Operations Research

Introduction

Input-Output Analysis (leontif)

- Internal demand (usage)

- Suppose the production of each dollar's worth of electricity requires \$ 0.30 worth of electricity and \$ 0.10 worth of water, and the production of each dollar's worth of water requires \$ 0.20 worth of electricity and \$ 0.40 worth of water.

outputs ↑

inputs →

$$\begin{array}{c|cc} & E & W \\ \hline E & 0.3 & 0.2 \\ W & 0.1 & 0.4 \end{array} \quad \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$$

0.3 → producing 1 dollar worth of output from Industry 1 requires 0.3 dollar's worth of input from industry 1 itself

$d_1 = 12$ million E
$d_2 = 8$ million W

internal demand (usage) + external demand (final consumers) = total production if $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} E \\ W \end{pmatrix}$

internal demand

$= 0.3(12) + 0.2(8) = 5.2$ M of electricity

$= 0.1(12) + 0.4(8) = 4.4$ M of water

total output = internal demand + external demand

$\bar{X} = MX + D \qquad X = MX + D$

$$\text{and } M = \begin{array}{cc} & \begin{array}{cc} E & W \end{array} \\ \begin{array}{c} E \\ W \end{array} & \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \end{array}$$

$MX = D - \bar{X}$

$X[M] = D - X$

$$\underbrace{\frac{1}{\det} \times [M]^{-1} \times [M]}_{\text{identity matrix}} = D - X \times \frac{1}{\det} \times [M]^{-1}$$

$X[I-M]=D$
where I = identity matrix. and $D=\begin{pmatrix}12\\8\end{pmatrix}$

$X\underline{\underline{[I-M][I-M]^{-1}}}=[I-M]^{-1}D$

$X=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}0.3&0.2\\0.1&0.4\end{pmatrix}\right]^{-1}\begin{pmatrix}12\\8\end{pmatrix}$

but recall $\rightarrow (I-M)^{-1}=\frac{1}{|I-M|}\,Adj\,(I-M)$

$X=\frac{1}{0.40}\cdot\begin{bmatrix}0.6&-0.2\\-0.1&0.7\end{bmatrix}\begin{pmatrix}12\\8\end{pmatrix}$

$X=\begin{bmatrix}1.5&-0.5\\-0.25&1.75\end{bmatrix}\begin{bmatrix}12\\8\end{bmatrix}$

$X=\begin{pmatrix}22\\17\end{pmatrix}=\begin{pmatrix}X1\\X2\end{pmatrix}$

hence $X1=22$ million units of F
$X2=17$ million units of W

* In 5 years time.

$X=\begin{pmatrix}1.5&0.5\\0.25&1.75\end{pmatrix}\begin{pmatrix}24\\16\end{pmatrix}$

Example 2

outputs

inputs

$$
\begin{array}{c c}
 & \begin{array}{ccc} A & F & M \end{array} \\
\begin{array}{c} A \\ E \\ M \end{array} & \begin{bmatrix} 0.2 & 0 & 0.10 \\ 0.40 & 0.2 & 0.10 \\ 0 & 0.40 & 0.30 \end{bmatrix}
\end{array}
$$

final demand
A = 20 billion
E = 10 billion
M = 30 billio

$$X = \begin{bmatrix} A \\ E \\ M \end{bmatrix} = [1 - M]^{-1}$$

$$[1-M]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0 & 0.10 \\ 0.4 & 0.2 & 0.10 \\ 0 & 0.4 & 0.30 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0 & -0.4 & 0.7 \end{bmatrix}$$

$$[I-M]^{-1} = \begin{bmatrix} 0.8 & 0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0 & -0.4 & 0.7 \end{bmatrix}$$

* Row reduction to row-reduced echelon form (rref) of augmented form
$(A|I) \to (I|A^{-1})^{-1}$

* Minor cofactor method
e.g $M_{11} = \begin{vmatrix} 0.8 & -0.1 \\ -0.4 & 0.7 \end{vmatrix}$ $M_{12} = \begin{vmatrix} -0.4 & -0.1 \\ 0 & 0.7 \end{vmatrix}$

cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} M_{ij}$$

get adjoint of matrix

$$(I-M)^{-1} = \frac{1}{(I-M)^{-1}} Adj(I-M)$$

$$\begin{vmatrix} 0.8 & 0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0 & -0.4 & 0.7 \end{vmatrix} \to \text{demand matrix}$$

$M_{11} = \begin{vmatrix} 0.8 & -0.1 \\ -0.4 & 0.7 \end{vmatrix} = 0.52$ $M_{12} = \begin{vmatrix} -0.4 & -0.1 \\ 0 & 0.7 \end{vmatrix}$

$= -0.28$

$M_{13} = \begin{vmatrix} -0.4 & 0.8 \\ 0 & -0.4 \end{vmatrix} = 0.16$ $M_{21} = \begin{vmatrix} 0 & -0.1 \\ -0.4 & 0.7 \end{vmatrix}$

$= -0.04$

$M_{22} = \begin{vmatrix} 0.8 & -0.1 \\ 0 & 0.7 \end{vmatrix} = 0.56$ $M_{23} = \begin{vmatrix} 0.8 & 0 \\ 0 & -0.4 \end{vmatrix}$

$= -0.32$

$M_{31} = \begin{vmatrix} 0 & -0.1 \\ 0.8 & -0.1 \end{vmatrix}$ $M_{32} = \begin{vmatrix} 0.8 & -0.1 \\ -0.4 & -0.1 \end{vmatrix} = -0.12$

$M_{33} = \begin{vmatrix} 0.8 & 0 \\ -0.4 & 0.8 \end{vmatrix} = 0.64$

$\det = 0.8(0.52) + 0(-0.28) + -0.1(0.18) = 0.4$

$-0.4(-0.04) + 0.8(0.56) + -0.1(-0.32 = 0.4$

$0(0.08) + -0.4(-0.12) + 0.7(0.84) = 0.4$

$I - M = \det = 0.4$

$|I-M|^{-1} = \frac{1}{\det} \operatorname{Adj}(I-M)$

$\operatorname{Adj}(I-M) = C^T$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$C = \begin{vmatrix} 0.52 & 0.28 & 0.16 \\ 0.04 & 0.56 & 0.32 \\ 0.08 & 0.12 & 0.54 \end{vmatrix}$$

if $C = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $C^T = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

$$= \frac{1}{0.4} \begin{vmatrix} 0.52 & 0.04 & 0.08 \\ 0.28 & 0.56 & 0.12 \\ 0.16 & 0.32 & 0.64 \end{vmatrix}$$

$$(I-M)^{-1} = \begin{vmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{vmatrix}$$

$$X = (I-M)^{-1} D$$

$$\begin{vmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{vmatrix} \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 33 \\ 37 \\ 64 \end{pmatrix}$$

- A **transition matrix** is a constant square matrix p of order n such that the entry in the i^{th} row and j^{th} column indicates the probability of the system moving from the i^{th} state to the j^{th} state on the next observation or trial.
- Sum of the entries in each row of a transition matrix $= 1$

$$P_{ij} = \begin{array}{cc} & \text{next} \\ & \begin{array}{cc} \text{loss} & \text{profit} \end{array} \\ \text{Now} \begin{array}{c} \text{loss} \\ \text{profit} \end{array} & \begin{bmatrix} P_{LL} & P_{PP} \\ P_{PL} & P_{LP} \end{bmatrix} \end{array}$$

$$P_{LL} + P_{PL} = 1$$
$$P[\text{loss} + \text{profit}] = 1$$

let E_j and EK be events

$$P[E_j \to EK] = P_{jK}$$
$$P[E_j \to E\theta \to EK] = P_{jK}^2$$
$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$
$$P(Y) = \frac{P(X \cap Y)}{P(X|Y)} \to \text{zero}^{th} \text{ / initial probability}$$

retention probability of state 1

loss by 1 but gain as 2

$$
P_{ij} = \text{now} \rightarrow
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ i \\ n
\end{array}
\begin{array}{c}
\text{next} \\
\begin{array}{cccccc} 1 & 2 & j & 3 & & n \end{array} \\
\begin{bmatrix}
P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\
P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\
P_{31} & P_{32} & P_{33} & \cdots & P_{3n} \\
\cdots & P_{ij} & \cdots & \cdots & P_{in} \\
\cdots & \cdots & \cdots & \cdots & P_{nn}
\end{bmatrix}
\end{array}
$$

retention prob.

- The transitional matrix is either finite (countable states) or finite (several states)

A toothpaste company markets a product (*Brand A*) that currently has 10 % of the toothpaste market. The company hires a market research firm to estimate the percentage of the market it might acquire in the future if it launches an aggressive sales campaign. The research firm finds that if a person is using *brand A*, the probability is 0.8 that this person will buy it again when he or she runs out of toothpaste. On the other hand, a person using *another brand* will switch to brand A with a probability of 0.6 when he or she runs out of toothpaste.
Thus each toothpaste consumer can be considered to be in one of the two possible states:
$A =$ Uses brand A
$A' =$ Uses another brand

two brands

A = sensodyne
A' = any other brand

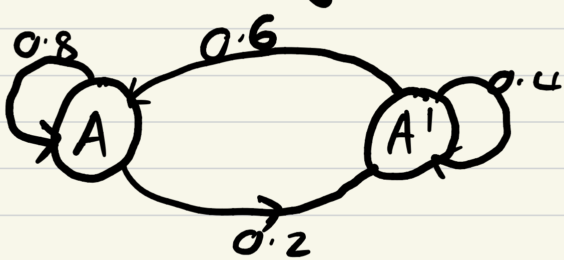
Initial distribution / zeroth dist^n ,

$$\begin{array}{cc} A & A^{-1} \end{array} \rightarrow \text{states}$$

$$s_0 = \underbrace{[0.1,\ 0.9]}_{\text{market share}}$$

$$P = \underbrace{\begin{array}{c|cc} & A & A' \\ \hline A & 0.8 & 0.2 \\ A' & 0.6 & 0.4 \end{array}}_{\substack{\text{transition probabilities} \\ (P)}}$$

→ transitional diagram to rep. the probabilities determined by the market research firm:



- To generate S_1 :

$$S_1 = S_0 P = [0.1 \quad 0.9] \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$S_1 = \begin{array}{c} \begin{array}{cc} A & A' \end{array} \\ [0.62,\ 0.38] \end{array}$$

$S_2 = ?$ $\qquad S_3 = ?$

$$S_2 = S_1 P = [0.62, 0.38]\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$= \underline{[0.724, 0.276]}$$

$$S_3 = S_2 P = [0.724 \;\; 0.276]\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$= \underline{[0.7448 \;\; 0.2552]}$$

$$S_4 = S_3 P = [0.7448 \;\; 0.2552]\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$= \underline{[0.74896 \;\; 0.25104]}$$

$$S_5 = S_4 P = [0.74896 \;\; 0.25104]\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$= \underline{[0.749792 \;\; 0.250208]}$$

$$S_6 = S_5 P = [0.749792 \;\; 0.250208]\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$= \underline{[0.7499584 \;\; 0.2500416]}$$

what about S_{60}: without doing it recursively

$S_1 = S_0 P$

$S_2 = S_1 P$: but $S_1 = S_0 P$

$= (S_0 P) P = S_0 P^2$

$$S_3 = S_2 P$$
$$= (S_0 P^2) P$$
$$= S_0 P^3$$

$$S_4 = S_3 P$$
$$= (S_0 P^3)(P)$$
$$= S_0 P^4$$

Therefore:

$$\boxed{\begin{aligned} S_{k-1} &= (S_{k-2} P) P \\ &= (S_0 P^{k-2}) P \\ &= S_0 P^{k-1} \\ S_k &= S_{k-1} P \\ &= (S_0 P^{k-1}) P \\ &= S_0 P^k \end{aligned}}$$

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$P^2 = P \times P$$
$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}$$

*what about S_{89}

$S_{89} = S_{88} P$ X

do $S_{89} = S_0 P^{89}$

- Steady state / long-run probabilities / invariant probabilities.

