Supplementary Methods

ZERO ENERGY MFS AT THE END OF A 1D KITAEV CHAIN

Consider a 1D Kitaev chain Hamiltonian

$$H_{eff} = \sum_n -\mu_n \hat{\psi}_n^\dagger \hat{\psi}_n + t_n(\hat{\psi}_{n+1}^\dagger \hat{\psi}_n + h.c) + \Delta_n(\hat{\psi}_{n+1}^\dagger \hat{\psi}_n^\dagger + h.c). \tag{S1}$$

In this sub-section we prove that if t_n, Δ_n are real and sign-ordered i.e.

$$\mathrm{sign}(\Delta_n t_n) = \mathrm{sign}(\Delta_{n+1} t_{n+1}), \tag{S2}$$

and $|\mu_n| < \max(|t_{n-1}|, |\Delta_{n-1}|)$, then the Kitaev chain has zero-energy MFs at the ends. As discussed in the main text, we can choose $|t_n| > |\Delta_n|$ and $t_n > 0$ without loss of generality. In the topological regime where Eq. S2 is satisfied, Δ_n has the same sign on all sites in the chain.

Defining a Nambu spinor $(\psi_n^\dagger, \psi_n)$, with a corresponding particle-hole symmetry operator $\Lambda = \tau_x K$, the BdG Hamiltonian corresponding to the above BCS Hamiltonian is written as

$$H_{BdG} = H_0 \tau_z + i\Delta\tau_y, \tag{S3}$$

where

$$H_0 = \sum_n -\mu_n |n\rangle\langle n| + t_n[|n\rangle\langle n+1| + h.c] \tag{S4}$$

$$\Delta = \sum_n \Delta_n[|n\rangle\langle n+1| - h.c]. \tag{S5}$$

Applying a unitary transformation $U = \frac{1-i\tau_y}{\sqrt{2}}$, H_{BdG} transforms into

$$UH_{BdG}U^\dagger = \begin{pmatrix} 0 & (H_0+\Delta) \\ (H_0+\Delta)^T & 0 \end{pmatrix}. \tag{S6}$$

The BdG equation

$$\begin{pmatrix} 0 & (H_0+\Delta) \\ (H_0-\Delta) & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = E \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \tag{S7}$$

is now written as

$$(H_0+\Delta)\zeta_2 = E\zeta_1 \tag{S8}$$

$$(H_0-\Delta)\zeta_1 = E\zeta_2. \tag{S9}$$

The matrices $H_0 \pm \Delta$ are written as

$$H_0 \pm \Delta = \sum_n -\mu_n |n\rangle\langle n| + \sum_n (t_n \pm \Delta_n)|n\rangle\langle n+1| + (t_{n-1} \mp \Delta_{n-1})|n\rangle\langle n-1|, \tag{S10}$$

where only sites from $n = 0, \ldots, N$ are included. The above Hamiltonians are non-Hermitean and display localization transitions similar to the Hamiltonians in previous work [37]. Zero energy (i.e. $E = 0$) modes are obtained as solutions of one of the two decoupled equations

$$(H_0 \pm \Delta)\zeta_{2,1} = 0. \tag{S11}$$

Using the definition of the unitary operator U and the particle-hole operator Λ it can be checked that solutions with either $\zeta_1 = 0$ or $\zeta_2 = 0$ are Majorana. The sign of Δ_n can be inverted by flipping the chain, therefore in this basis if one of the ends has a MF mode with $\zeta_2 = 0$ then the other end has an MF mode with $\zeta_1 = 0$. Since both signs of Δ occur in the above equation, we can assume that $\Delta_n > 0$ without loss of generality.

