HOMEWORK 1 - solutions:

[PH2202 ; Thermal Physics, Spr 20]

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1.

Look at the notes, and re-do the derivations on your own, and following your style of calculation.

2.

Number density, x-component, $u \to u+du$;

Given by: $$dn_u = n f(u)\, du$$
$$= n\, a \exp(-u^2/\alpha^2)\, du ;$$

n = overall number density ; $a, \alpha \to$ constants ;

Note, p is derived in Pg.14-15 of notes, considering the hemi-sphere above the small δS on the wall/bottom of the container.

You see from Pg.14 that velocities $\perp$ to the normal of the wall (i.e. velocities $\parallel$ to the wall) do not matter. So, you can find p by aligning one co-ordinate (say, x) to the wall-normal and make a 1d calculation instead of 3d (r, θ, ϕ) calculation done in class.

First, let us get a ;

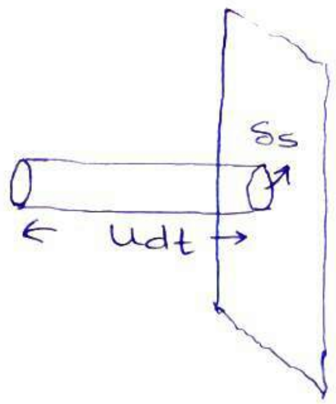
$$\int_{-\infty}^{\infty} dn_u = n = n \int a e^{-u^2/\alpha^2} du$$

$$\Rightarrow a = \frac{1}{\int_{-\infty}^{\infty} e^{-u^2/\alpha^2} du}$$

$$\left[\int_0^{\infty} e^{-u^2/\alpha^2} u^n du = \frac{\alpha}{2}\sqrt{\pi} \text{ for } n=0 \right.$$ (Pg. 38)

$$\Rightarrow a = \frac{1}{\alpha\sqrt{\pi}} ;$$

2. contd.



Momentum change in reflection of one molecule

$$= 2mu \; ;$$

No. of molecules hitting δs (perpendicular! that's the advantage if this derivation – no r, θ, ϕ)

$$= u\,dt \, . \, dn_u \, . \, \delta s$$

(volume of cylinder × density of $u \to u+du$)

So, ~~total~~ momentum change in time dt $(u \to u+du)$

$$= 2mu \, . \, u\,dt \, . \, dn_u \, \delta s$$

~~Force;~~ Total momentum change

$$\int_0^\infty 2mu \, . \, u\,dt \, . \, dn_u \, \delta s$$

[only (+ve) u matters]

Total force:

$$\int_0^\infty 2mu^2 \, dn_u \, \delta s$$

Total pressure:

$$p = 2mn \int_0^\infty f(u) \, u^2 \, du$$

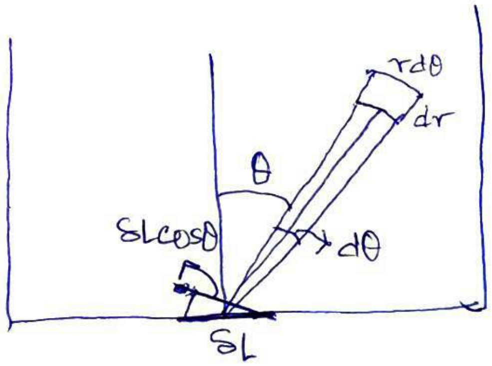
Use Pg. 38 table: $p = mn\alpha^2/2$

But, $p = nk_BT$

$$\Rightarrow \quad \alpha = \sqrt{\frac{2k_BT}{m}} \; ;$$

3. This is the problem of 2d ideal gas. While doing this problem, the main template should be all result we derived in class for 3d ideal gas, and then try to set up analogies.

a) number of impacts; (Pg. 13 for 3d)



The molecules are moving on a surface / plane, NOT in a volume.

So, $\delta S \rightarrow \delta L$

$\delta V \rightarrow \delta A = r\,dr\,d\theta$

n = surface density of molecules

4π solid angle $\rightarrow$ 2π (standard) angle.