- let S be a 1×2 matrix i.e $S = [s_1, s_2]$ and P be the transition probability matrix then the steady states of the markov chain (m.c) is given by:

$$S = SP \rightarrow [s_1 \ s_2] = [s_1 \ s_2] \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

where S is the stationary distribution of the ms

$$\left.\begin{aligned} s_1 &= 0.8 s_1 + 0.6 s_2 \quad \ldots (1) \\ s_2 &= 0.2 s_1 + 0.4 s_2 \quad \ldots (2) \end{aligned}\right\} \text{homogenous solution}$$

$s_1 = 0$
$s_2 = 0$

to solve, add the condition of pmf:
$s_1 + s_2 = 1$

therefore:

$s_1 = 0.8s_1 + 0.6s_2 \quad \ldots \text{ (1)}$

$s_2 = 0.2s_1 + 0.4s_2 \quad \ldots \text{ (2)}$

$s_1 + s_2 = 1 \quad \ldots \text{ (3)}$

from (3) $\Rightarrow s_1 = 1 - s_2 \quad \ldots \text{ (4)}$

put (4) in (1) or (2)

$0.2s_1 = 0.6s_2$

$s_1 = 3s_2 \quad \ldots \text{ (5)}$

but (4) = (5)

$= 3s_2 - 1 = 1 - s_2$

$4s_2 = 1$

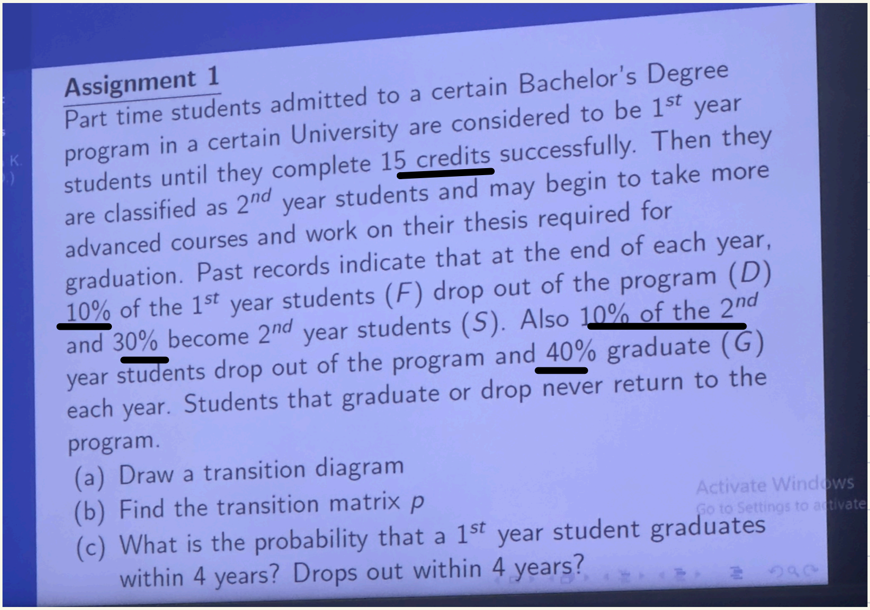
$\therefore \underline{s_2 = 0.25}$

and $s_1 = 1 - s_2$

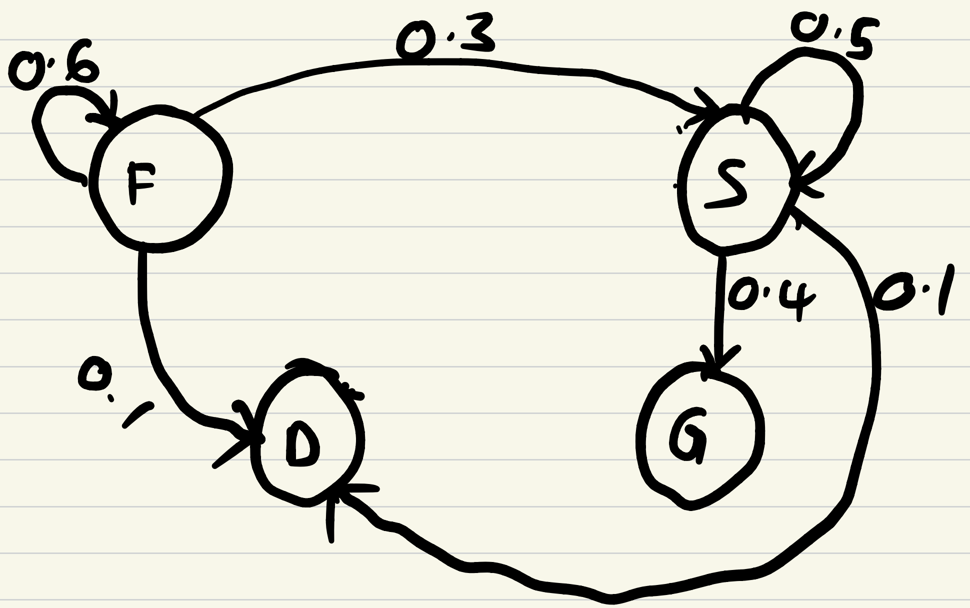
$= 1 - 0.25 = 0.75$

$\therefore \underline{s_1 = 0.75}$

$s = [s_1 \ \ s_2] = [0.75 \ \ 0.25]$



a) Transition diagram:



b) Transition matrix:

$$P = \begin{array}{c|cccc} & F & S & D & G \\ \hline F & 0.6 & 0.3 & 0.1 & 0 \\ S & 0 & 0.5 & 0.1 & 0.4 \\ D & 0 & 0 & 1 & 0 \\ G & 0 & 0 & 0 & 1 \end{array}$$

P

c) Prob that 1st year graduates within 4 years

$= P^4$

$$P^4 = \begin{bmatrix} 0.6 & 0.3 & 0.1 & 0 \\ 0 & 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^4$$

$P^2 = P \times P$

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 & 0 \\ 0 & 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.3 & 0.1 & 0 \\ 0 & 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.36 & 0.33 & 0.19 & 0.12 \\ 0 & 0.25 & 0.15 & 0.6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$P^2 \times P^2 = P^4$

$P^4 = ??$

$$\begin{bmatrix} 0.36 & 0.33 & 0.19 & 0.12 \\ 0 & 0.25 & 0.15 & 0.6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.36 & 0.33 & 0.19 & 0.12 \\ 0 & 0.25 & 0.15 & 0.6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^4 = \begin{array}{c} \\ F \\ S \\ D \\ G \end{array} \begin{array}{c} \begin{array}{cccc} F & S & D & G \end{array} \\ \begin{bmatrix} 0.1296 & 0.2013 & 0.3079 & 0.3612 \\ 0 & 0.0625 & 0.1875 & 0.75 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(i) Prob. that 1st Year graduate within 4 years

$FG = 0.3612$

$P_{F \to G} = 0.3612$

ii) Prob. that 1st Year student drops out within 4 years

$P_{F \to D}(4) = 0.3079$

Example 1

An insurance company found that on average, 23% of the drivers in a particular community who were involved in an accident one year were also involved in an accident the following year. They also found that only 11% of the drivers who were not involved in an accident one year were involved in an accident the following year. Use these percentages as empirical probabilities for the following:

(a) Draw a transition diagram to represent the given information

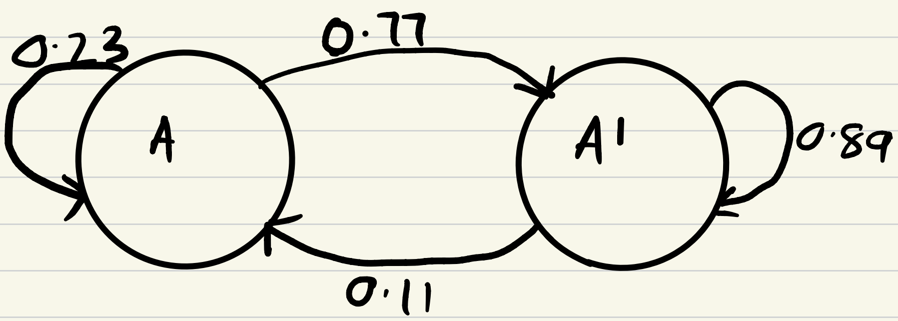
(b) Find the transition matrix p

(c) If 5% of the drivers in the community are involved in an accident this year, what is the probability that a driver chosen at random from the community will be involved in an accident next year? Year after next year?

→ let A be avg. of accidents and A', avg of no accidents

$$S_0 = \begin{array}{cc} A & A' \\ \end{array}$$
$$S_0 = \begin{bmatrix} 0.23 & 0.77 \end{bmatrix}$$

a) Transition Diagram



b) transition matrix :

$$\begin{array}{c|cc} & A & A' \\ \hline A & 0.23 & 0.77 \\ A' & 0.11 & 0.89 \end{array}$$

for this year $[0.05 \ 0.95] \rightarrow S_0$

for next year $= S_0 P$

$$= [0.05 \ 0.95] \begin{bmatrix} 0.23 & 0.77 \\ 0.11 & 0.89 \end{bmatrix}$$

$$S_1 = [0.116 \ 0.884]$$

$$S_2 = S_1 P$$

$$= [0.116 \ 0.884] \begin{bmatrix} 0.23 & 0.77 \\ 0.11 & 0.89 \end{bmatrix}$$

$$\begin{array}{cc} A & A' \end{array}$$

$$= [0.12392 \ 0.87608]$$

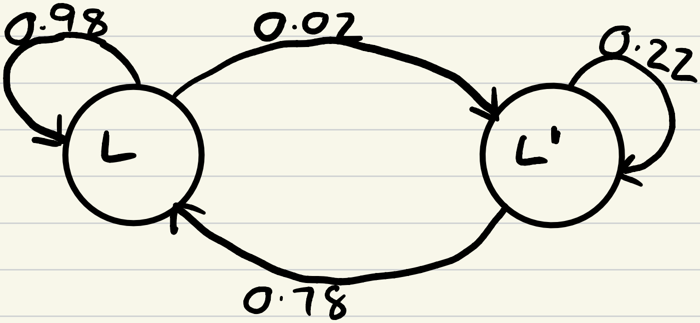
An insurance firm classifies drivers as low-risk if they are accident-free for 1 year. Past records indicate that 98% of the drivers in the low-risk category (L) one year will remain in that category the next year, and 78% of the drivers who are not in the low-risk (L') one year will be in the low-risk category the next year.

(a) Draw a transition diagram to represent the given information

(b) Find the transition matrix p

(c) If 90% of the drivers are in the low-risk this year, what is the probability that a driver chosen at random will be in the low-risk category next year? Year after next?

a) Transition diagram



b) Transition Matrix

$$= \begin{array}{c c} & \begin{array}{cc} L & L' \end{array} \\ \begin{array}{c} L \\ L' \end{array} & \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix} \end{array}$$

c) $S_0 = [0.9 \quad 0.1]$

$S_1 = S_0 P$

$$= [0.9 \quad 0.1] \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}$$

$S_1 = [0.96 \quad 0.04]$

$$S_2 = S_0 P = [0.96 \;\; 0.04] \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}$$

$$\begin{array}{cc} L & L' \end{array}$$

$$= [0.972 \;\; 0.028]$$

Higher Orders of Transition Probabilities and Classification of States of Markov chain

- State j is said to be accessible from state i if $P_{ij}^{n} > 0$ for some $n \geq 0$

- For example $P_{ij}^{(2)}$ is the prob. of moving from state E_j in 2 steps i.e
$P_{ij}^{(2)} = 2P[E_i \to E_k \to E_j]$

- $P_{ij}^{(n)} > 0$ implies that j is accessible from i iff starting in i, it is possible that the process will enter j. This is true since if j is not accessible from i

$P\{\text{ever}_{(j)} \text{ enter/start}_{(i)} \text{ in}\}$

$= P\{\bigcup_{n=0}^{\infty} \{X_n = j \mid X_0 = i\}\}$

$= \sum_{n=0}^{\infty} P\{X_n = j \mid X_0 = i\}$

$= \sum_{n=0}^{\infty} P_{ij}^{(n)} = 0$

- Since $P_{ij}^{(n)} \geq 0$, it is not always possible to move from state to state
- Two states i and j that are accessible to each other are said to communicate and its denoted by $i \longleftrightarrow j$.
- Any state communicates with itself since $P_{ii}^{(0)} = P\{X_0 = i \mid X_0 = i\} = 1$

- The state Ei (or E_j) is said to be accessible (or accessible) from state E_j (or E_i) if there exists some positive integer n such that $P_{ij}^{(n)} > 0$

- For $n = 0$, $P_{ii}^{(0)} = 1$, $P_{ij}^{(0)} = 0$ while $i \neq j$

- If $E_i \rightarrow E_K$ and $E_K \rightarrow E_j$, then there exist some positive integers m and n s h that: $P_{iK}^{(m)} > 0$, $P_{Kj}^{(n)} > 0$, therefore $P_{ij} > 0 \Rightarrow E_j$ is readable from i.

Three properties of Communication

(i) State i communicates with state i, $i \geq 0$

(ii) If state i communicates from j, then j communicates with i; i.e $i \longleftarrow j$

(iii) If $i \leftrightarrow k$ and $k \leftrightarrow j$ then $i \leftrightarrow j$

- Communication is a class property i.e two states that communicate are in the same class
- This follows from (i), (ii), (iii). The M.C is irreductible if there is only one class i.e all states communicate

Example

1. Consider the M.C consisting of 3 states 0, 1, 2 with transition prob.

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

$$0 \to 1 \to 2 \to 1 \to 0$$

Therefore the MC is irreductible since all states communicate with each other.