Writing the wave-function as $\psi = \sum_n \psi_n |n\rangle$, the equation for the zero-mode can be written as a transfer-matrix-like relation

$$-\mu_n\psi_n + (t_n+\Delta_n)\psi_{n+1} + (t_{n-1} - \Delta_{n-1})\psi_{n-1} = 0 \tag{S12}$$

for $n = 1, \ldots, N$, together with the boundary conditions $\psi_0 = \psi_{N+1} = 0$. Perfectly localized end MF mode solutions with $\psi_{n>1} = 0$ to this equation exist when $\mu_1 = \mu_2 = (t_1 - \Delta_1) = 0$. In this case, it is easy to check that

$$(t_1+\Delta_1)\psi_2 = (t_1+\Delta_1)0 = 0$$
$$-(t_1-\Delta_1)\psi_1 = -0\psi_1 = 0, \tag{S13}$$

so that $\psi_1 = 1$ and $\psi_{n>1} = 0$ is a solution. Therefore any QD array containing more than two sites can support localized MF modes at the ends. However, as shown later, only odd number size arrays have robust zero modes when $\mu_n = 0$.

Redefining

$$\psi_n \to \tilde{\psi}_n = g_n\psi_n$$
$$g_n = \prod_{1\le m<n} \sqrt{\frac{(t_m+\Delta_m)}{(t_m-\Delta_m)}} \tag{S14}$$

and dividing Eq. S12 by g_n, we obtain the equation as or equivalently as

$$-\mu_n\tilde{\psi}_n + \sqrt{(t_n+\Delta_n)}\tilde{\psi}_{n+1} + \sqrt{(t_{n-1}^2 - \Delta_{n-1}^2)}\tilde{\psi}_{n-1} = 0., \tag{S15}$$

which in turn can be re-written as

$$\tilde{\psi}_{n+1} = \frac{1}{\sqrt{(t_n^2 - \Delta_n)}}[\mu_n\tilde{\psi}_n - \sqrt{(t_{n-1}^2 - \Delta_{n-1}^2)}\tilde{\psi}_{n-1}], \tag{S16}$$

for $n = 1, \ldots, N$, where as before $\tilde{\psi}_1 = \tilde{\psi}_{N+1} = 0$. The above equation represents a zero-mode for a hermiteanized tight-binding Hamiltonian

$$\tilde{H}_0 = \sum_n -\mu_n \left|n\right\rangle \left\langle n\right| + \sum_n \sqrt{(t_n^2 - \Delta_n^2)} \left|n\right\rangle \left\langle n+1\right| + \sqrt{(t_{n-1}^2 - \Delta_{n-1}^2)} \left|n\right\rangle \left\langle n-1\right| . \tag{S17}$$

Let us start by considering the special case of the above Hamiltonian Eq. S17, where each gate in the QD has been tuned so that $\mu_n = 0$, so that $\tilde{H}_0$ has a chiral symmetry which supports zero-end modes similar to Su-schrieffer-Heeger model [39, 40]. The equation for $\tilde{\psi}_n$ then simplifies to

$$\tilde{\psi}_{n+1} = -\frac{\sqrt{(t_{n-1}^2 - \Delta_{n-1}^2)}}{\sqrt{(t_n^2 - \Delta_n^2)}} \tilde{\psi}_{n-1}, \tag{S18}$$

together with $\tilde{\psi}_0 = \tilde{\psi}_{N+1} = 0$. The above equations can be formally (i.e. if one ignores the boundary conditions) solved for $\tilde{\psi}_p$ as

$$\tilde{\psi}_{2n+1} = (-1)^n \prod_{p<n} \frac{\sqrt{(t_{2p+1}^2 - \Delta_{2p+1}^2)}}{\sqrt{(t_{2p}^2 - \Delta_{2p}^2)}} \tilde{\psi}_1$$
$$\tilde{\psi}_{2n} = 0. \tag{S19}$$

The above solution satisfies the boundary conditions if the number of sites is odd i.e. $N = 2m + 1$. Therefore for odd number of sites, the above chain always has an exact zero mode when $\mu_n = 0$. However, in the case of even sites and in the presence of μ_n fluctuations, the zero-modes are split by an amount that depends exponentially on the ratio of the length of the system to the coherence length of the zero-modes.

