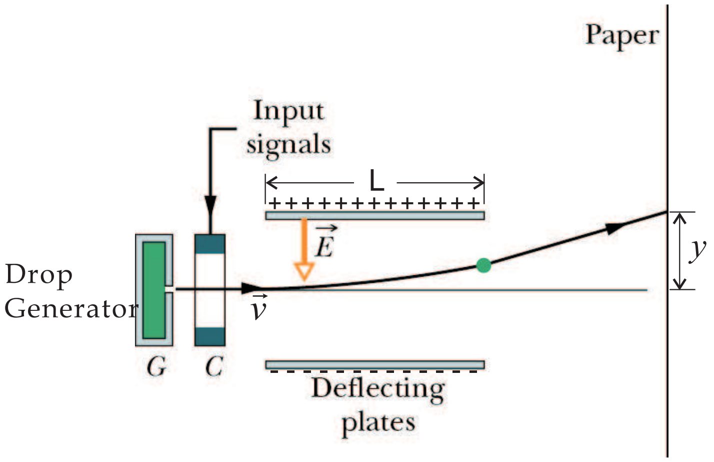
When we place a charge q in an E-field $\vec{E}$, the force experienced by the charge is

$$\boxed{\vec{F} = q\vec{E} = m\vec{a}}$$

Applications: *Ink-jet printer, TV cathoderay tube.*

Example:

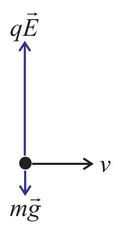
Ink particle has mass m, charge q ($q < 0$ here)



Assume that mass of inkdrop is small, what's the deflection y of the charge?

Solution:

First, the charge carried by the inkdrop is *negtive*, `i.e.` $q < 0$.



Note: $q\vec{E}$ points in opposite direction of $\vec{E}$.

Horizontal motion: Net force = 0

$$\therefore \; L = vt \tag{2.1}$$

Vertical motion: $|q\vec{E}| \gg |m\vec{g}|$, q is negative,

$$\therefore \text{ Net force} = -qE = ma \qquad \text{(Newton's 2nd Law)}$$

$$\therefore\ a = -\frac{qE}{m} \tag{2.2}$$

Vertical distance travelled:

$$y = \frac{1}{2}at^2$$

2.6 Dipole in E-field

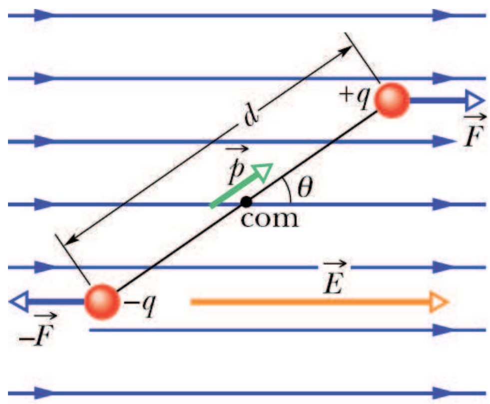
Consider the force exerted on the dipole in an *external* E-field:

Assumption: E-field from dipole doesn't affect the external E-field.

- Dipole moment:

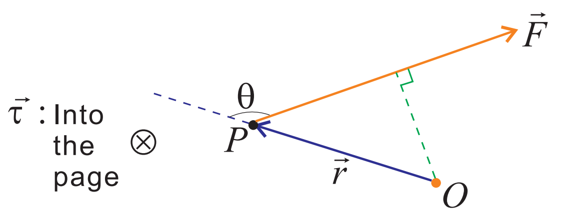
  $$\vec{p} = q\vec{d}$$

- Force due to the E-field on $+ve$ and $-ve$ charge are *equal and opposite in direction*. Total external force on dipole = 0.



BUT:	There is an external **torque** on the center of the dipole.

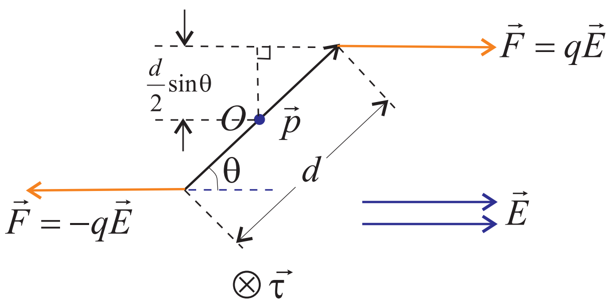
Reminder:



Force $\vec{F}$ exerts at point P. The force exerts a **torque** $\vec{\tau} = \vec{r} \times \vec{F}$ on point P with respect to point O.

Direction of the **torque vector $\vec{\tau}$** is determined from the **right-hand rule**.

Reference: Halliday Vol.1 Chap 9.1 (Pg.175) *torque*
Chap 11.7 (Pg.243) *work done*



Net torque $\vec{\tau}$

- direction: clockwise torque
- magnitude:

$$\begin{aligned}\tau &= \tau_{+ve} + \tau_{-ve} \\ &= F \cdot \frac{d}{2}\sin\theta + F \cdot \frac{d}{2}\sin\theta \\ &= qE \cdot d\sin\theta \\ &= pE\sin\theta\end{aligned}$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

Energy Consideration:

When the dipole $\vec{p}$ rotates $d\theta$, the E-field does work.

Work done by external E-field on the dipole:

$$dW = -\tau\, d\theta$$

Negative sign here because torque by E-field acts to *decrease* θ.

BUT: Because E-field is a **conservative force field** [1] [2], we can define a **potential energy** (U) for the system, so that

$$\boxed{dU = -dW}$$

$\therefore$ For the dipole in external E-field:

$$dU = -dW = pE\sin\theta\, d\theta$$

$$\begin{aligned}\therefore\ U(\theta) = \int dU &= \int pE\sin\theta\, d\theta \\ &= -pE\cos\theta + U_0\end{aligned}$$

[1] more to come in Chap.4 of notes
[2] ref. Halliday Vol.1 Pg.257, Chap 12.1

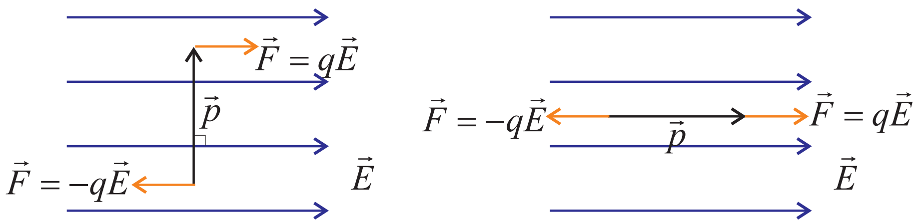
set $U(\theta = 90^\circ) = 0$,

$$\therefore\ 0 = -pE\cos 90^\circ + U_0$$

$$\therefore\ U_0 = 0$$

$\therefore$ Potential energy:

$$\boxed{U = -pE\cos\theta = -\vec{p}\cdot\vec{E}}$$



$\theta = 90^\circ$
Torque $|\vec{\tau}| = pE$

$U = 0$ (define)

$\theta = 0^\circ$
Torque $|\vec{\tau}| = 0$

$U = -pE$
(based on definition)

Minimum energy configuration

Chapter 3

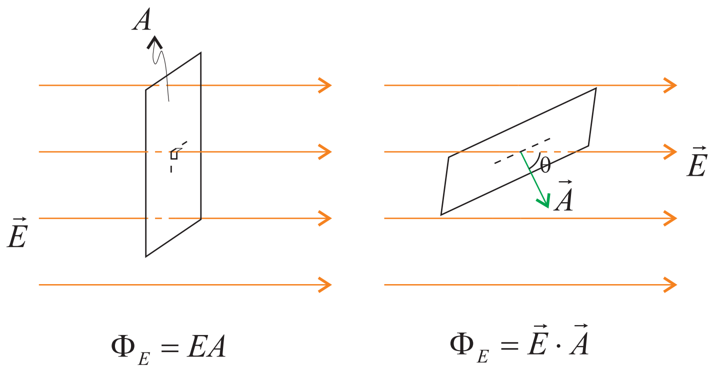
Electric Flux and Gauss' Law

3.1 Electric Flux

Latin: flux = "to flow"

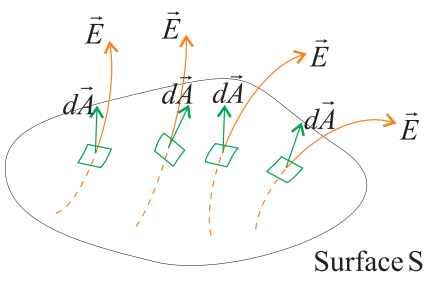
Graphically: Electric flux Φ_E represents the number of E-field lines crossing a surface.

Mathematically:



Reminder: Vector of the area $\vec{A}$ is perpendicular to the area A.

For non-uniform E-field & surface, direction of the area vector $\vec{A}$ is not uniform.



$d\vec{A} =$ Area vector for small area element dA

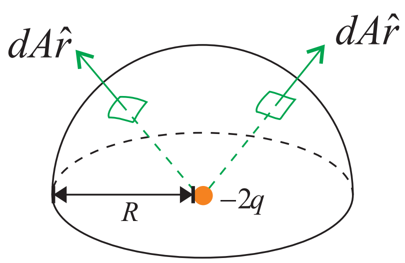
$$\therefore \text{ Electric flux} \qquad d\Phi_E = \vec{E}\cdot d\vec{A}$$

Electric flux of $\vec{E}$ through surface S: $\boxed{\Phi_E = \int_S \vec{E}\cdot d\vec{A}}$

$$\begin{aligned} \int_S &= \text{Surface integral over surface S} \\ &= \text{Integration of integral over all area elements on surface S} \end{aligned}$$

Example:

$S =$ hemisphere radius R



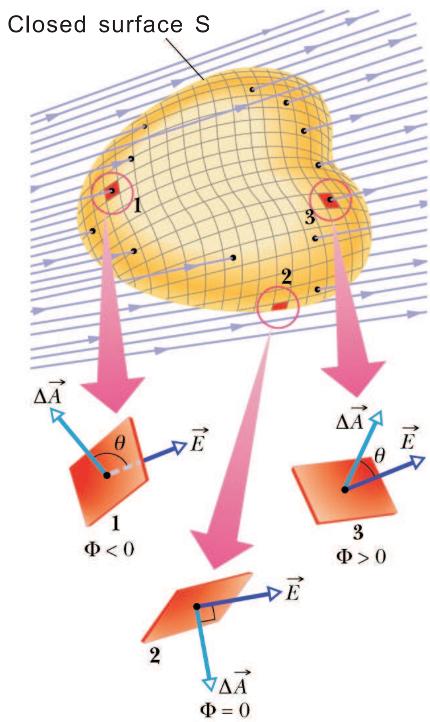
$$\boxed{\int_S dA = \text{Surface area of } S}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{-2q}{r^2}\,\hat{r} = \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r}$$

For a hemisphere, $d\vec{A} = dA\,\hat{r}$

$$\begin{aligned} \Phi_E &= \int_S \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r}\cdot(dA\,\hat{r}) \qquad (\because \hat{r}\cdot\hat{r} = 1) \\ &= -\frac{q}{2\pi\epsilon_0 R^2}\underbrace{\int_S dA}_{2\pi R^2} \\ &= \frac{-q}{\epsilon_0} \end{aligned}$$

For a closed surface:

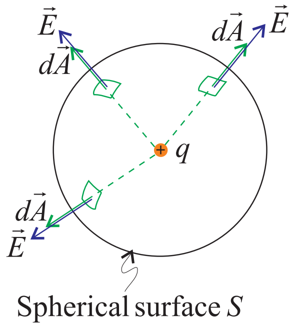


Recall: Direction of area vector $d\vec{A}$ goes *from inside to outside* of closed surface S.

Electric flux over closed surface S: $\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A}}$

$$\oint_S = \text{Surface integral over closed surface S}$$

Example:



Electric flux of charge q over closed spherical surface of radius R.

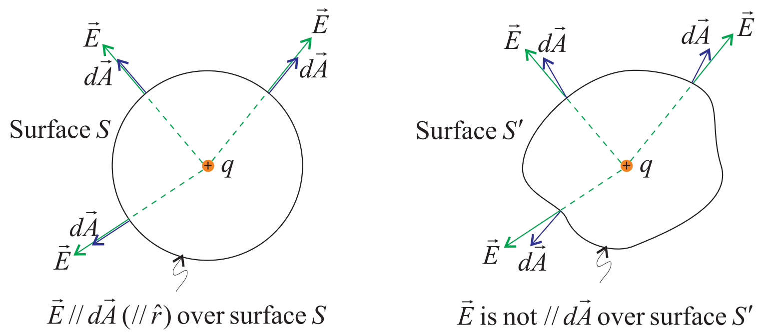
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}\,\hat{r} = \frac{q}{4\pi\epsilon_0 R^2}\,\hat{r} \qquad \text{at the surface}$$

Again, $d\vec{A} = dA \cdot \hat{r}$

$$\begin{aligned} \therefore \Phi_E &= \oint_S \overbrace{\frac{q}{4\pi\epsilon_0 R^2}\,\hat{r}}^{\vec{E}} \cdot \overbrace{dA\,\hat{r}}^{d\vec{A}} \\ &= \frac{q}{4\pi\epsilon_0 R^2} \underbrace{\oint_S dA}_{\text{Total surface area of S} = 4\pi R^2} \\ \Phi_E &= \frac{q}{\epsilon_0} \end{aligned}$$

IMPORTANT POINT:

If we remove the spherical symmetry of closed surface S, *the total number of E-field lines crossing the surface remains the same.*
$\therefore$ The electric flux Φ_E



$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \oint_{S'} \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

3.2 Gauss' Law

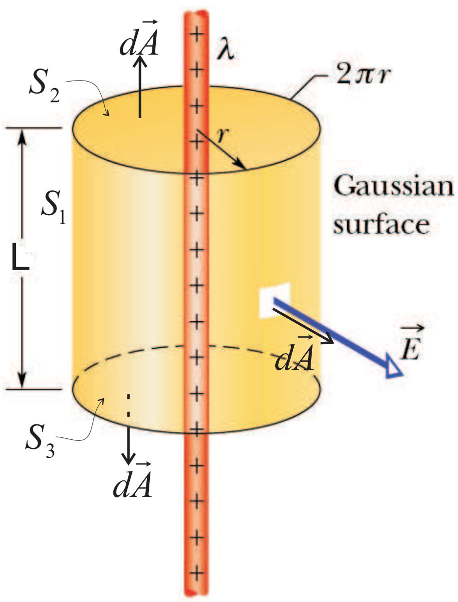
$$\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}} \quad \text{for any closed surface S}$$

And q is the net electric charge enclosed in closed surface S.

- Gauss' Law is valid for *all charge distributions and all closed surfaces. (Gaussian surfaces)*
- Coulomb's Law can be derived from Gauss' Law.
- For system with high order of *symmetry*, E-field can be easily determined if we construct *Gaussian surfaces with the same symmetry* and applies Gauss' Law

3.3 E-field Calculation with Gauss' Law

(A) Infinite line of charge



Linear charge density: λ
Cylindrical symmetry.
E-field directs radially outward from the rod.
Construct a Gaussian surface S in the shape of a **cylinder**, making up of a curved surface S_1, and the top and bottom circles S_2, S_3.

Gauss' Law: $$\oint_S \vec{E} \cdot d\vec{A} = \frac{\text{Total charge}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

$$\oint_S \vec{E}\cdot d\vec{A} = \underbrace{\int_{S_1} \vec{E}\cdot d\vec{A}}_{\vec{E}\parallel d\vec{A}} + \underbrace{\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A}}_{=0\ \because \vec{E}\perp d\vec{A}}$$

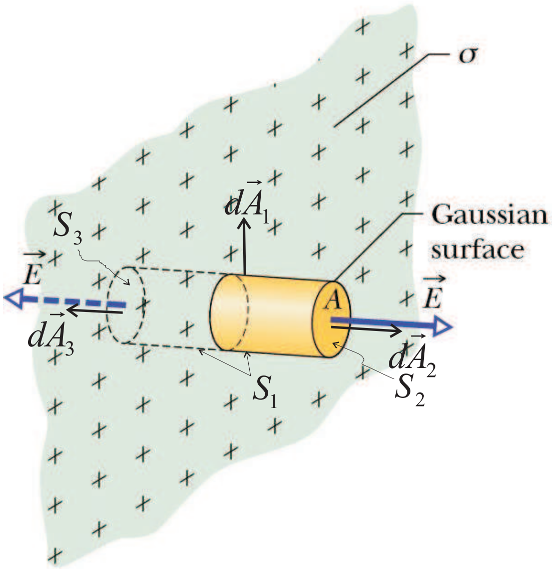
$$\therefore \quad E \underbrace{\int_{S_1} dA}_{\text{Total area of surface } S_1} = \frac{\lambda L}{\epsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

$$\therefore \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}} \qquad \text{(Compare with Chapter 2 note)}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$$

(B) Infinite sheet of charge



Uniform surface charge density: σ

Planar symmetry.

E-field directs perpendicular to the sheet of charge.

Construct Gaussian surface S in the shape of a **cylinder (pill box)** of cross-sectional area A.

Gauss' Law: $$\oint_S \vec{E}\cdot d\vec{A} = \frac{A\sigma}{\epsilon_0}$$

$$\int_{S_1} \vec{E}\cdot d\vec{A} = 0 \quad \because \vec{E}\perp d\vec{A} \text{ over whole surface } S_1$$

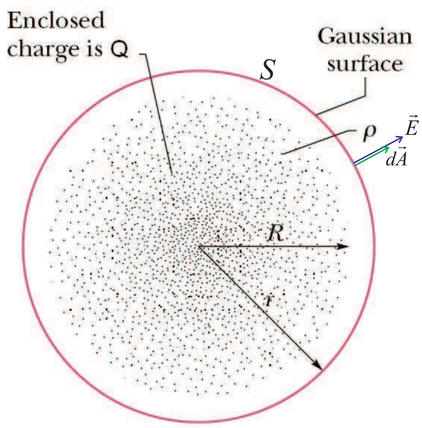
$$\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A} = 2EA \quad (\vec{E}\parallel d\vec{A}_2, \vec{E}\parallel d\vec{A}_3)$$

Note: For S_2, both $\vec{E}$ and $d\vec{A}_2$ point up
For S_3, both $\vec{E}$ and $d\vec{A}_3$ point down

$$\therefore \quad 2EA = \frac{A\sigma}{\epsilon_0} \quad \Rightarrow \quad \boxed{E = \frac{\sigma}{2\epsilon_0}} \quad \text{(Compare with Chapter 2 note)}$$

(C) Uniformly charged sphere
Total charge $= Q$
Spherical symmetry.

(a) For $r > R$:



Consider a spherical Gaussian surface S of radius r:

$$\vec{E} \parallel d\vec{A} \parallel \hat{r}$$

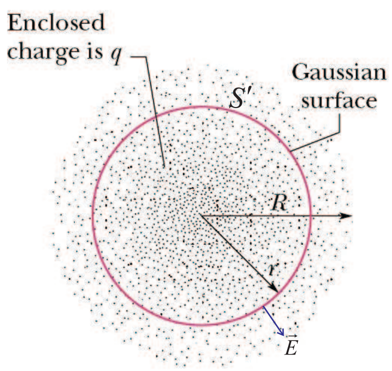
Gauss' Law: $$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oint_S E \cdot dA = \frac{Q}{\epsilon_0}$$

$$E \underbrace{\oint_S dA}_{\text{surface area of } S \,=\, 4\pi r^2} = \frac{Q}{\epsilon_0}$$

$$\therefore \quad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}\,; \qquad \text{for } r > R$$

(b) For $r < R$:



Consider a spherical Gaussian surface S' of radius $r < R$, then total charge included q is *proportional to the volume included by* S'

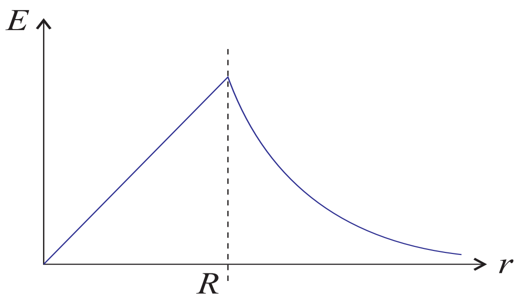
$$\therefore \quad \frac{q}{Q} = \frac{\text{Volume enclosed by } S'}{\text{Total volume of sphere}}$$

$$\frac{q}{Q} = \frac{4/3\,\pi r^3}{4/3\,\pi R^3} \quad\Rightarrow\quad q = \frac{r^3}{R^3}\,Q$$

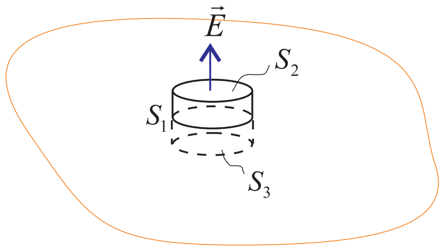
Gauss' Law: $\displaystyle\oint_{S'} \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$

$$E \underbrace{\oint_{S'} dA}_{\text{surface area of } S' \,=\, 4\pi r^2} = \frac{r^3}{R^3}\frac{1}{\epsilon_0}\cdot Q$$

$$\therefore \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R^3}\,r\,\hat{r}\,; \qquad \text{for } r \le R$$



3.4 Gauss' Law and Conductors



Cross-sectional area A

For *isolated* conductors, charges are free to move until *all charges lie outside the surface of the conductor.* Also, the *E-field at the surface of a conductor is perpendicular to its surface.* (`Why?`)

Consider Gaussian surface S of shape of cylinder:

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{\sigma A}{\epsilon_0}$$

BUT $\int_{S_1} \vec{E}\cdot d\vec{A} = 0 \quad (\because\ \vec{E} \perp d\vec{A}\,)$

$\int_{S_3} \vec{E}\cdot d\vec{A} = 0 \quad (\because\ \vec{E} = 0 \text{ inside conductor })$

$$\begin{aligned}\int_{S_2} \vec{E}\cdot d\vec{A} &= E \underbrace{\int_{S_2} dA}_{\text{Area of } S_2} \qquad (\because\ \vec{E} \parallel d\vec{A}\,)\\ &= EA\end{aligned}$$

$$\therefore \text{ Gauss' Law} \quad \Rightarrow \quad EA = \frac{\sigma A}{\epsilon_0}$$

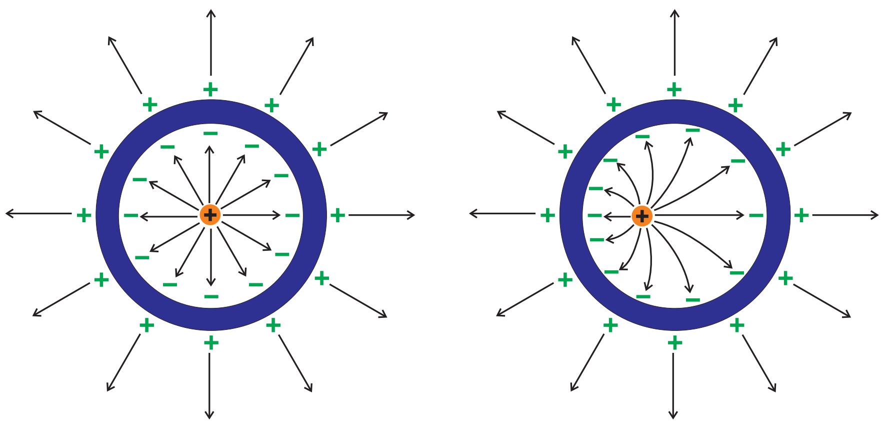
$$\therefore \boxed{\text{On conductor's surface} \quad E = \frac{\sigma}{\epsilon_0}}$$

BUT, there's no charge inside conductors.

$$\therefore \boxed{\text{Inside conductors} \quad E = 0} \quad \textit{Always!}$$

Notice: Surface charge density on a conductor's surface is *not uniform.*

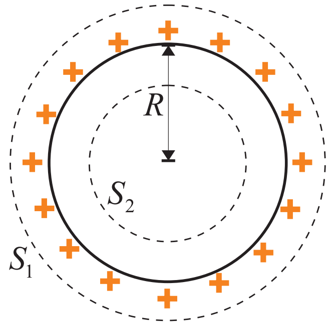
Example: Conductor with a charge inside
Note: This is not an isolated system (because of the charge inside).



Note: In BOTH cases, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{r}$ outside

Example:

I. Charge sprayed on a conductor sphere:



Total charge $= Q$

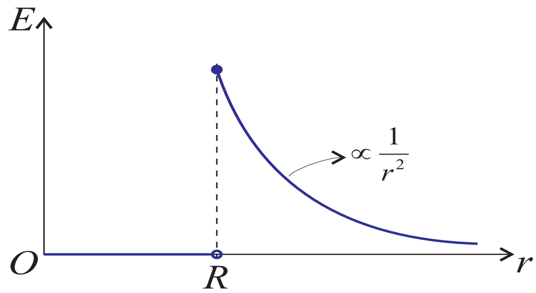
First, we know that charges all move to the *surface* of conductors.

(i) For $r < R$:
Consider Gaussian surface S_2

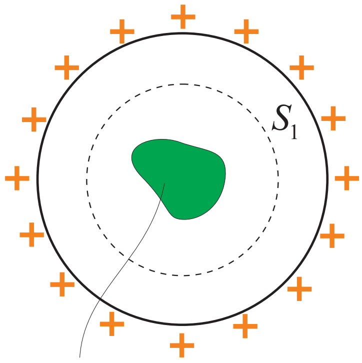
$$\oint_{S_2} \vec{E} \cdot d\vec{A} = 0 \quad (\because \text{ no charge inside })$$
$$\Rightarrow \quad E = 0 \quad \text{everywhere.}$$

(ii) For $r \geq R$:
Consider Gaussian surface S_1:

$$\oint_{S_1} \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$
$$E \underbrace{\oint_{S_1} d\vec{A}}_{4\pi r^2} = \frac{Q}{\epsilon_0} \qquad (\overbrace{\vec{E} \parallel d\vec{A}}^{\text{For a conductor}} \underbrace{\parallel \hat{r}}_{\text{Spherically symmetric}})$$
$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$



II. Conductor sphere with hole inside:

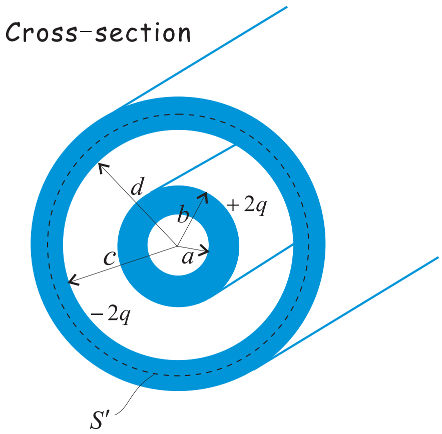


Consider Gaussian surface S_1: Total charge included = 0

$\therefore$ E-field = 0 inside

The E-field is identical to the case of a solid conductor!!

III. A long hollow cylindrical conductor:



Example:
Inside hollow cylinder ($+2q$)

$$\begin{cases} \text{Inner radius} & \text{a} \\ \text{Outer radius} & \text{b} \end{cases}$$

Outside hollow cylinder ($-3q$)

$$\begin{cases} \text{Inner radius} & \text{c} \\ \text{Outer radius} & \text{d} \end{cases}$$

Question: Find the charge on each surface of the conductor.

For the inside hollow cylinder, charges distribute only on the surface.

$\therefore$ Inner radius a surface, charge = 0
and Outer radius b surface, charge = $+2q$

For the outside hollow cylinder, charges do not distribute only on outside.

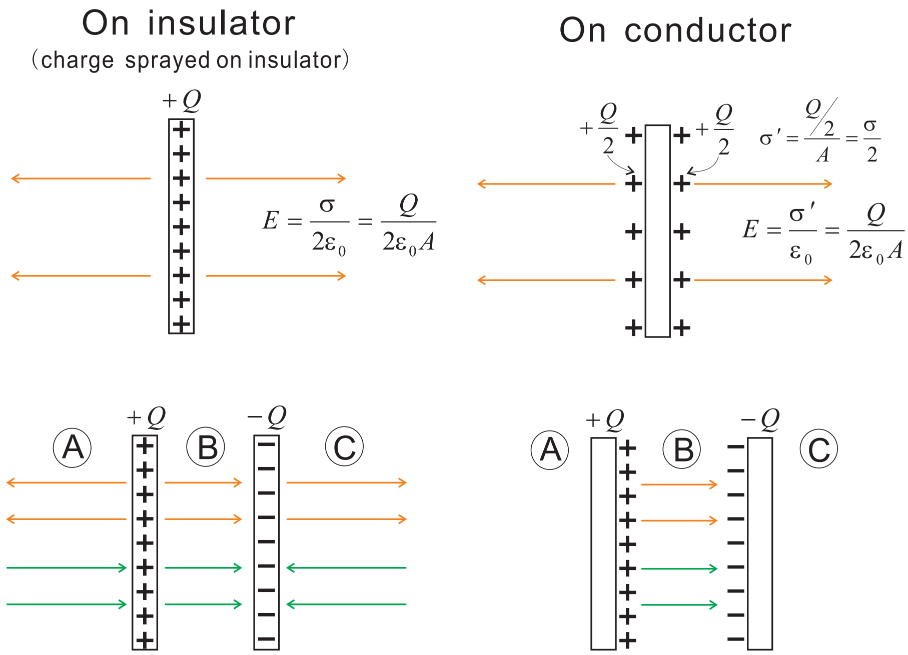
$\because$ It's not an isolated system. (There are charges inside!)