2. Consider the MC with 4 states:

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- The classes of this M.C are $\{0,1\}$, $\{2\}$ and $\{3\}$
- State 2 is not reachable from state 0 and 1
- State 3 is an abscibing state since once entered cannot be left. i.e $P_{33}=1$

- For any state i, let f_i denote the prob. that starting in state i, the process will ever re-enter state i.
- State i is reccurent if $f_i=1$ and transient if $f_i<1$.
- Therefore state i is recurrent iff, starting in state i, the expected number of time periods that the process is in state i is infinite

- Let $I_n \begin{cases} 1, \text{ if } X_n=0 \\ 0, \text{ if } X_n \neq 1 \end{cases}$

we have $\sum_{n=0}^{\infty} X_n$ represents the number of periods that the process is in state i.

- Besides $E[\sum_{n=0}^{\infty} I_n | X_0=i] = \sum_{n=0}^{\infty} E[I_n | X_0=i]$

$$= \sum_{n=0}^{\infty} P[X_n=i | X_0=i] = \sum_{n=0}^{\infty} P_{ii}^{n}$$

i.e

① State i is reccurent if $\sum_{n=0}^{\infty} P_{ii}^{n} = \infty$

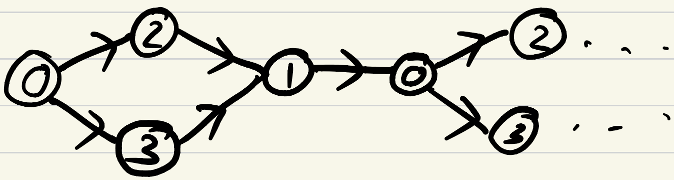
② State i is transient if $\sum_{n=0}^{\infty} P_{ii}^{n} < \infty$

Example:

- Let the MC, consisting of 4 states have P_{ij} given by:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Which states are reccurrent:



All states communicate and since this is a finite chain, all states are reccurent

- Consider the MC with states 0, 1, 2, 3, 4 and p_{ij}

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

Determine the recurrent and transient state:

$$0 \to 1 \to 0 \to 1 \qquad 2 \to 3 \to 2 \to 3$$

Classes: $\{0, 1\}$, $\{2, 3\}$, $\{4\}$

First 2 classes are recurrent.

<u>Relationship between p_{jj} and f_{jj}</u>

- $p_{jk}^{(n)}$ is the prob. of moving from state E_j to state E_k in n steps regardless of the number of entrances into state E_k prior to n
- Define $f_{jk}^{(n)}$ to be the prob. of entering state E_k from E_j in n steps for the first time

- Let the first return to state E_j take r steps.
- In this case, its the prob. will be $f_{jj}^{(r)}$. The prob. of the remaining $(n-r)$ steps will be $p_{jj}^{(n-r)}$

$$p_{ij}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} \cdot p_{jj}^{(n-r)}, \quad n \geq 1 \dots (1)$$

- In terms of generating functions, define:

$$P(s) = \sum_{n=0}^{\infty} p_{jj}^{(n)} s^n$$

- From (1) we have:

$$= \sum_{n=1}^{\infty} p_{jj}^{n} s^n = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} f_{jj}^{(r)} p_{jj}^{(n-r)} s^n$$

$$= \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} (f_{jj}^{(s)} s^{(r)})(p_{jj}^{(n-r)} s^{(n-r)})$$

$$= \sum_{n=1}^{\infty} p_{jj}^{n} s^n = \sum_{r=0}^{\infty} (p_{ij}^{(n-r)} s^{(n-r)})$$

<u>Recall</u>: $P(s) = \sum_{n=0}^{\infty} p_{jj}^{(n)} s^n$

$$P(s) - 1 = F(s)(P(s))$$

make $P(s)$ subject of the formula

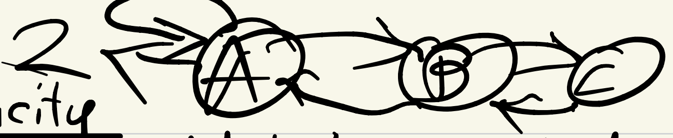
$\therefore P(s) - F(s)P(s) = 1$

$$P(s)[1 - F(s)] = 1 \rightarrow P(s) = \frac{1}{1 - F(s)}$$

Persistent and Transient States

- A state E_j is said to be persistent or recurrent if $F_{jj} = \sum_n f_{jj}^{(n)} = 1$
- A state E_j is transient (non-recurrent) if $F_{jj} = \sum_n f_{jj}^{(n)} < 1$
- If $F_{jj} = \sum_n f_{jj}^{(n)} = 1$, then f_{jj} is a probability distribution formula.
- Generally if $F_{jk} = \sum_n f_{jk}^{(n)} = 1$, then $f_{jk}^{(n)}$ is a probability distribution called <u>First passage distribution</u>
- The mean recurrence time for state E_j is given by

$P_j = \sum_n n f_{jj}^{(n)}$, if $P_j = \infty$, then E_j is null and non-null if $P_j < \infty$



$P_{AA} = 1$

Periodicity

- A state is said to be a period t if t is the GCD of $\{n; P_{jj}^{(n)} > 0\}$
- If $t = 1$ then the state is said to be <u>aperiodic</u>

Ergodicity

communicate
aperiodic
recurrent

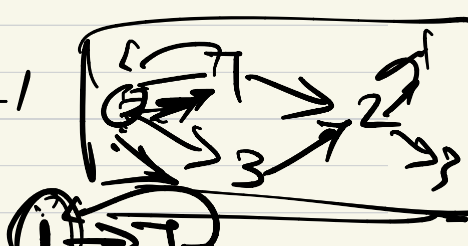
- A state that is persistent, non-null and aperiodic is said to be <u>ergodic</u>

Absorbing States

- Is such that once entered cannot be left e.g $P_{KK} = 1 \rightarrow E_K$ is an absorbing state.

Closed Sets

- A set C is said to be closed if no states outside C can be reached from any state in C i.e a set C is closed if each state in C communicates only with other states in C.
- So if $E_j \in C$ and $E_K \notin C \rightarrow P_{jK} = 0$
  $\rightarrow P_{jK}^{(n)} = 0$, for $n \geq 1$
- If $E_j, E_K \in C$, then $P_{jK} = 1$
  $\rightarrow \sum_K P_{jK}^{(n)} = 1$ for $n = 1$

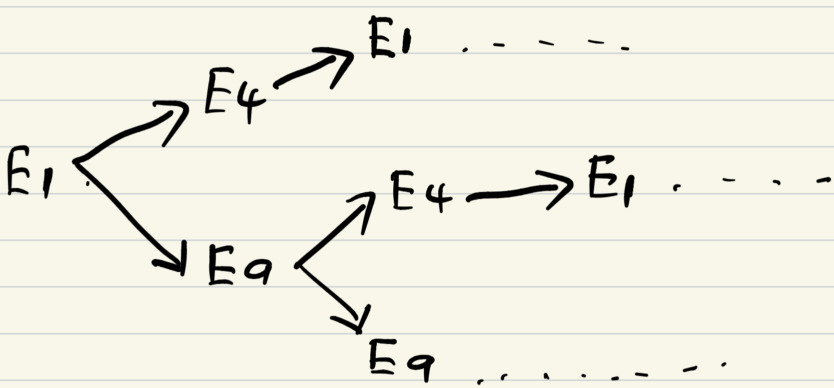


Example

1.

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \end{bmatrix} \end{array}$$

Where * entries showing $P_{jk} > 0$



$\rightarrow \{E_1, E_4\}$ form a <u>closed set</u> ($E_1 \longleftrightarrow E_4$)

$E_3 \nearrow E_8 \rightarrow E_3 \rightarrow E_8$

$\rightarrow \{E_3, E_8\}$ form a <u>closed set</u> ($E_3 \longleftrightarrow E_8$)

$E_2 \to E_2$

$E_2 \to E_3 \to E_8 \to E_3$

$E_2 \to E_5 \to E_5$

$E_2 \to E_9 \to E_4 \to E_1$

$E_9 \to E_9$

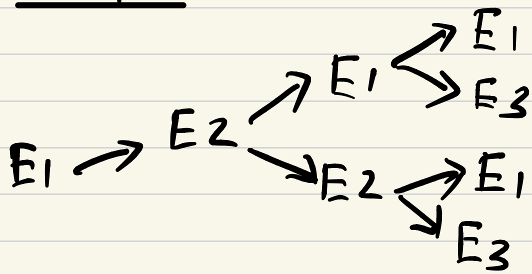
- Since $\{E_3, E_8\}$ and $\{E_1, E_4\}$ form closed sets, therefore $\{E_1, E_2, E_3, E_4, E_8\}$ cannot be a closed set by definition
- $\{E_5, E_5\}$ is a closed set ($E_5 \leftrightarrow E_5$)
- State 5 is an <u>absorbing state</u>

<u>**Theorem**</u>: All states of an irreducible MC are of the same type

2. $$P = \begin{array}{c|ccc} & E_1 & E_2 & E_3 \\ \hline E_1 & 0 & \frac{1}{2} & \frac{1}{2} \\ E_2 & \frac{1}{2} & 0 & \frac{1}{2} \\ E_3 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

Required: Classify the number of states

<u>Solution</u>



- The set $\{E_1, E_2, E_3\}$ form an irreductible chain. Therefore, all the states E_1, E_2, E_3 have the same properties. Therefore, only one state can be considered to represent the others, (say E_1)

→ Probability of moving from E_1 to E_1 in two steps

$f_{11}^{(2)} = P[E_1 \xrightarrow{\frac{1}{2}} E_2 \xrightarrow{\frac{1}{2}} E_1]$ or

$P[E_1 \xrightarrow{\frac{1}{2}} E_2 \xrightarrow{\frac{1}{2}} E_1]$

$= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

→ Probability of moving from E_1 to E_1 in three steps

$$f_{11}^{(3)} = P[E_1 \to E_2 \to E_3 \to E_1] \text{ or}$$

$$P[E_1 \to E_3 \to E_2 \to E_1]$$

$$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{8} + \frac{1}{8}$$

$$= \left(\frac{1}{2}\right)^2$$

→ for 4 steps:

$$f_{11}^{(4)} = \left(\frac{1}{2}\right)^3, \quad f_{11}^{(5)} = \left(\frac{1}{2}\right)^4$$

therefore:

$$\boxed{f_{11}^{(n)} = \left(\frac{1}{2}\right)^{n-1}}$$

$$F_{jj} = \sum_n f_{jj}^{(n)}$$

$$F_{11} = \sum_{n=2}^{\infty} f_{11}^{(n)}$$

$$F_{11} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

$$F_{11} = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \cdots\cdots$$

recall G.P $\rightarrow S_\infty = \frac{a}{1-r}$

$$r = \left(\frac{1}{2}\right)^3 \div \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

- Therefore, E_1 is persistent and therefore E_2 and E_3 are also persistent

- GCD of $\{2, 3, 4 \ldots\}$ is 1. E_1 is aperiodic i.e it has a period of 1

$\therefore$

$$P_j = \sum_n n f_{jj}^{(n)}$$

$$P_1 = \sum_n^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$

$$= 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 \dots$$

$$= 1 + 2(1/2) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \dots - 1$$

$$= \frac{1}{\left(1-\frac{1}{2}\right)^2} - 1$$

$$= 4 - 1 = 3 < \infty$$

- The states E_1, E_2 and E_3 are non-null

$\therefore$ E_1, E_2 and E_3 are ergodic states since they're persistent, aperiodic and non-null.

- Hence, the MC is irreducible & ergodic

Recall: $f'(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$

$\Rightarrow f(x) = x + x^2 + x^3 + x^4 + \dots$

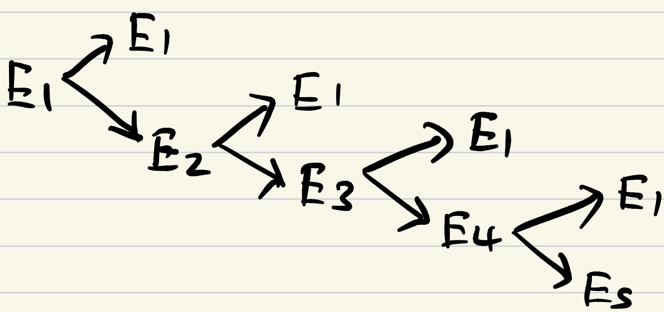
$$= \frac{x}{1-x} + C$$

if $k(x) = \dfrac{f(x)}{g(x)} = k'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Example

Classify the states of this infinite chain

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & \cdots & E_n \end{array} \\
\begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \\ E_n \end{array} &
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots & \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots & \\
\vdots & & & & & & & \\
\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots &
\end{bmatrix}
\end{array}
$$



$f_{11}^{(1)} = P[E_1 \to E_1] = \frac{1}{2}$

$$f_{11}^{(2)} = P[E_1 \to E_2 \to E_1] = \left(\frac{1}{2}\right)^2$$

$$f_{11}^{(3)} = P[E_1 \to E_2 \to E_3 \to E_1] = \left(\frac{1}{2}\right)^3$$

$$\boxed{= f_{11}^{(n)} = \frac{1}{2}^{(n)}}$$

$$F_{11} = \sum_n f_{11}^{(n)} = \sum_{n=1}^{\infty} (1/2)^n$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 .$$

$$= \frac{1/2}{1 - 1/2} = 1$$

- All states are persistent
- All states are aperiodic since its HCF , of $\{1, 2, 3, \dots\} = 1$

$$P_1 = \sum_n n f_{11}^{(n)}$$

$$= \sum_{n=1}^{\infty} n \left(\tfrac{1}{2}\right)^n$$

$$= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^3 + \cdots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{1}{2}(4) = 2 < \infty$$

- Thus all the states are non-null. Therefore, all the states are ergodic, since they're all non-null, persistent and aperiodic.