So, $\delta n = \dfrac{n\,\delta A\,\delta L\cos\theta}{2\pi r}$ [see Pg. 13 - mid]

And number of impacts in time Δt;

$$\int_{r=0}^{c\Delta t}\int_{-\pi/2}^{\pi/2} \frac{n\,\delta A\,\delta L\cos\theta}{2\pi r}$$

$$= \frac{nc}{\pi}\Delta t;$$

b) pressure; (Pg. 14-15 for 3d)

$$p = \int \delta n \; 2mc\cos\theta \;/(\Delta t\,\Delta L)$$

Integral $\rightarrow$ $\dfrac{n\,\delta L \cdot 2mc}{2\pi}\displaystyle\int_{r=0}^{c\Delta t} dr \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$

$$= \frac{1}{2} m n c^2 . \delta L . \Delta t$$

$p = F/\Delta L$, a special definition and unit!

So, $p = \frac{1}{2} m n c^2$;

3. cond.

c) Maxwellian velocity distribution: (Pg. 33-44 for 3d)

$u, v, w \rightarrow u, v \Rightarrow$ rest fall in place;

$$dN = N\, f(u) f(v)\, dA \quad \text{(see Pg. 35 - top)}$$

All arguments remain the same till Pg. 37 - mid to take us to:

$$f(u) = a\, e^{-u^2/\alpha^2}$$
$$f(v) = a\, e^{-v^2/\alpha^2} \qquad \Big|\ c^2 = u^2 + v^2$$

$$a = \frac{1}{\alpha\sqrt{\pi}} \quad \text{(Pg. 38 - mid)}$$

[no change for 2d]

$$dn = \frac{n}{\alpha^2 \pi}\, e^{-c^2/\alpha^2}\, c\, d\theta\, dc \quad \text{(Pg. 40 - mid)}$$

[$\theta = 0 \rightarrow 2\pi$, like ϕ in 3d!]

or, $$dn = \frac{2n}{\alpha^2}\, e^{-c^2/\alpha^2}\, c\, dc$$

(see, Pg. 40 - bot)

d) Average (mean) speed: (Pg. 41 - a))

$$\bar{c} = \frac{1}{n} \int_0^\infty c\, dn$$

$$= \frac{\alpha}{2}\sqrt{\pi} \quad \text{(Use Table. Pg. 38)}$$

e) RMS speed: (Pg. 41 - b))

$$\overline{c^2} = \frac{1}{n} \int_0^\infty c^2\, dn$$

$$= \alpha^2\, ; \Rightarrow \sqrt{\overline{c^2}} = \alpha\, ;$$

f) Most probable speed: (Pg. 42 - top)

$\rightarrow$ Next

$$\frac{d}{dc}\left(\frac{dn}{dc}\right)_{c_m} = 0$$

$$\Rightarrow \frac{d}{dc}\left(e^{-c^2/\alpha^2} c\right)_{c_m} = 0$$

$$\Rightarrow e^{-c^2/\alpha^2} + c \cdot e^{-c^2/\alpha^2} \cdot \left(-\frac{2c}{\alpha^2}\right) = 0$$

$$\Rightarrow c_m^2 = \alpha^2/2 \Rightarrow c_m = \frac{\alpha}{\sqrt{2}}$$

So, ratio: $\bar{c} : \sqrt{\overline{c^2}} : c_m = \frac{\sqrt{\pi}}{2} : 1 : \frac{1}{\sqrt{2}}$

$= 1.25 : 1.41 : 1$

- what is the value of α? (compare Pg. 43)

$$p = \frac{1}{2} m n \overline{c^2}$$

$$pV = \frac{1}{2} m N \overline{c^2} = N\epsilon = E \qquad \left[\epsilon = \frac{1}{2} m \overline{c^2}\right]$$

[Pg. 16 - top]

Now, $\epsilon = \frac{1}{2} k_B T \times 2$ (2d)