Consider Gaussian surface S' inside the conductor:
E-field always = 0
$\therefore$ Need charge $-2q$ on radius c surface to balance the charge of inner cylinder.
So charge on radius d surface = $-q$. (`Why?`)

IV. Large sheets of charge:
Total charge Q on sheet of area A,

$$\therefore \quad \text{Surface charge density } \sigma = \frac{Q}{A}$$

By principle of superposition



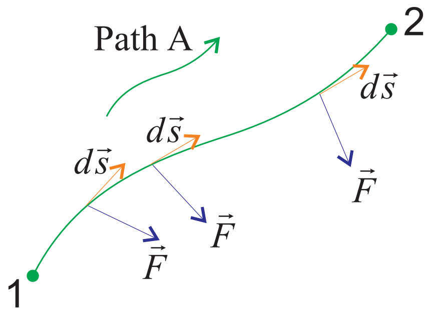
	On insulator	On conductor
Region A:	$E = 0$	$E = 0$
Region B:	$E = \dfrac{Q}{\epsilon_0 A}$	$E = \dfrac{Q}{\epsilon_0 A}$
Region C:	$E = 0$	$E = 0$

Chapter 4

Electric Potential

4.1 Potential Energy and Conservative Forces

(Read Halliday Vol.1 Chap.12)
Electric force is a **conservative force**



Work done by the electric force $\vec{F}$ as a charge moves an infinitesimal distance $d\vec{s}$ along *Path A* $= dW$

Note: $d\vec{s}$ is in the *tangent* direction of the curve of *Path A*.

$$dW = \vec{F} \cdot d\vec{s}$$

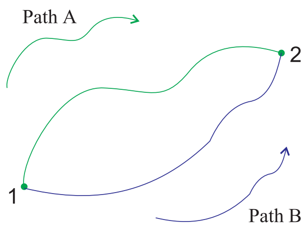
$\therefore$ Total work done W by force $\vec{F}$ in moving the particle from Point 1 to Point 2

$$\boxed{W = \int_{\substack{1 \\ \text{Path A}}}^{2} \vec{F} \cdot d\vec{s}}$$

$$\int_{\substack{1 \\ \text{Path A}}}^{2} = \textit{Path Integral}$$

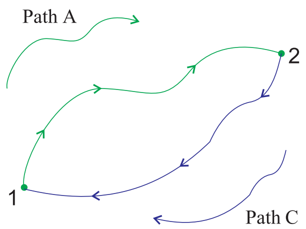
$$= \text{Integration over Path A from Point 1 to Point 2.}$$

DEFINITION: A force is **conservative** *if* the work done on a particle by the force is *independent of the path taken.*



$\therefore$ For conservative forces,

$$\underset{\text{Path A}}{\int_1^2} \vec{F}\cdot d\vec{s} \quad = \quad \underset{\text{Path B}}{\int_1^2} \vec{F}\cdot d\vec{s}$$



Let's consider a path starting at point 1 to 2 through *Path A* and from 2 to 1 through *Path C*

$$\begin{aligned} \text{Work done} &= \underset{\text{Path A}}{\int_1^2} \vec{F}\cdot d\vec{s} + \underset{\text{Path C}}{\int_2^1} \vec{F}\cdot d\vec{s} \\ &= \underset{\text{Path A}}{\int_1^2} \vec{F}\cdot d\vec{s} - \underset{\textbf{Path B}}{\int_{\mathbf{1}}^{\mathbf{2}}} \vec{F}\cdot d\vec{s} \end{aligned}$$

DEFINITION: The work done by a **conservative force** on a particle when it *moves around a closed path returning to its initial position* is *zero.*

MATHEMATICALLY, $\vec{\nabla}\times\vec{F} = 0$ everywhere for conservative force $\vec{F}$

Conclusion: Since the work done by a conservative force $\vec{F}$ is *path-independent*, we can define a quantity, **potential energy**, that depends only on the *position* of the particle.

Convention: We define **potential energy** U such that

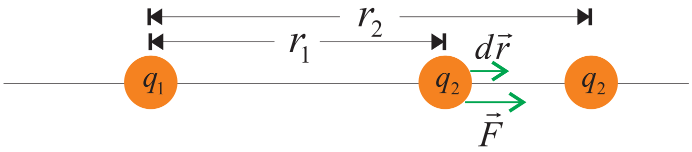
$$\boxed{dU = -W = -\int \vec{F}\cdot d\vec{s}}$$

$\therefore$ For particle moving from 1 to 2

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F}\cdot d\vec{s}$$

where U_1, U_2 are **potential energy** at position 1, 2.

Example:



Suppose charge q_2 moves from point 1 to 2.

$$\begin{aligned}
\text{From definition:}\quad U_2 - U_1 &= -\int_1^2 \vec{F}\cdot d\vec{r} \\
&= -\int_{r_1}^{r_2} F\,dr \quad (\because\ \vec{F}\parallel d\vec{r}\,) \\
&= -\int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0}\,\frac{q_1q_2}{r^2}\,dr \\
\left(\because\ \int\frac{dr}{r^2} = -\frac{1}{r} + C\ \right)\qquad &= \frac{1}{4\pi\epsilon_0}\,\frac{q_1q_2}{r}\bigg|_{r_1}^{r_2} \\
-\Delta W = \Delta U &= \frac{1}{4\pi\epsilon_0}\,q_1q_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)
\end{aligned}$$

Note:

(1) This result is generally true for 2-Dimension or 3-D motion.

(2) If q_2 moves away from q_1,
then $r_2 > r_1$, we have

- If q_1, q_2 are of *same* sign,
  then $\Delta U < 0$, $\Delta W > 0$
  (ΔW = Work done by electric *repulsive* force)
- If q_1, q_2 are of *different* sign,
  then $\Delta U > 0$, $\Delta W < 0$
  (ΔW = Work done by electric *attractive* force)

(3) If q_2 moves towards q_1,
then $r_2 < r_1$, we have

- If q_1, q_2 are of *same* sign,
  then $\Delta U \quad 0$, $\Delta W \quad 0$
- If q_1, q_2 are of *different* sign,
  then $\Delta U \quad 0$, $\Delta W \quad 0$

(4) Note: It is the *difference* in potential energy that is important.

$$\text{REFERENCE POINT:} \quad U(r=\infty)=0$$

$$\therefore\ U_\infty - U_1 = \frac{1}{4\pi\epsilon_0}\, q_1 q_2 \left(\underset{\substack{\downarrow \\ \infty}}{\frac{1}{r_2}} - \frac{1}{r_1} \right)$$

$$\boxed{U(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r}}$$

If q_1, q_2 *same sign*, then $U(r) > 0$ for all r
If q_1, q_2 *opposite sign*, then $U(r) < 0$ for all r

(5) Conservation of Mechanical Energy:
For a system of charges with no external force,

$$E = \underset{\substack{\swarrow \\ \text{(Kinetic Energy)}}}{K} + \underset{\substack{\searrow \\ \text{(Potential Energy)}}}{U} = \text{Constant}$$

$$\text{or} \quad \boxed{\Delta E = \Delta K + \Delta U = 0}$$

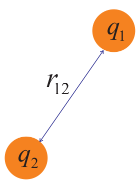
<u>Potential Energy of A System of Charges</u>

Example: P.E. of 3 charges q_1, q_2, q_3

Start: q_1, q_2, q_3 all at $r=\infty, U=0$

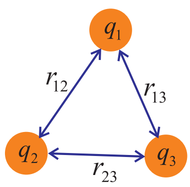
Step1: 

Move q_1 from ∞ to its position $\Rightarrow\ U=0$

Step2: 

Move q_2 from ∞ to new position $\Rightarrow$

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

Step3: 

Move q_3 from ∞ to new position $\Rightarrow$ Total P.E.

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Step4: What if there are 4 charges?

4.2 Electric Potential

Consider a charge q at center, we consider its effect on test charge q_0

DEFINITION: We define electric potential V so that

$$\boxed{\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}}$$

$(\therefore\ V$ is the P.E. per unit charge)

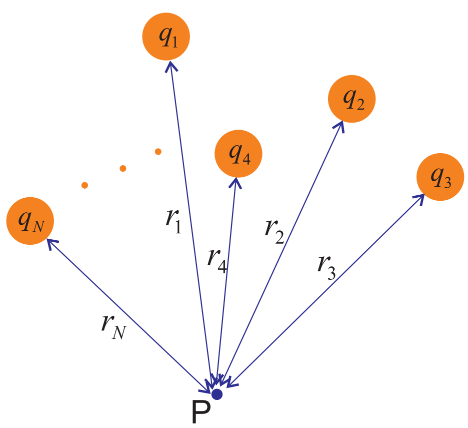
- Similarly, we take $V(r = \infty) = 0$.
- Electric Potential is a **scalar**.
- Unit: $Volt(V) = Joules/Coulomb$
- For a single point charge:

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}}$$

- Energy Unit: $\Delta U = q\Delta V$

$$electron - Volt(eV) = \underbrace{1.6 \times 10^{-19}}_{\text{charge of electron}}\ J$$

Potential For A System of Charges



For a total of N point charges, the potential V at any point P can be derived from the **principle of superposition**.

Recall that potential due to q_1 at point P: $V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r_1}$

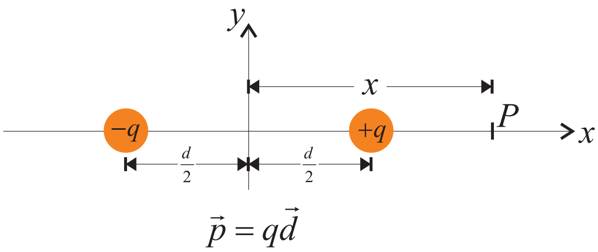
$\therefore$ Total potential at point P due to N charges:

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_N \quad \text{(principle of superposition)} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_N}{r_N}\right] \end{aligned}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}}$$

Note: For $\vec{E}, \vec{F}$, we have a sum of vectors
For V, U, we have a sum of scalars

Example: Potential of an electric dipole



Consider the potential of point P at distance $x > \frac{d}{2}$ from dipole.

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{x-\frac{d}{2}} + \frac{-q}{x+\frac{d}{2}}\right]$$

Special Limiting Case: $x \gg d$

$$\frac{1}{x \mp \frac{d}{2}} = \frac{1}{x}\cdot\frac{1}{1 \mp \frac{d}{2x}} \simeq \frac{1}{x}\left[1 \pm \frac{d}{2x}\right]$$

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{x}\left[1 + \frac{d}{2x} - (1 - \frac{d}{2x})\right]$$

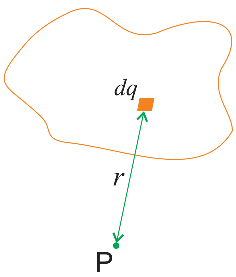
$$V = \frac{p}{4\pi\epsilon_0 x^2} \qquad \text{(Recall } p = qd\text{)}$$

For a point charge $E \propto \frac{1}{r^2}$ $V \propto \frac{1}{r}$

For a dipole $E \propto \frac{1}{r^3}$ $V \propto \frac{1}{r^2}$

For a quadrupole $E \propto \frac{1}{r^4}$ $V \propto \frac{1}{r^3}$

Electric Potential of Continuous Charge Distribution



For any charge distribution, we write the electrical potential dV due to infinitesimal charge dq:

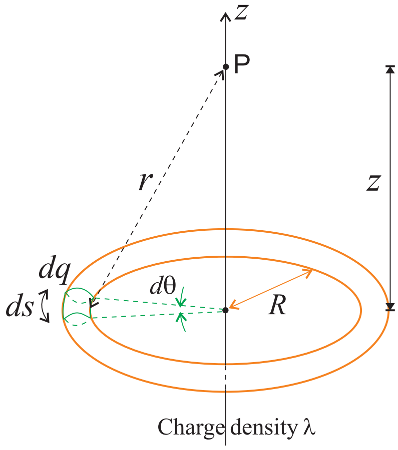
$$dV = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r}$$

$$\therefore \quad \boxed{V = \int_{\substack{\text{charge}\\\text{distribution}}} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r}}$$

Similar to the previous examples on E-field, for the case of *uniform* charge distribution:

1-D	$\Rightarrow$	long rod	$\Rightarrow$	$dq = \lambda\, dx$
2-D	$\Rightarrow$	charge sheet	$\Rightarrow$	$dq = \sigma\, dA$
3-D	$\Rightarrow$	uniformly charged body	$\Rightarrow$	$dq = \rho\, dV$

Example (1): Uniformly-charged ring



Length of the infinitesimal ring element $= ds = Rd\theta$

$$\therefore \quad \text{charge } dq = \lambda\, ds = \lambda R\, d\theta$$

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

The integration is around the entire ring.

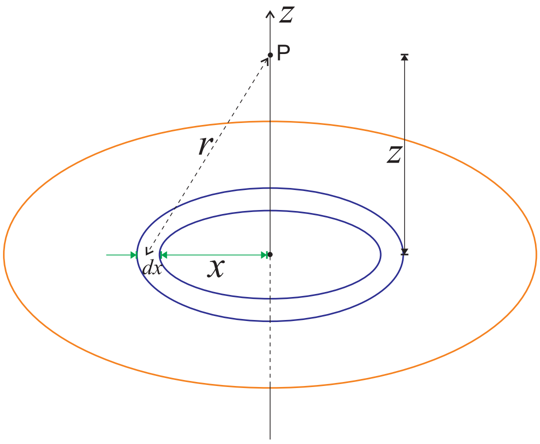
$$\begin{aligned} \therefore \quad V &= \int_{\text{ring}} dV \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2+z^2}} \\ &= \frac{\lambda R}{4\pi\epsilon_0\sqrt{R^2+z^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \end{aligned}$$

Total charge on the ring $= \lambda \cdot (2\pi R)$

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{R^2+z^2}}$$

LIMITING CASE: $\quad z \gg R \;\Rightarrow\; V = \dfrac{Q}{4\pi\epsilon_0\sqrt{z^2}} = \dfrac{Q}{4\pi\epsilon_0|z|}$

Example (2): Uniformly-charged disk



Using the **principle of superposition**, we will find the potential of a disk of uniform charge density by integrating the potential of *concentric rings.*

$$\therefore \quad dV = \frac{1}{4\pi\epsilon_0}\int_{\text{disk}} \frac{dq}{r}$$

Ring of radius x: $\quad dq = \sigma\, dA = \sigma\,(2\pi x dx)$

$$\begin{aligned}
\therefore \quad V &= \int_0^R \frac{1}{4\pi\epsilon_0}\cdot\frac{\sigma 2\pi x\, dx}{\sqrt{x^2+z^2}} \\
&= \frac{\sigma}{4\epsilon_0}\int_0^R \frac{d(x^2+z^2)}{(x^2+z^2)^{1/2}} \\
V &= \frac{\sigma}{2\epsilon_0}\,(\sqrt{z^2+R^2}-\sqrt{z^2}) \\
&= \frac{\sigma}{2\epsilon_0}\,(\sqrt{z^2+R^2}-|z|)
\end{aligned}$$

Recall: $|x| = \begin{cases} +x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

Limiting Case:

(1) If $|z| \gg R$

$$\begin{aligned}
\sqrt{z^2+R^2} &= \sqrt{z^2\Big(1+\frac{R^2}{z^2}\Big)} \\
&= |z|\cdot\Big(1+\frac{R^2}{z^2}\Big)^{\frac{1}{2}} && (\,(1+x)^n \approx 1+nx \text{ if } x \ll 1\,) \\
&\simeq |z|\cdot\Big(1+\frac{R^2}{2z^2}\Big) && (\,\frac{|z|}{z^2} = \frac{1}{|z|}\,)
\end{aligned}$$

$\therefore$ At large z, $V \simeq \dfrac{\sigma}{2\epsilon_0}\cdot\dfrac{R^2}{2|z|} = \dfrac{Q}{4\pi\epsilon_0|z|}$ (like a point charge)

where Q = total charge on disk $= \sigma\cdot\pi R^2$

(2) If $|z| \ll R$

$$\begin{aligned} \sqrt{z^2+R^2} &= R\cdot\left(1+\frac{z^2}{R^2}\right)^{\frac{1}{2}} \\ &\simeq R\left(1+\frac{z^2}{2R^2}\right) \end{aligned}$$

$$\therefore \quad V \simeq \frac{\sigma}{2\epsilon_0}\left[R-|z|+\frac{z^2}{2R}\right]$$

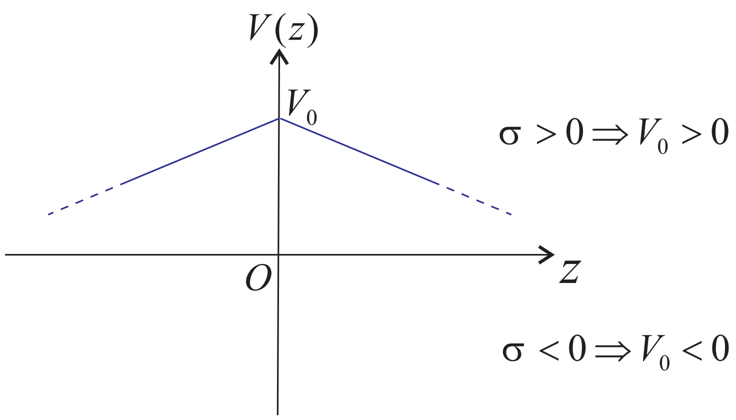
At $z=0$, $V=\dfrac{\sigma R}{2\epsilon_0}$; Let's call this V_0

$$\begin{aligned} \therefore \quad V(z) &= \frac{\sigma R}{2\epsilon_0}\left[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\right] \\ V(z) &= V_0\left[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\right] \end{aligned}$$

The *key* here is that it is the difference between potentials of two points that is important.
$\Rightarrow$ A convenience reference point to compare in this example is the potential of the charged disk.
$\therefore$ The important quantity here is

$$V(z)-V_0 = -\frac{|z|}{R}V_0 + \underbrace{\cancel{\frac{z^2}{2R^2}V_0}}_{\text{neglected as } z\ll R}$$

$$\boxed{V(z)-V_0 = -\frac{V_0}{R}|z|}$$



4.3 Relation Between Electric Field E and Electric Potential V

(A) To get V from $\vec{E}$:
Recall our definition of the potential V:

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0}$$

where ΔU is the change in P.E.; W_{12} is the work done in bringing charge q_0 from point 1 to 2.

$$\therefore \quad \Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F} \cdot d\vec{s}}{q_0}$$

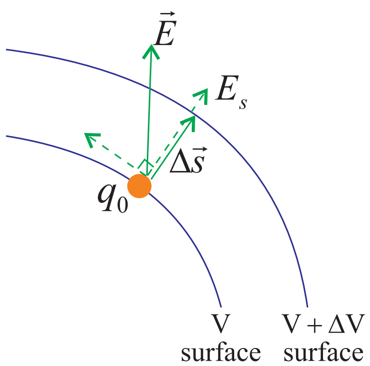
However, the definition of E-field: $\vec{F} = q_0 \vec{E}$

$$\therefore \quad \boxed{\Delta V = V_2 - V_1 = -\int_1^2 \vec{E} \cdot d\vec{s}}$$

Note: The integral on the right hand side of the above can be calculated *along any path from point 1 to 2.* (**Path-Independent**)

Convention: $V_\infty = 0 \quad \Rightarrow \quad \boxed{V_P = -\int_\infty^P \vec{E} \cdot d\vec{s}}$

(B) To get $\vec{E}$ from V:



(i.e. Potential = V on the surface)

Again, use the definition of V:

$$\Delta U = q_0 \Delta V = \underbrace{-W}_{\text{Work done}}$$

However,

$$W = \underbrace{q_0 \vec{E}}_{\text{Electric force}} \cdot \Delta\vec{s} = q_0 \, E_s \, \Delta s$$

where E_s is the E-field component along the path $\Delta\vec{s}$.

$$\therefore \quad q_0 \Delta V = -q_0 E_s \Delta s$$

$$\therefore \quad E_s = -\frac{\Delta V}{\Delta s}$$

For infinitesimal Δs,

$$\therefore \quad \boxed{E_s = -\frac{dV}{ds}}$$

Note: (1) Therefore the E-field component along *any direction* is the negtive derivative of the potential *along the same direction.*

(2) If $d\vec{s} \perp \vec{E}$, then $\Delta V = 0$

(3) ΔV is biggest/smallest if $d\vec{s} \parallel \vec{E}$

Generally, for a potential $V(x, y, z)$, the relation between $\vec{E}(x, y, z)$ and V is

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are **partial derivative**s

For $\frac{\partial}{\partial x} V(x, y, z)$, everything y, z are treated like a *constant* and we only take derivative with respect to x.

Example: If $V(x, y, z) = x^2 y - z$

$$\frac{\partial V}{\partial x} =$$

$$\frac{\partial V}{\partial y} =$$

$$\frac{\partial V}{\partial z} =$$

For other co-ordinate systems

(1) Cylindrical:

$$V(r, \theta, z) \qquad \begin{cases} E_r = -\frac{\partial V}{\partial r} \\ E_\theta = -\frac{1}{r} \cdot \frac{\partial V}{\partial \theta} \\ E_z = -\frac{\partial V}{\partial z} \end{cases}$$

(2) Spherical:

$$V(r,\theta,\phi) \qquad \begin{cases} E_r &= -\dfrac{\partial V}{\partial r} \\[2ex] E_\theta &= -\dfrac{1}{r}\cdot\dfrac{\partial V}{\partial \theta} \\[2ex] E_\phi &= -\dfrac{1}{r\sin\theta}\cdot\dfrac{\partial V}{\partial \phi} \end{cases}$$

Note: Calculating V involves summation of *scalars*, which is easier than adding *vectors* for calculating E-field.

$\therefore$ To find the E-field of a general charge system, we first calculate $\boldsymbol{V}$, and then derive $\vec{\boldsymbol{E}}$ from the partial derivative.

Example: Uniformly charged disk

From potential calculations:

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - |z|\,\right) \qquad \text{for a point along the z-axis}$$

For $\quad z > 0, \quad |z| = z$

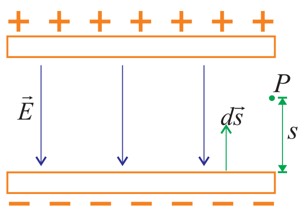
$$\therefore \quad E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right] \qquad \text{(Compare with Chap.2 notes)}$$

Example: Uniform electric field

(`e.g.` Uniformly charged $+ve$ and $-ve$ plates)

Consider a path going from the $-ve$ plate to the $+ve$ plate

Potential at point P, V_P can be deduced from definition.



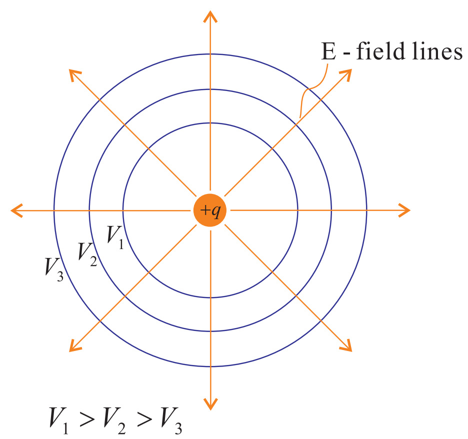
$$\begin{aligned} \texttt{i.e.} \qquad V_P - V_- &= -\int_0^s \vec{E}\cdot d\vec{s} && (V_- = \text{Potential of } -ve \text{ plate}) \\ &= -\int_0^s (-E\,ds) && \because \ \vec{E}, d\vec{s} \text{ pointing opposite directions} \\ &= E\int_0^s ds = Es \end{aligned}$$

Convenient reference: $\qquad V_- = 0$

$$\therefore \quad \boxed{V_P = E\cdot s}$$

4.4 Equipotential Surfaces

Equipotential surface is a surface on which the *potential is constant*.
$\Rightarrow \quad (\Delta V = 0)$



$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

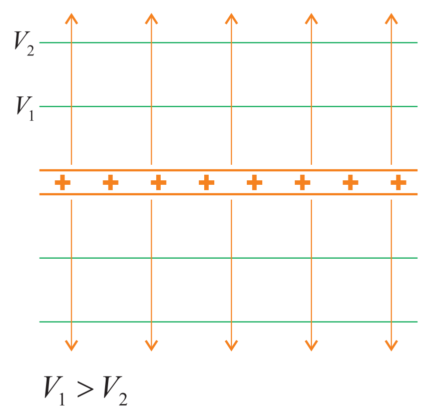
$\Rightarrow \quad r = const$

$\Rightarrow$ Equipotential surfaces are *circles/spherical surfaces*

Note: (1) A charge can move freely on an equipotential surface without any work done.

(2) The **electric field lines** must be *perpendicular* to the **equipotential surfaces**. (`Why?`)
On an equipotential surface, $V = constant$
$\Rightarrow \Delta V = 0 \Rightarrow \vec{E} \cdot d\vec{l} = 0$, where $d\vec{l}$ is *tangent* to equipotential surface
$\therefore \quad \vec{E}$ must be *perpendicular* to equipotential surfaces.

Example: Uniformly charged surface (infinite)

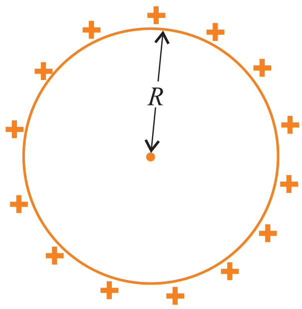


Recall $\qquad V = V_0 - \dfrac{\sigma}{2\epsilon_0}|z|$
$\uparrow$
Potential at $z = 0$

Equipotential surface means

$$V = const \Rightarrow V_0 - \frac{\sigma}{2\epsilon_0}|z| = C$$
$$\Rightarrow |z| = constant$$

Example: Isolated spherical charged conductors



Recall:

(1) E-field inside $= 0$

(2) charge distributed on the *outside* of conductors.

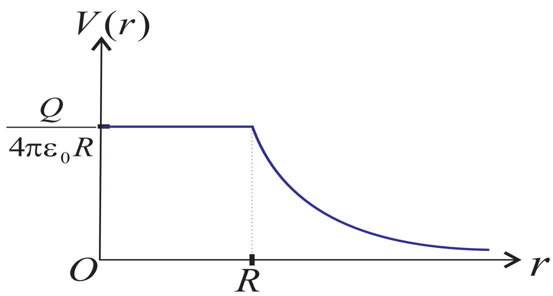
(i) Inside conductor:

$$\begin{aligned} E = 0 \quad &\Rightarrow \quad \Delta V = 0 \text{ everywhere in conductor} \\ &\Rightarrow \quad V = \textit{constant} \text{ everywhere in conductor} \\ &\Rightarrow \quad \text{The entire conductor is at the same potential} \end{aligned}$$

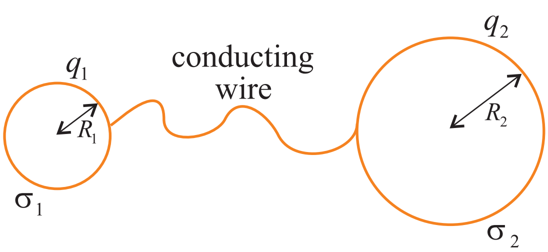
(ii) Outside conductor:

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

$\because$ Spherically symmetric (Just like a point charge.)
BUT not true for conductors of arbitrary shape.



Example: Connected conducting spheres



Two conductors connected can be seen as a *single conductor*

$\therefore$ Potential everywhere is identical.