- The chain is an irreductible and ergodic markov chain

Question 1

- Classify the states of the following MC

1.

$$\begin{array}{c|c} & \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$E_1 \to E_4 \to E_3 \to E_1$

$E_3 \to E_2 \to E_4$

$E_4 \to E_4$

② $\begin{array}{c c} & \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$

③ $\begin{array}{c c} & \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$

④ $\begin{array}{c c} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

⑤ $\begin{array}{c c} & \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

Linear Optimization (LP)

- LP is a mathematical procedule for minimizing or maximizing a linear function of sevelal vaiables, subject to a finite no. of linear restrictions on these variables.

Components of LP

1. Objective function

- What is to be maximized or minimized

2. Decision variables

- Controllable inputs
- Denoted by X_j's i.e $j = 1, 2, 3$
- $X_{ij} \geq 0$ (non-negative) → non-negativity condition
- Eg units produced : hours/duration, no. of pple/machine

3. Parameters

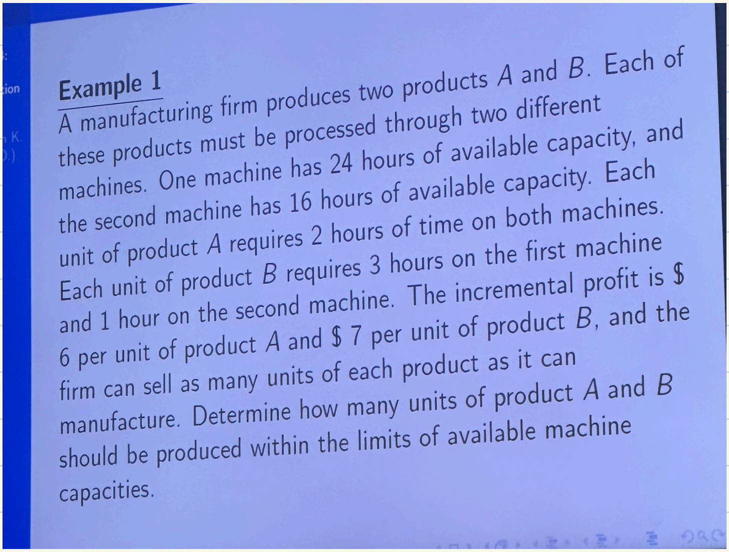
- Uncontrollable inputs
- Co-efficients of X_j's

4. Constiaints

- Restrictions/requirements faced while trying to achieve an objective
- Example: limited raw materials
  limited supply/demand

Problems facing LP

1. Formulation
2. Deciding on x_1 and x_2 to max. profits
3. Optimal solution should be cornered
   → if not could lead to an ∞ no. of solutions



Example 1

A manufacturing firm produces two products A and B. Each of these products must be processed through two different machines. One machine has 24 hours of available capacity, and the second machine has 16 hours of available capacity. Each unit of product A requires 2 hours of time on both machines. Each unit of product B requires 3 hours on the first machine and 1 hour on the second machine. The incremental profit is \$ 6 per unit of product A and \$ 7 per unit of product B, and the firm can sell as many units of each product as it can manufacture. Determine how many units of product A and B should be produced within the limits of available machine capacities.

24, 16 → constraints
2, 3, 1 →
6, 7 →

Formulate the Linear program:

Profit

A	B
6\$	7\$

Constraints: machine capacities.

Machine 1 ≤ 24

Machine 2 ≤ 16

let x_1 = no. of units of A produced

x_2 = no. of units of B produced

		A	B	Capacity
Machine	1	2	3	24
	2	2	1	16
profit maximization		6	7	

max: $P = 6x_1 + 7x_2$

subject to: $2x_1 + 3x_2 \leq 24$

$2x_1 + x_2 \leq 16$

$x_1 \geq 0, x_2 \geq 0$ (non-negativity)

① Formulation

② Solution - a) Graphical method.

b) Simplex method.

a) Graphical Solution

- Represented constraints using straight lines
- Identify the solution space (feasible region)
- Choose optimal solution located at one of the corners (vertices) of the feasible region

* Corner-point solution criterion

① $2x_1 + 3x_2 = 24$

x_1	0	12
x_2	8	0

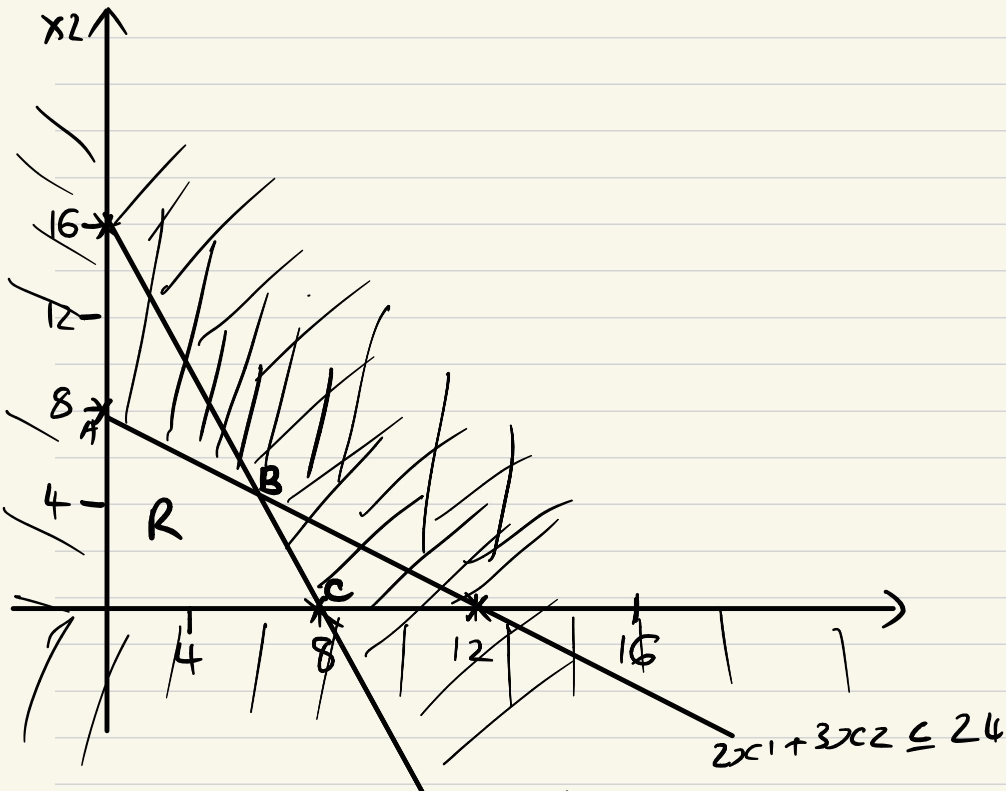
(0,8) (12,0)

② $2x_1 + x_2 = 16$

x_1	0	8
x_2	16	0

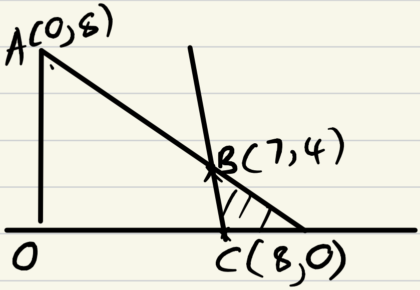
$(0, 16)$ $(8, 0)$

Sketch: $R \rightarrow$ solution space



Let A, B, C to be the corner of the solution space

$A(0,8)$ $B(7,4)$ $C(8,0)$



vertex	coordinates x_1, x_2	objective function $P = 6x_1 + 7x_2$
O	(0,0)	0
A	(0,8)	56
B	(7,4)	70
C	(8,0)	48

Optimal solution

$P \geq 70$

$x_1 = 7, x_2 = 4$

Simplex Method

- Used when you have 2 or more decision variables.
- Uses row-reduced echelon form.
- The equations should be written in standard form:

In standard form:

1. The objective function is of the maximization or minimization type. Minimization of a function is equivalent to the maximization of the negative expression of this function.
2. All constraints must be stated/expressed as equations except the non-negativity constraints.
3. The right hand sides of the constraints are non-negative.
4. All the decision variables are restricted to non-negative values.

Type I ($\leq$) Inequalities

- In ($\leq$) constraints, the RHS represents the limit on the availability of a resource in which case the LHS would represent the usage of this limited resource by the activities (variables) of the model.
- For each ($\leq$) constraint, a non-negative variable, called a **slack variable** is added to the LHS of the constraint.
- **Slack variable** serves the function by balancing the two sides of the equation. It represents unused amount of the resource.

Example 1

A manufacturing firm produces two products A and B. Each of these products must be processed through two different machines. One machine has 24 hours of available capacity, and the second machine has 16 hours of available capacity. Each unit of product A requires 2 hours of time on both machines. Each unit of product B requires 3 hours on the first machine and 1 hour on the second machine. The incremental profit is $ 6 per unit of product A and $ 7 per unit of product B, and the firm can sell as many units of each product as it can manufacture. Determine how many units of product A and B should be produced within the limits of available machine capacities.

		A	B	Capacity
Machine	1	2	3	24
	2	2	1	16
profit maximization		6	7	

x_1 = no. of units of A
x_2 = no. of units of B

Max: $P = 6x_1 + 7x_2$

subject to: $2x_1 + 3x_2 \leq 24$

$2x_1 + x_2 \leq 16$

$x_1, x_2 \geq 0$

$S_{①} \rightarrow$ subscript rep. equation

standard form:

max: $P = 6x_1 + 7x_2 + 0s_1 + 0s_2$

$P - 6x_1 - 7x_2 - 0s_1 - 0s_2 = 0$

$2x_1 + 3x_2 + s_1 = 24$

$2x_1 + x_2 + s_2 = 16$

Maximize: $Z = 5x_1 + 6x_2$

subject to: $3x_1 + 2x_2 \leq 120$

$4x_1 + 6x_2 \leq 260$

$x_1, x_2 \geq 0$ $\Rightarrow$ non-negativity equation

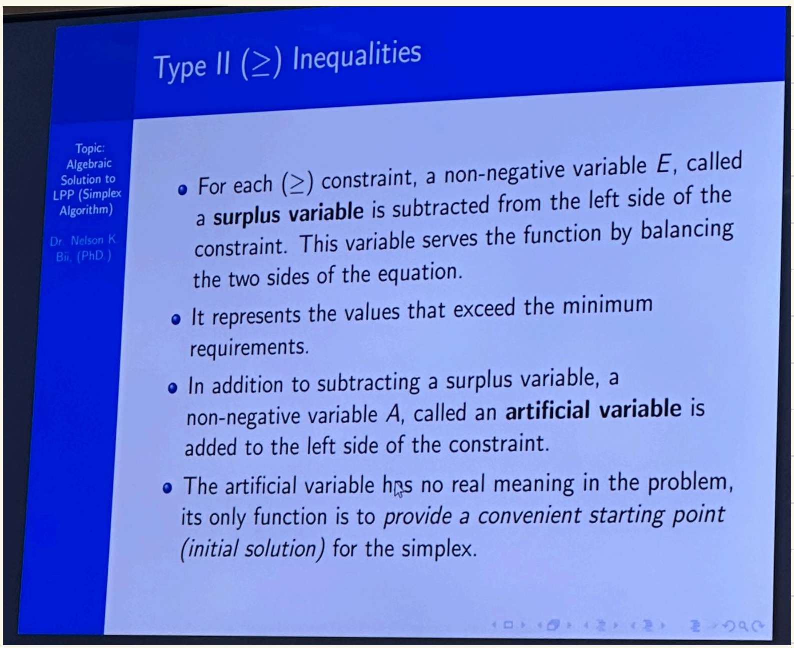
In standard form

Maximize: $Z = 5x_1 + 6x_2 + 0s_1 + 0s_2$

$Z - 5x_1 - 6x_2 - 0s_1 - 0s_2 = 0$

$3x_1 + 2x_2 + 0s_1 = 120$

$4x_1 + 6x_2 + 0s_2 = 260$



$\leq \Rightarrow$ no surplus variable

Assume in Example 1:

x_1 = no. of units of A

x_2 = no. of units of B

$$\max: P = 6x_1 + 7x_2$$

subject to: $2x_1 + 3x_2 \leq 24$

$2x_1 + x_2 \leq 16$

$x_1, x_2 \geq 0$

$$\max: P = 6x_1 + 7x_2 + 0S_1 + 0S_2 + 0E_3 + 0 \cdot A_3$$

$$P - 6x_1 - 7x_2 - 0S_1 - 0S_2 - 0E_3 - 0A_3 = 0$$

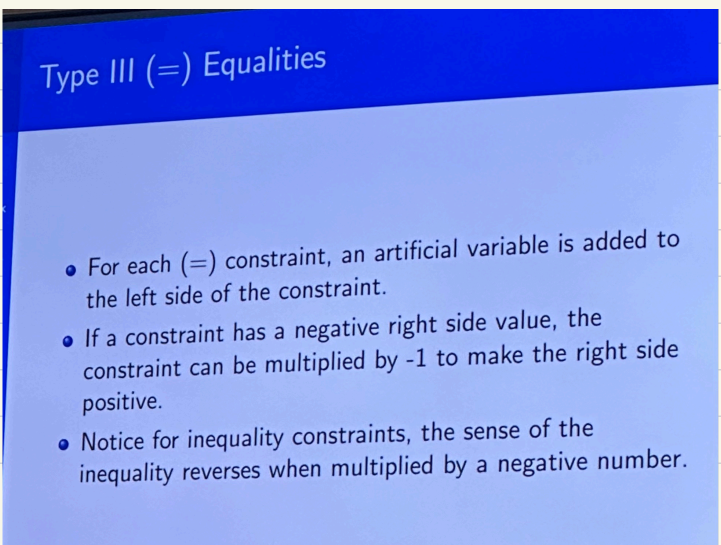
$$2x_1 + 3x_2 + S_1 = 24$$

$$2x_1 + x_2 + S_2 = 16$$

$$x_1 + x_2 - E_3 + A_3 = 6$$

$$x_1, x_2, S_1, S_2, E_3, A_3 \geq 0$$

*Subscript rep. equation number



Example:

Maximize $Z = 3x_1 + 2x_2 + x_3$

subject to : $2x_1 - 3x_2 \leq 3$

$x_1 + 2x_2 + 3x_3 \geq 5$

$3x_1 + 2x_3 \leq 0$

$x_1, x_2, x_3 \geq 0$

Solution

max $Z = 3x_1 + 2x_2 + x_3 + 0S_1 + 0E_2 + 0A_2 + 0S_3$

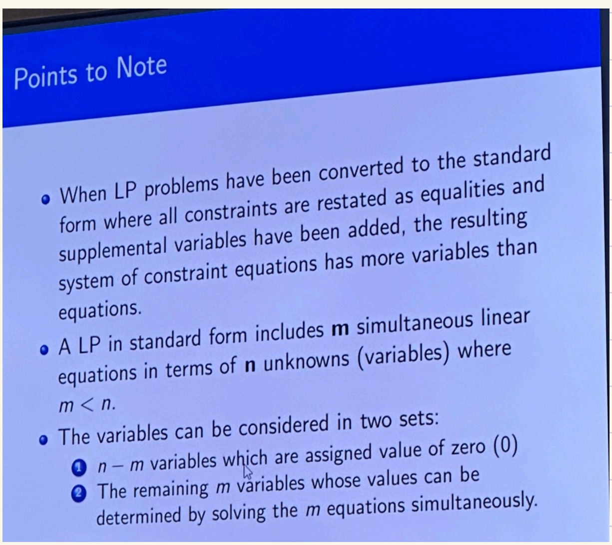
$Z - 3x_1 - 2x_2 - x_3 - 0S_1 - 0E_2 - 0A_2 - 0S_3$

st: $2x_1 - 3x_2 + S_1 = 3$

$x_1 + 2x_2 + 3x_3 - E_2 + A_2 = 5$

$3x_1 + 2x_3 + S_3 = 2$

$x_1, x_2, S_1, E_2, A_2, S_3 \geq 0$



There are 3 types of solutions:
1. Feasible solution
2. Basic solution
3. Feasible basic solution.

1. Feasible Solution * satisfies non-negative constraints
- It is any set of values for the n variables which satisfies the standard equations & the non-negative constraints.

- For standard equations:

$$2x_1 + 3x_2 + s_1 = 24 \quad \text{---(i)} \quad \text{constraint} \to 24$$
$$2x_1 + x_2 + s_2 = 16 \quad \text{---(ii)} \quad \text{constraint} \to 16$$

- Suppose $x_1 = 5$ and $x_2 = 2$, then $s_1 = 8$ and $s_2 = 4$
- One feasible solution for the system would be: $x_1 = 5$, $x_2 = 2$, $s_1 = 8$ and $s_2 = 4$.

2. Basic solution * $(n-m) = 0$ & basic variables rem (m) are solved
- Any solution by setting $(n-m)$ variables equal to zero(0) & solving the system of equations for the values of the remaining m variables
- The m variables are called basic variables
- The remaining values that are assigned value (zero) are called non-basic variables.

→ example: $2x_1 + 3x_2 + s_1 = 24$ (machine 1)
$2x_1 + x_2 + s_2 = 16$ (machine 2)

→ no. of equations = 2 = m
→ no. of variables = 4 = n
*(2*4) system*

→ Suppose x_1 and s_1 are set to zero:

$2(0) + 3x_2 + 0 = 24$ (machine 1)
$3x_2 = 24$
$x_2 = 8$

$2(0) + x_2 + s_2 = 16$ (machine 2)
$8 + s_2 = 16 \Rightarrow s_2 = 8$

→ The basic solution for the system is:
$x_1 = 0$, $s_1 = 0$, $x_2 = 8$, $s_2 = 8$ where:

x_1, s_1 are non-basic variables
x_2, s_2 are basic variables.

→ You can try setting x_1 and s_2 to 0, and get the basic solution.

3. Basic Feasible Solution

- A basic solution is a solution which satisfies the non-negativity constraint/restriction.

* no of basic variables = no of equations

Example:

Machine:

$$2x_1 + 3x_2 + s_1 = 24 \quad -- (i)$$
$$2x_1 + x_2 + s_2 = 16 \quad -- (ii)$$

- Setting x_1 and x_2 equal to zero gives $s_1 = 24$, $s_2 = 16$ $\therefore$ $x_1 = 0$, $x_2 = 0$, $s_1 = 24$, $s_2 = 16$ is a basic solution
- Since all four variables satisfy the non-negativity constraint/restriction, it is a basic feasible solution

Note:
- In the initial simplex tableau, the initial (first) basic variables are typically the slack, surplus and artificial variables (s_1, E_2 for example)
- The initial basic variables are the original decision variables of the problem.

Example:

Maximize $Z = 5x_1 + 6x_2$

subject to: $3x_1 + 2x_2 \leq 120$

$4x_1 + 6x_2 \leq 260$

$x_1, x_2 \geq 0$

Solution

set $x_1 = x_2 = 0$ } non-basic variables

$s_1 = 120$ } basic variables

$s_2 = 260$

$$
\left.\begin{aligned}
Z &= 5x_1 + 6x_2 + 0s_1 + 0s_2 \\
Z - 5x_1 - 6x_2 - 0s_1 - 0s_2 &= 0 \\
3x_1 + 2x_2 + s_1 &= 120 \\
4x_1 + 6x_2 + s_2 &= 260 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}\right\} \text{standardized solution}
$$

1

Basic variables	Z	x_1	x_2	s_1	s_2	RHS of solution	
Z	1	-5	-6	0	0	0	→ R_0
s_1	0	3	2	1	0	120	
s_2	0	4	6	0	1	260	

We want to maximize for solution to be optimal, thus all variables should be greater than 0 (choose a non-basic variable to enter as basic variable)

R_0 → basic solution / row zero

- x_2 enters in the next iteration since it has the most negative co-efficient.
- Choose where x_2 can enter; either through <u>S1 or S2</u>
- Can be computed through $= \frac{RHS}{\text{coefficients of } x_1}$

ratios $= \frac{120}{2} = 60$, $\frac{260}{6} = 43.3$

- Choose smaller ratio to leave the solution

(2)

Basic variables	z	x_1	x_2	s1	s2	solution RHS
z	1	-5	-6	0	0	0
S1	0	3	2	1	0	120
x_2	0	4	6	0	1	260

make into 1 by dividing by 6

	Basic variables	z	x_1	x_2	s1	s2	solution RHS
R0	z	1	-5	-6	0	0	0
R1	S1	0	3	2	1	0	120
R2	x_2	0	4	6	0	1	260

$R_2 : R2 \div 6$

make this into 1 and the rest of the column 0

Basic variables	Z	x_1	x_2	S_1	S_2	solution RHS
Z	1	-5	-6	0	0	0
S_1	0	3	2	1	0	120
x_2	0	4/6	1	0	$\frac{1}{6}$	$\frac{260}{6}$

$R_0 : R_0 + 6R_2$
$R_1 : R_1 - 2R_2$

Basic variables	Z	x_1	x_2	S_1	S_2	solution RHS
Z	1	-1	0	0	1	260
S_1	0	5/3	0	1	-1/3	100/3
x_2	0	4/6	1	0	1/6	260/3

→ x_1 enters since it has the most negative coefficient

→ what leaves: $\frac{RHS}{\text{co-efficient of } X}$

$$= \left(\frac{100}{3} \div \frac{5}{3}, \frac{260}{6} \div \frac{4}{6}\right) = \min(20, 65)$$

min ratio = 20 hence S_1 leaves

make cell 5/3 into 1

Basic variables	Z	x_1	x_2	S_1	S_2	solution RHS
Z	1	-1	0	0	1	260
x_1	0	5/3	0	1	-1/3	100/3
x_2	0	4/6	1	0	1/6	260/3

$R_1 : R_1 \div \frac{5}{3}$

Basic variables	Z	x_1	x_2	S_1	S_2	solution RHS	
Z	1	-1	0	0	1	260	$R_0: R_0 + R_1$
x_1	0	1	0	3/5	$-\frac{1}{3}$	20	
x_2	0	4/6	1	0	1/6	260/3	$R_2: R_2 - \frac{4}{6} R_1$

make the other cells in the column x_1 that is not 1 into 0

Basic variables	Z	x_1	x_2	S_1	S_2	solution RHS	
Z	1	0	0	3/5	4/5	280	
x_1	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	20	
x_2	0	0	1	$-\frac{2}{5}$	$\frac{3}{10}$	$\frac{180}{6}$	

The solution is optimal since we do not have negative co-efficients in R_0

$Z = 280$ $\quad S_1 = 0$
$x_1 = 20$ $\quad S_2 = 0$
$x_2 = 30$

$Z = 5x_1 + 6x_2$
$280 = 5(20) + 6(30)$
$280 = 280$

Hence solution is optimal

Example 2

Max : $Z = 40x_1 + 30x_2$

s/t: $x_1 + x_2 \leq 12$

$2x_1 + x_2 \leq 16$

$x_1, x_2 \geq 0$

Solve using simplex method:

Solution

1. Express in standard form:

Max: $Z = 40x_1 + 30x_2 + 0s_1 + 0s_2$

$Z - 40x_1 - 30x_2 - 0s_1 - 0s_2 = 0$

$x_1 + x_2 + s_1 = 12$

$2x_1 + x_2 + s_2 = 16$

$x_1, x_2, s_1, s_2 \geq 0$

2. Generate basic feasible solution (bfs)

decision

set : $x_1 = x_2 = 0$ → non-basic variables

$s_1 = 12, s_2 = 16$ → basic variables

slack

Basic variables	Z	x_1	x_2	s_1	s_2	RHS
Z	1	-40	-30	0	0	0
s_1	0	1	1	1	0	12
s_2	0	2	1	1	0	16

$R_2 : R_2/2$

pivot column

not optimal since we have negative co-efficients.