In the case where the hopping amplitudes are statistically independent of each other, the pre-factor

$$\prod_{p<n} \frac{\sqrt{(t_{2p+1}^2 - \Delta_{2p+1}^2)}}{\sqrt{(t_{2p}^2 - \Delta_{2p}^2)}} \sim e^{\pm\sqrt{n}\sigma^2} \tag{S20}$$

in the large n limit where

$$\sigma^2 = \mathrm{var}(\log(\sqrt{(t_p^2 - \Delta_p^2)})), \tag{S21}$$

where the variance var is taken with respect to the random distribution. Since the pre-factor for $\tilde{\psi}_{2n+1}$ scales as $e^{\sqrt{n}}$, and the factor g_n in the definition of $\tilde{\psi}_n$ scales as $g_n \sim e^{-n\zeta_1}$ where

$$\zeta_1 = -E\log\left\{\frac{|t_n - \Delta_n|}{t_n + \Delta_n}\right\}, \tag{S22}$$

and E is the expectation value with respect to the distribution of t_n, Δ_n, it follows that

$$\psi_{2n} \sim e^{\pm\sqrt{n}\sigma^2 - 2n\zeta_1} \sim e^{-2n\zeta_1}, \tag{S23}$$

so that the zero-mode wave-function ψ_{2n+1} is localized near the end of the chain and is normalizable. This shows that, as claimed in the main text that localized zero-energy MF modes exist for $\mu_n = 0$, for independently ordered bonds. However, one should note that such localized MFs don't exist for every configuration of t_n, Δ_n. For example, a dimerized configuration with $t_{2n-1} > t_{2n}$ would not lead to zero modes below a critical value of Δ_n.

Now we discuss the more general case for $\mu_n \neq 0$. In this case the solution $\tilde{\psi}_n$ of the transfer matrix relation Eq. S16 is expected to scale as

$$\tilde{\psi}_n \sim e^{n\zeta_0} \tag{S24}$$

where ζ_0^{-1} is the localization length for Eq. S17. Using the definition of ψ_n in terms of $\tilde{\psi}_n$ we find that

$$\psi_n \sim e^{n\zeta} \tag{S25}$$

where $\zeta = (\zeta_1 - \zeta_0)$, proving the topological condition in the main-text so that ψ_n is normalized and localized when $\zeta < 0$. This regime corresponds to the delocalized phase of Ref. [37]. Note that this condition is also necessary, since in the localized phase one will generically obtain low-energy localized states in $\tilde{H}_0$, which will lead to similar energy states in $H_0 + \Delta$ [37]. This establishes the conditions where one can rigorously expect localized end modes with a gap. However, the numerics show that even for short chains and $\mu_n \neq 0$, localized MFs exist.

Lower bound on the gaps of sign-ordered Kitaev chains

While we have shown the existence of Majorana fermions at the ends of the Kitaev chain for a large set of parameters, the thermal robustness of the topological phase is determined by the quasiparticle gap in the system with periodic boundary conditions. To estimate the smallest possible (worst case) gap of H_{BCS}, we need to find a lower bound on E^2 to the solutions of Eq. S9 for sign-ordered Kitaev chains in the delocalized phase.

It follows from Eq. S9 that E^2 are eigenvalues of both the real-symmetric matrices

$$(H_0 + \Delta)^T (H_0 + \Delta)\zeta_2 = E^2\zeta_2 \tag{S26}$$
$$(H_0 + \Delta)(H_0 + \Delta)^T \zeta_1 = E^2\zeta_1. \tag{S27}$$

Since it is easier to prove bounds upper bounds on the maximum eigenvalues then lower bounds on the minimum eigenvalue, we consider the inverse matrices

$$(H_0 + \Delta)^{-1,T}(H_0 + \Delta)^{-1}\zeta_1 = E^{-2}\zeta_1 \tag{S28}$$
$$(H_0 + \Delta)^{-1}(H_0 + \Delta)^{-1,T}\zeta_2 = E^{-2}\zeta_2. \tag{S29}$$