Potential of radius R_1 sphere $V_1 = \dfrac{q_1}{4\pi\epsilon_0 R_1}$

Potential of radius R_2 sphere $V_2 = \dfrac{q_2}{4\pi\epsilon_0 R_2}$

$$\mathbf{V_1 = V_2}$$

$$\Rightarrow \quad \frac{q_1}{R_1} = \frac{q_2}{R_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{R_1}{R_2}$$

Surface charge density

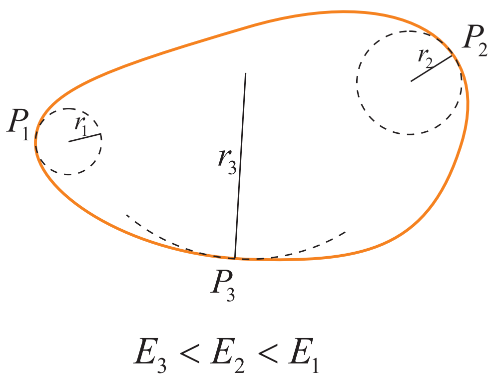
$$\sigma_1 = \underbrace{\frac{q_1}{4\pi R_1^2}}_{\text{Surface area of radius } R_1 \text{ sphere}}$$

$$\therefore \quad \frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2} \cdot \frac{R_2^2}{R_1^2} = \frac{R_2}{R_1}$$

$$\therefore \quad \text{If } R_1 < R_2, \text{ then } \sigma_1 > \sigma_2$$

And the surface electric field $E_1 > E_2$

For arbitrary shape conductor:



At every point on the conductor, we fit a *circle*. The radius of this circle is the *radius of curvature*.

Note: Charge distribution on a conductor does **not** have to be uniform.

Chapter 5

Capacitance and DC Circuits

5.1 Capacitors

A **capacitor** is a system of *two conductors* that carries *equal and opposite charges*. A capacitor *stores charge and energy* in the form of electro-static field.

We define **capacitance** as

$$\boxed{C = \frac{Q}{V}} \qquad \text{Unit: Farad(F)}$$

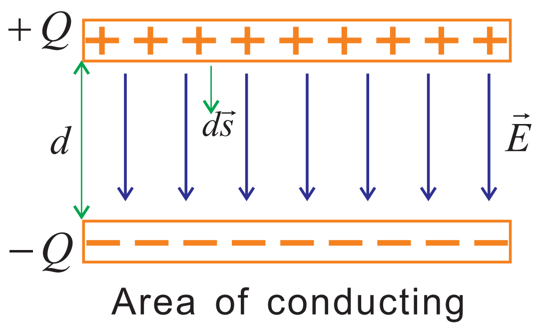
where

$$\begin{aligned} Q &= \text{Charge on one plate} \\ V &= \text{Potential difference between the plates} \end{aligned}$$

Note: The C of a capacitor is a *constant* that depends only on its shape and material.
`i.e.` If we increase V for a capacitor, we can increase Q stored.

5.2 Calculating Capacitance

5.2.1 Parallel-Plate Capacitor



(1) Recall from Chapter 3 note,

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

(2) Recall from Chapter 4 note,

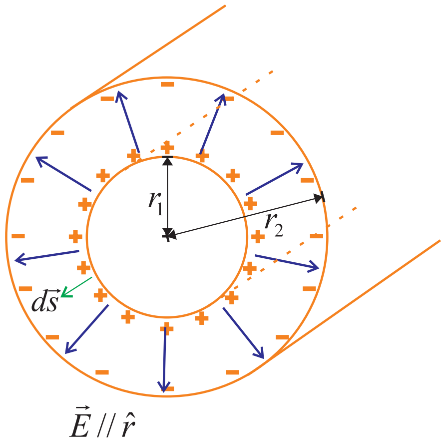
$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{s}$$

Again, notice that this integral is independent of the path taken.
$\therefore$ We can take the path that is parallel to the $\vec{E}$-field.

$$\begin{aligned} \therefore \quad \Delta V &= \int_+^- \vec{E} \cdot d\vec{s} \\ &= \int_+^- E \cdot ds \\ &= \frac{Q}{\epsilon_0 A} \underbrace{\int_+^- ds}_{\text{Length of path taken}} \\ &= \frac{Q}{\epsilon_0 A} \cdot d \end{aligned}$$

$$(3) \ \therefore \quad \boxed{C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}}$$

5.2.2 Cylindrical Capacitor



Consider two concentric cylindrical wire of innner and outer radii r_1 and r_2 respectively. The length of the capacitor is L where $r_1 < r_2 \ll L$.

(1) Using Gauss' Law, we determine that the E-field between the conductors is (`cf.` Chap3 note)

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \cdot \frac{\lambda}{r}\,\hat{r} = \frac{1}{2\pi\epsilon_0} \cdot \frac{Q}{Lr}\,\hat{r}$$

where λ is charge per unit length

(2)

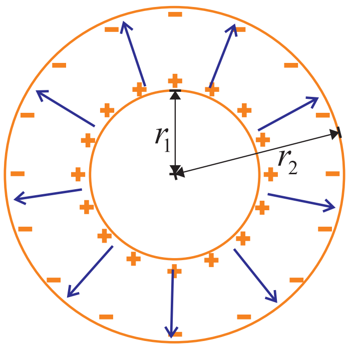
$$\Delta V = \int_+^- \vec{E} \cdot d\vec{s}$$

Again, we choose the path of integration so that $d\vec{s} \parallel \hat{r} \parallel \vec{E}$

$$\therefore \quad \Delta V = \int_{r_1}^{r_2} E\,dr = \frac{Q}{2\pi\epsilon_0 L} \underbrace{\int_{r_1}^{r_2} \frac{dr}{r}}_{\ln(\frac{r_2}{r_1})}$$

$$\therefore \quad \boxed{C = \frac{Q}{\Delta V} = 2\pi\epsilon_0 \frac{L}{\ln(r_2/r_1)}}$$

5.2.3 Spherical Capacitor



$\vec{E}\,//\,\hat{r}$

Choose $d\vec{s}\,//\,\hat{r}$

For the space between the two conductors,

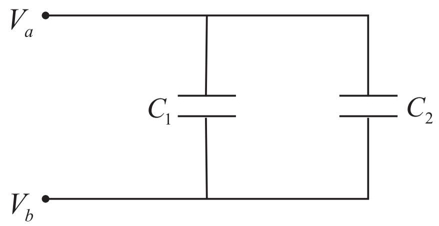
$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}; \qquad r_1 < r < r_2$$

$$\begin{aligned} \Delta V &= \int_+^- \vec{E} \cdot d\vec{s} \\ \text{Choose } d\vec{s} \parallel \hat{r} \qquad &= \int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}\,dr \\ &= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \end{aligned}$$

$$\boxed{C = 4\pi\epsilon_0 \left[\frac{r_1 r_2}{r_2 - r_1}\right]}$$

5.3 Capacitors in Combination

(a) Capacitors in Parallel



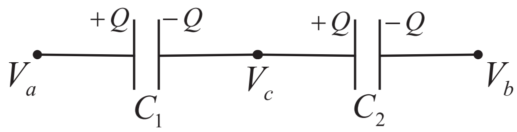
In this case, it's the *potential difference* $V = V_a - V_b$ that is the same across the capacitor.

BUT: *Charge* on each capacitor different

$$\begin{aligned} \text{Total charge} \quad Q &= Q_1 + Q_2 \\ &= C_1 V + C_2 V \\ Q &= \underbrace{(C_1 + C_2)}_{\text{Equivalent capacitance}} V \end{aligned}$$

$\therefore$ For capacitors in parallel: $\boxed{C = C_1 + C_2}$

(b) Capacitors in Series



The *charge across capacitors* are the same.

BUT: *Potential difference* (P.D.) across capacitors different

$$\begin{aligned} \Delta V_1 &= V_a - V_c = \frac{Q}{C_1} \qquad \text{P.D. across } C_1 \\ \Delta V_2 &= V_c - V_b = \frac{Q}{C_2} \qquad \text{P.D. across } C_2 \end{aligned}$$

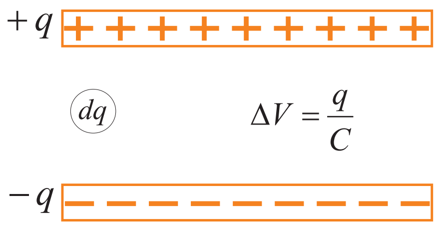
$\therefore$ Potential difference

$$\begin{aligned} \Delta V &= V_a - V_b \\ &= \Delta V_1 + \Delta V_2 \\ \Delta V &= Q\,(\frac{1}{C_1} + \frac{1}{C_2}) = \frac{Q}{C} \end{aligned}$$

where C is the **Equivalent Capacitance**

$$\therefore \quad \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

5.4 Energy Storage in Capacitor



In charging a capacitor, *positive charge* is being moved from the *negative plate* to the *positive plate*.
$\Rightarrow$ NEEDS WORK DONE!

Suppose we move charge dq from $-ve$ to $+ve$ plate, *change in potential energy*

$$dU = \Delta V \cdot dq = \frac{q}{C}\, dq$$

Suppose we keep putting in a total charge Q to the capacitor, the *total potential energy*

$$U = \int dU = \int_0^Q \frac{q}{C}\, dq$$

$$\therefore \quad \boxed{U = \frac{Q^2}{2C} = \frac{1}{2} C \Delta V^2} \qquad (\because Q{=}C\Delta V)$$

The energy stored in the capacitor is stored in the **electric field** between the plates.

Note : In a parallel-plate capacitor, the *E-field is constant between the plates.*

$\therefore$ We can consider the E-field energy

$$\text{density u} = \frac{\text{Total energy stored}}{\text{Total volume with E-field}}$$

$$\therefore \quad u = \frac{U}{\underbrace{Ad}_{\text{Rectangular volume}}}$$

Recall

$$\begin{cases} C &= \dfrac{\epsilon_0 A}{d} \\ E &= \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$$

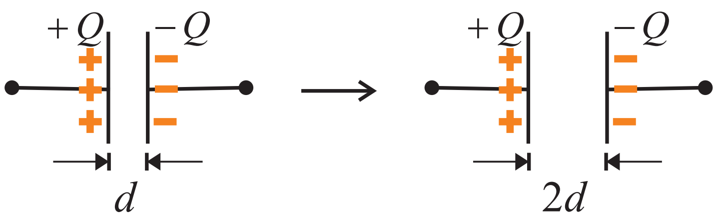
$$\therefore \quad u = \frac{1}{2} \big(\overbrace{\frac{\epsilon_0 A}{d}}^{C} \big) \cdot \big(\overbrace{Ed}^{(\Delta V)^2} \big)^2 \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{Volume}}$$

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$

Energy per unit volume of the electrostatic field

$\uparrow$ can be generally applied

Example : Changing capacitance



(1) Isolated Capacitor:
Charge on the capacitor plates remains *constant*.

<u>BUT</u>: $C_{new} = \dfrac{\epsilon_0 A}{2d} = \dfrac{1}{2} C_{old}$

$\therefore \quad U_{new} = \dfrac{Q^2}{2C_{new}} = \dfrac{Q^2}{2C_{old}/2} = 2U_{old}$

$\therefore$ In pulling the plates apart, work done $W > 0$

Summary :

$$\begin{array}{rcccccccc} & Q & \rightarrow & Q & & C & \rightarrow & C/2 & \\ (V = \frac{Q}{C}) \Rightarrow & V & \rightarrow & 2V & & E & \rightarrow & E & (E = \frac{V}{d}) \\ \frac{1}{2}\epsilon_0 E^2 = & u & \rightarrow & u & & U & \rightarrow & 2U & (U = u \cdot volume) \end{array}$$

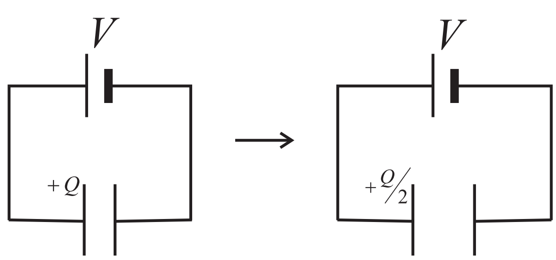
(2) Capacitor connected to a battery:
Potential difference between capacitor plates remains *constant*.

$U_{new} = \dfrac{1}{2} C_{new} \Delta V^2 = \dfrac{1}{2} \cdot \dfrac{1}{2} C_{old} \Delta V^2 = \dfrac{1}{2} U_{old}$

$\therefore$ In pulling the plates apart, work done by battery < 0

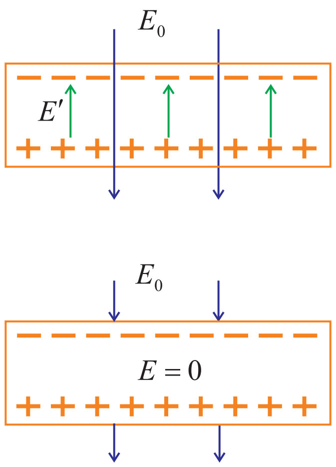
Summary :

$$\begin{array}{ccccccc} Q & \rightarrow & Q/2 & & C & \rightarrow & C/2 \\ V & \rightarrow & V & & E & \rightarrow & E/2 \\ u & \rightarrow & u/4 & & U & \rightarrow & U/2 \end{array}$$



5.5 Dielectric Constant

We first recall the case for a *conductor* being placed in an *external E-field* E_0.



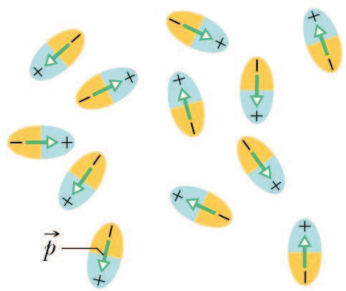
In a conductor, charges are free to move inside so that the *internal E-field* E' set up by these charges

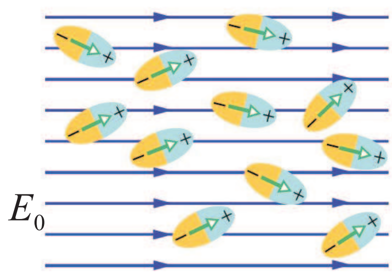
$$E' = -E_0$$

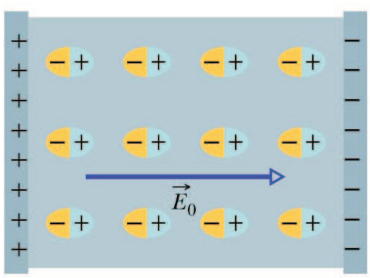
so that E-field inside conductor = 0.

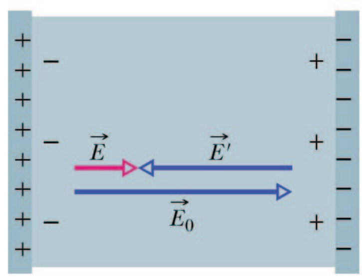
Generally, for **dielectric**, the atoms and molecules behave like a **dipole** in an E-field.

Or, we can envision this so that in the absence of E-field, the *direction of dipole in the dielectric are* randomly distributed.









The aligned dipoles will generate an *induced E-field* E', where $|E'| < |E_0|$.
We can observe the aligned dipoles in the form of *induced surface charge*.

Dielectric Constant : When a dielectric is placed in an external E-field E_0, the E-field inside a dielectric is *induced*.
E-field in dielectric

$$\boxed{E = \frac{1}{K_e} E_0}$$

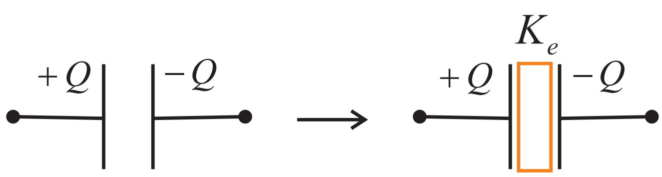
$$K_e \;=\; \textbf{dielectric constant} \quad \geq 1$$

Example :

Vacuum	$K_e = 1$
Porcelain	$K_e = 6.5$
Water	$K_e \sim 80$
Perfect conductor	$K_e = \infty$
Air	$K_e = 1.00059$

5.6 Capacitor with Dielectric

Case I :



Again, the *charge remains constant* after dielectric is inserted.
BUT: $E_{new} = \dfrac{1}{K_e} E_{old}$

$\therefore \quad \Delta V = Ed \quad \Rightarrow \quad \Delta V_{new} = \dfrac{1}{K_e} \Delta V_{old}$

$\therefore \quad C = \dfrac{Q}{\Delta V} \quad \Rightarrow \quad C_{new} = K_e \, C_{old}$

For a parallel-plate capacitor with dielectric:

$$\boxed{C = \frac{K_e \epsilon_0 A}{d}}$$

We can also write $\quad C = \dfrac{\epsilon A}{d}\quad$ in general with

$$\epsilon = K_e\,\epsilon_0 \qquad \text{(called \textbf{permittivity of dielectric})}$$

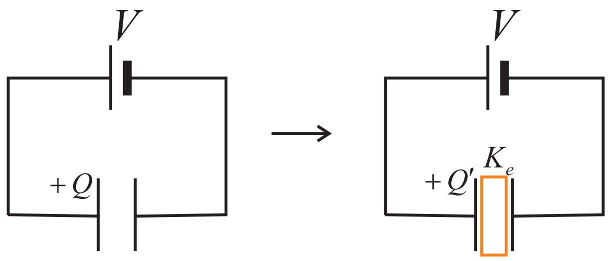
(Recall ϵ_0 = **Permittivity of free space**)

Energy stored $\quad U = \dfrac{Q^2}{2C}$;

$$\therefore \quad U_{new} = \frac{1}{K_e} U_{old} \quad < \quad U_{old}$$

$$\boxed{\therefore \quad \text{Work done in inserting dielectric} < 0}$$

Case II : Capacitor connected to a battery



Voltage across capacitor plates *remains constant* after insertion of dielectric.
In both scenarios, the E-field inside capacitor remains constant
($\because\ E = V/d$)

BUT: How can E-field remain constant?
ANSWER: By having extra charge on capacitor plates.

Recall: For conductors,

$$E = \frac{\sigma}{\epsilon_0} \qquad \text{(Chapter 3 note)}$$

$$\Rightarrow \quad E = \frac{Q}{\epsilon_0 A} \qquad (\sigma = \text{charge per unit area} = Q/A)$$

After insertion of dielectric:

$$E' = \frac{E}{K_e} = \frac{Q'}{K_e \epsilon_0 A}$$

But E-field remains constant!

$$\therefore \quad E' = E \quad \Rightarrow \quad \frac{Q'}{K_e\epsilon_0 A} = \frac{Q}{\epsilon_0 A}$$

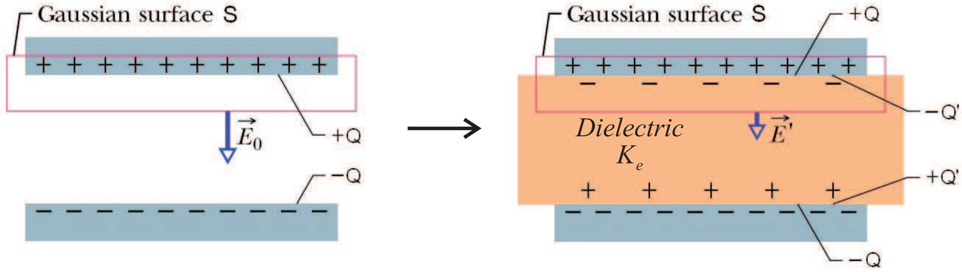
$$\Rightarrow \quad Q' = K_e Q \ > \ Q$$

$\therefore$ Capacitor $C = Q/V \Rightarrow C' \rightarrow K_e C$
Energy stored $U = \frac{1}{2}CV^2 \Rightarrow U' \rightarrow K_e U$
(i.e. $U_{new} > U_{old}$)

$$\boxed{\therefore \text{ Work done to insert dielectric} > 0}$$

5.7 Gauss' Law in Dielectric

The Gauss' Law we've learned is applicable in *vacuum only*. Let's use the capacitor as an example to examine Gauss' Law in dielectric.



Free charge on plates	$\pm Q$	$\pm Q$
Induced charge on dielectric	0	$\mp Q'$

Gauss' Law

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \quad E_0 = \frac{Q}{\epsilon_0 A} \qquad (1)$$

Gauss' Law:

$$\oint_S \vec{E}' \cdot d\vec{A} = \frac{Q - Q'}{\epsilon_0}$$

$$\therefore \quad E' = \frac{Q - Q'}{\epsilon_0 A} \qquad (2)$$

However, we define $\quad E' = \dfrac{E_0}{K_e} \qquad (3)$

From (1), (2), (3) $\quad \therefore \dfrac{Q}{K_e \epsilon_0 A} = \dfrac{Q}{\epsilon_0 A} - \dfrac{Q'}{\epsilon_0 A}$

$$\therefore \quad \boxed{\text{Induced charge density } \sigma' = \frac{Q'}{A} = \sigma\left(1 - \frac{1}{K_e}\right) < \sigma}$$

where σ is free charge density.

Recall Gauss' Law in Dielectric:

$$\underbrace{\epsilon_0 \oint_S \vec{E}' \cdot d\vec{A}}_{\text{E-field in dielectric}} = \underbrace{Q}_{\text{free charge}} - \underbrace{Q'}_{\text{induced charge}}$$

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = Q - Q\Big[1 - \frac{1}{K_e}\Big]$$

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = \frac{Q}{K_e}$$

$$\boxed{\oint_S K_e \vec{E}' \cdot d\vec{A} = \frac{Q}{\epsilon_0}}$$ Gauss' Law in dielectric

Note :

(1) This goes back to the Gauss' Law in vacuum with $E = \dfrac{E_0}{K_e}$ for dielectric

(2) Only *free charges* need to be considered, even for dielectric where there are *induced charges*.

(3) Another way to write:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon}$$

where $\vec{E}$ is E-field in dielectric, $\epsilon = K_e\epsilon_0$ is Permittivity

Energy stored with dielectric:

Total energy stored: $U = \dfrac{1}{2}CV^2$

With dielectric, recall $C = \dfrac{K_e\epsilon_0 A}{d}$

$$V = Ed$$

$\therefore$ Energy stored per unit volume:

$$\boxed{u_e = \frac{U}{Ad} = \frac{1}{2}K_e\epsilon_0 E^2}$$

and $u_{dielectric} = K_e u_{vacuum}$

$\therefore$ More energy is stored per unit volume in dielectric than in vacuum.

5.8 Ohm's Law and Resistance

ELECTRIC CURRENT is defined as the flow of electric charge through a cross-sectional area.

$$\boxed{i = \frac{dQ}{dt}}$$ Unit: Ampere (A)
= C/second

Convention :

(1) Direction of current is the direction of *flow of positive charge.*

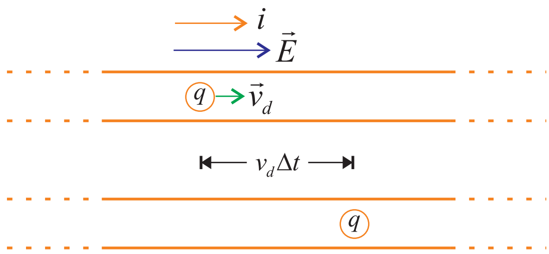
(2) Current is NOT a vector, but the **current density** is a **vector**.

$$\vec{j} = \text{charge flow per unit time per unit area}$$

$$\boxed{i = \int \vec{j} \cdot d\vec{A}}$$

Drift Velocity :

Consider a current i flowing through a cross-sectional area A:



$\therefore$ In time Δt, total charges passing through segment:

$$\Delta Q = q \underbrace{A(V_d \Delta t)}_{\substack{\text{Volume of charge} \\ \text{passing through}}} n$$

where q is charge of the current carrier, n is density of charge carrier per unit volume

$$\therefore \quad \text{Current:} \quad \boxed{i = \frac{\Delta Q}{\Delta t} = nqAv_d}$$

$$\text{Current Density:} \quad \boxed{\vec{j} = nq\vec{v}_d}$$

Note : For metal, the charge carriers are the free electrons inside.
$\therefore$ $\vec{j} = -ne\vec{v}_d$ for metals
$\therefore$ Inside metals, $\vec{j}$ and $\vec{v}_d$ are in *opposite direction.*

We define a general property, **conductivity** ($\boldsymbol{\sigma}$), of a material as:

$$\boxed{\vec{j} = \boldsymbol{\sigma}\vec{E}}$$

Note : In general, σ is NOT a constant number, but rather a *function of position and applied E-field.*

A more commonly used property, **resistivity** ($\boldsymbol{\rho}$), is defined as $\quad \rho = \dfrac{1}{\sigma}$

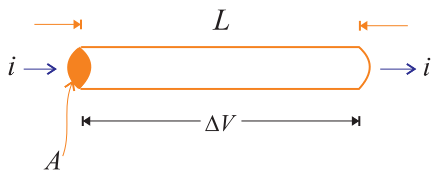
$$\therefore \quad \vec{E} = \rho \vec{j}$$

Unit of ρ : Ohm-meter (Ωm)
where Ohm (Ω) = Volt/Ampere

<u>OHM'S LAW:</u>
Ohmic materials have resistivity that are *independent of the applied electric field.*
`i.e.` metals (in not too high E-field)

Example :

Consider a **resistor** (ohmic material) of length L and cross-sectional area A.



$\therefore$ Electric field inside conductor:

$$\Delta V = \int \vec{E} \cdot d\vec{s} = E \cdot L \quad \Rightarrow \quad E = \frac{\Delta V}{L}$$

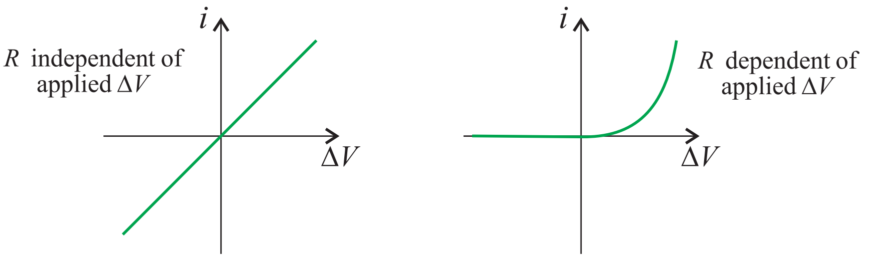
Current density: $\quad j = \dfrac{i}{A}$

$$\begin{aligned} \therefore \quad \rho &= \frac{E}{j} \\ \rho &= \frac{\Delta V}{L} \cdot \frac{1}{i/A} \end{aligned}$$

$$\boxed{\frac{\Delta V}{i} = R = \rho \frac{L}{A}}$$

where R is the **resistance** of the conductor.

Note: $\Delta V = iR$ is NOT a statement of Ohm's Law. It's just a definition for resistance.

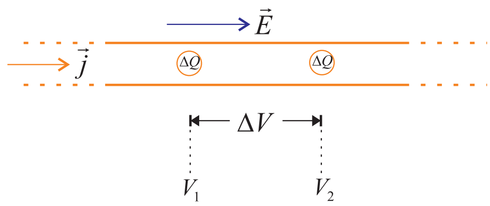


Ohmic material Non-ohmic material

(Read Chap. 29-4 of Halliday Vol 2)

ENERGY IN CURRENT:

Assuming a charge ΔQ enters with potential V_1 and leaves with potential V_2 :



$\therefore$ Potential energy lost in the wire:

$$\begin{aligned} \Delta U &= \Delta Q\, V_2 - \Delta Q\, V_1 \\ \Delta U &= \Delta Q (V_2 - V_1) \end{aligned}$$

$\therefore$ Rate of energy lost per unit time

$$\frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t}(V_2 - V_1)$$

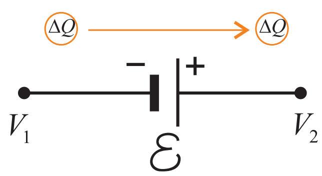
Joule's heating $\boxed{P = i \cdot \Delta V = \text{Power dissipated in conductor}}$

For a resistor R, $P = i^2 R = \dfrac{\Delta V^2}{R}$

5.9 DC Circuits

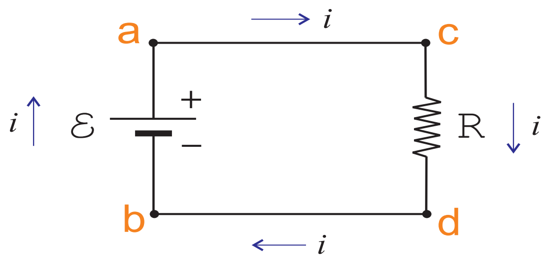
A **battery** is a device that *supplies electrical energy* to maintain a current in a circuit.