3. x_1 enters the solution (most negative decision variable

4 Ratios:

$$\min\left(\overset{S1}{\frac{12}{1}}, \overset{S2}{\frac{16}{1}}\right) = 8$$

hence S2 leaves

Basic variables	Z	x_1	x_2	s_1	s_2	RHS	RO
Z	1	-40	-30	0	0	0	$R_0: R_0 + 40R_2$
S1	0	1	1	1	0	12	$R_1: R_1 - R_2$
x_1	0	1	1/2	0	1/2	8	

B·V	Z	X1	X2	S1	S2	RHS	R.O
Z	1	0	-10	0	20	320	R_0
S_1	0	0	1/2	1	$-\frac{1}{2}$	4	$R_1: R_1 \times 2$
x_1	0	1	1/2	0	1/2	8	

pivot column

5. X2 enters:

$$\min\left(\frac{4}{\frac{1}{2}}, \frac{8}{\frac{1}{2}}\right) = 8$$

S1 leaves

allocation of basic variables

B·V	Z	x_1	x_2	S1	S2	RHS	R·O
Z	1	0	-10	0	20	320	$R_0: R_0 + 10R_1$
x_1	0	0	1	2	-1	8	
x_2	0	1	1/2	0	1/2	8	$R_2: 2R_2 - 1R_1$

B·V	Z	x_1	x_2	S1	S2	RHS	R·O
Z	1	0	0	20	10	400	
x_1	0	0	1	2	-1	8	
x_2	0	1	0	-2	2	8	

The solution is optimal since all the decision variables (x_1, x_2) are positive

$Z = 400$ → maximum possible values

$x_2 = 8$

$x_1 = 4$

$S_1 = 0$

$S_2 = 0$

Exercise

Max: $P = 6x_1 + 3x_2$

s/t: $-2x_1 + 3x_2 \leq 9$

$-x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0$

solution

1· Standard form: $P = 6x_1 + 3x_2 + 0s_1 + 0s_2$

$P - 6x_1 - 3x_2 - 0s_1 - 0s_2 = 0$

$-2x_1 + 3x_2 + s_1 = 9$

$-x_1 + 3x_2 + s_2 = 12$

2· Generate bfs

$x_1, x_2 = 0$ → non basic variables

$\left.\begin{array}{l} s_1 = 9 \\ s_2 = 12 \end{array}\right\}$ basic variables

B·V	P	x_1	x_2	s_1	s_2	RHS	Ro
P	1	-6	-3	0	0	0	
s_1	0	-2	3	1	0	9	
s_2	0	-1	3	0	1	12	Ro: 2Ro − R1

$\min\left(\frac{9}{-2}, \frac{12}{-1}\right)$ $-4.5, -6$

$= -6$

solution is unbounded ↗ since ratio $= \frac{RHS}{+\text{coefficients}}$

Example 2:

Max: $z = 3x_1 + 2x_2$

s/t: $2x_1 + x_2 \le 18$

$2x_1 + 3x_2 \le 42$

$3x_1 + x_2 \le 24$

$x_1, x_2 \le 24$

solution

Standard form:

$z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

$z - 3x_1 - 2x_2 - 0s_1 - 0s_2 = 0$

$2x_1 + x_2 + s_1 = 18$

$2x_1 + 3x_2 + s_2 = 42$

$3x_1 + x_2 + s_3 = 24$

B.V	z	x_1	x_2	s_1	s_2	s_3	RHS	
z	1	-3	-2	0	0	0	0	
s_1	0	2	1	1	0	0	18	
s_2	0	2	3	0	1	0	42	
s_3	0	3	1	0	0	1	24	$R_3 : R_3/3$

$x_1 \rightarrow$ pivot column (most negative)

$$\text{ratio} = \frac{\text{RHS}}{\text{co-efficient of } x_1} = \min\left(\frac{18}{2}, \frac{42}{2}, \frac{24}{3}\right)$$

$$= \min(9, 21, 8)$$

s_3 leaves: least ratio

BV	Z	x1	x2	s1	s2	s3	RHS	Rn
Z	1	-3	-2	0	0	0	0	Ro: Ro + 3R3
S1	0	2	1	1	0	0	18	R1: R1 - 2R3
S2	0	2	3	0	1	0	42	R2: R2 - 2R3
x1	0	1	1/3	0	0	1/3	8	

B.V	Z	x1	x2	S1	s2	s3	RHS	Rn
Z	1	0	-1	0	0	1	24	
S1	0	0	-1/3	1	0	-2/3	2	
S2	0	0	2	0	1	-2/3	26	
x1	0	1	1/3	0	0	1/3	8	

solution is not optimal, breaks rule of non-negativity

Duality of LPP

- Associated with any LP(primal/original) is another LP called the dual LP
- Primal LP is the original LP(min or max) Dual LP(max or min)
- If primal LP is to max then the dual LP will be to min. so that max for primal = min for dual

Primal LP

$$\max: \sum_j c_i x_j$$

$$s/t: \sum_j a_{ij} \leq b_i \ \forall_j$$
$$x_j \geq 0, \ \forall_j$$

Dual LP

$$\min: \sum_i u_i b_i$$
$$s/t: \sum_i u_i a_{ij} \geq c_j, \ \forall_i$$
$$u_i \geq 0, \ \forall_j$$

Primal Problem

$$\max: Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$
$$s/t: a_{11} x_{1} + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots a_{mn} x_n \leq b_m$$
$$x_1, x_2, \ldots, x_n \geq 0$$

→ there are m constraints, n variables

Dual LP

- Transpose the primal LP

$$
\left|\begin{array}{c|c}
a_{11}, a_{12} \dots a_{1n} & b_1 \\
a_{21}, a_{22} \dots a_{2n} & b_2 \\
\vdots \qquad\qquad \vdots & \vdots \\
a_{31}, a_{32} \dots a_{mn} & b_m \\
\hline
c_1, c_2 \dots c_n &
\end{array}\right|
\xrightarrow{\text{transpose}}
\left|\begin{array}{cc|c}
a_{11} & a_{21} & c_1 \\
a_{12} & a_{22} & c_2 \\
\vdots & \vdots & \vdots \\
a_{1n} & a_{2n} & c_n \\
\hline
b_1 & b_2 \dots & b_m
\end{array}\right.
$$

Dual LP: minimize Q:

$$
\min Q = b_1y_1 + b_2y_2 + \dots + b_my_m
$$

s/t: $a_{11}y_1 + a_{12}y_2 + \dots + a_{31}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \dots + a_{32}y_m \geq c_2$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{nm}y_m \geq c_n$

where $y_1, y_2, \dots y_m \geq 0$, such that $y_1 \neq x_1$, $y_2 \neq x_2, \dots, y_j \neq x_j$

Example

$\max: Z = 3x_1 + 2x_2$

$s/t: 2x_1 + x_2 \leq 18$

$2x_1 + 3x_2 \leq 42$

$3x_1 + x_2 \leq 24$

$x_1, x_2 \geq 0$

Primal LP

2	1	18
2	3	42
3	1	24
3	2	

$\xrightarrow{\text{transpose}}$

2	2	3	3
1	3	1	2
18	42	24	

Dual LP

$\min Q = 18y_1 + 42y_2 + 24y_3$

$s/t: 2y_1 + 2y_2 + 3y_2 \geq 3$

$y_1 + 3y_2 + y_3 \geq 2$

$y_1, y_2, y_3 \geq 0$

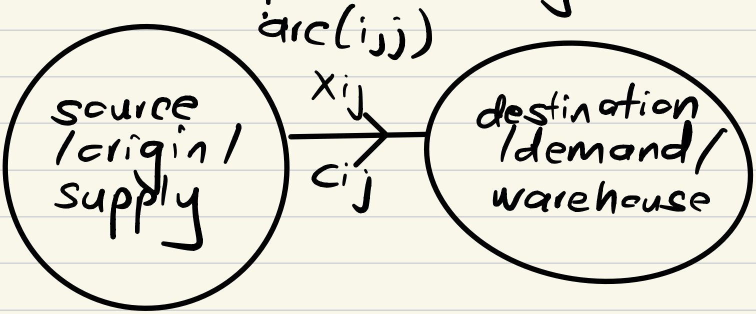
$y_i \neq x_j$

Transportation Problem

- An LP dealing with shipment of a commodity from scuices (origins) to destinations e.g warehouses or points of demand.
- Each origin represents a source of supply for the commodity.
- Each destination reps. its point of demand (market)
- The model assumes that shipping cost is proportional to the number of units shipped on a given route.
- The problem is to determine the number of units from a given scuice to a given destination so that the cost of transportation is minimum.

Transportation Mode

- This is represented by a network diagram



- Source is connected to the destination with an arc(i,j) which carries two pieces of info.

1. $x_{ij} \rightarrow$ units transported from source i to destination j.

2. $C_{ij} \rightarrow$ cost of transporting 1 unit from i to j.

- Let a_i be the amount of supply at source i.
- Let b_j be the amount of demand at destination j.

TP Model

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$$ $\rightarrow m$ sources, n destination.

subject : $\sum_{j=1}^{n} X_{ij} \leq a_i, \; i = 1, 2, \ldots, m$ units supplied

whose : $\sum^{m} x_{ij} \geq b_j, \; j = 1, 2, \ldots, n$ units demanded

Types of Transportation Problems (TP)

1. Balanced TP where amount of supply = amount of demand
2. Unbalanced TP where $a_i \neq b_j \; \forall i, j$

e.g Balanced TP:

$$\min \sum_{i} \sum_{j} c_{ij} x_{ij}$$ where $a_i = b_j$

s/t : $\sum x_{ij} = a_i, \; i = 1, 2, \ldots m$, $x_{ij} \geq 0 \quad \forall i, j$

$\sum x_{ij} = b_j, \; j = 1, 2, \ldots n$

Note : To solve unbalanced TP, introduce dummy rows and columns with zero cost of transportation.

Example

A leading processor of sugar has two plants which supply 4 warehouses. Let X_{ij}, denote the number of units shipped from plant i to warehouse j. Formulate a LP such that the plant & warehouse requirements are satisfied

		1	2	3	4	warehouse supply
plant	1	20	15	10	25	2800
	2	30	25	20	15	3500
Warehouse demand		1400	1600	1000	1500	

Solution

X_{ij} ← no. of units transported from i to j

X_{ij}, $i = 1, 2$

$j = 1, 2, 3, 4$

$$\min C = \sum_{i=1}^{2} \sum_{j=1}^{4} c_{ij} X_{ij}$$

$$\min C = 20x_{11} + 15x_{12} + 10x_{13} + 25x_{14} + 30x_{21} + 25x_{22} + 20x_{23} + 15x_{24}$$

s/t: $\left.\begin{array}{l} x_{11} + x_{12} + x_{13} + x_{14} \leq 2800 \\ x_{21} + x_{22} + x_{23} + x_{24} \leq 3500 \end{array}\right\}$ supply constraints

since you cannot transport more than you supply

$$x_{11} + x_{21} \geq 1400$$
$$x_{11} + x_{22} \geq 1600$$
$$x_{13} + x_{23} \geq 1000$$
$$x_{14} + x_{24} \geq 1500$$

$$x_{ij} \geq 0, \; i = 1, 2$$
$$j = 1, 2, 3, 4$$

Introduce a dummy column if supply $\neq$ demand

		1	2	3	4	Dummy	supply
plant	1	20	15	10	25	0	2800
	2	30	25	20	15	0	3500
Warehouse demand		1400	1600	1000	1500	800	supply=demand 6600

Basic Feasible Solution for a Transportation Problem

1. Northwest Corner method
2. Least/minimum cost cell method
3. Vogel's method

1. Northwest Corner Method (N-W Corner Rule

- Begin upper left corner of the tableau and set x_{11} as large as possible
- x_{11} cannot be larger than the smaller of s_1 (supply) and d_1 (demand)

- If $x_{11} = s_1$, cross out its first row of tableau, then change d_1 to $d_1 - s_1$
- If $x_{11} = d_1$, cross-out the first column of tableau, then change s_1 to $s_1 - d_1$
- If $x_{11} = d_{11} = s_1$, cross-out either row 1 or column 1 but not both.
- Cross out r_1 then change d_1 to 0
- Cross out c_1 then change s_1 to 0.

- No costs are needed in this method.

Example

Obtain initial bfs for the following TP

		D1	D2	D3	D4	Supply
Source	1	3 250	1 50	7 -	4 -	300
	2	2 -	6 300	5 100	9	400
	3	8 -	3 -	3	2	500
	demand	250	350	400	200	

Solution

x_{ij}, $i = 1, 2, 3$
$j = 1, 2, 3, 4$

$$\min C = 3x_{11} + x_{12} + 7x_{13} + 4x_{14} + 2x_{21} + 6x_{22} + 5x_{23} + 9x_{24} + 8x_{31} + 3x_{32} + 3x_{33} + 2x_{34}$$

s/t:
$$x_{11} + x_{12} + x_{13} + x_{14} \le 300$$
$$x_{21} + x_{22} + x_{23} + x_{24} \le 400$$
$$x_{31} + x_{32} + x_{33} + x_{34} \le 500$$
$$x_{11} + x_{21} + x_{31} \ge 250$$
$$x_{12} + x_{22} + x_{23} \ge 350$$
$$x_{13} + x_{23} + x_{33} \ge 400$$
$$x_{14} + x_{24} + x_{34} \ge 200$$

		D1	D2	D3	D4	Supply	
source	1	3 (250)	1 (50)	7	4	~~300~~ ~~50~~ X	
	2	2	6 (300)	5 (100)	9	~~400~~ ~~100~~ X	
	3	8	3	3 (300)	2 (200)	~~500~~ ~~300~~ 0	
	demand	~~250~~ X	~~350~~ ~~300~~ X	~~400~~ ~~300~~	~~200~~ X	1200	

Allocations for Basic Feasible Solution

X_{ij}	C_{ij}	Total Cost ($C_{ij}X_{ij}$)
$X_{11} = 250$	3	750
$X_{12} = 50$	1	50
$X_{22} = 300$	6	1800
$X_{23} = 100$	5	500
$X_{33} = 300$	3	900
$X_{34} = 200$	2	400

$$t = \sum_{i=1}^{3} \sum_{j=1}^{4} C_{ij} X_{ij}$$

$$= \underline{\underline{4400}}$$

2. Least cost method

- Start from the cell with the minimum cost of transportation then follow the same procedure as in the N-W corner rule.

