α 振幅

$\vec{F} = (-T\cos\varphi + mg,\ -T\sin\varphi)$

$\vec{S} = (x(t),\ y(t)) = (l\cos\varphi,\ l\sin\varphi)$

$$\vec{a} = \frac{d^2\vec{S}}{dt^2} = \left(\frac{d^2x(t)}{dt^2},\ \frac{d^2y(t)}{dt^2}\right) =$$

$$= \frac{d}{dt}\left(-l\dot{\varphi}\sin\varphi,\ l\dot{\varphi}\cos\varphi\right)$$

$$= \left(-l\ddot{\varphi}\sin\varphi - l\dot{\varphi}^2\cos\varphi,\ l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2\sin\varphi\right)$$

$\vec{F} = m\vec{a} \Rightarrow$

$$ml\begin{bmatrix}-\sin\varphi\\\cos\varphi\end{bmatrix}\ddot{\varphi} - ml\begin{bmatrix}\cos\varphi\\-\sin\varphi\end{bmatrix}\dot{\varphi}^2 = \begin{bmatrix}-T\cos\varphi + mg\\-T\sin\varphi\end{bmatrix}$$

完整运
动方程

x 分量 $\times (-\sin\varphi) + y$ 分量 $\times (\cos\varphi)$

$$ml\ddot{\varphi} = -mg\sin\varphi \xrightarrow[\omega = \sqrt{\frac{g}{l}}]{} \frac{d^2\varphi}{dt^2} = -\omega^2 s_i$$

切线方向的运动方程 $\left(\begin{array}{c}切向加速度\\a_\tau = l\ddot{\varphi}\end{array}\right)$

φ 较小时，$\sin\varphi \approx \varphi$，方程简化为

$$\frac{d^2\varphi}{dt^2} = -\omega^2\varphi \implies \varphi(t) = C_1\cos\omega t + C_2\sin\omega t$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{\ell}}} = 2\pi\sqrt{\frac{\ell}{g}}$$

简谐
振动

精确解：
$$\boxed{\frac{d^2\varphi}{dt^2} + \omega^2\sin\varphi = 0}$$

两边同乘 $\frac{d\varphi}{dt}$，得

$$\frac{d^2\varphi}{dt^2}\cdot\frac{d\varphi}{dt} + \omega^2\sin\varphi\frac{d\varphi}{dt} = 0 \quad 即$$

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - \omega^2\cos\varphi\right] = 0 \quad (初积分$$

$$\implies E = \frac{m\ell}{2}\left(\frac{d\varphi}{dt}\right)^2 - mg\ell\cos\varphi = 常数 \quad \text{运动积}$$

总能量　　动能　　势能　　求能量积

$$\varphi(t_0) = \alpha, \quad \dot\varphi(t_0) = 0, \implies$$

$$\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - \omega^2\cos\varphi = -\omega^2\cos\alpha \quad \Longleftrightarrow$$

$$\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 = \omega^2(\cos\varphi - \cos\alpha) = 2\omega^2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}\right)$$

$$|\varphi| \leq \alpha < \pi \Rightarrow |\sin\frac{\varphi}{2}| \leq$$
$$\leq \sin\frac{\alpha}{2}$$

$$\Downarrow$$

$$\frac{d\varphi}{dt} = 2\omega\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}}$$

令 $k := \sin\frac{\alpha}{2}$, 记 Θ 有 $\sin\frac{\varphi}{2} = k\sin\Theta$ 为新变量.

则方程变为 $\frac{d\varphi}{dt} = \frac{d\varphi}{d\Theta}\frac{d\Theta}{dt} = 2\omega\sqrt{k^2 - k^2\sin^2\Theta}$

$$= 2k\omega\cos\Theta$$

又对 $\sin\frac{\varphi}{2} = k\sin\Theta$ 两边对 Θ 求导, 得

$$k\cos\Theta = \frac{1}{2}\cos\frac{\varphi}{2}\frac{d\varphi}{d\Theta} = \frac{\sqrt{1 - k^2\sin^2\Theta}}{2}\frac{d\varphi}{d\Theta}.$$

$$\Rightarrow \frac{d\varphi}{d\Theta} = \frac{2k\cos\Theta}{\sqrt{1 - k^2\sin^2\Theta}}$$

$$\frac{2k\cos\Theta}{\sqrt{1 - k^2\sin^2\Theta}} = 2k\omega.$$

$$\frac{2k\cos\Theta}{\sqrt{1 - k^2\sin\Theta}}\frac{d\Theta}{dt} = 2k\omega\cos\Theta$$

$$\Rightarrow \frac{d\Theta}{\sqrt{1 - k^2\sin\Theta}} = \omega\,dt \Rightarrow \int_{\Theta(0)}^{\Theta(t)}\frac{d\Theta}{\sqrt{1 - k^2\sin^2\Theta}} = \omega t$$

设 $\Theta(0)=0$, $(\varphi(0)=0)$. 则为

$$t(\Theta)=\frac{1}{\omega}\int_0^\Theta \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad \cancel{\cos t}$$

单摆周期 $T=4\times(\Theta$ 从 0 到 $\frac{\pi}{2}$ 的时间)（Θ 从 0 到 $\frac{\pi}{2}$ 的时间）

$$=4\cdot\frac{1}{\omega}\int_0^{\frac{\pi}{2}}\frac{d\Theta}{\sqrt{1-k^2\sin^2\Theta}}=$$

$$=2\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6+\cdots\right]\approx 2\pi\sqrt{\frac{l}{g}}$$

$$k=\sin\frac{\alpha}{2} \quad (\alpha\ 小时)$$

$$\approx\frac{\alpha}{2}\approx 0$$