In moving from point 1 to 2, electric potential energy increase by $\Delta U = \Delta Q(V_2 - V_1) =$ Work done by $\mathcal{E}$



Define $\mathcal{E} =$ Work done/charge $= V_2 - V_1$

Example :



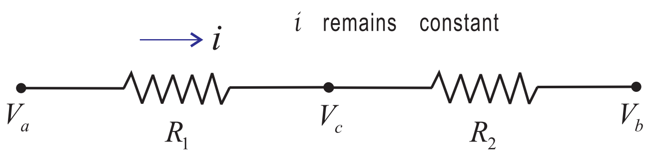
$$\left.\begin{array}{lcl} V_a & = & V_c \\ V_b & = & V_d \end{array}\right\} \quad \text{assuming}^{(1)} \text{ perfect conducting wires.}$$

$$\text{By Definition:} \quad \begin{array}{lcl} V_c - V_d & = & iR \\ V_a - V_b & = & \mathcal{E} \end{array}$$

$$\therefore \quad \mathcal{E} = iR \quad \Rightarrow \quad i = \frac{\mathcal{E}}{R}$$

Also, we have assumed$^{(2)}$ zero resistance inside battery.

Resistance in combination :

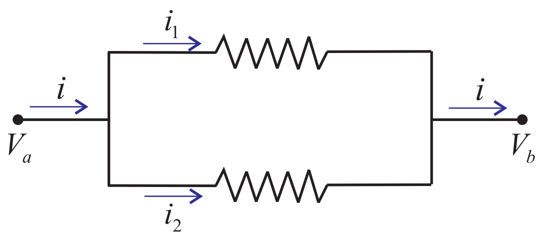


Potential differece (P.D.)

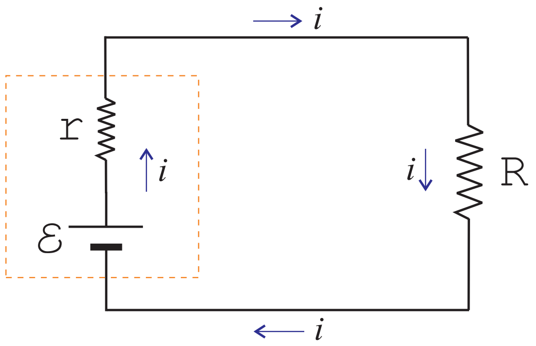
$$\begin{array}{rcl} V_a - V_b & = & (V_a - V_c) + (V_c - V_b) \\ & = & iR_1 + iR_2 \end{array}$$

$\therefore$ Equivalent Resistance

$$\begin{array}{ll} R = R_1 + R_2 & \text{for resistors in series} \\ \dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} & \text{for resistors in parallel} \end{array}$$



Example :



For real battery, there is an **internal resistance** that we cannot ignore.

$$\begin{aligned} \therefore \quad \mathcal{E} &= i(R+r) \\ i &= \frac{\mathcal{E}}{R+r} \end{aligned}$$

Joule's heating in resistor R :

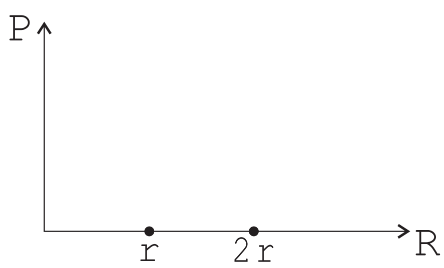
$$\begin{aligned} P &= i \cdot (\text{P.D. across resistor } R) \\ &= i^2 R \\ P &= \frac{\mathcal{E}^2 R}{(R+r)^2} \end{aligned}$$

Question: What is the value of R to obtain *maximum* Joule's heating?

Answer: We want to find R to *maximize* P.

$$\frac{dP}{dR} = \frac{\mathcal{E}^2}{(R+r)^2} - \frac{\mathcal{E}^2\, 2R}{(R+r)^3}$$

$$\begin{aligned} \text{Setting} \quad \frac{dP}{dR} = 0 \quad &\Rightarrow \quad \frac{\mathcal{E}^2}{(R+r)^3}\,[(R+r) - 2R] = 0 \\ &\Rightarrow \quad r - R = 0 \\ &\Rightarrow \quad R = r \end{aligned}$$

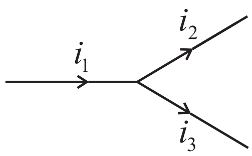


ANALYSIS OF COMPLEX CIRCUITS:

KIRCHOFF'S LAWS:

(1) First Law (Junction Rule):
Total current entering a junction equal to the total current leaving the junction.

Conservation of charge

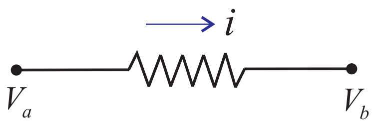


$\Rightarrow i_1 = i_2 + i_3$

(2) Second Law (Loop Rule):
The sum of potential differences around a complete circuit loop is zero.

Convention :

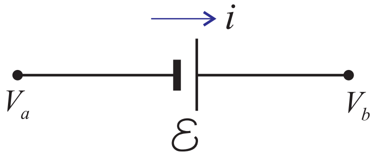
(i)



$$V_a > V_b \quad \Rightarrow \quad \text{Potential difference} = -iR$$

`i.e.` Potential *drops* across resistors

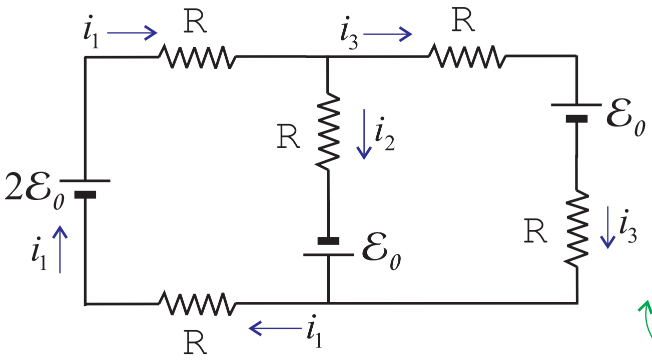
(ii)



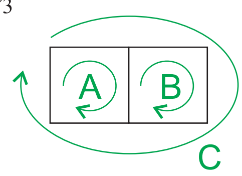
$$V_b > V_a \quad \Rightarrow \quad \text{Potential difference} = +\mathcal{E}$$

`i.e.` Potential *rises* across the negative plate of the battery.

Example :



Find i_1, i_2, i_3



By junction rule:

$$i_1 = i_2 + i_3 \tag{5.1}$$

By loop rule:

$$\begin{aligned}
\text{Loop A} \quad &\Rightarrow \quad 2\mathcal{E}_0 - i_1R - i_2R + \mathcal{E}_0 - i_1R = 0 && (5.2)\\
\text{Loop B} \quad &\Rightarrow \quad -i_3R - \mathcal{E}_0 - i_3R - \mathcal{E}_0 + i_2R = 0 && (5.3)\\
\text{Loop C} \quad &\Rightarrow \quad 2\mathcal{E}_0 - i_1R - i_3R - \mathcal{E}_0 - i_3R - i_1R = 0 && (5.4)
\end{aligned}$$

BUT: (5.4) = (5.2) + (5.3)
General rule: Need only 3 equations for 3 current

$$\begin{aligned}
i_1 &= i_2 + i_3 && (5.1)\\
3\mathcal{E}_0 - 2i_1R - i_2R &= 0 && (5.2)\\
-2\mathcal{E}_0 + i_2R - 2i_3R &= 0 && (5.3)
\end{aligned}$$

Substitute (5.1) into (5.2) :

$$\begin{aligned}
&3\mathcal{E}_0 - 2(i_2 + i_3)R - i_2R = 0\\
\Rightarrow \quad &3\mathcal{E}_0 - 3i_2R - 2i_3R = 0 && (5.4)
\end{aligned}$$

Subtract (5.3) from (5.4), `i.e.` (5.4)−(5.3)

$$3\mathcal{E}_0 - (-2\mathcal{E}_0) - 3i_2R - i_2R = 0$$

$$\Rightarrow \quad \boxed{i_2 = \frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}}$$

Substitute i_2 into (5.3) :

$$-2\mathcal{E}_0 + \Big(\frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}\Big)R - 2i_3R = 0$$

$$\Rightarrow \qquad \boxed{i_3 = -\frac{3}{8} \cdot \frac{\mathcal{E}_0}{R}}$$

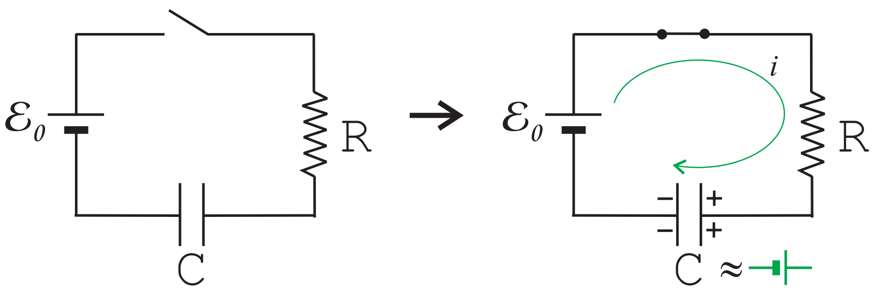
Substitute i_2, i_3 into (5.1) :

$$\boxed{i_1 = \left(\frac{5}{4} - \frac{3}{8}\right)\frac{\mathcal{E}_0}{R} = \frac{7}{8} \cdot \frac{\mathcal{E}_0}{R}}$$

Note: A *negative* current means that it is flowing in *opposite direction* from the one assumed.

5.10 RC Circuits

(A) *Charging* a capacitor with battery:



Using the loop rule:

$$+\mathcal{E}_0 \quad - \underbrace{iR}_{\substack{\text{P.D.}\\ \text{across R}}} \quad - \underbrace{\frac{Q}{C}}_{\substack{\text{P.D.}\\ \text{across C}}} \quad = 0$$

Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *increasing*.

$$\therefore \quad \mathcal{E} = R\overbrace{\frac{dQ}{dt}}^{i} + \frac{Q}{C} \qquad \text{1st order differential eqn.}$$

$$\Rightarrow \qquad \frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{RC}$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= 0$

$$\int_0^Q \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC}$$

$$
\begin{aligned}
& -\ln(\mathcal{E}C - Q)\Big|_0^Q = \frac{t}{RC}\Big|_0^t \\
\Rightarrow \quad & -\ln(\mathcal{E}C - Q) + \ln(\mathcal{E}C) = \frac{t}{RC} \\
\Rightarrow \quad & \ln\Big(\frac{1}{1 - \frac{Q}{\mathcal{E}C}}\Big) = \frac{t}{RC} \\
\Rightarrow \quad & \frac{1}{1 - \frac{Q}{\mathcal{E}C}} = e^{t/RC} \\
\Rightarrow \quad & \frac{Q}{\mathcal{E}C} = 1 - e^{-t/RC} \\
\Rightarrow \quad & \boxed{Q(t) = \mathcal{E}C(1 - e^{-t/RC})}
\end{aligned}
$$

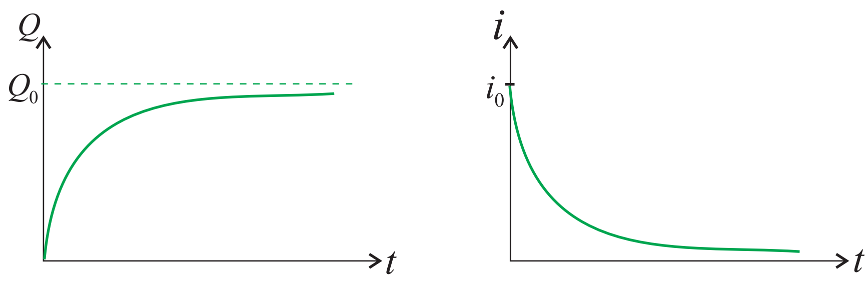
Note: (1) At $t = 0$, $Q(t = 0) = \mathcal{E}C(1 - 1) = 0$

(2) As $t \to \infty$, $Q(t \to \infty) = \mathcal{E}C(1 - 0) = \mathcal{E}C$
$=$ Final charge on capacitor (Q_0)

(3) Current:

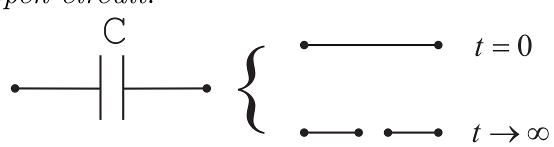
$$
\begin{aligned}
i &= \frac{dQ}{dt} \\
&= \mathcal{E}C\Big(\frac{1}{RC}\Big)e^{-t/RC} \\
i(t) &= \frac{\mathcal{E}}{R}e^{-t/RC}
\end{aligned}
$$

$$
\begin{cases}
i(t = 0) = \dfrac{\mathcal{E}}{R} = \text{Initial current} = i_0 \\
i(t \to \infty) = 0
\end{cases}
$$



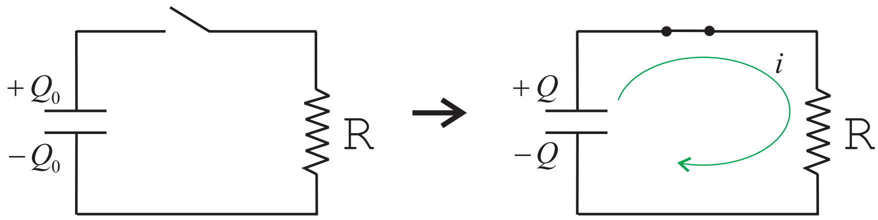
(4) At time $= 0$, the capacitor acts like *short circuit* when there is *zero charge on the capacitor*.

(5) As time $\to \infty$, the capacitor is *fully charged* and current $= 0$, it acts like a *open circuit*.



(6) $\tau_c = RC$ is called the **time constant**. It's the time it takes for the charge to reach $\quad (1 - \frac{1}{e})\, Q_0 \simeq 0.63 Q_0$

(B) *Discharging* a charged capacitor:



Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *decreasing*.

$$\therefore \quad i = -\frac{dQ}{dt}$$

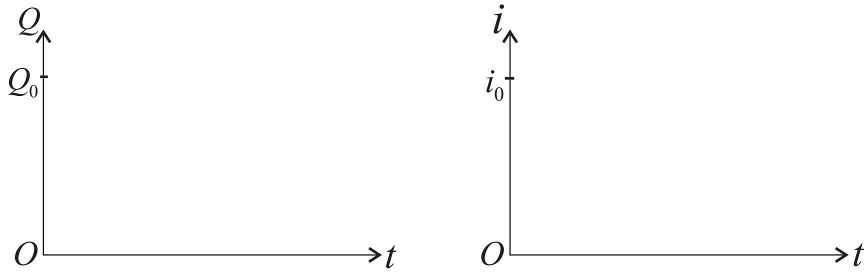
Loop Rule:

$$\begin{aligned} & V_c - iR = 0 \\ \Rightarrow \quad & \frac{Q}{C} + \frac{dQ}{dt} R = 0 \\ \Rightarrow \quad & \frac{dQ}{dt} = -\frac{1}{RC}\, Q \end{aligned}$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= Q_0$

$$\begin{aligned} & \int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC} \int_0^t dt \\ \Rightarrow \quad & \ln Q - \ln Q_0 = -\frac{t}{RC} \\ \Rightarrow \quad & \ln\Big(\frac{Q}{Q_0}\Big) = -\frac{t}{RC} \\ \Rightarrow \quad & \frac{Q}{Q_0} = e^{-t/RC} \\ \Rightarrow \quad & Q(t) = Q_0\, e^{-t/RC} \\ (i = -\frac{dQ}{dt}) \qquad \Rightarrow \quad & i(t) = \frac{Q_0}{RC}\, e^{-t/RC} \\ (\text{At } t = 0) \qquad \Rightarrow \quad & i(t=0) = \frac{1}{R} \cdot \underbrace{\frac{Q_0}{C}}_{\text{Initial P.D. across capacitor}} \end{aligned}$$

$$i_0 = \frac{V_0}{R}$$



At $\quad t = RC = \tau \qquad\qquad Q(t = RC) \; = \; \frac{1}{e} Q_0 \; \simeq \; 0.37 Q_0$

Chapter 6

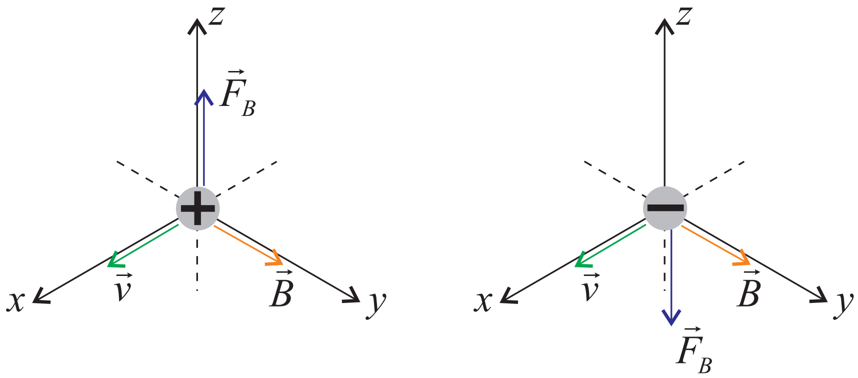
Magnetic Force

6.1 Magnetic Field

For stationary charges, they experienced an **electric force** in an **electric field**. For moving charges, they experienced a **magnetic force** in a **magnetic field**.

Mathematically, $\vec{F}_E = q\vec{E}$ (electric force)
$\vec{F}_B = q\vec{v} \times \vec{B}$ (magnetic force)

Direction of the magnetic force determined from *right hand rule*.



Magnetic field $\vec{B}$: Unit = Tesla (T)
1T = 1C moving at 1m/s experiencing 1N

Common Unit: 1 Gauss (G) = 10^{-4}T $\approx$ magnetic field on earth's surface

Example: What's the force on a 0.1C charge moving at velocity $\vec{v} = (10\hat{j} - 20\hat{k})ms^{-1}$ in a magnetic field $\vec{B} = (-3\hat{i} + 4\hat{k}) \times 10^{-4}T$

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\begin{aligned} &= +0.1\,(10\hat{j} - 20\hat{k}) \times (-3\hat{i} + 4\hat{k}) \times 10^{-4} N \\ &= 10^{-5}\,(-30 \cdot -\hat{k} + 40\hat{i} + 60\hat{j} + 0) N \end{aligned}$$

Effects of magnetic field is usually quite small.

$$\begin{aligned} \vec{F} &= q\vec{v} \times \vec{B} \\ |\vec{F}| &= qvB\sin\theta, \quad \text{where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{B} \end{aligned}$$

$\therefore$ Magnetic force is *maximum* when $\theta = 90°$ (`i.e.` $\vec{v} \perp \vec{B}$)
Magnetic force is *minimum* (0) when $\theta = 0°$, $180°$ (`i.e.` $\vec{v} \parallel \vec{B}$)

Graphical representation of B-field: **Magnetic field lines**
Compared with **Electric field lines**:

Similar characteristics :

(1) Direction of E-field/B-field indicated by *tangent* of the field lines.

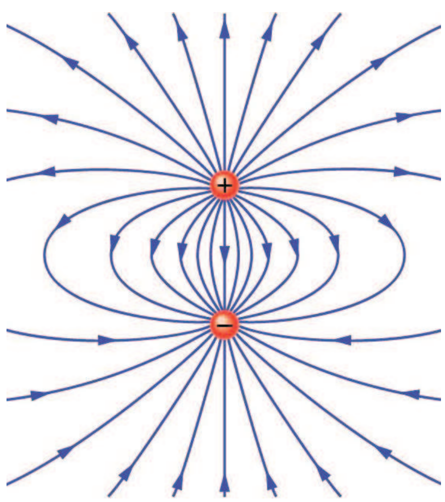
(2) Magnitude of E-field/B-field indicated by *density* of the field lines.

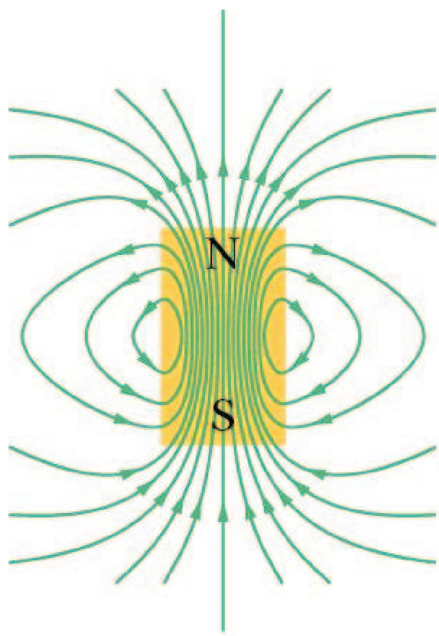
Differeces :

(1) $\vec{F}_E \parallel$ E-field lines; $\vec{F}_B \perp$ B-field lines

(2) E-field line begins at positive charge and ends at negative charge; B-field line forms a closed loop.

Example : Chap35, Pg803 Halliday



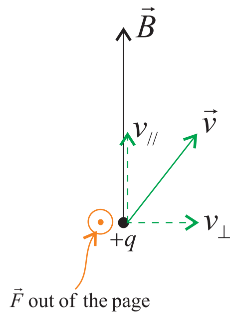


Note: Isolated magnetic monopoles do not exist.

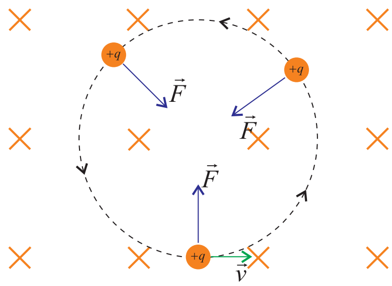
6.2 Motion of A Point Charge in Magnetic Field

Since $\vec{F}_B \perp \vec{v}$, therefore B-field only changes the *direction* of the velocity but not its *magnitude*.

Generally, $\vec{F}_B = q\vec{v} \times \vec{B} = q\, v_\perp B$,
$\therefore$ We only need to consider the motion component $\perp$ to B-field.



We have *circular motion*. Magnetic force provides the *centripetal force* on the moving charge particles.



$$\begin{aligned} \therefore \quad F_B &= m\,\frac{v^2}{r} \\ |q|\,vB &= m\,\frac{v^2}{r} \\ \therefore \quad r &= \frac{mv}{|q|B} \end{aligned}$$

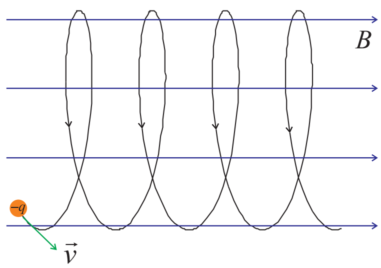
where r is radius of circular motion.

Time for moving around one orbit:

$$\boxed{T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}} \quad \text{Cyclotron Period}$$

(1) Independent of v (non-relativistic)

(2) Use it to measure m/q



Generally, charged particles with constant velocity moves in **helix** in the presence of constant B-field.

Note :

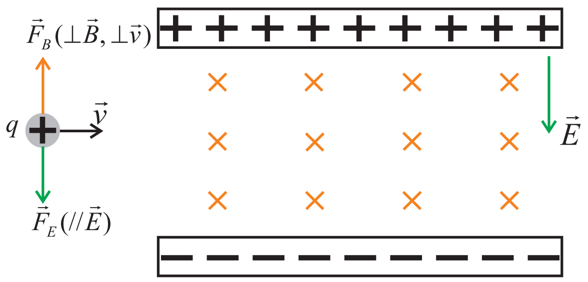
(1) B-field does NO work on particles.

(2) B-field does NOT change K.E. of particles.

Particle Motion in Presence of E-field & B-field:

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} \qquad \text{Lorentz Force}$$

Special Case : $\vec{E} \perp \vec{B}$



When $|\vec{F}_E| = |\vec{F}_B|$

$qE = qvB$

$$\Rightarrow \quad \boxed{v = \frac{E}{B}}$$

$\therefore$ For charged particles moving at $v = E/B$, they will pass through the *crossed E and B fields* without vertical displacement.

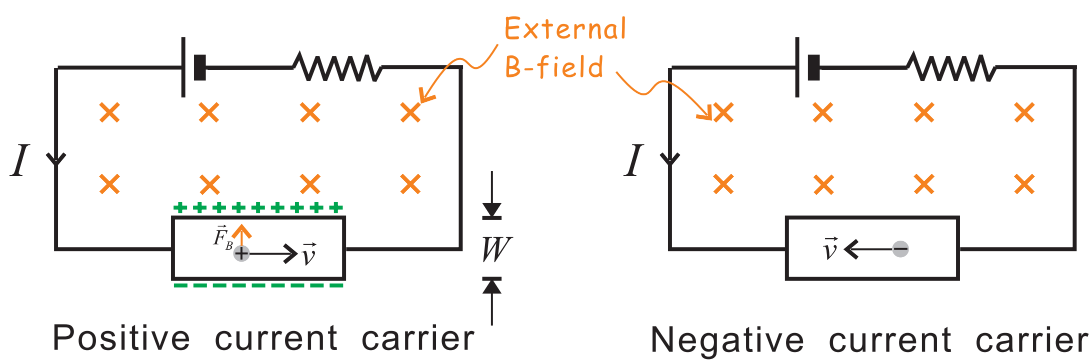
$\Rightarrow$ **velocity selector**

Applications :

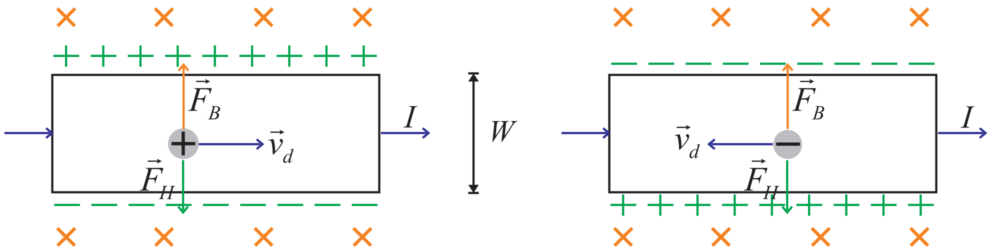
- Cyclotron (Lawrence & Livingston 1934)
- Measuring e/m for electrons (Thomson 1897)
- Mass Spectrometer (Aston 1919)

6.3 Hall Effect

Charges travelling in a conducting wire will be *pushed to one side of the wire by the external magnetic field.* This separation of charge in the wire is called the **Hall Effect**.



The separation will stop when F_B experienced by the current carrier is *balanced* by the force $\vec{F}_H$ caused by the E-field set up by the separated charges.



Define :

$$\begin{aligned} \Delta V_H &= \textbf{Hall Voltage} \\ &= \text{Potential difference across the conducting strip} \end{aligned}$$

$\therefore$ E-field from separated charges: $E_H = \dfrac{\Delta V_H}{W}$
where W = *width* of conducting strip

In equilibrium: $q\vec{E}_H + q\vec{v}_d \times \vec{B} = 0$, where $\vec{v}_d$ is drift velocity

$$\therefore \quad \frac{\Delta V_H}{W} = v_d B$$

Recall from Chapter 5,

$$i = nqAv_d$$

where n is density of charge carrier,
A is cross-sectional area = width × thickness = $W \cdot t$

$$\therefore \quad \frac{\Delta V_H}{W} = \frac{i}{nqWt} B$$

$$\Rightarrow \quad \boxed{n = \frac{iB}{qt\Delta V_H}}$$ To determine density of charge carriers

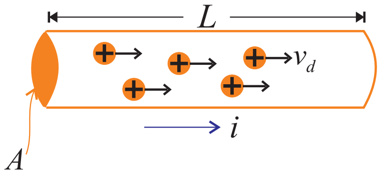
Suppose we determine n for a particular metal ($\therefore\ q = e$), then we can *measure B-field strength by measuring the Hall voltage*:

$$\boxed{B = \frac{net}{i}\Delta V_H}$$

6.4 Magnetic Force on Currents

Current = many charges moving together

Consider a wire segment, length L, carrying current i in a magnetic field.