$= k_B T$

So, $pV = N k_B T \Rightarrow p = n k_B T$ (same as 3d)

[check: $p = 2mn \int_0^\infty f(u)\, u^2\, du = n k_B T$ as well]

So, $\epsilon = \frac{1}{2} m \overline{c^2} = k_B T$

$$\alpha^2 = \overline{c^2} = \frac{2k_B T}{m} \Rightarrow \alpha = \sqrt{\frac{2k_B T}{m}};$$

Hence, velocity distribution $= \frac{m}{k_B T} e^{-\frac{mc^2}{2k_B T}} c\, dc$

$$\bar{c} = \sqrt{\frac{\pi k_B T}{2m}}; \quad \sqrt{\overline{c^2}} = \sqrt{\frac{2k_B T}{m}}; \quad c_m = \sqrt{\frac{k_B T}{m}};$$

9)

$$f(c)\,dc = \frac{m}{K_BT} e^{-mc^2/2K_BT}\, c\, dc$$

$$\epsilon = \frac{1}{2} m c^2$$

$$d\epsilon = mc\,dc \Rightarrow dc = \frac{d\epsilon}{\sqrt{2m\epsilon}};$$

So, $$f(\epsilon)\,d\epsilon = \frac{m}{K_BT} e^{-\epsilon/K_BT} \cdot \sqrt{\frac{2\epsilon}{m}} \cdot \frac{d\epsilon}{\sqrt{2m\epsilon}}$$

$$= \frac{1}{K_BT} e^{-\epsilon/K_BT} d\epsilon\,;$$

This is the energy distribution;

[Pg. 44 - mid]

4.

This one led to some confusion;

If one uses $p = nK_BT$, then

$T = 27°C = 300\,K$

$K_B = 1.38 \times 10^{-23}\ m^2\,Kg\,s^{-2}\,K^{-1}$ (SI)

$p = 40\ mm\ Hg = 0.04 \times 13600 \times 9.81\ N/m^2$

Then, $n = 12.89 \times 10^{23}/m^3$

$\simeq 1.3 \times 10^{18}/cc$

In this case you do not need the value of energy of a molecules.

However, assume that you donot know the value of K_B;

Then, $\epsilon = \frac{3}{2} K_BT$;

$n = \frac{3p}{2\epsilon}$ (you do not need to know T as well!)

$= \frac{3}{2} \times 0.04 \times 13600 \times 9.81 \big/ 8 \times 10^{-21}\ /m^3$

$\simeq 10^{24}/m^3 \simeq 10^{18}/cc$

5.

To find the number of collisions in unit time we need the mean free path and mean speed.

Collision frequency = $\frac{\bar{c}}{\lambda}$;

$\bar{c} = \sqrt{\frac{8K_BT}{\pi m}}$ (Maxwell's distribution)

$\lambda = \frac{1}{\sqrt{2}\pi\sigma^2 n}$ (Maxwell's distribution and basic derivation Pg. 22)

So, collision frequency =

$\sqrt{\frac{8K_BT}{\pi m}} \times \sqrt{2}\,\pi\sigma^2 n$

$= \sqrt{16\pi K_BT/m} \cdot \sigma^2 n$

$= \sqrt{\frac{16 \times 3.14 \times 1.38 \times 10^{-23} \times 300}{3.32 \times 10^{-27}}}$

$\times (1.9 \times 10^{-10})^2 \times 2.45 \times 10^{25}/s$

$= 7.00 \times 10^9/s$

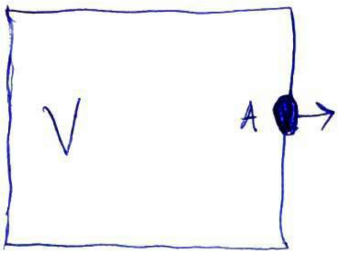
$N_A = 6.023 \times 10^{23}\ H_2$
$= 2\ gm$
$= 0.002\ kg$
$\Rightarrow$ 1 H_2 molecule
$= 3.32 \times 10^{-27}$ kg

$n = p/K_BT$

$= \frac{0.76 \times 13600 \times 9.81}{1.38 \times 10^{-23} \times 300}\ /m^3$

$= 2.45 \times 10^{25}/m^3$

6.