		D1	D2	D3	D4	Supply
Source	1	3	1 (300)	7	4	~~300~~ X
	2	2 (250)	6	5 (150)	9	~~400~~ ~~150~~ X
	3	8	3	3 (250)	2 (200)	~~500~~ ~~300~~ ~~250~~ X
	demand	~~250~~ X	~~350~~ ~~50~~ X	~~400~~ ~~150~~ X	~~200~~ X	

X_{ij}	C_{ij}	$C_{ij}X_{ij}$
$X_{12}=300$	1	300
$X_{21}=250$	2	500
$X_{23}=150$	5	750
$X_{32}=50$	3	150
$X_{33}=250$	3	750
$X_{34}=200$	2	400

= Initial bfs = 2850

N.W:

	D_1	D_2	D_3	D_4	Supply
1	250 (3)	50 (1)	– (7)	– (4)	~~300~~ ~~50~~ 0
2	– (12)	300 (6)	100 (5)	– (9)	~~400~~ 0
3	– (18)	– (3)	300 (3)	200 (2)	~~500~~ 0
demand	~~250~~ 0	~~350~~ 0	~~400~~ 0	~~200~~ 0	

$$\text{cost} = (250\times 3)+(50\times 1)+(300\times 6)+(100\times 5)+(300\times 3)$$
$$+(200\times 2)$$
$$=$$

	D_1	D_2	D_3	D_4	Supply
1	– (3)	300 (1)	– (7)	– (4)	~~300~~ 0
2	250 (12)	– (6)	150 (5)	– (9)	~~400~~ 150 0
3	– (18)	50 (3)	250 (3)	200 (2)	~~500~~ ~~300~~ 250
demand	~~250~~ 0	~~350~~ ~~50~~ 0	~~400~~ 150 0	~~200~~ 0	

$$(250\times 2)+(300\times 1)+(50\times 3)+(250\times 3)+(150\times 5)$$
$$+(200\times 2) =$$

Assignment Problem

- It is a special class of transportation problem with 1 supply and 1 demand
- Every supplier is assigned to one destination every destination is assigned to one supplier.
- There's a group of n 'applicants' applying for 'n' jobs with non-negative cost C_{ij} of assigning the j^{th} job to the i^{th} person.

Note: The objective is to assign one job to one person (machine) such a way that the total cost of assignment is minimal.

Assumptions

- Each resource is assigned to exactly one task.
- Each task is assigned to one resource.
- No. of tasks = no. of resources.

- Define binary variable.

$$X_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to the } i^{th} \text{ person.} \\ 0, & \text{elsewhere.} \end{cases}$$

LP for Assignment Problem

$$\text{Min } C = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_j$$

Subject to:

$$\sum_{j=1}^{n} X_{ij} = 1, \quad i = 1, 2, \ldots, n$$

(1 person assigned 1 job)

$$\sum_{i=1}^{n} X_{ij} = 1, \quad j = 1, 2, \ldots, n$$

(1 job assigned to 1 person)

$$X_{ij} = 0 \text{ or } 1 \quad \forall i, j$$

If the cost of doing the j^{th} work by the i^{th} person is C_{ij}, then the cost matrix is given by:

persons \ jobs	1	2	3 ⋯	j ⋯	n
1	C_{11}	C_{12}	C_{13} ⋯	C_{ij}	C_{1n}
2	C_{21}	C_{22}	C_{23} ⋯	C_{2j} ⋯	C_{2n}
⋮	⋮	⋮	⋮	⋮	⋮
i	C_{i1}	C_{i2}	C_{i3} ⋯	C_{ij} ⋯	C_{in}
⋮	⋮	⋮	⋮	⋮	⋮
n	C_{n1}	C_{n2}	C_{n3} ⋯	C_{nj} ⋯	C_{nn}

$C_{11} \rightarrow$ cost of assigning the 1st job to the first person

$C_{nn} \rightarrow$ cost of assigning the n^{th} job to the n^{th} person.

Cost matrix $\rightarrow$ square matrix (jobs = persons)

Solutions to Assignment Problem

1. Hungarian method
- Applies to a $m \times m$ assignment.

Steps

1. Row Reduction:
- Find the minimum element in each row of the $m \times m$ cost matrix.
- Construct a new matrix by subtracting from each cost the minimum cost on its row.

2. Column Reduction
- Find the minimum cost in each column. Subtract this minimum cost from every element in its column.

3. Draw the minimum number of lines (horizontal and/or vertical) that will cover all the zeros. If the minimum number of lines required is less than m, then assignment cannot be done; instead proceed to step 4

4. Select the smallest uncrossed-out element. Subtract this element, add this element to every element of the intersection of 2 lines

5. Return to step 3.

Example 1

4 persons A, B, C, D are to be assigned 4 jobs I, II, III, IV. The cost matrix is given as:

	Person			
Jobs	A	B	C	D
I	8	10	17	9
II	3	8	5	6
III	10	12	11	9
IV	6	13	9	7

Solution

I: Row Reduction

$$\begin{bmatrix} 0 & 2 & 9 & 1 \\ 0 & 5 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 0 & 7 & 3 & 1 \end{bmatrix}$$

II. Column Reduction

$$\begin{bmatrix} 0 & 0 & 7 & 1 \\ 0 & 3 & 0 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 1 \end{bmatrix}$$

no. of lines = dimensions (m)

Assignment

Jobs	Persons	Cost
I	B	10
II	C	5
III	D	9
IV	A	6

Example 2

- There are 5 machines and 5 jobs to be assigned. The associated cost matrix is as follows. Find the proper assignment.

	Machines				
Jobs	I	II	III	IV	V
A	6	12	3	11	15
B	4	2	7	1	10
C	8	11	10	7	11
D	16	19	12	23	21
E	9	5	7	6	10

Solution

Row Reduction

$$\begin{bmatrix} 3 & 9 & 0 & 8 & 12 \\ 3 & 1 & 6 & 0 & 9 \\ 1 & 4 & 3 & 0 & 4 \\ 4 & 7 & 0 & 11 & 9 \\ 4 & 0 & 2 & 1 & 5 \end{bmatrix}$$

Column Reduction

$$\begin{bmatrix} 2-1 & 9 & 0 & 8 & 8-1 \\ 2-1 & 1 & 6 & 0 & 5-1 \\ 0 & 4+1 & 3+1 & 0+1 & 0 \\ 3-1 & 7 & 0 & 11 & 5-1 \\ 3-1 & 0 & 2 & 1 & 1-1 \end{bmatrix}$$

- no. of lines < m : check smallest element uncrossed = 1
- add smallest no. with what is on the intersection

$$\begin{bmatrix} 1 & 9 & 0 & 8 & 7 \\ 1 & 1 & 6 & 0 & 4 \\ 0 & 5 & 4 & 1 & 0 \\ 2 & 7 & 0 & 11 & 4 \\ 2 & 0 & 2 & 1 & 0 \end{bmatrix}$$

repeat process

$$\begin{bmatrix} 0 & 8 & 0 & 8 & 6 \\ 0 & 0 & 6 & 0 & 3 \\ 0 & 5 & 5 & 2 & 0 \\ 1 & 6 & 0 & 1 & 3 \\ 2 & 0 & 2 & 1 & 0 \end{bmatrix}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & \checkmark & \checkmark & \end{array} \\
\begin{bmatrix} 0 & 8 & \boxed{0} & 8 & 6 \\ 0 & 0 & 6 & \boxed{0} & 3 \\ 0 & 5 & 5 & 2 & \boxed{0} \\ 1 & 6 & \boxed{0} & 1 & 3 \\ 2 & \boxed{0} & 2 & 1 & 0 \end{bmatrix}
\begin{array}{l} 2 \\ \cancel{3}\ 2 \\ \cancel{2}\ 1 \\ 1 \\ 2 \end{array} \\
\begin{array}{ccccc} \cancel{3} & 2 & \cancel{2} & 1 & 2 \\ & & 1 & & \end{array}
\end{array}$$

*

Assignment

Jobs	Machines	Cost
A	I	6
B	IV	1
C	V	11
D	III	12
E	II	5
		35

$$\sum C_{ij} = 35$$

Exercise

1.

	A	B	C	D
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Solve the minimal assignment problem

Solution

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 8 \\ 3 & 5 & 8 & 4 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

optimal

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 5 & 1 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{column}} \begin{array}{c|l} \begin{bmatrix} 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 \\ \boxed{0} & 0 & 0 & 0 \\ 0 & 1 & 3 & \boxed{0} \end{bmatrix} & \begin{array}{l} \cancel{3}\ 2 \\ \cancel{3}\ \cancel{2}\ 1 \\ \cancel{4}\ \cancel{3}\ 1 \\ \cancel{2}\ 1 \end{array} \\ \begin{array}{cccc} \cancel{4} & \cancel{3} & \cancel{3} & \cancel{2} \\ \cancel{3} & \cancel{2} & \cancel{2} & \cancel{1} \\ 2 & \cancel{1} & 1 & 0 \\ & 0 & & \end{array} & \end{array}$$

$$\text{final solution} \rightarrow \begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 0 & \boxed{0} & 0 & 2 \\ 0 & 0 & \boxed{0} & 2 \\ \boxed{0} & 0 & 0 & 0 \\ 0 & 1 & 3 & \boxed{0} \end{bmatrix} \end{array}$$

Jobs	Machines	Cost
A	III	7
B	I	3
C	II	9
D	IV	4

2. Consider 2 workers and tasks to be performed w/ effectiveness matrix given by:

	I	II	III	IV
A	5	23	14	8
B	10	25	1	23
C	35	16	15	12
D	16	23	21	7

Obtain proper assignment

8|3
− 7
6

Solution

$$\begin{bmatrix} 5 & 23 & 14 & 8 \\ 10 & 25 & 1 & 23 \\ 35 & 16 & 15 & 12 \\ 16 & 23 & 21 & 7 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} \begin{bmatrix} 0 & 18 & 9 & 3 \\ 9 & 24 & 0 & 22 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 18 & 9 & 3 \\ 9 & 24 & 0 & 22 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{column}} \begin{bmatrix} 0 & 14 & 9 & 3 \\ 9 & 20 & 0 & 22 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{bmatrix} = 3$$

4x4 = 4   4 −

$$\begin{bmatrix} 0 & 14 & 9 & 3 \\ 9 & 20 & 0 & 22 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{bmatrix}$$

optimal since
lines = m

$$
\begin{array}{c}
\begin{bmatrix}
\boxed{0} & 14 & 9 & 3 \\
9 & 20 & 0 & 22 \\
23 & \boxed{0} & 3 & 0 \\
9 & 12 & 14 & \boxed{0}
\end{bmatrix}
\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array}
\\
\begin{array}{cccc} 1 & 1 & 1 & 2 \end{array}
\end{array}
\qquad
\begin{array}{c|cccc|c}
 & A & B & C & D & \\
\hline
I & \boxed{0} & 14 & 9 & 3 & 1 \\
II & 9 & 20 & \boxed{0} & 22 & 1 \\
III & 23 & \boxed{0} & 3 & \boxed{\cancel{0}} & 2 \\
IV & 9 & 12 & 14 & \boxed{0} & 1 \\
\hline
 & 1 & 1 & 1 & 2 &
\end{array}
\quad \times
$$

Jobs	Resource	Cost
A	I	
B		
C	III	
D	IV	

Project Analysis using Networks

- A project is made up of integrated activities that must be executed in a certain order before the entire task can be completed.
- An activity in a project refers to a task requiring time and resources for its completion.

Three phases of the project

1. Planning

- Partitioning the project into different activities
- Determination of duration of each activity is done.
- Network diagram of the project is developed

2. Scheduling

- Time schedule is determined i.e start and finish times.
- Critical activities are also determined at this stage.

3. Controlling

- In values analysis of perfomance i.e actual vs the ex ected.