Total magnetic force = $\Big(\underbrace{q\vec{v}_d \times \vec{B}}_{\text{force on one charge carrier}}\Big) \cdot \underbrace{n\,A\,L}_{\text{Total number of charge carrier}}$

Recall $i = nqv_dA$

$$\therefore \quad \boxed{\text{Magnetic force on current } \vec{F} = i\vec{L} \times \vec{B}}$$

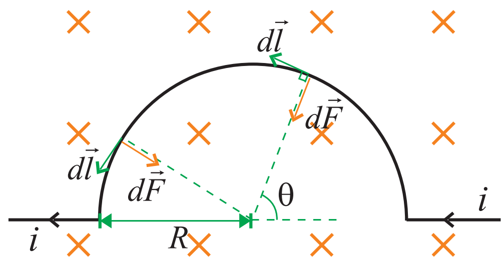
where $\vec{L}$ = Vector of which: $|\vec{L}|$ = length of current segment; direction = direction of current

For an infinitesimal wire segment $d\vec{l}$

$$\boxed{d\vec{F} = i\,d\vec{l} \times \vec{B}}$$

Example 1: Force on a semicircle current loop

$$\begin{aligned} d\vec{l} &= \text{Infinitesimal} \\ &\quad\ \text{arc length element} \perp \vec{B} \\ \therefore \quad dl &= R\,d\theta \\ \therefore \quad dF &= iRB\,d\theta \end{aligned}$$



By symmetry argument, we only need to consider vertical forces, $dF \cdot \sin\theta$

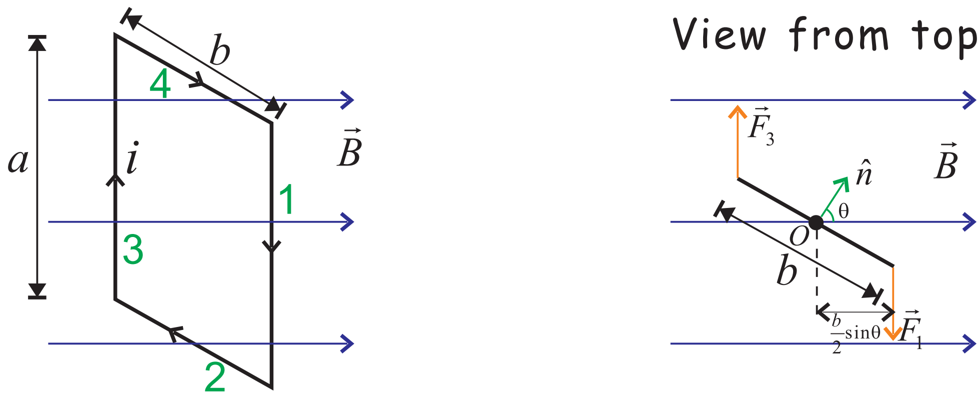
$$\begin{aligned} \therefore \quad \text{Net force } F &= \int_0^\pi dF\ \sin\theta \\ &= iRB \int_0^\pi \sin\theta\,d\theta \\ F &= 2iRB \quad \text{(downward)} \end{aligned}$$

Method 2: Write $d\vec{l}$ in $\hat{\imath}, \hat{\jmath}$ components

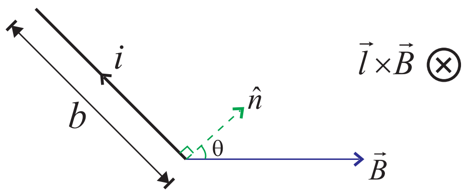
$$\begin{aligned}
d\vec{l} &= -dl\sin\theta\,\hat{\imath} + dl\cos\theta\,\hat{\jmath} \\
&= R\,d\theta\,(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\
\vec{B} &= -B\,\hat{k} \quad \text{(into the page)} \\
\therefore \quad d\vec{F} &= i\,d\vec{l} \times \vec{B} \\
&= -iRB\sin\theta\,d\theta\,\hat{\jmath} - iRB\cos\theta\,\hat{\imath}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \vec{F} &= \int_0^\pi d\vec{F} \\
&= -iRB\left[\int_0^\pi \sin\theta\,d\theta\,\hat{\jmath} + \int_0^\pi \cos\theta\,d\theta\,\hat{\imath}\right] \\
&= -2iRB\,\hat{\jmath}
\end{aligned}$$

Example 2: Current loop in B-field

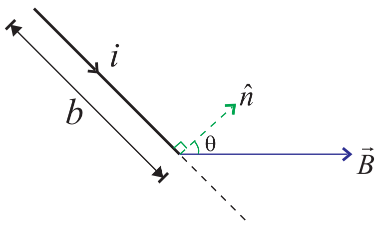


For segment2:



$$F_2 = ibB\sin(90^\circ + \theta) = ibB\cos\theta \qquad \text{(pointing downward)}$$

For segment4:



$$F_2 = ibB\sin(90^\circ - \theta) = ibB\cos\theta \qquad \text{(pointing upward)}$$

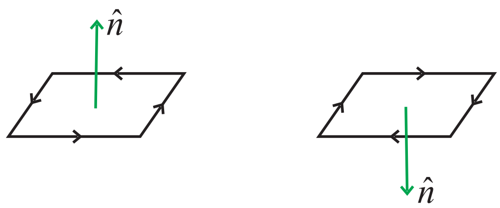
For segment1: $F_1 = iaB$
For segment3: $F_3 = iaB$
$\therefore$ Net force on the current loop $= 0$
But, net torque on the loop about O

$$\begin{aligned} &= \tau_1 + \tau_3 \\ &= iaB \cdot \frac{b}{2} \sin\theta + iaB \cdot \frac{b}{2} \sin\theta \\ &= i \underbrace{ab}_{A \,=\, \text{area of loop}} B \sin\theta \end{aligned}$$

Suppose the loop is a coil with N turns of wires:

$$\text{Total torque } \boxed{\tau = NiAB\sin\theta}$$

Define: Unit vector $\hat{n}$ to represent the area-vector (using right hand rule)



Then we can rewrite the torque equation as

$$\boxed{\vec{\tau} = NiA\,\hat{n} \times \vec{B}}$$

Define: $NiA\,\hat{n} = \vec{\mu} =$ Magnetic dipole moment of loop

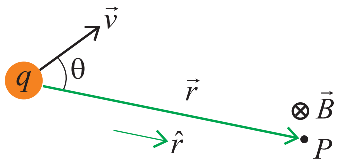
$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

Chapter 7

Magnetic Field

7.1 Magnetic Field

A *moving charge* $\begin{cases} \text{experiences magnetic force in B-field.} \\ \text{can generate B-field.} \end{cases}$



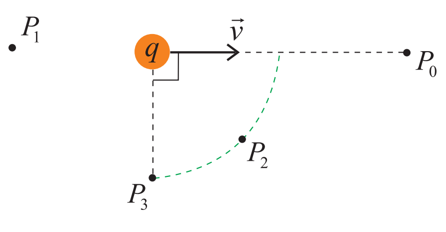
Magnetic field $\vec{B}$ due to moving point charge:

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \vec{r}}{r^3}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Tm/A (N/A^2)

Permeability of free space (Magnetic constant)

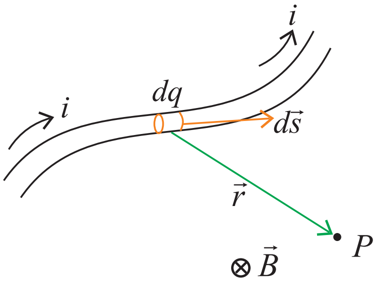
$$|\vec{B}| = \frac{\mu_0}{4\pi} \cdot \frac{qv\sin\theta}{r^2} \qquad \begin{cases} \textit{maximum} & \text{when } \theta = 90^\circ \\ \textit{minimum} & \text{when } \theta = 0^\circ/180^\circ \end{cases}$$



$\vec{B}$ at $P_0 = 0 = \vec{B}$ at P_1
$\vec{B}$ at $P_2 < \vec{B}$ at P_3

However, a single moving charge will NOT generate a steady magnetic field.

stationary charges	generate	steady E-field.
steady currents	generate	steady B-field.



Magnetic field at point P can be obtained by *integrating* the contribution from individual current segments. (**Principle of Superposition**)

$$\therefore \quad d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{dq\,\vec{v} \times \hat{r}}{r^2}$$

Notice: $\quad dq\,\vec{v} = dq \cdot \dfrac{d\vec{s}}{dt} = i\,d\vec{s}$

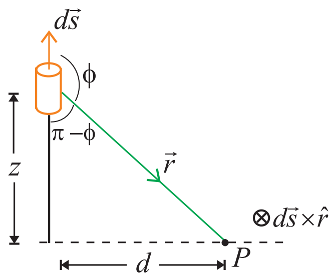
$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{i\,d\vec{s} \times \hat{r}}{r^2}} \qquad \textbf{Biot-Savart Law}$$

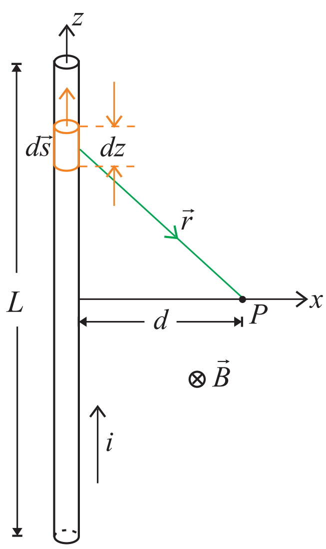
For current around a whole circuit:

$$\vec{B} = \int_{\substack{\text{entire}\\\text{circuit}}} d\vec{B} = \int_{\substack{\text{entire}\\\text{circuit}}} \frac{\mu_0}{4\pi} \cdot \frac{i\,d\vec{s} \times \hat{r}}{r^2}$$

Biot-Savart Law is to *magnetic field* as
Coulomb's Law is to *electric field.*
Basic element of E-field: *Electric charges* dq
Basic element of B-field: *Current element* $i\,d\vec{s}$

Example 1 : Magnetic field due to straight current segment





$$
\begin{aligned}
\therefore\ |d\vec{s} \times \hat{r}| &= dz \sin\phi \\
&= dz \sin(\pi - \phi) \qquad \text{(Trigonometry Identity)} \\
&= dz \cdot \frac{d}{r} = \frac{d \cdot dz}{\sqrt{d^2 + z^2}}
\end{aligned}
$$

$$dB = \frac{\mu_0}{4\pi} \cdot \frac{i\,dz}{r^2} \cdot \frac{d}{r} = \frac{\mu_0 i}{4\pi} \cdot \frac{d}{(d^2+z^2)^{3/2}}\,dz$$

$$\therefore \quad B = \int_{-L/2}^{L/2} dB = \frac{\mu_0 i d}{4\pi} \int_{-L/2}^{+L/2} \frac{dz}{(d^2+z^2)^{3/2}}$$

$$
\begin{aligned}
B &= \frac{\mu_0 i}{4\pi d} \cdot \left. \frac{z}{(z^2+d^2)^{1/2}} \right|_{-L/2}^{+L/2} \\
B &= \frac{\mu_0 i}{4\pi d} \cdot \frac{L}{(\frac{L^2}{4}+d^2)^{1/2}}
\end{aligned}
$$

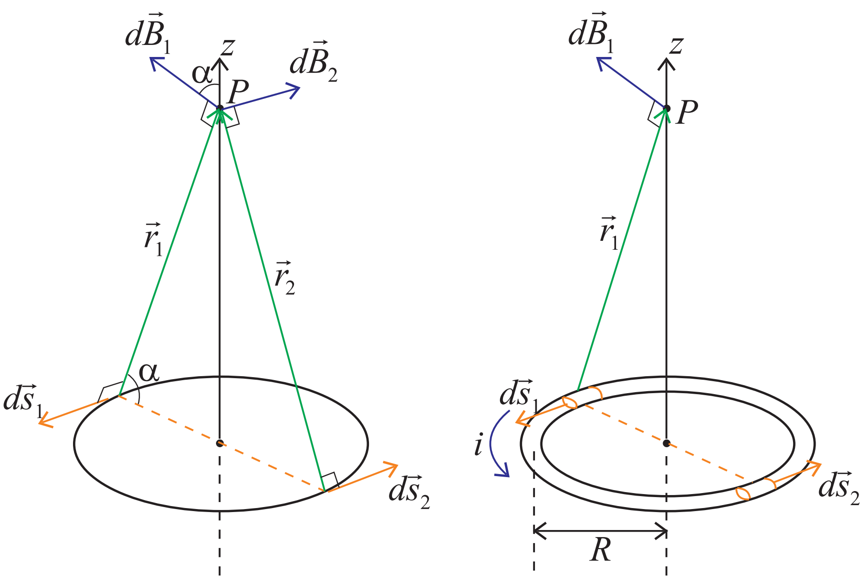
Limiting Cases : When $L \gg d$ (B-field due to long wire)

$$\left(\frac{L^2}{4} + d^2\right)^{-1/2} \approx \left(\frac{L^2}{4}\right)^{-1/2} = \frac{2}{L}$$

$$\therefore \quad \boxed{B = \frac{\mu_0 i}{2\pi d}\ ; \quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

Recall : $E = \dfrac{\lambda}{2\pi\epsilon_0 d}$ for an infinite long line of charge.

Example 2 : A circular current loop



Notice that for every current element $id\vec{s}_1$, generating a magnetic field $d\vec{B}_1$ at point P, there is an opposite current element $id\vec{s}_2$, generating B-field $d\vec{B}_2$ so that

$$d\vec{B}_1 \sin\alpha = -d\vec{B}_2 \sin\alpha$$

$\therefore$ Only vertical component of B-field needs to be considered at point P.

$$dB = \frac{\mu_0}{4\pi} \cdot \frac{i\,ds\ \sin \overbrace{90^\circ}^{\because d\vec{s} \perp \hat{r}}}{r^2}$$

$\therefore$ B-field at point P:

$$B = \underbrace{\int}_{\substack{\text{around}\\ \text{circuit}}} dB \quad \underbrace{\cos\alpha}_{\substack{\text{consider vertical}\\ \text{component}}}$$

$$\begin{aligned}
\therefore \quad B &= \int_0^{2\pi} \frac{\mu_0 i \cos\alpha}{4\pi r^2} \cdot \underbrace{ds}_{Rd\theta} \\
&= \frac{\mu_0 i}{4\pi} \cdot \frac{R}{r^3} \quad \underbrace{\int_0^{2\pi} ds}_{\substack{\text{Integrate around circum-}\\ \text{ference of circle } = 2\pi R}} \\
\therefore \quad B &= \frac{\mu_0 i R^2}{2r^3}
\end{aligned}$$

$$\boxed{B = \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}\ ; \quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

Limiting Cases :

(1) B-field at center of loop:

$$z = 0 \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 i}{2R}}$$

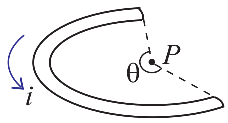
(2) For $z \gg R$,

$$B = \frac{\mu_0 i R^2}{2z^3\left(1 + \frac{R^2}{z^2}\right)^{3/2}} \approx \frac{\mu_0 i R^2}{2z^3} \propto \frac{1}{z^3}$$

Recall E-field for an electric dipole: $\quad E = \dfrac{p}{4\pi\epsilon_0 x^3}$

$\therefore$ A circular current loop is also called a **magnetic dipole**.

(3) A current arc:



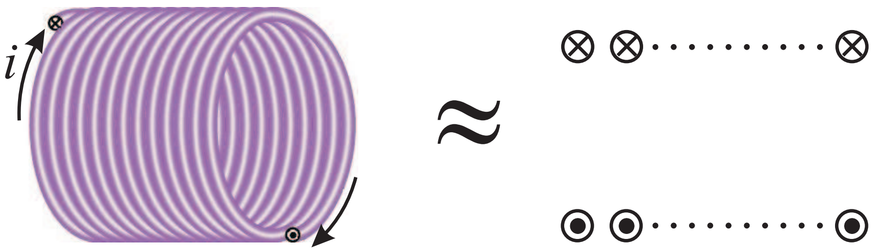
$$B = \int_{\substack{\text{around}\\\text{circuit}}} dB \underbrace{\cos\alpha}_{\substack{z=0 \;\Rightarrow\\ \alpha=0 \;\text{ here.}}}$$

$$= \frac{\mu_0 i}{4\pi} \cdot \underbrace{\frac{R}{r^3}}_{\substack{R=r\\ \text{when } \alpha=0}} \cdot \overbrace{\int_0^\theta \underbrace{ds}_{R d\theta}}^{R\theta \,=\, \text{length of arc}}$$

$$B = \frac{\mu_0 i\, \theta}{4\pi R}$$

Example 3 : Magnetic field of a solenoid

Solenoid is used to produce a *strong and uniform* magnetic field inside its coils.



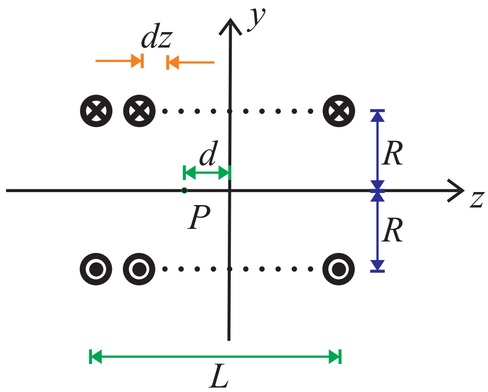
Consider a solenoid of length L consisting of N turns of wire.

Define: n = Number of turns per unit length $= \dfrac{N}{L}$

Consider B-field at distance d from the center of the solenoid:

For a segment of length dz, number of current turns $= ndz$

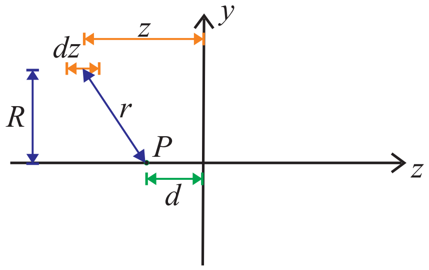
$\therefore$ Total current $= ni\,dz$



Using the result from one coil in Example 2, we get B-field from coils of length dz at distance z from center:

$$dB = \frac{\mu_0 (ni\,dz) R^2}{2r^3}$$

However $r = \sqrt{R^2 + (z-d)^2}$



$$\begin{aligned} \therefore \quad B &= \int_{-L/2}^{+L/2} dB \qquad \text{(Integrating over the entire solenoid)} \\ &= \frac{\mu_0 n i R^2}{2} \int_{-L/2}^{+L/2} \frac{dz}{[R^2 + (z-d)^2]^{3/2}} \\ B &= \frac{\mu_0 n i}{2} \left[\frac{\frac{L}{2} + d}{\sqrt{R^2 + (\frac{L}{2}+d)^2}} + \frac{\frac{L}{2} - d}{\sqrt{R^2 + (\frac{L}{2}-d)^2}} \right] \end{aligned}$$

along negative z direction

Ideal Solenoid :

$$L \gg R$$

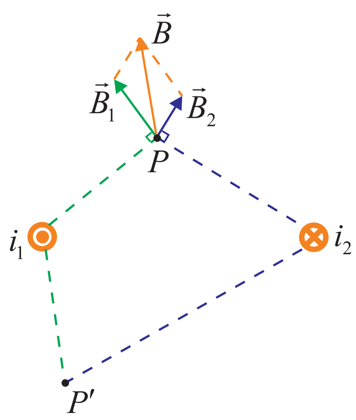
$$\text{then} \quad B = \frac{\mu_0 n i}{2} [1+1]$$

$$\therefore \quad \boxed{B = \mu_0 n i \,; \quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

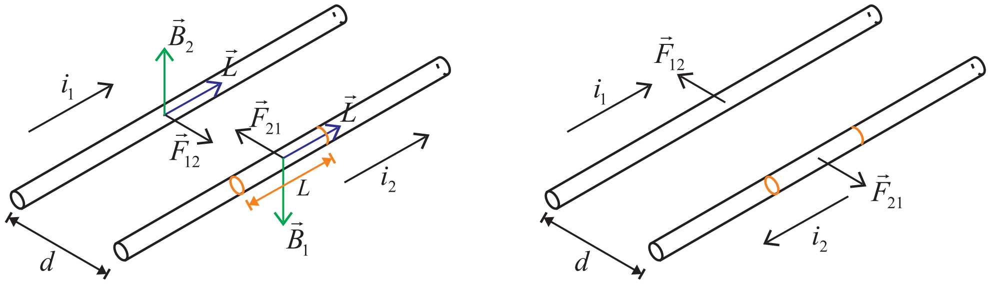
Question : What is the B-field at the end of an ideal solenoid? $B = \frac{\mu_0 n i}{2}$

7.2 Parallel Currents

Magnetic field at point P $\vec{B}$ due to two currents i_1 and i_2 is the *vector sum* of the $\vec{B}$ fields $\vec{B}_1$, $\vec{B}_2$ due to individual currents. **(Principle of Superposition)**



Force Between Parallel Currents :



Consider a segment of length L on i_2 :

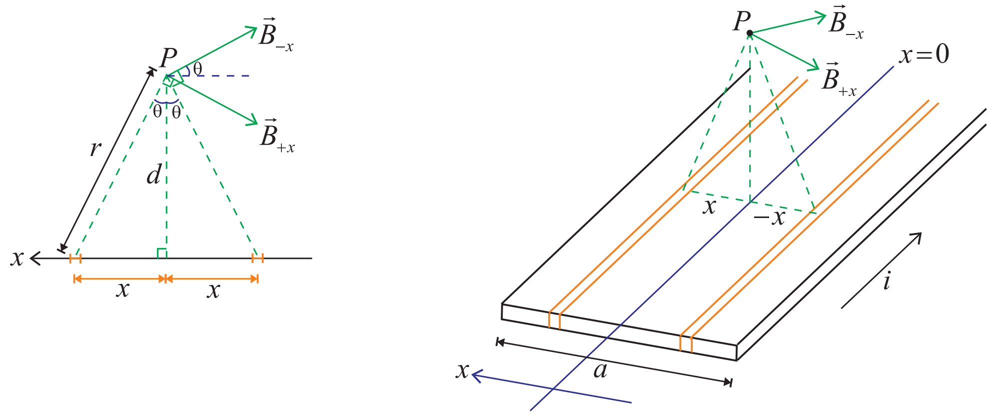
$$\vec{B}_1 = \frac{\mu_0 i_1}{2\pi d} \quad \text{(pointing down)} \qquad\qquad \vec{B}_2 = \frac{\mu_0 i_1}{2\pi d} \quad \text{(pointing up)}$$

Force on i_2 coming from i_1:

$$|\vec{F}_{21}| = i_2\vec{L}\times\vec{B}_1 = \frac{\mu_0 L i_1 i_2}{2\pi d} = |\vec{F}_{12}| \qquad \textit{(Def'n of ampere, A)}$$

$\therefore$ *Parallel currents attract, anti-parallel currents repel.*

Example : Sheet of current



Consider an infinitesimal wire of width dx at position x, there exists another element at $-x$ so that vertical $\vec{B}$-field components of $\vec{B}_{+x}$ and $\vec{B}_{-x}$ cancel.
$\therefore$ Magnetic field due to dx wire:

$$dB = \frac{\mu_0 \cdot di}{2\pi r} \qquad \text{where } di = i\Big(\frac{dx}{a}\Big)$$

$\therefore$ Total B-field (*pointing along* $-x$ *axis*) at point P:

$$B = \int_{-a/2}^{+a/2} dB\cos\theta = \int_{-a/2}^{+a/2} \frac{\mu_0 i}{2\pi a}\cdot\frac{dx}{r}\cdot\cos\theta$$

Variable transformation (Goal: change r, x to d, θ, then integrate over θ):

$$\begin{cases} d = r\cos\theta & \Rightarrow \quad r = d\sec\theta \\ x = d\tan\theta & \Rightarrow \quad dx = d\sec^2\theta\, d\theta \end{cases}$$

Limits of integration: $-\theta_0$ to θ_0, where $\tan\theta_0 = \dfrac{a}{2d}$

$$\begin{aligned} \therefore \quad B &= \frac{\mu_0 i}{2\pi a}\int_{-\theta_0}^{\theta_0} \frac{d\sec^2\theta\, d\theta}{d\sec\theta}\cdot\cos\theta \\ &= \frac{\mu_0 i}{2\pi a}\int_{-\theta_0}^{\theta_0} d\theta \\ B &= \frac{\mu_0 i\theta_0}{\pi a} = \frac{\mu_0 i}{\pi a}\tan^{-1}\Big(\frac{a}{2d}\Big) \end{aligned}$$

Limiting Cases :

(1) $d \gg a$

$$\tan\theta = \frac{a}{2d} \quad \Rightarrow \quad \theta \approx \frac{a}{2d}$$

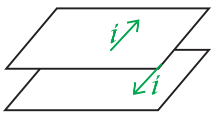
$$\therefore \quad B = \frac{\mu_0 i}{2\pi a}$$ B-field due to infinite long wire

(2) $d \ll a$ (*Infinite sheet of current*)

$$\tan\theta = \frac{a}{2d} \to \infty \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

$$\therefore \quad B = \frac{\mu_0 i}{2a} \qquad Constant!$$

Question : Large sheet of opposite flowing currents.



What's the B-field between & outside the sheets?

7.3 Ampère's Law

In our study of electricity, we notice that the **inverse square force law** leads to **Gauss' Law**, which is useful for *finding E-field for systems with high level of symmetry.*
For magnetism, Gauss' Law is simple

$$\oiint_S \qquad \boxed{\oint_S \vec{B} \cdot d\vec{A} = 0}$$

$\because$ There is no magnetic monopole.