$\frac{n\bar{c}}{4}$ molecules hit unit area in unit time.

So, $\frac{n\bar{c}A}{4}dt \rightarrow$ molecules hit 'A' area in dt time

[Note! n is a time-dependent quantity; so, you have to take infinitesimal time $\rightarrow dt$, to consider n to be constant]

So,
$$dN = -\frac{n\bar{c}A}{4}dt$$ (−ve, because molecules going out)

$$dn = \frac{dN}{V} = -\frac{n\bar{c}A}{4V}dt;$$

$$\Rightarrow \frac{dn}{n} = -\frac{\bar{c}A}{4V}dt;$$

Again $p \propto n$ [as, $p = nk_BT$; T is constant here]

So, $\frac{dn}{n} = \frac{dp}{p}$ (Check!)

So, $\frac{dp}{p} = -\frac{\bar{c}A}{4V}dt$

Take, $p = p_0$ at $t = 0$;

Integrate to get;

$$p = p_0 e^{-\frac{\bar{c}A}{4V}t};$$

And, $p = p_0/e$ at $t = \frac{4V}{\bar{c}A}$;

This time is called the characteristic / relaxation time in an exponential system.

7.

It talks about the probability distribution, so write that down first.

$$dn_c = 4\pi n \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-mc^2/2K_BT} c^2 dc$$

(Pg. 43 bottom)

Again, this is density of particles with speed $c \to c + dc$;

RMS speed: $\sqrt{\overline{c^2}} = \sqrt{\frac{3K_BT}{m}}$ (Pg. 43/44)

If this becomes the unit, then,

$$c' = \frac{c}{\sqrt{3K_BT/m}} \Rightarrow c = c'\sqrt{\frac{3K_BT}{m}};$$

Just replace c in the above expression:

$$dn_{c'} = 4\pi n \left(\frac{m}{2\pi K_BT}\right)^{3/2} e^{-\frac{m}{2K_BT}\cdot c'^2 \frac{3K_BT}{m}}$$

$$\times\ c'^2 \frac{3K_BT}{m} \cdot dc' \sqrt{\frac{3K_BT}{m}}$$

$$= 4\pi n \cdot \left(\frac{3}{2\pi}\right)^{3/2} e^{-\frac{3}{2}c'^2} c'^2 dc'$$

[Reduced form of Maxwell's distribution]

- This is independent of temperature; (and independent of type of gas)
- Note that the extra $T^{1/2}$ factor came from dc;
- Further, either mean $\bar{c}$ or most probable velocity (speed) c_m (see Pg. 44-top) can be used as the unit of speed; In all cases, the probability distribution would be independent of T;

8.

this one is easy + just substitute the numbers/expressions in the Maxwell's speed distribution.

# of molecules with speed between c_r → $c_r + dc$:

$$dN_r = 4\pi N \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-\frac{m}{2K_BT} \cdot c_r^2} c_r^2 \, dc$$

[Pg. 43 - bottom]

.. and between c_m → $c_m + dc$:

$$dN_m = 4\pi N \left(\frac{m}{2\pi K_B T}\right)^{3/2} e^{-\frac{m}{2K_BT} c_m^2} c_m^2 \, dc$$