It is well-known from linear-algebra [41], that the maximum value of E^{-2} is related to the maximum row sum of the matrix $(H_0+\Delta)^{-1}$,i.e.

$$\begin{aligned} E^{-2} &= \frac{\zeta_1^T (H_0+\Delta)^{-1,T}(H_0+\Delta)^{-1}\zeta_1}{\zeta_1^T\zeta_1} \\ &< \max_j \sum_i |((H_0+\Delta)^{-1})_{i,j}| \max_i \sum_j |((H_0+\Delta)^{-1})_{i,j}|. \end{aligned} \tag{S30}$$

Therefore we get the lower bound on the magnitude of the eigenvalues

$$|E| > \sqrt{\left(\max_j \sum_i |((H_0+\Delta)^{-1})_{i,j}| \max_i \sum_j |((H_0+\Delta)^{-1})_{i,j}|\right)}. \tag{S31}$$

Matrices whose inverses decay exponentially and are finite have energies that are bounded away from zero.

The inverse of $(H_0+\Delta)$ can be computed using the transfer matrix form for Eq. S16 can be written as a matrix recursion relation

$$\Psi_n = \begin{pmatrix} 0 \\ \frac{\tilde{\phi}_n}{\sqrt{t_n^2-\Delta_n^2}} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{(t_{n-1}^2-\Delta_{n-1}^2)}}{\sqrt{t_n^2-\Delta_n^2}} & -\frac{\mu_n}{\sqrt{t_n^2-\Delta_n^2}} \end{pmatrix} \Psi_{n-1}, \tag{S32}$$

where $\Psi_n = (\tilde{\psi}_n, \tilde{\psi}_{n+1})^T$. To calculate the inverse of $H_0+\Delta$ for a chain with periodic boundary conditions, we can label any site to be 0 and choose $\phi_0 = 1$ and $\phi_{n\neq 0} = 0$. Furthermore, we will make the ansatz $\Psi_{-1} = 0$. This leads to the sequence of equations

$$\begin{aligned} \Psi_0 &= \begin{pmatrix} 0 \\ \frac{1}{\sqrt{t_0^2-\Delta_0^2}} \end{pmatrix} \\ \Psi_{n>0} &= \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{(t_{n-1}^2-\Delta_{n-1}^2)}}{\sqrt{t_n^2-\Delta_n^2}} & -\frac{\mu_n}{\sqrt{t_n^2-\Delta_n^2}} \end{pmatrix} \Psi_{n-1}. \end{aligned} \tag{S33}$$

Using arguments from the theory of 1D localization using transfer matrices $\Psi_{n>0}$ is expected to grow as $\Psi_n \sim e^{n\zeta_0}$, where ζ_0^{-1} is the localization length of the hermitean Hamiltonian in Eq. S17. By transforming back to the original ψ_n variables using Eq. S14, we find using Eq. S22 that in the delocalized phase

$$\psi_n \sim \frac{1}{\sqrt{t_0^2-\Delta_0^2}} e^{n\zeta} \tag{S34}$$

with $\zeta < 0$ so that for long Kitaev chains the corrections related to our ansatz $\Psi_{-1} = 0$ are exponentially small. Furthermore this proves a bound of $\sim \frac{1}{\Delta_0}$ on the row-sums of $\max_i \sum_j |((H_0+\Delta)^{-1})_{i,j}|$ in Eq. S31. On the other hand, by considering the transpose $(H_0+\Delta)^T$, one can obtain a similar bound on the column sum. Substituting in Eq. S31, we obtain a gap of order $\sim |\Delta_n|$ for the Kitaev chain. Of course this is a rough estimate since we have not carefully estimated values for the localization length ζ_0^{-1}. However, the numerical results in the main text seem to be consistent with these expectations.

Supplementary References

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