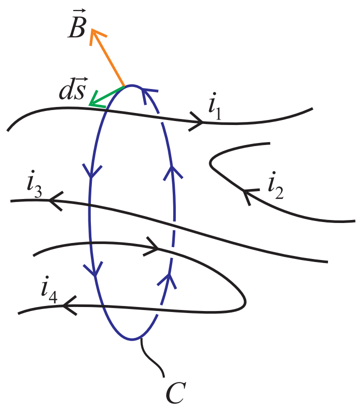
A more useful law for calculating B-field for highly symmetric situations is **the Ampère's Law**:

$$\oint_C \qquad \boxed{\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i}$$

$\oint_C$ = *Line intefral* evaluated around a *closed loop* C **(Amperian curve)**

i = *Net current* that penetrates the area bounded by curve C^* (topological property)

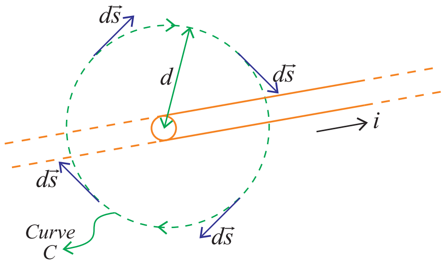
Convention : Use the **right-hand screw rule** to determine the *sign* of current.



$$\begin{aligned} \oint_C \vec{B} \cdot d\vec{s} &= \mu_0(i_1 - i_3 + i_4 - i_4) \\ &= \mu_0(i_1 - i_3) \end{aligned}$$

Applications of the Ampere's Law :

(1) Long-straight wire



Construct an Amperian curve of radius d:

By symmetry argument, we know $\vec{B}$-field only has *tangential component*

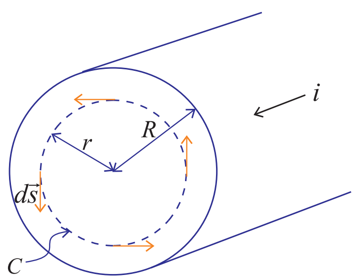
$$\therefore \quad \oint_C \vec{B} \cdot d\vec{s} = \mu_0 i$$

Take $d\vec{s}$ to be the tangential vector around the circular path:

$$\begin{aligned}
\therefore \quad \vec{B}\cdot d\vec{s} &= B\,ds \\
B\underbrace{\oint_C ds}_{\substack{\text{Circumference} \\ \text{of circle} = 2\pi d}} &= \mu_0 i \\
\therefore \quad B(2\pi d) &= \mu_0 i
\end{aligned}$$

B-field due to long, straight current $\boxed{B = \dfrac{\mu_0 i}{2\pi d}}$ (Compare with 7.1 Example 1)

(2) Inside a current-carrying wire

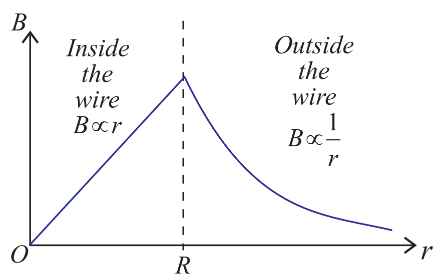


Again, symmetry argument implies that $\vec{B}$ is *tangential* to the Amperian curve and $\vec{B} \to B(r)\hat{\theta}$

Consider an Amperian curve of radius $r(< R)$

$$\oint_C \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi r) = \mu_0 i_{included}$$

But $i_{included} \propto$ cross-sectional area of C

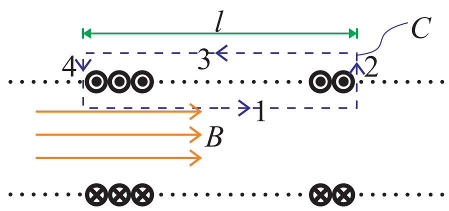


$$\begin{aligned}
\therefore \quad \frac{i_{included}}{i} &= \frac{\pi r^2}{\pi R^2} \\
\therefore \quad i_{included} &= \frac{r^2}{R^2}\, i \\
\therefore \quad B &= \frac{\mu_0 i}{2\pi R^2}\cdot r \propto r
\end{aligned}$$

Recall: Uniformly charged infinite long rod

(3) Solenoid (Ideal)

Consider the rectangular Amperian curve 1234.



$$\oint_C \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} + \cancel{\int_2 \vec{B}\cdot d\vec{s}} + \cancel{\int_3 \vec{B}\cdot d\vec{s}} + \cancel{\int_4 \vec{B}\cdot d\vec{s}}$$

$$\int_2 = \int_4 = 0 \quad \because \quad \begin{cases} \vec{B}\cdot d\vec{s} &= 0 \quad \text{inside solenoid} \\ \vec{B} &= 0 \quad \text{outside solenoid} \end{cases}$$

$$\int_3 = 0 \quad \because \quad \vec{B} = 0 \quad \text{outside solenoid}$$

$$\therefore \quad \oint_C \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} = Bl = \mu_0 i_{tot}$$

But $i_{tot} = \underbrace{nl}_{\text{Number of coils included}} \cdot i$

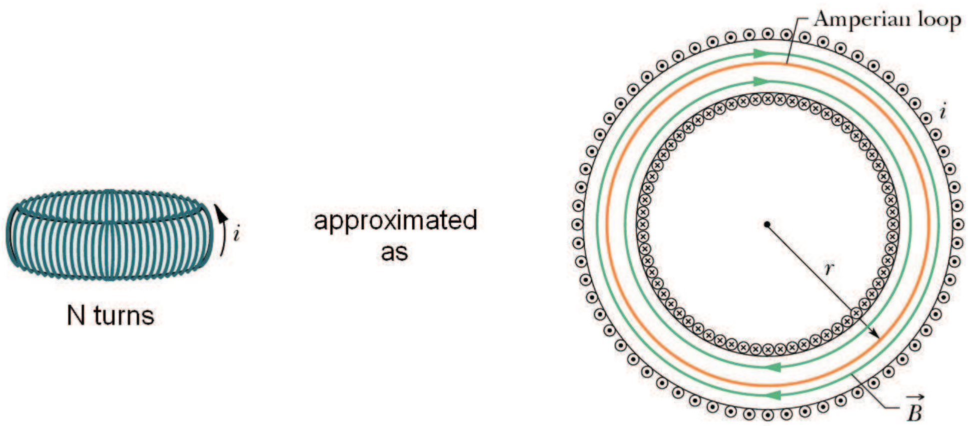
$$\therefore \boxed{B = \mu_0 n i}$$

Note :

(i) The assumption that $\vec{B} = 0$ outside the ideal solenoid is only *approximate*. (Halliday, Pg.763)

(ii) B-field everywhere inside the solenoid is a *constant*. (for ideal solenoid)

(4) Toroid (A *circular solenoid*)



By symmetry argument, the B-field lines form *concentric circles inside the toroid.*

Take Amperian curve C to be a circle of radius r inside the toroid.

$$\oint_C \vec{B}\cdot d\vec{s} = B \oint_C ds = B \cdot 2\pi r = \mu_0 (Ni)$$

$$\therefore \quad B = \frac{\mu_0 N i}{2\pi r} \qquad \text{inside toroid}$$

Note :

(i) $B \neq$ constant inside toroid

(ii) Outside toroid:
Take Amperian curve to be circle of radius $r > R$.

$$\oint_C \vec{B} \cdot d\vec{s} = B \oint_C ds = B \cdot 2\pi r = \mu_0 \cdot i_{incl} = 0$$

$$\therefore \quad B = 0$$

Similarly, in the central cavity $B = 0$

7.4 Magnetic Dipole

Recall from §6.4, we define the **magnetic dipole moment** of a rectangular current loop

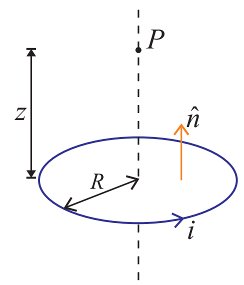
$$\boxed{\vec{\mu} = N i A \hat{n}}$$

where $\hat{n}$ = area unit vector with direction determined by the right-hand rule
N = Number of turns in current loop
A = Area of current loop

This is actually a *general definition* of a magnetic dipole, `i.e.` we use it for current loops of <u>all</u> shapes.

A common and symmetric example: circular current.

Recall from §7.1 Example 2, magnetic field at point P (height z above the ring)



$$\vec{B} = \frac{\mu_0 i R^2 \hat{n}}{2(R^2 + z^2)^{3/2}} = \frac{\mu_0 \vec{\mu}}{2\pi (R^2 + z^2)^{3/2}}$$

At distance $z \gg R$,

$$\vec{B} = \frac{\mu_0 \vec{\mu}}{2\pi z^3}$$

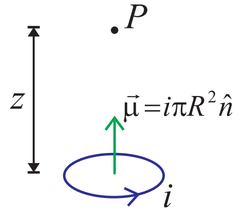
due to *magnetic dipole*

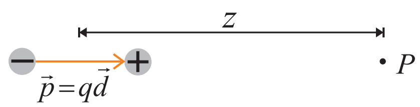
(for $z \gg R$)

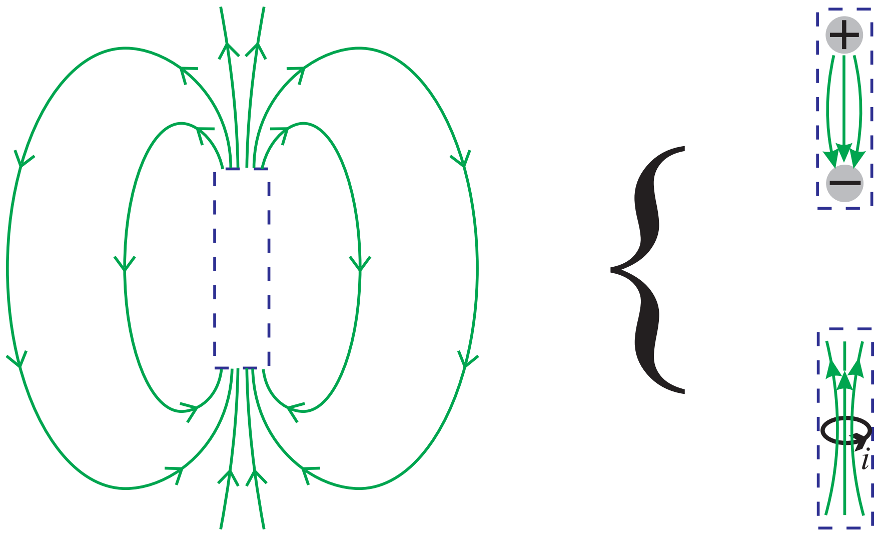
$$\vec{E} = \frac{\vec{p}}{4\pi\epsilon_0 z^3}$$

due to *electric dipole*

(for $z \gg d$)







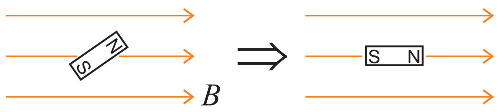
Also, notice

$$\begin{aligned} \vec{\mu} &= \text{magnetic dipole moment} \quad \left[\begin{array}{ll} \text{Unit:} & \text{Am}^2 \\ & J/T \end{array} \right] \\ \mu_0 &= \text{Permeability of free space} \\ &= 4\pi \times 10^{-7} \text{Tm/A} \end{aligned}$$

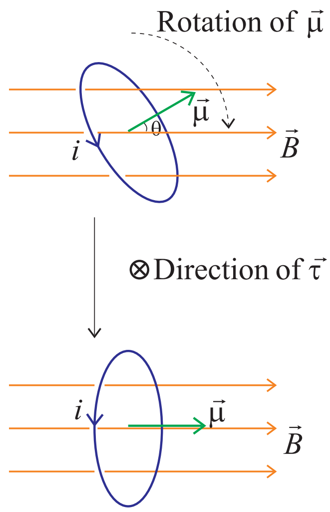
7.5 Magnetic Dipole in A Constant B-field

In the presence of a constant magnetic field, we have shown for a *rectangular current loop*, it experiences a **torque** $\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$. It applies to any magnetic dipole in general.

$\therefore$ External magnetic field aligns the magnetic dipoles.



Similar to electric dipole in a E-field, we can consider the work done in rotating the magnetic dipole. (refer to Chapter 2)



$$dW = -dU, \qquad \text{where } U \text{ is potential energy of dipole}$$

$$\boxed{U = -\vec{\mu} \cdot \vec{B}}$$

Note :

(1) We cannot define the potential energy of a magnetic field in general. However, we can define the potential energy of a magnetic dipole in a constant magnetic field.

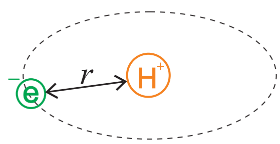
(2) In a *non-uniform external B-field*, the magnetic dipole will *experience a net force (not only net torque)*

7.6 Magnetic Properties of Materials

Recall intrinsic electric dipole in molecules:

Intrinsic dipole (magnetic) in atoms:

In our classical model of atoms, electrons revolve around a positive nuclear.



$$\therefore \quad \text{"Current" } i = \frac{e}{P}, \quad \text{where } P \text{ is period of one orbit around nucleus}$$

$$P = \frac{2\pi r}{v}, \quad \text{where } v \text{ is velocity of electron}$$

$\therefore$ **Orbit magnetic dipole** of atom:

$$\mu = iA = \left(\frac{ev}{2\pi r}\right)(\pi r^2) = \frac{erv}{2}$$

Recall: angular momentum of rotation $l = mrv$

$$\therefore \quad \mu = \frac{e}{2m} \cdot l$$

In *quantum mechanics*, we know that

$$l \text{ is quantized, } \texttt{i.e.} \quad l = N \cdot \frac{h}{2\pi}$$

$$\begin{aligned} \text{where } N &= \text{Any positive integer } (1,2,3, \dots) \\ h &= \text{Planck's constant } (6.63 \times 10^{-34} J \cdot s) \end{aligned}$$

$\therefore$ Orbital magnetic dipole moment

$$\mu_l = \underbrace{\frac{eh}{4m\pi}}_{\textbf{Bohr's magneton } \mu_B = 9.27\times 10^{-24} J/T} \cdot N$$

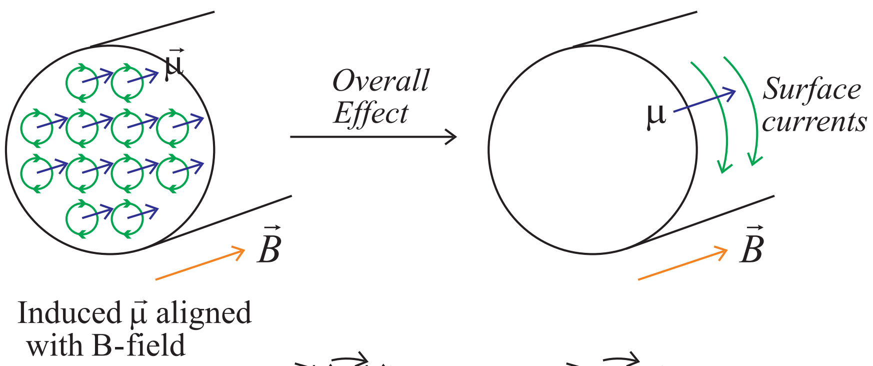
There is another source of intrinsic magnetic dipole moment inside an atom:

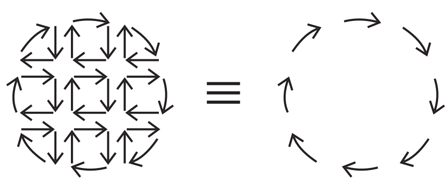
Spin dipole moment: coming from the intrinsic "spin" of electrons.

Quantum mechanics suggests that e^- are *always* spinning and it's either an "*up*" spin or a "*down*" spin

$$\mu_e = 9.65 \times 10^{-27} \approx \mu_B$$

So can there be induced magnetic dipole?





Recall: for electric field

$$\boxed{E_{dielectric} = K_e E_{vacuum}\ ; \quad K_e \geq 1}$$

For magnetic field in a material:

$$\vec{B}_{net} = \underset{\substack{\uparrow \\ \text{applied} \\ \text{B-field}}}{\vec{B}_0} + \underset{\substack{\uparrow \\ \text{B-field produced} \\ \text{by induced dipoles}}}{\vec{B}_M}$$

In many materials (except ferromagnets),

$$\vec{B}_M \propto \vec{B}_0$$

Define :

$$\boxed{\vec{B}_M = \chi_m \vec{B}_0}$$

χ_m is a *number* called **magnetic susceptibility**.

$$\begin{aligned} \therefore \quad \vec{B}_{net} &= \vec{B}_0 + \chi_m \vec{B}_0 \\ &= (1 + \chi_m)\,\vec{B}_0 \end{aligned}$$

$$\boxed{\vec{B}_{net} = \kappa_m \vec{B}_0\ ; \quad \kappa_m = 1 + \chi_m}$$

Define : κ_m is a *number* called **relative permeability**.

One more term ......

Define : the **Magnetization** of a material:

$$\vec{M} = \frac{d\vec{\mu}}{dV} \qquad \text{where } \vec{\mu} \text{ is magnetic dipole moment, } V \text{ is volume}$$

(or, the net magnetic dipole moment per unit volume)

In most materials (except ferromagnets),

$$\boxed{\vec{B}_M = \mu_0 \vec{M}}$$

Three types of magnetic materials:

(1) **Paramagnetic**:

$$\begin{matrix} \kappa_m \geq 1 \\ (\chi_m \geq 0) \end{matrix}\ , \quad \text{induced magnetic dipoles } \textit{aligned} \text{ } \textit{with} \text{ the applied B-field.}$$

`e.g.` Al ($\chi_m \doteq 2.2 \times 10^{-5}$), Mg ($1.2 \times 10^{-5}$), O_2 (2.0×10^{-6})

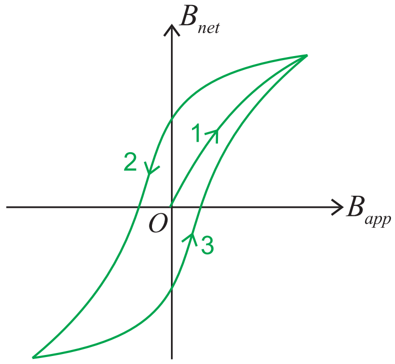
(2) **Diamagnetic**:

$$\begin{matrix} \kappa_m \leq 1 \\ (\chi_m \leq 0) \end{matrix} , \quad \begin{matrix} \text{induced magnetic dipoles } \textit{aligned} \\ \textit{opposite with} \text{ the applied B-field.} \end{matrix}$$

`e.g.` Cu ($\chi_m \approx -1 \times 10^{-5}$), Ag ($-2.6 \times 10^{-5}$), N_2 (-5×10^{-9})

(3) **Ferromagnetic**:

`e.g.` Fe, Co, Ni
Magnetization not linearly proportional to applied field.
$\Rightarrow \dfrac{B_{net}}{B_{app}}$ not a constant (can be as big as $\sim 5000 - 100,000$)

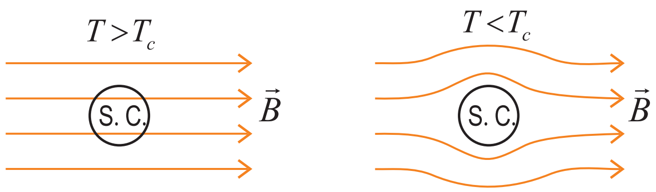


hysteresis curve
(hysteros: [Greek!] later, behind)

Interesting Case : *Superconductors*

$$\chi_m = -1$$

A perfect diamagnetic.
NO magnetic field inside.



Chapter 8

Faraday's Law of Induction

8.1 Faraday's Law

In the previous chapter, we have shown that *steady electric current* can give *steady magnetic field* because of the symmetry between electricity & magnetism.

We can ask: *Steady magnetic field* can give *steady electric current.* $\times$

OR *Changing magnetic field* can give *steady electric current.* $\checkmark$

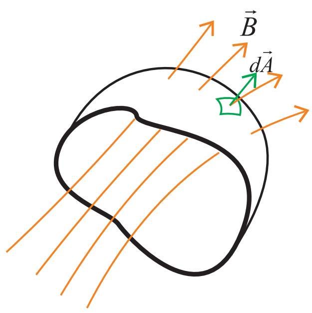
Define :

(1) Magnetic flux through surface S:

$$\boxed{\Phi_m = \int_S \vec{B} \cdot d\vec{A}}$$

Unit of Φ_m : Weber (Wb)

$1\text{Wb} = 1\text{Tm}^2$



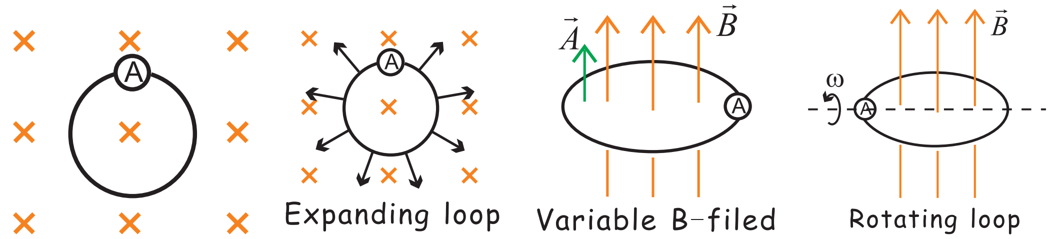
(2) Graphical:

$$\Phi_m = \text{Number of magnetic field lines passing through surface } S$$

Faraday's law of induction:

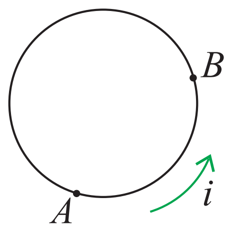
$$\boxed{\text{Induced emf } |\mathcal{E}| = N \left| \frac{d\Phi_m}{dt} \right|}$$

where N = Number of coils in the circuit.



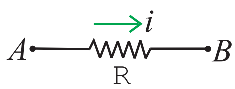
$\vec{B}$ = Constant	$\vec{B}$ = Constant	$\hat{B}$ = Constant	$\vec{B}$ = Constant
$\vec{A}$ = Constant	$\hat{A}$ = Constant	$dB/dt \neq 0$	A = Constant
	$dA/dt \neq 0$	$\vec{A}$ = Constant	$d\hat{A}/dt \neq 0$
$\mathcal{E} = 0$	$\therefore\ \lvert\mathcal{E}\rvert > 0$	$\therefore\ \lvert\mathcal{E}\rvert > 0$	$\therefore\ \lvert\mathcal{E}\rvert > 0$

Note : The *induced emf* drives a current throughout the circuit, similar to the function of a *battery*. However, the difference here is that the induced emf is *distributed throughout the circuit*. The consequence is that *we cannot define a potential difference between any two points in the circuit.*



Suppose there is an *induced current* in the loop, can we define ΔV_{AB}?

Recall:



$$\Delta V_{AB} = V_A - V_B = iR > 0$$
$$\Rightarrow \quad V_A > V_B$$

Going *anti-clockwise (same as i)*,

If we start from A, going to B, then we get $V_A > V_B$.
If we start from B, going to A, then we get $V_B > V_A$.

$$\boxed{\therefore \text{ We cannot define } \Delta V_{AB} \text{ !!}}$$

This situation is like when we study *the interior of a battery*.

$$\left.\begin{array}{l}\text{A battery}\\ \\ \text{The loop}\end{array}\right\}\ \begin{array}{l}\text{provides the energy needed to drive the}\\ \text{charge carriers around the circuit by}\end{array}\ \left\{\begin{array}{l}\text{chemical reactions.}\\ \\ \text{changing magnetic flux.}\end{array}\right.$$

sources of emf *non-electric means*

8.2 Lenz' Law

(1) The flux of the magnetic field due to induced current *opposes* the change in flux that causes the induced current.

(2) The induced current is in such a direction as to *oppose* the changes that produces it.

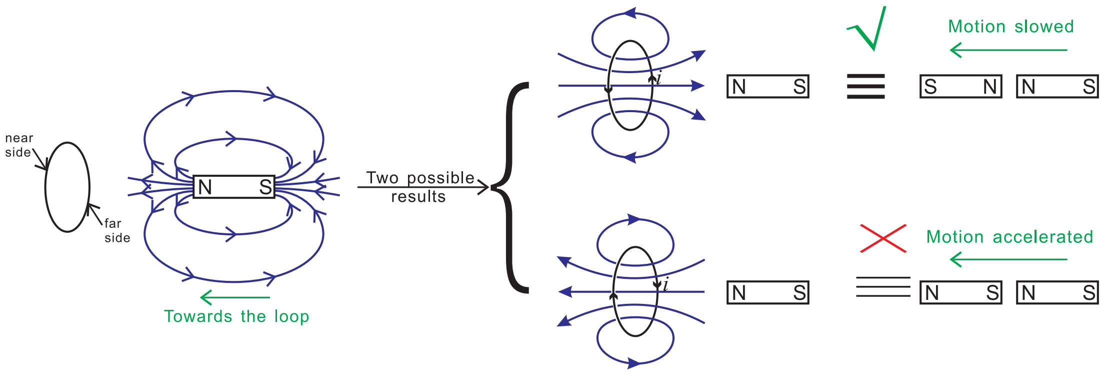
(3) Incorporating Lentz' Law into Faraday's Law:

$$\boxed{\mathcal{E} = -N\,\frac{d\Phi_m}{dt}}$$

If $\dfrac{d\Phi_m}{dt} > 0,\ \Phi_m \uparrow$ $\Rightarrow$ $\mathcal{E}$ appears $\Rightarrow$ Induced current appears.

$\Rightarrow$ $\vec{B}$-field due to induced current $\Rightarrow$ change in Φ_m $\overset{so\ that}{\Longrightarrow}$ $\Phi_m \downarrow$

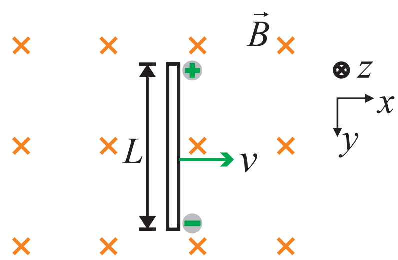
(4) Lenz' Law is a consequence from *the principle of conservation of energy.*



8.3 Motional EMF

Let's try to look at a special case when the *changing magnetic flux* is carried by *motion in the circuit wires.*

Consider a conductor of length L moving with a velocity v in a magnetic field $\vec{B}$.



Hall Effect for the charge carriers in the rod:

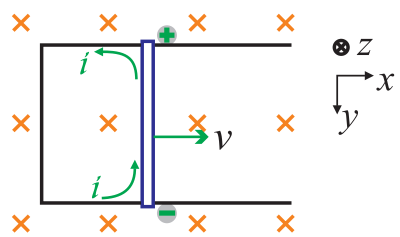
$$\begin{aligned} & \vec{F}_E + \vec{F}_B = 0 \\ \Rightarrow\quad & q\vec{E} + q\vec{v} \times \vec{B} = 0 \qquad \text{(where } \vec{E} \text{ is Hall electric field)} \\ \Rightarrow\quad & \vec{E} = -\vec{v} \times \vec{B} \end{aligned}$$

Hall Voltage inside rod:

$$\begin{aligned} \Delta V &= -\int_0^L \vec{E} \cdot d\vec{s} \\ \Delta V &= -EL \end{aligned}$$

$$\therefore \quad \text{Hall Voltage : } \boxed{\Delta V = vBL}$$

Now, suppose the moving wire *slides without friction* on a stationary U-shape conductor. The motional emf can drive an electric current i in the U-shape conductor.
$\Rightarrow$ Power is dissipated in the circuit.
$\Rightarrow$ $P_{out} = Vi$ (Joule's heating)
(see Lecture Notes Chapter 4)



What is the source of this power?
Look at the forces acting on the conducting rod:

- Magnetic force:

$$\begin{aligned} \vec{F}_m &= i\vec{L} \times \vec{B} \\ F_m &= iLB \quad \text{(pointing left)} \end{aligned}$$

- For the rod to continue to move at constant velocity v, we need to *apply an external force*:

$$\vec{F}_{ext} = -\vec{F}_m = iLB \quad \text{(pointing right)}$$

$\therefore$ *Power* required to keep the rod moving:

$$\begin{aligned} P_{in} &= \vec{F}_{ext} \cdot \vec{v} \\ &= iBLv \\ &= iBL\,\frac{dx}{dt} \\ &= iB\,\frac{d(xL)}{dt} \qquad \text{(} xL = A\text{, area enclosed by circuit)} \\ &= i\,\frac{d(BA)}{dt} \qquad \text{(} BA = \Phi_m\text{, magnetic flux)} \end{aligned}$$

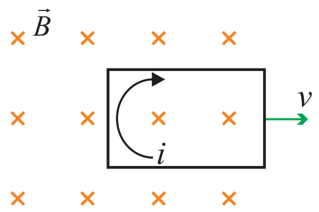
Since energy is not being stored in the system,

$$\begin{aligned} \therefore \quad P_{in} + P_{out} &= 0 \\ iV + i\frac{d\Phi_m}{dt} &= 0 \end{aligned}$$

We "prove" Faraday's Law $\Rightarrow \boxed{V = -\frac{d\Phi_m}{dt}}$

Applications :

(1) Eddy current: moving conductors in presence of magnetic field

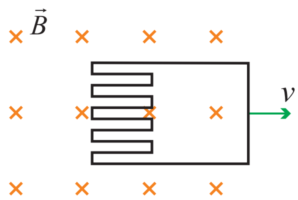


Induced current

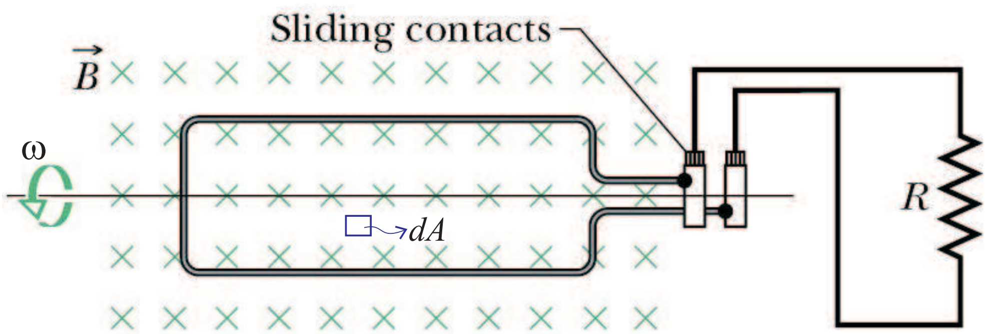
$\Rightarrow$ Power lost in Joule's heating $\left(\frac{\mathcal{E}^2}{R}\right)$

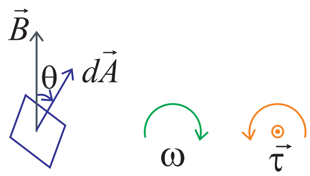
$\Rightarrow$ Extra power input to keep moving

To reduce Eddy currents:



(2) Generators and Motors:
Assume that the circuit loop is *rotating at a constant angular velocity* ω, (Source of rotation, `e.g.` steam produced by burner, water falling from a dam)





Magnetic flux through the loop

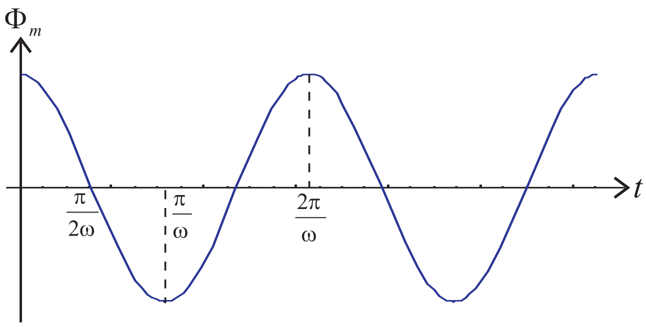
$$\Phi_B = \overbrace{N}^{\text{Number of coils}} \int_{loop} \vec{B} \cdot d\vec{A} = NBA \underbrace{\cos\theta}_{\text{changes with time! } \theta = \omega t}$$

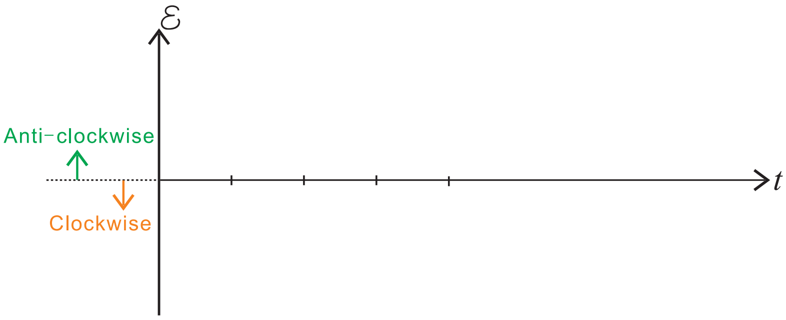
$$\therefore \quad \Phi_B = NBA\cos\omega t$$

$$\text{Induced emf:} \quad \mathcal{E} = -\frac{d\Phi_B}{dt} \quad = \quad -NBA\frac{d}{dt}(\cos\omega t)$$
$$= \quad NBA\omega\sin\omega t$$

$$\text{Induced current:} \quad i = \frac{\mathcal{E}}{R} \quad = \quad \frac{NBA\omega}{R}\sin\omega t$$

Alternating current (AC) voltage generator



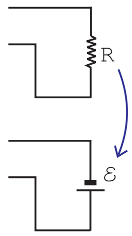


Power has to be provided by the source of rotation to overcome the torque acting on a current loop in a magnetic field.

$$\vec{\tau} \quad = \quad \overbrace{Ni\vec{A}}^{\vec{\mu}} \times \vec{B}$$
$$\therefore \ \tau \quad = \quad NiAB\sin\theta$$

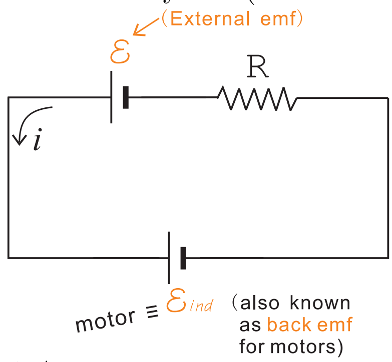
The net effect of the torque is to *oppose* the rotation of the coil.