We know, $c_r = \sqrt{\frac{3K_BT}{m}}$; $c_m = \sqrt{\frac{2K_BT}{m}}$

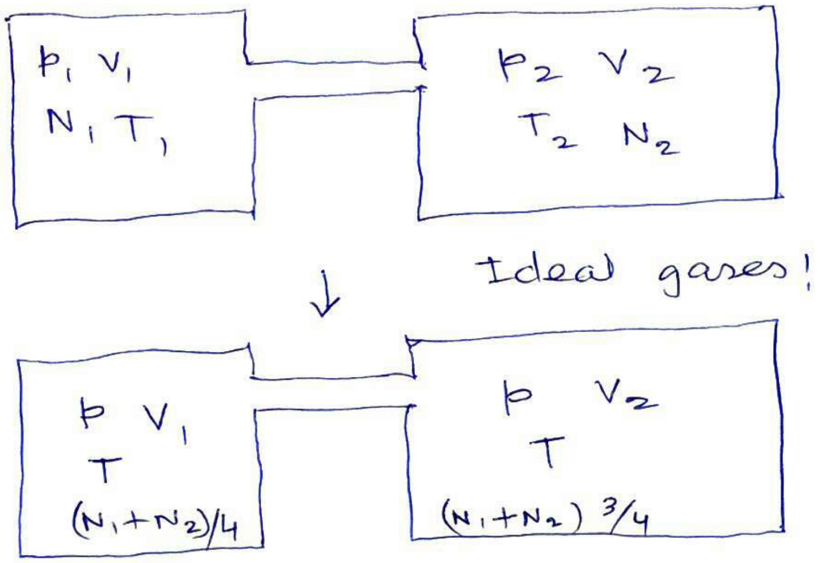
Then, the required ratio: $\frac{dN_r}{dN_m}$

$$= \frac{e^{-3/2}}{e^{-1}} \cdot \frac{3}{2} = 0.9097;$$

- Note this ratio is independent of temperature and type of gas.

Q: From the sketch of the distribution (Pg. 42) and location of c_m, $\bar{c}$, c_r - can you predict the above result (which one will be higher) without having knowledge of the actual distribution? In other words, is the above result dependent on the actual mathematical form of the distribution?

9.



$p_1 = 0.5$ atmos
$p_2 = 1.5$ atmos
$V_1 = 1$ litre
$V_2 = 2$ litre
$T_1 = 273$ K
$T_2 = 373$ K

→ Although it may look difficult at first sight, it is the easiest.

- Two containers are exchanging quantities. You need to know what they can be.
- Particles (N) → Yes; Energy (E) → Yes; Volume (V) → No!
- No other thing in bulk can be exchanged.

Now, just use Ideal gas law and get the answers (equilibrium p, T);

$$p_1 V_1 = N_1 k_B T_1 \quad ; \quad p_2 V_2 = N_2 k_B T_2$$

$$p(V_1 + V_2) = (N_1 + N_2) k_B T \, ;$$

- Four unknowns (N_1, N_2, p, T), and we have three equations. Fourth one? It is the energy conservation:

$$\frac{5}{2} N_1 k_B T_1 + \frac{3}{2} N_2 k_B T_2 = \frac{5}{2} N_1 k_B T + \frac{3}{2} N_2 k_B T.$$

(thinking about energy exchange reminds you of the above equation)

9. contd.
~~Well,~~

Well, the problem is over; Rest are just algebra and numerics.

From fourth:

$$T = \frac{\frac{5}{2} N_1 K_B T_1 + \frac{3}{2} N_2 K_B T_2}{\frac{5}{2} N_1 K_B + \frac{3}{2} N_2 K_B}$$

Using ideal gas laws (first three),

$$T = \frac{\frac{5}{2} p_1 V_1 + \frac{3}{2} p_2 V_2}{\frac{5}{2} \frac{p_1 V_1}{T_1} + \frac{3}{2} \frac{p_2 V_2}{T_2}}$$

and,

$$p = \frac{N_1 K_B + N_2 K_B}{V_1 + V_2} T$$

$$= \frac{\frac{p_1 V_1}{T_1} + \frac{p_2 V_2}{T_2}}{V_1 + V_2} T \; ;$$

Putting the numbers (do it yourself - you may use any unit, but why?),

$T = 352.8$ K, $p = 1.22$ atmos;

Note 1: Nitrogen is diatomic (dof = 5), Argon is monatomic (dof = 3);

Note 2: ~~One of~~ Had the volume changed for any one of containers, you had to consider the energy spent/gained for the expansion/contraction. As the volumes did not change, specific heat is c_v here; Only molecular K.E. based on temperature → sufficient.