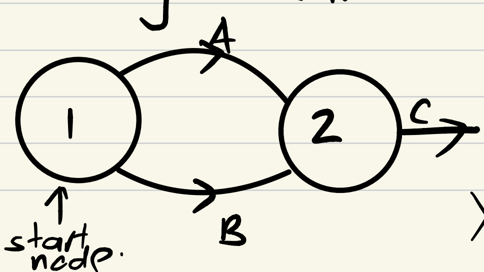
Note: A project can be represented by:

1. A number of events (nodes) and activities (arcs) in form of a network.
2. Space all the activities directed to the event must be completed before the event occurs. The time at which an event occurs is the max. duration of the paths directed to the event.
3. Event (nodes) do not consume any resources. They signify the end of some activities $ beginning of others.

Rules of Constructing a Project Network

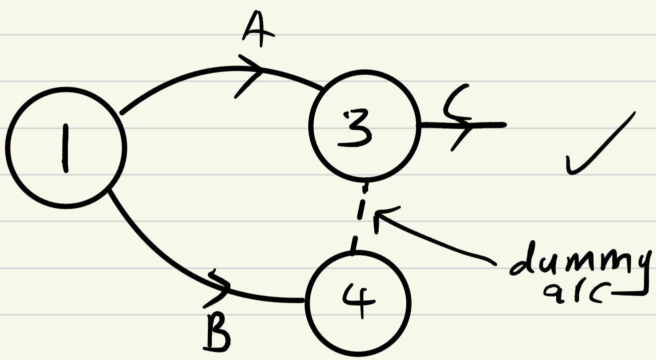
1. First node represents start of the project. An arc emanating from the node rep. activities with no predecessors
2. The left node (finish node) reps the completion of the project $ must be included in the network diagram.
3. An activity should not be represented by any more than one arc in the network
4. Two nodes can be connected by at most one arc.

NB To avoid violation of rules 3 and 4, a dummy arc can be used



A, B reps predecessor arc

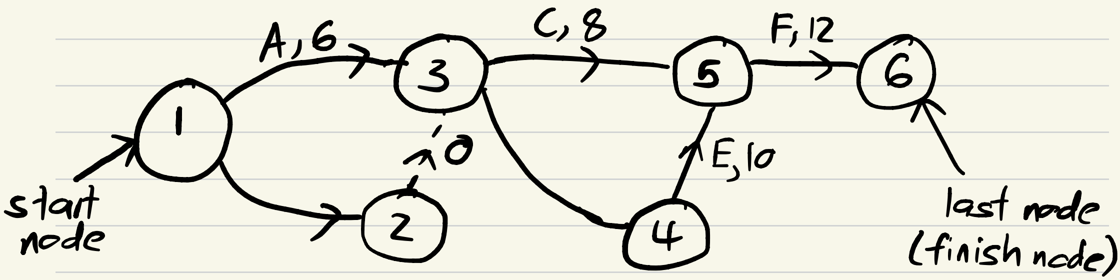
X violates rule 3 and 4



Example 1

Activity	Predecessor	Duration
A- Train workers	—	6
B- Purchase of raw materials	—	9
C- Produce Product 1	A, B	8
D-Product product 2	A, B	7
E-Test product 2	D	10
F-Assemble product 1,2	C, E	12

Construct a project network

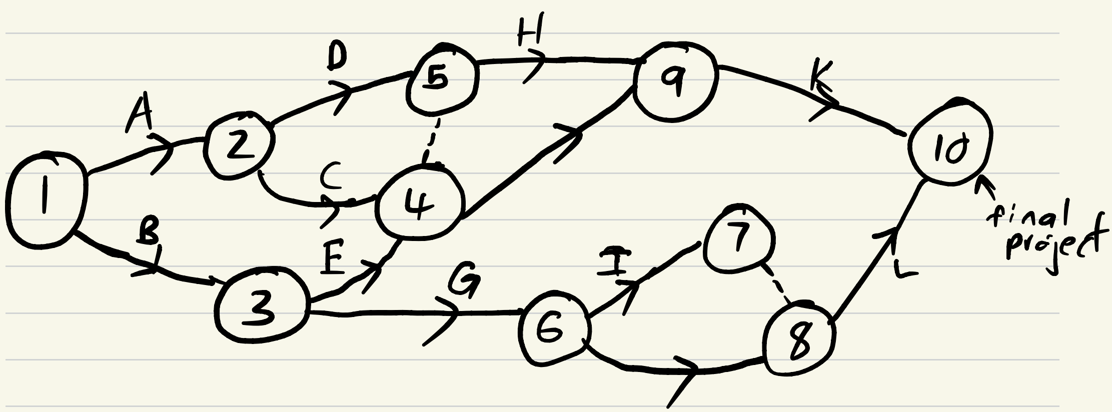


Example 2

Construct a network diagram consisting of the activities A, B, ..., L given as:

Concurrent activities A and B begin the project
" " C and D suceed A
" " E and G suceed B
Activity F suceeds both C and E
" " " " C and D
Activity I and J suceeds G
Activity K suceeds H and F.
Activity L suceeds I and J.
Activity L and K completes the project

Solution

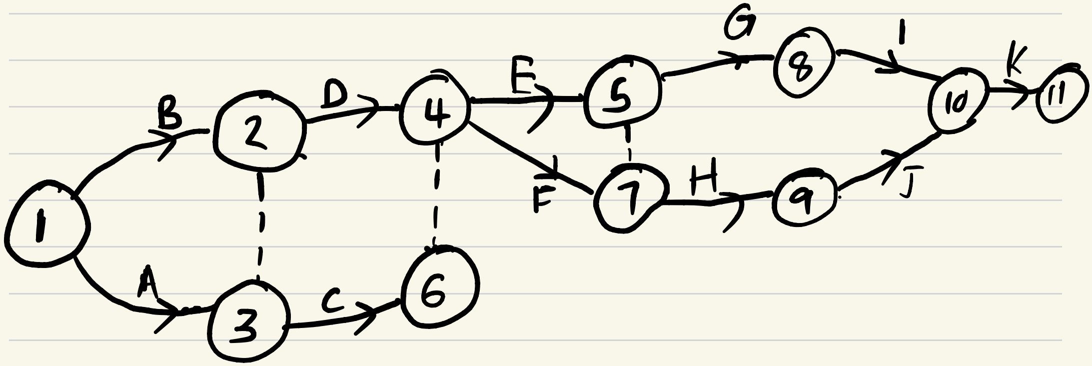


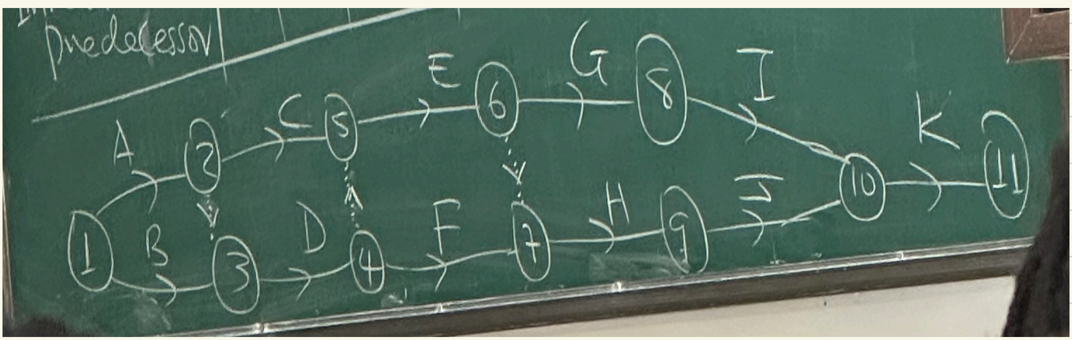
Example 3

Develop a network diagram with the precedence r/ships given below

Activity	A	B	C	D	E	F	G	H	I	J	K
Immediate Precedence	-	-	A	A,B	C,D	D	E	E,F	G	H	I,J

Solution





Critical Path

- Note: A critical activity(ies) can cause the delay in the completion of the project if these activities are delayed.

- A critical path is the chain of critical activities that connect the start & finish nodes i.e the path with the longest duration.

Determination of Critical Path-

1. Critical Path Method (CPM)
2. Project Evaluation & Review Technique (PERT)

Note: CPM is used if the duration of each activity is known with certainity i.e CPM is deterministic.
PERT is used if the duration of each activity is unknown i.e PERT is probabilistic

Critical Path Method

- Done in two phases

1. Forward Pass Computation (FPC)

- Computation begins from the start node and moves towards the finish node.
- At each node, earliest start time is computed

2. Backward Pass Computation (BPC)

- Computation begin from finish node and moves towards the start node.
- Latest completion time for all activities coming into a given node are obtained

Example

Calculate:
BPC
FPC

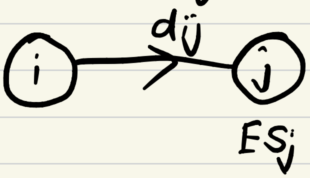
Solution

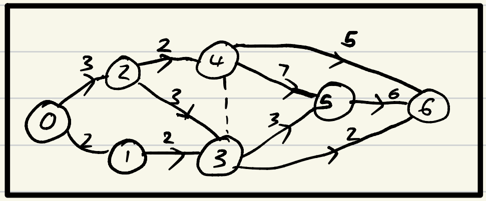
Forward Pass Computations

- Let ES_i be the earliest start time for all activities leading from event i
- If $i=0$, the start event, then $ES_0 = 0$
- Let d_{ij} be the duration of activity (i,j)
- The FPC formula is given by:

$$ES_j = Max_{(i)}[ES_i + d_{i,j}] \forall_{ij}$$

Back to question:





$$ES_1 = ES_0 + d_{0,1} = 0 + 1$$

$$ES_2 = ES_0 + d_{0,2} = 0+3=3$$

$$ES_3 = \underset{\{i=1,2\}}{Max} [ES_i + d_{i,3}]$$
$$= Max[2+2, 3+3]$$
$$= 6$$

$$ES_4 = \underset{\{i=2,3\}}{Max} [ES_i + d_{i,4}]$$
$$= Max[3+2, 6+0]$$
$$= 6$$

$$ES_5 = \underset{i=\{3,4\}}{Max} [ES_i + d_{i,5}]$$
$$= Max[6+3, 6+7]$$
$$= 13$$

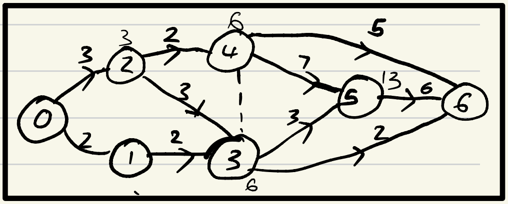
$$ES_6 = \underset{\{i=3,4,5\}}{Max} [ES_i + d_{i,6}]$$
$$= Max[6+2, 6+6, 13+6]$$
$$= 19$$

Duration of the project is 19 units of time.

2. Backward Pass Computation

- Let LC_i be the latest computation time for all the activities coming into node i
- If $i = n$ is the end event then $LC_n = ES_n$ initiates the BPC where ES_n is the earliest start time at node n
- Generally, for any node i,

$$LC_i = \min_{\{j\}} [LC_j - d_{i,j}] \quad \forall_{ij} \ni \text{activities defined}$$



$$LC_5 = LC_6 - d_{5,6}$$
$$= 19 - 6$$
$$= 13$$

$$LC_4 = \min_{\{j=5,6\}} [LC_j - d_{4,j}]$$
$$= \min[13 - 7, 19 - 5]$$
$$= 6$$

$$LC_3 = \min_{\{j=4,5,6\}} [LC_j - d_{3,j}]$$
$$= \min[6-0, 13-3, 19-2]$$

$$LC_2 = \min_{\{j=3,4\}} [LC_j - d_{2,j}]$$
$$= \min[6-3, 6-2]$$
$$= 3$$

$$LC_1 = LC_3 - d_{1,3}$$
$$= 6 - 2$$
$$= 4$$

$$LC_0 = \min[4-2,\ 3-3]$$
$$= 0$$

- Critical Path (CP): An activity (i,j) lies on the critical path, if it satisfies the conditions:

① $ES_i = LC_i$
② $ES_j = LC_j$
③ $ES_j - ES_i = LC_j - LC_i = d_{i,j}$

- The critical path is given by:

$$0 \xrightarrow{3} 2 \xrightarrow{3} 3 \xrightarrow{0} 4 \xrightarrow{7} 5 \xrightarrow{6} 6$$

$(0,2), (2,3), (3,4) (4,5), (5,6)$

i.e $3+3+0+7+6 = 19$: total duration of the project.

Exercise

Activity	Immediate Predecessor	Duration
A	—	3
B	A	3
C	A	3
D	C	5
E	B, C	6
F	D, E	9
G	E	5
H	G	3
I	G, F	5
J	I, H	6
K	J	4

Develop CPM network and determine CP