An *electric motor* is simply a *generator operating in reverse.*
$\Rightarrow$ Replace the load resistance R with a battery of emf $\mathcal{E}$.



With the battery, there is a current in the coil, and it experiences a torque in the B-field.
$\Rightarrow$ Rotation of the coil leads to an induced emf, $\mathcal{E}_{ind}$, in the direction opposite of that of the battery. (Lenz' Law)

$$\therefore \quad i = \frac{\mathcal{E} - \mathcal{E}_{ind}}{R}$$

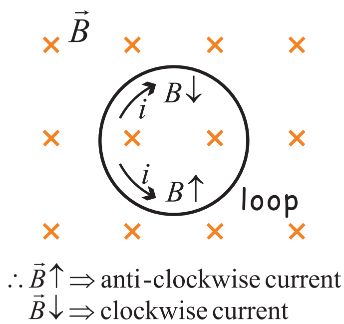


$\Rightarrow$ As motor speeds up, $\mathcal{E}_{ind} \uparrow$, $\quad \therefore\ i \downarrow$
$\therefore$ mechanical power delivered = torque delivered = $NiAB\sin\theta \downarrow$
In conclusion, we can show that

$$\boxed{\underbrace{P_{electric}}_{\text{Electric power input}} = i^2R + \underbrace{P_{mechanical}}_{\text{Mechanical power delivered}}}$$

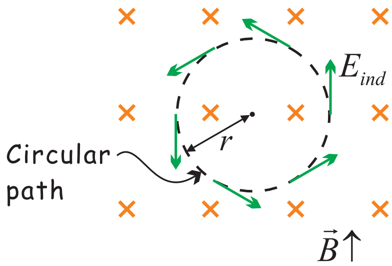
8.4 Induced Electric Field

So far we have discussed that a *change* in magnetic flux will lead in an induced emf distributed in the loop, resulting from an induced E-field.



However, even in the *absence* of the loop (so that there is no induced current), the induced E-field will still accompany a change in magnetic flux.

$\therefore$ Consider a circular path in a region with changing magnetic field.



The induced E-field only has tangential components. (`i.e.` radial E-field = 0)
`Why?`
Imagine a point charge q_0 travelling around the circular path.
Work done by induced E-field = $\underbrace{q_0 E_{ind}}_{force} \cdot \underbrace{2\pi r}_{distance}$
Recall work done also equals to $q_0\mathcal{E}$, where $\mathcal{E}$ is induced emf

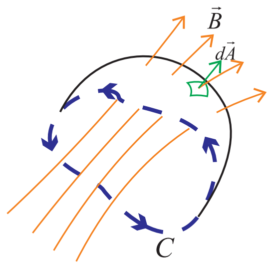
$$\therefore \quad \mathcal{E} = E_{ind} 2\pi r$$

Generally,

$$\mathcal{E} = \oint \vec{E}_{ind} \cdot d\vec{s}$$

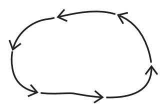
where $\oint$ is line integral around a closed loop, $\vec{E}_{ind}$ is induced E-field, $\vec{s}$ is tangential vector of path.
$\therefore$ Faraday's Law becomes

$$\boxed{\oint_C \vec{E}_{ind} \cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{A}}$$



L.H.S. = Integral around a closed loop C
R.H.S. = Integral over a surface bounded by C

Direction of $d\vec{A}$ determined by direction of line integration C (Right-Hand Rule)

"Regular" E-field	Induced E-field
created by charges	created by changing B-field
E-field lines start from $+ve$ and end on $-ve$ charge	E-field lines form closed loops
	
can define electric potential so that we can discuss potential difference between two points	Electric potential cannot be defined (or, potential has no meaning)
$\Downarrow$	$\Downarrow$
Conservative force field	Non-conservative force field

The classification of electric and magnetic effects *depend on the frame of reference of the observer*. `e.g.` For motional emf, observer in the reference frame of the moving loop, will NOT see an induced E-field, just a "regular" E-field.
(Read: Halliday Chap.33-6, 34-7)

Chapter 9

Inductance

9.1 Inductance

An *inductor* stores energy in the *magnetic field* just as a *capacitor* stores energy in the *electric field*.

We have shown earlier that *a changing B-field will lead to an induced emf in a circuit*.

Question : If a circuit generates a changing magnetic field, does it lead to an induced emf in the same circuit? YES! **Self-Inductance**

The **inductance L** of any current element is

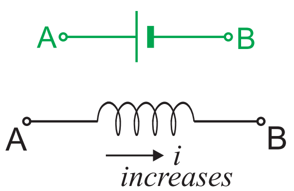
$$\boxed{\mathcal{E}_L = \Delta V_L = -L\frac{di}{dt}}$$

The negative sign comes from Lenz Law.

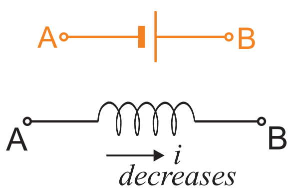
Unit of L: Henry(H) $\quad$ $1\text{H}=1\cdot\dfrac{\text{Vs}}{\text{A}}$

- All circuit elements (including resistors) have some inductance.
- Commonly used inductors: solenoids, toroids
- circuit symbol: 

Example : Solenoid



$$\mathcal{E}_L = V_B - V_A = -L\frac{di}{dt} < 0$$
$$\therefore \quad V_B < V_A$$



$$\mathcal{E}_L = V_B - V_A = -L\frac{di}{dt} > 0$$
$$V_B > V_A$$

Recall *Faraday's Law*,

$$\mathcal{E}_L = -N\,\frac{d\Phi_B}{dt} = -\frac{d}{dt}\,(N\Phi_B)$$

where Φ_B is magnetic flux, $N\Phi_B$ is flux linkage.
$\therefore$ Alternative definition of Inductance:

$$-\frac{d}{dt}\,(N\Phi_B) = -L\frac{di}{dt} \quad \Rightarrow \quad \boxed{L = \frac{N\Phi_B}{i}}$$

$\therefore$ Inductance is also *flux linkage per unit current.*

Calculating Inductance:

(1) Solenoid:

To first order approximation,

$$B = \mu_0 n i$$

where $n = N/L =$ Number of coils per unit length.

Consider a subsection of length l of the solenoid:

$$\begin{aligned} \text{Flux linkage} &= N\,\Phi_B \\ &= nl \cdot BA \end{aligned}$$

where A is cross-sectional area

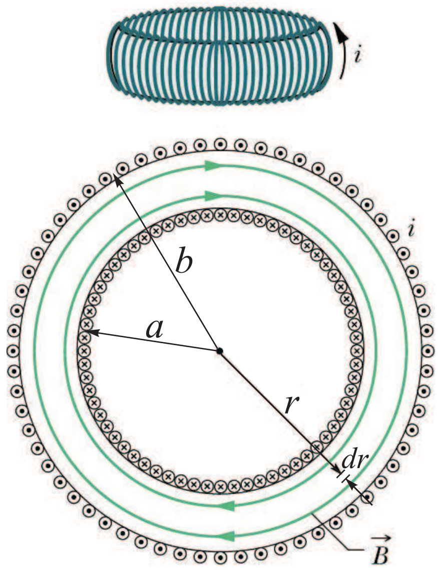
$$\therefore \boxed{\begin{aligned} &L = \frac{N\Phi_B}{i} = \mu_0 n^2 l A \\ &\frac{L}{l} = \mu_0 n^2 A = \textit{Inductance per unit length} \end{aligned}}$$

Notice :

(i) $L \propto n^2$

(ii) The inductance, like the capacitance, depends only on geometric factors, not on i.

(2) Toroid:



Recall: B-field lines are concentric circles.
Inside the toroid:

$$B = \frac{\mu_0 i N}{2\pi r}$$
(NOT a constant)

where r is the distance from center.
Outside the toroid:

$$B = 0$$

Flux linkage through the toroid

$$\begin{aligned} N\Phi_B &= N\int \vec{B}\cdot d\vec{a} \qquad \left\{ \begin{array}{ll} \text{Notice} & \vec{B} \parallel d\vec{a} \\ \text{Write} & da = h\,dr \end{array} \right\rangle \boxed{\text{KEY}} \\ &= \frac{\mu_0 i N^2}{2\pi}\int_a^b \frac{h\,dr}{r} \\ &= \frac{\mu_0 i N^2 h}{2\pi}\ln\Big(\frac{b}{a}\Big) \end{aligned}$$

$\therefore$ Inductance $\quad L = \dfrac{N\Phi_B}{i} = \dfrac{\mu_0 N^2 h}{2\pi}\ln\Big(\dfrac{b}{a}\Big)$

Again, $\quad L \propto N^2$

Inductance with magnetic materials :
We showed earlier that for capacitors:

$$\left\{ \begin{array}{lll} \vec{E} & \rightarrow & \vec{E}/\kappa_e \\ C & \rightarrow & \kappa_e C \end{array} \right. \qquad \text{(after insertion of dielectric } \kappa_e > 1)$$

For inductors, we first know that

$$\vec{B} \rightarrow \kappa_m \vec{B} \qquad \text{(after insertion of magnetic material)}$$

Inductance $\quad L = \dfrac{N\Phi_B}{i}$

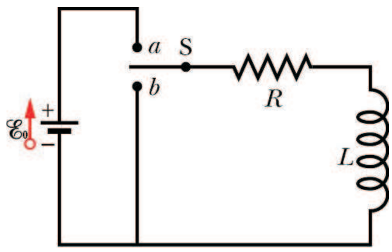
However $\quad \Phi_B = \displaystyle\int \vec{B}\cdot d\vec{A} \rightarrow \kappa_m \Phi_B$

$$\therefore \quad \boxed{L \longrightarrow \kappa_m L}$$ (after insertion of magnetic material)

$\therefore$ To increase inductance, fill the interior of inductor with *ferromagnetic materials*. $(\times 10^3 - 10^4)$

9.2 LR Circuits

(A) "Charging" an inductor



When the switch is adjusted to position a,
By *loop rule* (clockwise) :

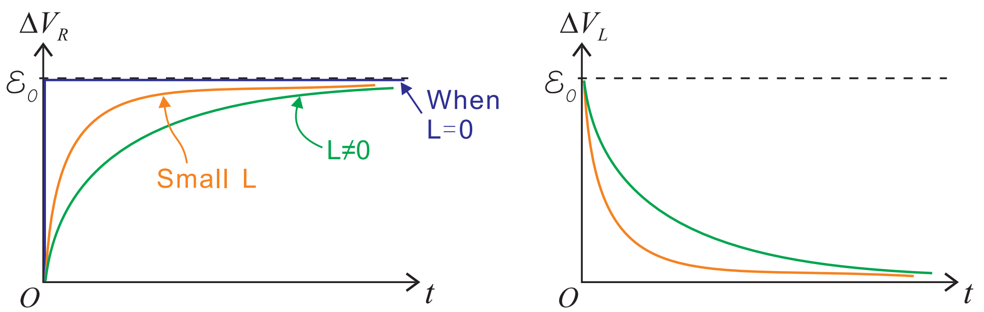
$$\begin{array}{ccccccc} \mathcal{E}_0 & - & \Delta V_R & + & \Delta V_L & = & 0 \\ & & \downarrow & & \downarrow & & \\ \mathcal{E}_0 & - & iR & - & L\dfrac{di}{dt} & = & 0 \end{array}$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}\, i = \frac{\mathcal{E}_0}{L}$$ First Order Differential Equation

Similar to the equation for charging a capacitor! (Chap5)

$$\boxed{\begin{array}{l} \text{Solution: } i(t) = \dfrac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau_L}\right) \\ \quad \text{where } \tau_L = \textit{Inductive time constant} = L/R \end{array}}$$

$$\begin{array}{lcccccc} \therefore \quad |\Delta V_R| & = & iR & = & \mathcal{E}_0(1 - e^{-t/\tau_L}) & & \\ \quad\;\; |\Delta V_L| & = & L\dfrac{di}{dt} & = & L \cdot \dfrac{\mathcal{E}_0}{R} \cdot \dfrac{1}{\tau_L} \cdot e^{-t/\tau_L} & = & \mathcal{E}_0\, e^{-t/\tau_L} \end{array}$$

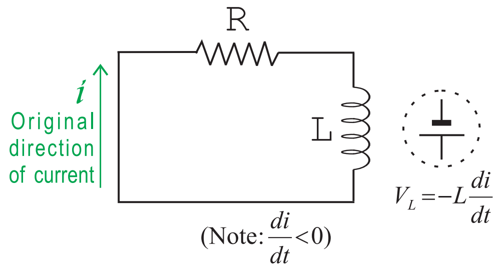


(B) "Discharging" an inductor
When the switch is adjusted at position b after the inductor has been "*charged*" (`i.e.` current $i = \mathcal{E}_0/R$ is flowing in the circuit.).

By *loop rule*:

$$\begin{array}{ccccc} \Delta V_L & - & \Delta V_R & = & 0 \\ \downarrow & & \downarrow & & \\ -L\dfrac{di}{dt} & - & iR & = & 0 \end{array}$$

(Treat inductor as source of emf)

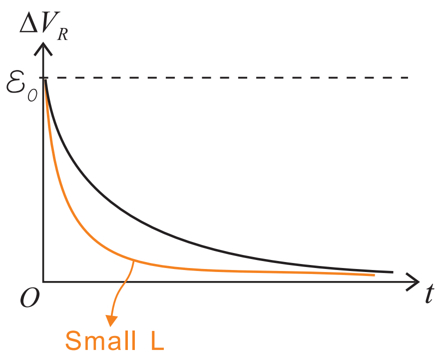


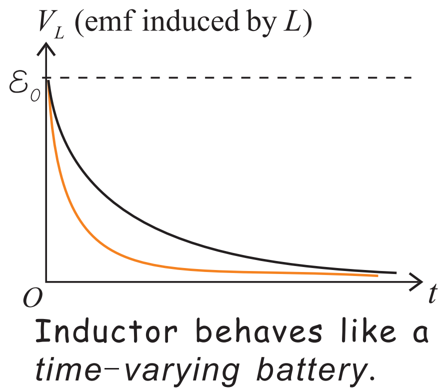
$$\therefore \quad \frac{di}{dt} + \frac{R}{L}\, i = 0$$

Discharging a capacitor (Chap5)

$$\boxed{i(t) = i_0\, e^{-t/\tau_L}}$$

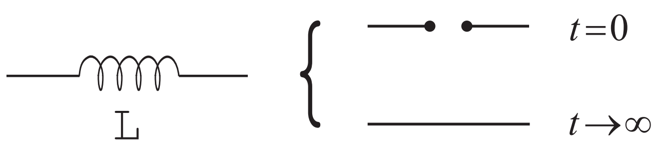
where $i_0 = i(t=0) =$ Current when the circuit just switch to position b.





Summary : During charging of inductor,

1. At $t = 0$, inductor acts like *open circuit* when *current flowing is zero.*
2. At $t \to \infty$, inductor acts like *short circuit* when *current flowing is stablized at maximum.*



3. Inductors are used everyday in switches for safety concerns.

9.3 Energy Stored in Inductors

Inductors stored *magnetic energy* through the *magnetic field* stored in the circuit. Recall the equation for charging inductors:

$$\mathcal{E}_0 - iR - L\,\frac{di}{dt} = 0$$

Multiply both sides by i :

$$\underbrace{\mathcal{E}_0 i}_{\substack{\text{Power input by emf}\\ \text{(Energy supplied to}\\ \text{one charge } = q\mathcal{E}_0)}} \quad = \quad \underbrace{i^2 R}_{\substack{\text{Joule's heating}\\ \text{(Power dissipated}\\ \text{by resistor)}}} \quad + \quad \underbrace{Li\,\frac{di}{dt}}_{\substack{\text{Power stored}\\ \text{in inductor}}}$$

$\therefore$ Power stored in inductor $= \dfrac{dU_B}{dt} = Li\,\dfrac{di}{dt}$

Integrating both sides and use initial condition

$$\text{At } t = 0, \quad i(t=0) \;=\; U_B(t=0) \;=\; 0$$

$$\therefore \quad \boxed{\text{Energy stored in inductor: } U_B \;=\; \frac{1}{2}\,Li^2}$$

Energy Density Stored in Inductors :

Consider an *infinitely long* solenoid of cross-sectional area A.
For a portion l of the solenoid, we know from §8.1,

$$L = \mu_0 n^2 l A$$

$\therefore$ Energy stored in inductor:

$$U_B = \frac{1}{2}\,Li^2 = \frac{1}{2}\,\mu_0 n^2 i^2 \underbrace{lA}_{\substack{\text{Volume of}\\ \text{solenoid}}}$$

$\therefore$ **Energy density** (= Energy stored per unit volume) inside inductor:

$$u_B = \frac{U_B}{lA} = \frac{1}{2}\,\mu_0 n^2 i^2$$

Recall magnetic field inside solenoid (Chap7)

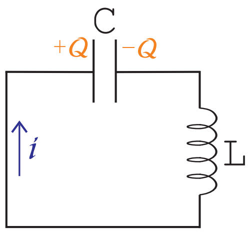
$$B = \mu_0 n i$$

$$\therefore \quad \boxed{u_B = \frac{B^2}{2\mu_0}}$$

This is a *general result* of the *energy stored in a magnetic field.*

9.4 LC Circuit (Electromagnetic Oscillator)

Initial charge on capacitor $= Q$
Initial current $= 0$
No battery.



Assume current i to be in the direction that *increases* charge on the *positive capacitor plate.*

$$\Rightarrow \qquad i = \frac{dQ}{dt} \tag{9.1}$$

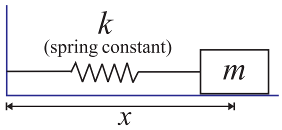
By *Lenz Law*, we also know the "poles" of the inductor.

$$\begin{aligned} \text{Loop rule:} \qquad V_C + V_L &= 0 \\ -\frac{Q}{C} - L\frac{di}{dt} &= 0 \end{aligned} \tag{9.2}$$

Combining equations (9.1) and (9.2), we get

$$\boxed{\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0}$$

This is similar to the equation of motion of a *simple harmonic oscillator*:



$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Another approach (*conservation of energy*)
Total energy stored in circuit:

$$\begin{aligned} U &= U_E + U_B \\ &\quad\ \downarrow \qquad \downarrow \\ U &= \frac{Q^2}{2C} + \frac{1}{2}Li^2 \end{aligned}$$

Since the resistance in the circuit is zero, *no energy is dissipated in the circuit.*
$\therefore$ Energy contained in the circuit is *conserved.*

$$\therefore \qquad \frac{dU}{dt} = 0$$

$$\Rightarrow \quad \frac{Q}{C} \cdot \frac{d\cancel{Q}}{\cancel{dt}} + L\cancel{i}\frac{di}{dt} = 0 \qquad (\because\ i = \frac{dQ}{dt})$$

$$\Rightarrow \quad L\frac{di}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow \quad \boxed{\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0}$$

The solution to this differential equation is in the form

$$\begin{aligned}
Q(t) &= Q_0\cos(\omega t + \phi) \\
\therefore \quad \frac{dQ}{dt} &= -\omega Q_0 \sin(\omega t + \phi) \\
\frac{d^2Q}{dt^2} &= -\omega^2 Q_0 \cos(\omega t + \phi) \\
&= -\omega^2 Q
\end{aligned}$$

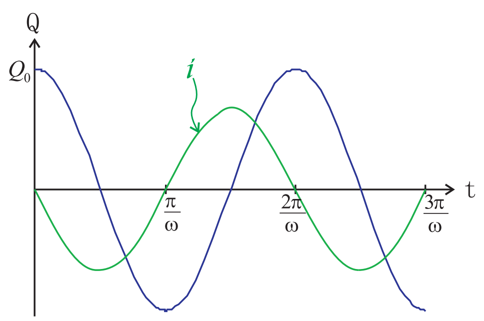
$$\therefore \quad \frac{d^2Q}{dt^2} + \omega^2 Q = 0$$

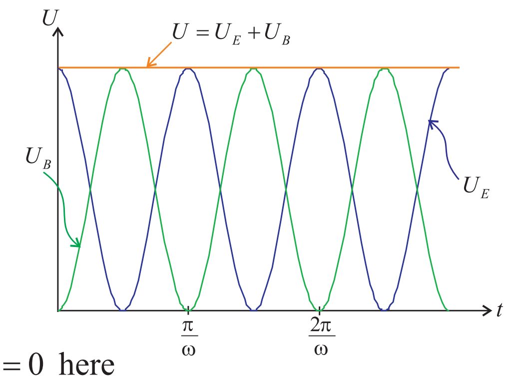
$$\therefore \quad \boxed{\omega^2 = \frac{1}{LC}}$$

Angular frequency of the LC oscillator

Also, Q_0, ϕ are constants derived from the initial conditions. (Two initial conditions, `e.g.` $Q(t = 0)$, and $i(t = 0) = \left.\frac{dQ}{dt}\right|_{t=0}$ are required.)

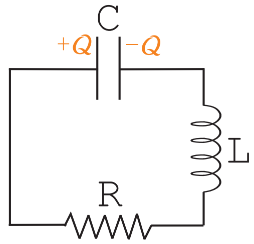
$$\begin{aligned}
\text{Energy stored in } C = \frac{Q^2}{2C} &= \frac{Q_0^2}{2C}\cos^2(\omega t + \phi) \\
\text{Energy stored in } L = \frac{1}{2}Li^2 &= \frac{1}{2}L\omega^2 Q_0^2 \sin^2(\omega t + \phi) \\
\boxed{\because \quad L\omega^2 = \frac{1}{C}} &= \frac{Q_0^2}{2C}\sin^2(\omega t + \phi) \\
\therefore \quad \text{Total energy stored} &= \frac{Q_0^2}{2C} \\
&= \textit{Initial energy stored in capacitor}
\end{aligned}$$





Assume $\phi = 0$ here

9.5 RLC Circuit (Damped Oscillator)



In real life circuit, there's *always* resistance.
In this case, energy stored in the LC oscillator is *NOT* conserved,

$$\text{and} \quad \frac{dU}{dt} = \text{Power dissipated in the resistor} = -i^2 R \qquad (\textit{Joule's heating})$$

Negative sign shows that energy U is *decreasing*.

$$\therefore \quad Li\frac{di}{dt} + \frac{Q}{C}\cdot\overbrace{\frac{dQ}{dt}}^{i} = -i^2R$$

$$\Rightarrow \quad \boxed{\frac{d^2Q}{dt^2} + \frac{R}{L}\cdot\frac{dQ}{dt} + \frac{1}{LC}Q = 0}$$

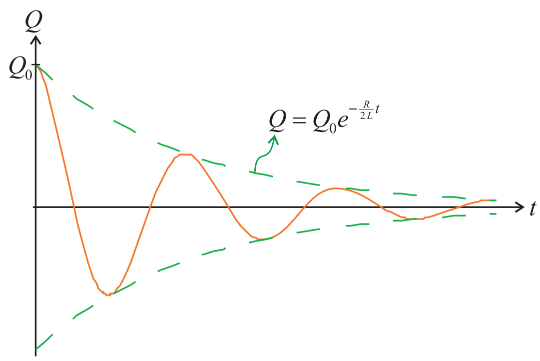
This is similar to the equation of motion of a *damped harmonic oscillator* (e.g. if a mass-spring system faces a frictional force $\vec{F} = -b\vec{v}$).
Solution to the equation is in the form $\quad Q(t) = e^{\lambda t}$
If damping is not too big (i.e. R not too big), solution would become

$$\boxed{Q(t) = Q_0 \underbrace{e^{-\frac{R}{2L}t}}_{\substack{\text{exponential}\\ \text{decay term}}} \underbrace{\cos(\omega_1 t + \phi)}_{\substack{\text{oscillating}\\ \text{term}}}}$$

$$\text{where} \qquad \omega_1^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

$$\boxed{\omega_1^2 = \omega^2 - \left(\frac{R}{2L}\right)^2}$$

Damped oscillator always oscillates at a *lower* frequency than the *natural frequency* of the oscillator. (Refer to *Halliday*, Vol1, Chap17 for more details.)



Check this at home: What is $U_E(t) + U_B(t)$ for the case when damping is small? (i.e. $R \ll \omega$)

Chapter 10

AC Circuits

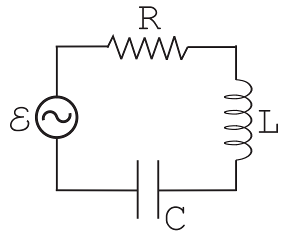
10.1 Alternating Current (AC) Voltage

Recall that an *AC generator* described in Chapter 9 generates a *sinusoidal emf.*

$$\texttt{i.e.} \quad \mathcal{E} = \mathcal{E}_m \sin(\omega t + \delta)$$

Note :

This circuit is the RLC circuit with one *additional element : the time varying AC power supply.* This is similar to a *driven (damped) oscillator.*



$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{1}{C}\,Q = \mathcal{E}_m \sin(\omega t + \delta)$$

The general solution consists of two parts:

transient : rapidly dies away in a few cycles (not interesting)

steady state : $Q(t), i(t)$ varies *sinusoidally* with the same frequency as input

Note : Current does NOT vary at frequency $\omega_1^2 = \dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2$

Since we only concern about the *steady state solution*, therefore we can take any time as starting reference time = 0
For convenience, we can write

$$\boxed{\mathcal{E} = \mathcal{E}_m \sin \omega t}$$

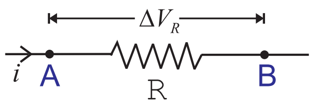
And we can write

$$\boxed{i = i_m \sin(\omega t - \phi)}$$

where i_m is current amplitude, ϕ is phase constant.
Our goal is to determine i_m and ϕ.

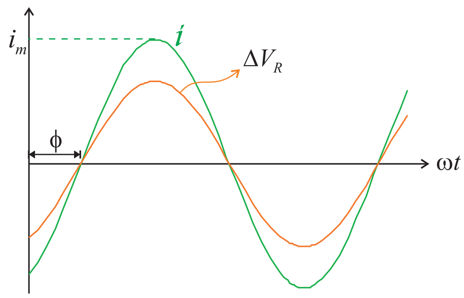
10.2 Phase Relation Between i, V for R,L and C

(A) Resistive Element



$$\Delta V_R = V_A - V_B = iR$$
$$\therefore \quad \Delta V_R = i_m R \sin(\omega t - \phi)$$

ΔV_R and i are *in phase*, `i.e.` what's <u>inside the "sine bracket"</u> (**phase**) is the same for ΔV_R and i.