Note 3: Why can you write $N_1' = \frac{N_1 + N_2}{4}$?

10.

Talks about # of molecules hitting unit area — we straight go to Notes: Pg. 13:

$$\delta n = \frac{n\,\delta v\,\delta s\cos\theta}{4\pi r^2}$$; # of molecules running ~~towards~~ δs;

Use the Maxwells distribution: (Pg. 40 - top)

$$dn_c = \frac{n}{\alpha^3\pi^{3/2}} e^{-c^2/\alpha^2} c^2 \sin\theta\, d\theta\, d\phi\, dc$$

(is the # w/ speed between $c \to c+dc$, or w/ speed c, for angles θ, ϕ)

$$= \frac{4n}{\alpha^3\pi^{1/2}} e^{-c^2/\alpha^2} c^2 dc$$ (for all angles)

so,

$$\delta n = \frac{dn_c\,\delta v\,\delta s\cos\theta}{4\pi r^2}$$; $[\delta v = r^2\sin\theta\, d\theta\, d\phi\, dr]$

~~Total~~ Kinetic energy:

$= \delta n \cdot \frac{1}{2} m c^2$; (for this set of molecules w/ speed c)

Therefore, total kinetic energy in time Δt; (for molecules w/ all velocities) [see, Pg. 13 - bot, same logic]

~~$= \frac{1}{2}mc^2 \frac{dn_c}{4\pi} \cdot \delta s \int_{r=0}^{c\Delta t} dr \int_{\theta=}$~~

$$= \int_{c=0}^{\infty} \frac{1}{2}mc^2 \cdot \frac{dn_c}{4\pi} \int_{r=0}^{c\Delta t} dr \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\phi \cdot \delta s$$

$$= \int_{c=0}^{\infty} \frac{1}{2}mc^2 \cdot \frac{1}{4\pi} \frac{4n}{\alpha^3\sqrt{\pi}} \cdot e^{-c^2/\alpha^2} c^2 \cdot c\Delta t \cdot \frac{1}{2} \cdot 2\pi \cdot \delta s$$

At unit area in unit time,

$$= \frac{mn}{2\sqrt{\pi}\alpha^3} \int_0^{\infty} e^{-c^2/\alpha^2} c^5\, dc$$

10. contd.

Use the table in Pg. 38:

$\Rightarrow$ Total K.E

$$= \frac{mn}{2\sqrt{\pi}\,\alpha^3} \cdot \alpha^6$$

$$= \frac{mn}{2\sqrt{\pi}}\,\alpha^3 ; \qquad \left[\alpha = \sqrt{\frac{2K_BT}{m}} \;;\; \text{Pg. 44-top}\right]$$

$$= \frac{mn}{2\sqrt{\pi}} \cdot \frac{2\sqrt{2}\,(K_BT)^{3/2}}{m^{3/2}} = \frac{\cancel{m}\, n\sqrt{2}}{\sqrt{\pi}.\sqrt{m}} \cdot (K_BT)^{3/2} ;$$

11. Asks for mean free path, so first write the formula of λ;

$$\lambda = \frac{1}{n\pi\sigma^2} ; \qquad [\text{Pg. 22-bot, 23}]$$

σ is given, need to know n;
p is asked for, T is given; Then,

$$n = \frac{p}{K_BT} ;$$

So,
$$\lambda = \frac{K_BT}{p\pi\sigma^2} ,$$

or,
$$p = \frac{K_BT}{\lambda\pi\sigma^2} ;$$

$T = 273\,K$; $\sigma = 3\times10^{-10}\,m$;
$\lambda = 0.1\,m$;

So, $p = 0.133\,N/m^2 = 10^{-4}$ cm of Hg

[Note: 1 atmos: $76\times13.6\times981$ dyne/cm^2
$= 76\times13.6\times981\times10^4/10^5\ N/m^2$
$= 1.01396\times10^5\ N/m^2$]