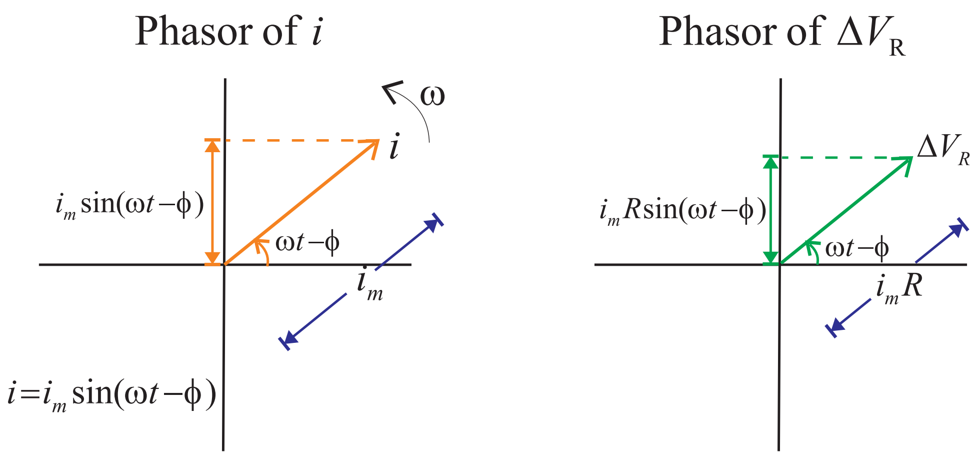


Graphically, we introduce **phasor diagrams** properties of **phasors**:

(1) *Length* of a phasor is proportional to the *maximum value.*

(2) *Projection* of a phasor *onto the vertical axis* gives *instantaneous value.*

(3) **Convention**: Phasors rotate *anti-clockwise* in a uniform circular motion with *angular velocity.*

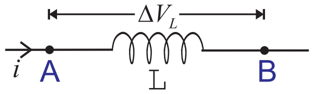


$$\therefore \quad \Delta V_R = (\Delta V_R)_m \sin(\omega t - \phi)$$

$$\boxed{(\Delta V_R)_m = i_m R}$$

"Ohm's Law like" relation for AC resistor

(B) The Inductive Element



Potential drop across inductor

$$\Delta V_L = V_A - V_B = -\mathcal{E}_L = L\,\frac{di}{dt}$$

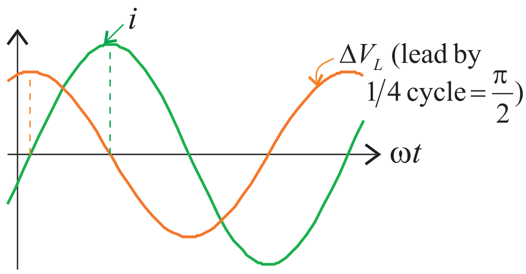
$$\begin{aligned}
\therefore\quad \Delta V_L &= L i_m \omega \cos(\omega t - \phi) \\
&= L i_m \omega \sin(\omega t - \phi + \frac{\pi}{2}) \qquad [\because\ \cos\theta = \sin(\theta + \frac{\pi}{2})] \\
&= i_m X_L \sin(\omega t - \phi + \frac{\pi}{2})
\end{aligned}$$

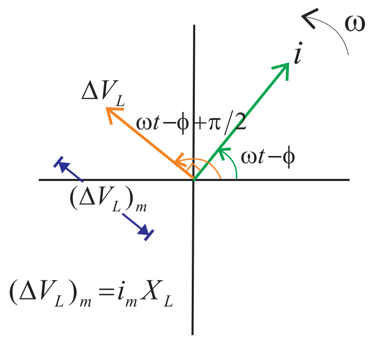
$$\boxed{(\Delta V_L)_m = i_m X_L}$$

"Ohm's Law like" relation for AC inductor

where $\quad X_L = \textit{Inductive Reactance}$

$$\boxed{X_L = \omega L}$$

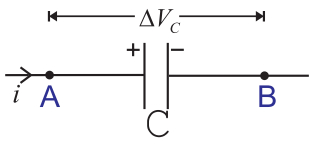




$$\begin{array}{lllll}
\text{As} & i\uparrow, & V_A > V_B & \therefore & \Delta V_L > 0 \\
 & i\downarrow, & V_A < V_B & \therefore & \Delta V_L < 0
\end{array}$$

$$\boxed{\begin{array}{lllll}
\Delta V_L & \textit{leads} & i & \text{by} & \frac{\pi}{2} \\
i & \textit{lags} & \Delta V_L & \text{by} & \frac{\pi}{2}
\end{array}}$$

(C) Capacitive Element



$$\Delta V_C = V_A - V_B = \frac{Q}{C}$$

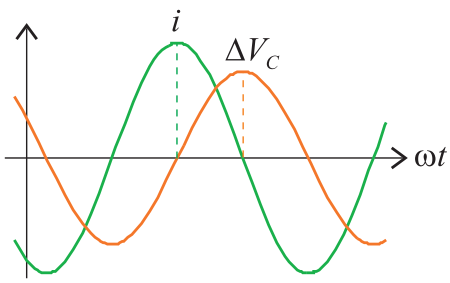
where Q = charge on the positive plate of the capacitor.

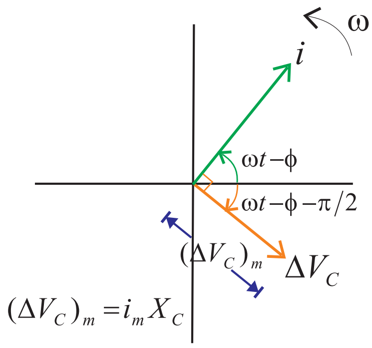
$$\begin{aligned} \therefore \quad i = \frac{dQ}{dt} \quad \Rightarrow \quad Q &= \int i\,dt \\ &= \int i_m \sin(\omega t - \phi)\,dt \\ &= -\frac{i_m}{\omega}\cos(\omega t - \phi) \end{aligned}$$

$$\begin{aligned} \therefore \quad \Delta V_C &= -\frac{i_m}{\omega C}\cos(\omega t - \phi) \\ &= i_m X_C \sin(\omega t - \phi - \frac{\pi}{2}) \qquad [\because\ -\cos\theta = \sin(\theta - \frac{\pi}{2})] \end{aligned}$$

$$\therefore \quad \boxed{(\Delta V_C)_m = i_m X_C}$$ "Ohm's Law like" relation for AC capacitor

where $\boxed{X_C = \frac{1}{\omega C}}$ = *Capacitive Reactance*

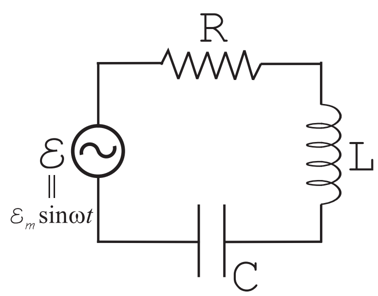




ΔV_C	*lags*	i	by	$\frac{\pi}{2}$
i	*leads*	ΔV_C	by	$\frac{\pi}{2}$

10.3 Single Loop RLC AC Circuit

Given that $\mathcal{E} = \mathcal{E}_m \sin\omega t$, we want to find i_m and ϕ so that we can write $i = i_m \sin(\omega t - \phi)$



$$\begin{aligned} \text{Loop rule:} \qquad & \mathcal{E} - \Delta V_R - \Delta V_L - \Delta V_C = 0 \\ \Rightarrow \quad & \mathcal{E} = \Delta V_R + \Delta V_L + \Delta V_C \end{aligned}$$

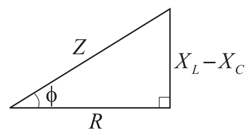
Using results from the previous section, we can write

$$\begin{aligned}
\mathcal{E}_m \sin\omega t &= \quad i_m R\sin(\omega t-\phi) \\
&\quad +i_m X_L\cos(\omega t-\phi) - i_m X_C\cos(\omega t-\phi) \\
\mathcal{E}_m \sin\omega t &= i_m\Big[R\sin(\omega t-\phi)+(X_L-X_C)\cos(\omega t-\phi)\Big]
\end{aligned}$$

Answer :

1. Take $\boxed{\tan\phi = \dfrac{X_L-X_C}{R}}$

2. Define $\boxed{Z=\sqrt{R^2+(X_L-X_C)^2}}$



   as the **impedance** of the circuit.

3. Then

$$\boxed{i_m = \frac{\mathcal{E}_m}{Z} \quad \text{or} \quad \mathcal{E}_m = i_m Z}$$

"Ohm's Law like" relation for AC RLC circuits

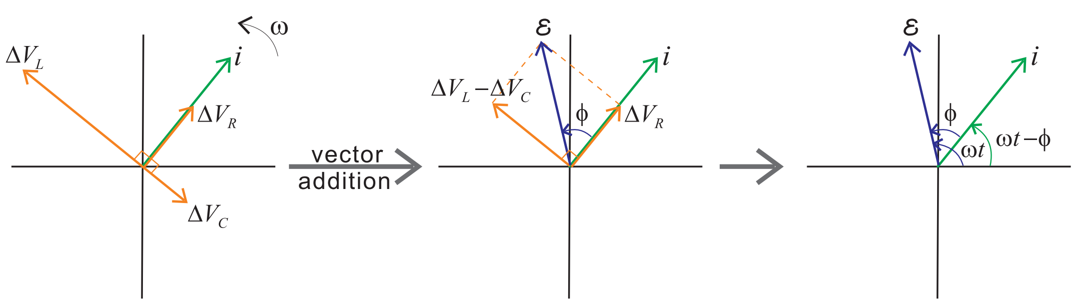
Check :

$$\begin{aligned}
R.H.S. &= i_m Z\left[\frac{R}{Z}\sin(\omega t-\phi)+\frac{X_L-X_C}{Z}\cos(\omega t-\phi)\right] \\
&= i_m Z\Big[\cos\phi\sin(\omega t-\phi)+\sin\phi\cos(\omega t-\phi)\Big]
\end{aligned}$$

$$\left(\begin{array}{l}
\text{Use the relation:} \\
\sin(A+B)=\sin A\cos B+\cos A\sin B \\
\text{Here:}\quad A=\omega t-\phi,\quad B=\phi
\end{array}\right)$$

$$\begin{aligned}
&= i_m Z\ \sin(\omega t-\phi+\phi) \\
&= i_m z\ \sin\omega t \\
&= L.H.S. \quad \text{if} \quad \mathcal{E}_m = i_m Z \qquad\qquad \text{QED.}
\end{aligned}$$

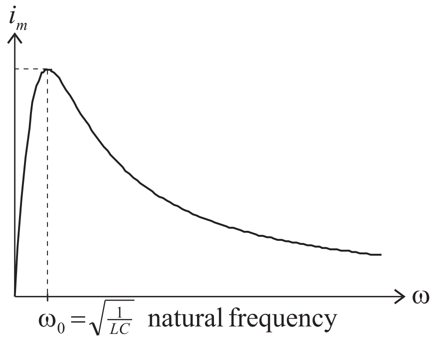
Phasor Approach :



10.4 Resonance

$i_m = \dfrac{\mathcal{E}_m}{Z}$ is at *maximum* for an AC circuit of *fixed input frequency* ω when Z is at *minimum*.

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \Big(\omega L - \frac{1}{\omega C}\Big)^2}$$



is at a minimum for a fixed ω when

$$\begin{aligned} X_L - X_C \;&=\; \omega L - \frac{1}{\omega C} \;=\; 0 \\ \Rightarrow \quad \omega L \;&=\; \frac{1}{\omega C} \\ \Rightarrow \quad \boxed{\omega^2 \;=\; \frac{1}{LC}} & \end{aligned}$$

same as that for a RLC circuit

In Hong Kong, the AC power input is `50Hz`.
(In US, as mentioned in *Halliday*, is `60Hz`.)

$$\therefore \quad \omega = 2\pi f = 314.2 s^{-1}$$

10.5 Power in AC Circuits

Consider the *Power dissipated by R* in an AC circuit:

$$P = i^2 R = i_m^2 R \sin^2(\omega t - \phi)$$

The *average* power dissipated in each cycle:

$$P_{ave} = \frac{\int_0^{2\pi/\omega} P\,dt}{2\pi/\omega} \qquad \Big(\frac{2\pi}{\omega} \text{ is period of each cycle}\Big)$$

$$\begin{aligned} \int_0^{2\pi/\omega} P\,dt \;&=\; i_m^2 R \int_0^{2\pi/\omega} \sin^2(\omega t - \phi)\,dt \\ &=\; i_m^2 R \int_0^{2\pi/\omega} \frac{1}{2}\Big[1 - \cos 2(\omega t - \phi)\Big]\,dt \\ &=\; i_m^2 R \cdot \Big[\frac{t}{2} - \cancel{\frac{\sin^2(\omega t - \phi)}{4\omega}}\Big]\Big|_0^{2\pi/\omega} \\ &=\; i_m^2 R \cdot \frac{1}{2} \cdot \frac{2\pi}{\omega} \end{aligned}$$

$$\therefore \quad \boxed{\begin{array}{l} P_{ave} \;=\; \dfrac{i_m^2}{2}\,R \;=\; i_{rms}^2 R \\ \quad \text{where} \quad i_{rms} = \textit{root-mean-square current} \end{array}}$$

$$i_{rms} \;=\; \frac{i_m}{\sqrt{2}} \qquad \because \text{ Current is a sinusoidal func.}$$

Symbol : $\langle P \rangle = P_{ave} =$ Average of P over time

For sine and cosine functions of time:

Average : $\langle \sin \omega t \rangle = \langle \cos \omega t \rangle = 0$

Amplitude : Peak value, e.g. $\mathcal{E}_m$, i_m, $(\Delta V_R)_m$, $\cdots$

Root-Mean-Square(RMS) : It's a measure of the *"time-averaged"* deviation from zero.

$$x_{rms} \;=\; \sqrt{\langle x^2 \rangle}$$

For sines and cosines, for whatever quantity x:

$$x_{rms} = \frac{x_m}{\sqrt{2}} \qquad (x_m \text{ is amplitude})$$

For an AC resistor circuit:

$$\boxed{\langle P \rangle \;=\; i_{rms}^2 R \;=\; \frac{\mathcal{E}_{rms}^2}{R}}$$

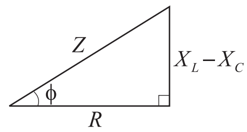
Laws for DC circuits can be used to describe AC circuits if we *use rms values for i and $\mathcal{E}$.*

For general AC circuits:

$$\begin{aligned} P \;=\; \mathcal{E} i \;&=\; \overbrace{\mathcal{E}_m \sin \omega t}^{\mathcal{E}} \cdot \overbrace{i_m \sin(\omega t - \phi)}^{i} \\ &=\; \mathcal{E}_m i_m \sin \omega t \,[\sin \omega t \cos \phi - \cos \omega t \sin \phi] \\ P \;&=\; \mathcal{E}_m i_m \,[\,\underbrace{\sin^2 \omega t}_{\frac{1}{2}} \cos \phi - \underbrace{\sin \omega t \cos \omega t}_{\substack{0 \\ \text{(check this!)}}} \sin \phi \,] \\ \langle P \rangle \;&=\; \frac{\mathcal{E}_m i_m}{2} \cos \phi \end{aligned}$$

$$\boxed{\langle P \rangle \;=\; \mathcal{E}_{rms} i_{rms} \underbrace{\cos \phi}_{\text{power factor}}}$$

$$\text{Recall} \qquad \tan\phi = \frac{X_L - X_C}{R}$$
$$\therefore \quad \cos\phi = \frac{R}{Z}$$



Maximum power dissipated in circuit when

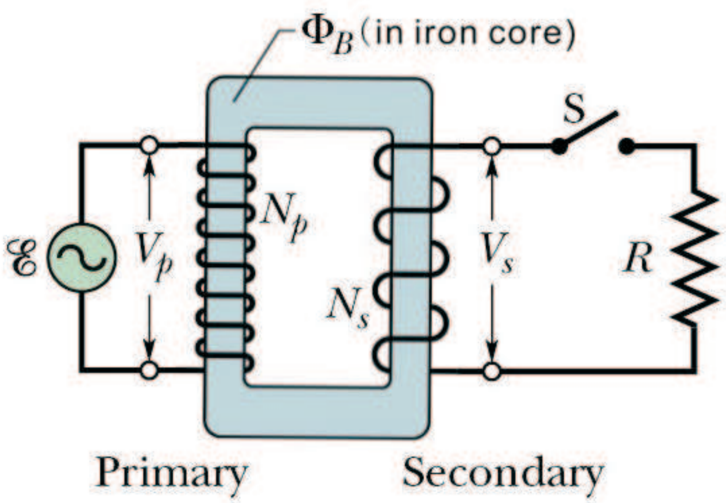
$$\cos\phi = 1$$

Two possibilities:

(1) $X_L = X_C = 0$

(2) $X_L - X_C = 0 \quad \Rightarrow \quad X_L = X_C \quad \Rightarrow \quad \omega L = \dfrac{1}{\omega C} \quad \Rightarrow \quad \omega^2 = \dfrac{1}{LC}$
(Resonance Condition)

10.6 The Transformer



Power dissipated in resistor

$$\langle P \rangle = i_{rms}^2 R$$

$\therefore$ For power transmission, we'd like to *keep i_{rms} at minimum.*
$\Rightarrow$ HIGH potential difference across transmission wires. (So that total power transmitted $P = i_{rms}\mathcal{E}_{rms}$ is constant.)
However, for home safety, we would like LOW emf supply.

Solution : **Transformers**

Primary : Number of winding $= N_P$

Secondary : Number of winding $= N_S$

In primary circuit, $R_P \approx C_P \approx 0$

$$\therefore \quad \text{Pure inductive}$$

$$\text{Power factor :} \quad \cos\phi = \frac{R}{Z} \approx 0$$

$$\therefore \quad \text{No power delivered from emf to transformer.}$$

The *varying* current ($\because$ AC!) in the primary produces an *induced emf* in the secondary coils. *Assuming* perfect magnetic flux linkage:

$$\begin{aligned} & \text{emf per turn in primary} \\ = \; & \text{emf per turn in secondary} \\ = \; & -\frac{d\Phi_B}{dt} \end{aligned}$$

$$\begin{aligned} \text{emf per turn in primary} &= \frac{\Delta V_P}{N_P} \quad (\Delta V_P \text{ is P.D. across primary}) \\ \text{emf per turn in secondary} &= \frac{\Delta V_S}{N_S} \end{aligned}$$

$$\Rightarrow \quad \boxed{\frac{\Delta V_P}{\Delta V_S} = \frac{N_P}{N_S}}$$

$$\begin{array}{llll} \text{If} \quad N_P > N_S\,, & \text{then} \quad \Delta V_P > \Delta V_S & \textit{Step-Down} \\ \text{If} \quad N_P < N_S\,, & \text{then} \quad \Delta V_P < \Delta V_S & \textit{Step-Up} \end{array}$$

Consider power in circuit:

$$i_P \Delta V_P = i_S \Delta V_S$$

In the secondary, we have

$$\Delta V_S = i_S R$$

Combining the 3 equations, we have

$$\boxed{\Delta V_P = \left(\frac{N_P}{N_S}\right)^2 R \cdot i_P}$$

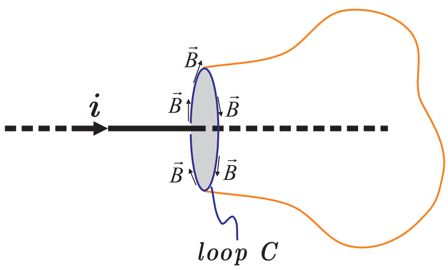
$$\text{"Equivalence Resistor"} = \left(\frac{N_P}{N_S}\right)^2 R$$

Chapter 11

Displacement Current and Maxwell's Equations

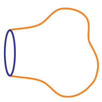
11.1 Displacement Current

We saw in Chap.7 that we can use **Ampère's law** to calculate magnetic fields due to currents.
We know that the integral $\oint_C \vec{B} \cdot d\vec{s}$ around any close loop C is equal to $\mu_0 i_{incl}$, where i_{incl} = *current passing an area bounded by the closed curve C.*



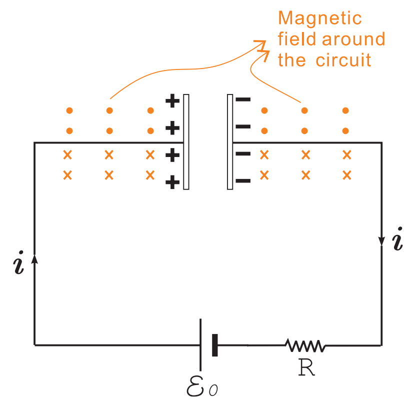
e.g.

 = Flat surface bounded by loop C

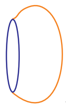
 = Curved surface bounded by loop C

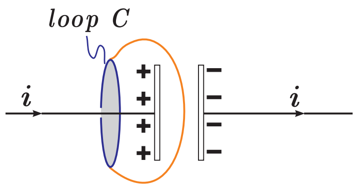
If **Ampère's law** is true all the time, then the i_{incl} *determined should be independent of the surface chosen.*

Let's consider a simple case: *charging a capacitor*.
From Chap.5, we know there is a current flowing $i(t) = \frac{\mathcal{E}_i}{R}e^{-t/RC}$, which leads to a magnetic field observed $\vec{B}$. With Ampère's law, $\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{incl}$.
BUT WHAT IS i_{incl}?



If we look at , $i_{incl} = i(t)$

If we look at , $i_{incl} = 0$



($\because$ There is no charge flow between the capacitor plates.)
$\therefore$ Ampère's law is either WRONG or INCOMPLETE.

Two observations:

1. While there is no current between the capacitor's plates, there is a *time-varying electric field between the plates of the capacitor.*
2. We know *Ampère's law is mostly correct from measurements of B-field around circuits.*

$$\Downarrow$$

$$\boxed{\text{Can we revise Ampère's law to fix it?}}$$

Electric field between capacitor's plates: $E = \dfrac{\sigma}{\varepsilon_0} = \dfrac{Q}{\varepsilon_0 A}$, where Q = *charge on capacitor's plates*, A = *Area of capacitor's plates.*

$$\therefore \qquad Q = \varepsilon_0 \underbrace{E\cdot A}_{\text{Electric flux}} = \varepsilon_0 \Phi_E$$

$\therefore$ We can define

$$\frac{dQ}{dt} = \boxed{\varepsilon_0 \frac{d\Phi_E}{dt} = i_{disp}}$$

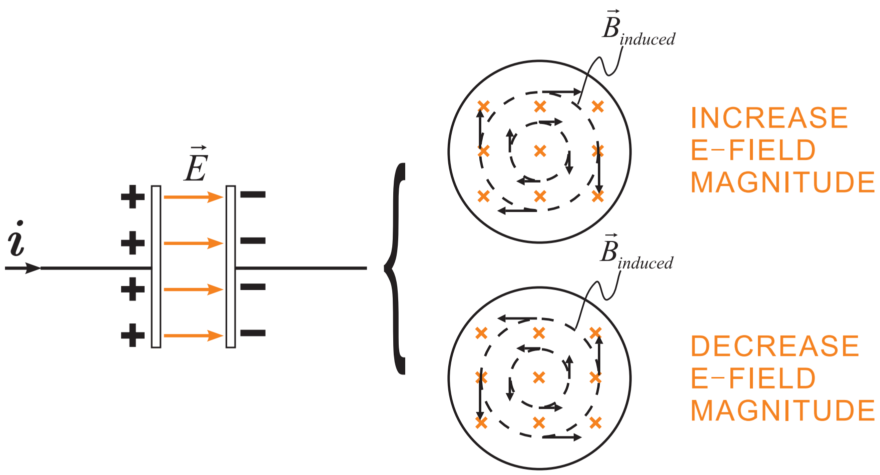
where i_{disp} is called **Displacement Current** (first proposed by Maxwell).
Maxwell first proposed that this is the missing term for the Ampère's law:

$$\boxed{\oint_C \vec{B}\cdot d\vec{s} = \mu_0 (i_{incl} + \varepsilon_0 \frac{d\Phi_E}{dt})} \qquad \textbf{Ampère-Maxwell law}$$

Where i_{incl} = current through any surface bounded by C,
Φ_E = electric flux through that *same surface bounded by curve* C, $\Phi_E = \int_S \vec{E} \cdot d\vec{a}$.

11.2 Induced Magnetic Field

We learn earlier that electric field can be generated by
$$\begin{cases} \textit{charges} \\ \textit{changing magnetic flux} \end{cases}.$$
We see from Ampère-Maxwell law that a magnetic field can be generated by
$$\begin{cases} \textit{moving charges (current)} \\ \textit{changing electric flux} \end{cases}.$$
That is, a change in electric flux through a surface bounded by C can lead to an *induced magnetic field along the loop* C.



Notes The induced magnetic field is along the *same direction* as caused by the *changing electric flux.*

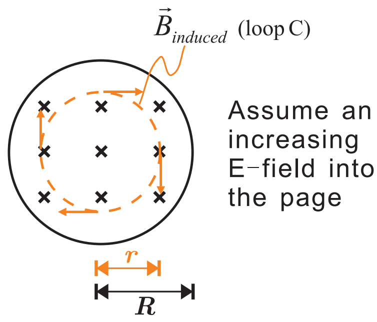
Example What is the magnetic field strength inside a circular plate capacitor of radius R with a current $I(t)$ charging it?

Answer Electric field of capacitor

$$E = \frac{Q}{\varepsilon_0 A} = \frac{Q}{\varepsilon_0 \pi R^2}$$

Electric flux inside capacitor through a loop C of radius r:

$$\Phi_E = E \cdot \pi r^2 = \frac{Qr^2}{\varepsilon_0 R^2}$$



Ampère-Maxwell Law inside capacitor:

$$\underbrace{\oint_C \vec{B} \cdot d\vec{s}}_{\because \vec{B}_{induced} \| d\vec{s}} = \mu_0(\cancel{i_{incl}} + \varepsilon_0 \frac{d\Phi_E}{dt})$$

$$\begin{aligned} \underbrace{2\pi r}_{\text{Length of loop } C} B_{induced} &= \mu_0 \varepsilon_0 \frac{d}{dt}\Big(\frac{Qr^2}{\varepsilon_0 R^2}\Big) \\ &= \mu_0 \frac{r^2}{R^2} \underbrace{\frac{dQ}{dt}}_{I(t)} \end{aligned}$$

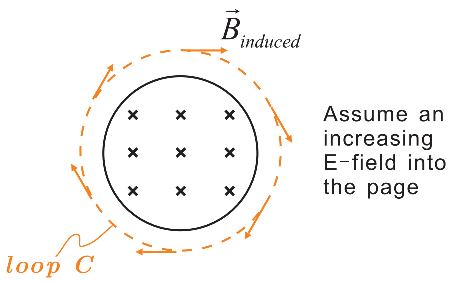
$$\therefore B_{induced} = \frac{\mu_0 r}{2\pi R^2} I(t) \qquad \text{for } r < R$$

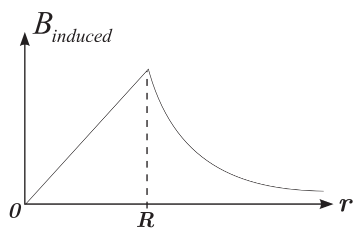
Outside the capacitor plate:
Electric flux through loop C: $\Phi_E = E \cdot \pi R^2 = \dfrac{Q}{\varepsilon_0}$

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0(i_{incl} + \varepsilon_0 \frac{d\Phi_E}{dt})$$

$$2\pi r B_{induced} = \mu_0 \varepsilon_0 \Big(\frac{1}{\varepsilon_0} \cdot \frac{dQ}{dt}\Big)$$

$$\therefore\ B_{induced} = \frac{\mu_0 I(t)}{2\pi r}$$





11.3 Maxwell's Equations

The four equations that *completely* describe the behaviors of electric and magnetic fields.

$$\boxed{\begin{aligned}
\oint_S \vec{E}\cdot d\vec{a} &= \frac{Q_{incl}}{\varepsilon_0} \\
\oint_S \vec{B}\cdot d\vec{a} &= 0 \\
\oint_C \vec{E}\cdot d\vec{s} &= -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \\
\oint_C \vec{B}\cdot d\vec{s} &= \mu_0 i_{incl} + \mu_0\varepsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}
\end{aligned}}$$

The one equation that describes *how matter reacts to electric and magnetic fields.*

$$\boxed{\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})}$$

Features of Maxwell's equations:

(1) There is a high level of *symmetry* in the equations. That's why the study of electricity and magnetism is also called **electromagnetism**.
There are *small asymmetries* though:

i) There is *NO point "charge" of magnetism / NO magnetic monopole.*

ii) Direction of induced E-field *opposes to* B-flux change.
Direction of induced B-filed *enhances* E-flux change.

(2) Maxwell's equations predicted the existence of propagating waves of E-field and B-field, known as **electromagnetic waves** (**EM waves**).

Examples of EM waves: visible light, radio, TV signals, mobile phone signals, X-rays, UV, Infrared, gamma-ray, microwaves...

(3) Maxwell's equations are *entirely consistent with the special theory of relativity.* This is *not* true for Newton's laws!