hw_6

必做题：

1. 设 $f(x)$ 在 $\mathbb{R}$ 上可导，利用导数的定义计算下列各式的值

a) $\lim_{x\to 0}\frac{f(x)}{x}$，**其中** $f(0)=0$**;**

解:

原式 = $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)$

b) $\lim_{\Delta x\to 0}\frac{f(x_0-\Delta x)-f(x_0)}{\Delta x}$

解:

原式 = $\lim_{\Delta x\to 0}\frac{f(x_0-\Delta x)-f(x_0)}{\Delta x}=-f'(x_0)$

c) $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{h}$**;**

解:
原式 = $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{h}=2f'(x_0)$

d) $\lim_{h\to 0}\frac{f^2(x_0+3h)-f^2(x_0-h)}{h}$

解:
原式 =
$\lim_{h\to 0}\frac{f^2(x_0+3h)-f^2(x_0-h)}{h}=\lim_{h\to 0}\frac{[f(x_0+3h)-f(x_0-h)][f(x_0+3h)+f(x_0-h)]}{h}=4f'(x_0)\cdot 2f(x_0)=8f(x_0)f'(x_0)$

2. 求下列函数在 x_0 处的左、右导数，并指出它在该点的可导性。

a)

$f(x)=\begin{cases}\sin x, & x\ge 0\\ x^2, & x<0\end{cases}\ x_0=0$

解:

$$f'_-(0)=\lim_{x\to 0^-}\frac{x^2-0}{x-0}=0$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{\sin x-0}{x-0}=\cos 0=1$$

$$f'_-(0)\neq f'_+(0)\quad \text{不可导}$$

b)

$$f(x)=\begin{cases}\frac{x}{1+e^{\frac{1}{x}}}, & x\neq 0,\\ 0, & x=0\end{cases}\quad x_0=0$$

解:

$$f'_-(0)=\lim_{x\to 0^-}\frac{\frac{x}{1+e^{1/x}}-0}{x-0}=\lim_{x\to 0^-}\frac{1}{1+e^{1/x}}=1$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{\frac{x}{1+e^{1/x}}-0}{x-0}=\lim_{x\to 0^+}\frac{1}{1+e^{1/x}}=0$$

$$f'_-(0)\neq f'_+(0)\quad \text{不可导}$$

c)

$$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}\quad x_0=0$$

解:

$$f'_-(0)=\lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0}=\lim_{x\to 0^-}x\sin\frac{1}{x}=0$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0}=\lim_{x\to 0^+}x\sin\frac{1}{x}=0$$

$$f'_-(0)=f'_+(0)=0\quad \text{可导}$$

d)

$$f(x)=\begin{cases}x^2, & x\leq 1\\ ax+b, & x>1\end{cases}\quad x_0=1$$

解:

$$f'_-(1)=\lim_{x\to 1^-}\frac{x^2-1}{x-1}=2$$

$$f'_+(1)=\lim_{x\to 1^+}\frac{ax+b-1}{x-1}=a$$

$a=2, b=-1$ 时可导，否则不可导

3. 求导数

a)

$f(x)=\begin{cases}x^2e^{-x^2}, & |x|\le 1,\\ \frac{1}{e}, & |x|>1\end{cases}$

解:
a) $|x|>1$
$f'(x)=0$

$|x|\le 1$

$$\begin{aligned}f'(x)&=2xe^{-x^2}+x^2(-2x)e^{-x^2}\\&=(2x-2x^3)e^{-x^2}\\&=2x(1-x^2)e^{-x^2}\end{aligned}$$

$$f'(x)=\begin{cases}2x(1-x^2)e^{-x^2}, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

b)

$f(x)=\arccos\frac{1}{|x|}$

解: $\frac{1}{|x|}\in(0,1]$
当 $x>1$

$$f'(x)=-\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)$$

$$=\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}=\frac{1}{\sqrt{x^4-x^2}}$$

当 $x<-1$

$$f'(x)=-\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(\frac{1}{x^2}\right)=-\frac{1}{\sqrt{x^4-x^2}}$$

$$f'(x)=\begin{cases}\frac{1}{\sqrt{x^4-x^2}}, & x>1\\ -\frac{1}{\sqrt{x^4-x^2}}, & x<-1\end{cases}$$

$x=\pm 1$ 时不可导

4. 按定义求导数

a) $f(x) = x^2 + 4x + 200$;

解:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 4(x+\Delta x) + 200 - (x^2 + 4x + 200)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + 4 + \Delta x)$$

$$= 2x + 4$$

b) $f(x) = x \sin x$

解:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)\sin(x+\Delta x) - x\sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[x \cdot \frac{\sin(x+\Delta x) - \sin x}{\Delta x} + \sin(x+\Delta x) \right]$$

$$= x\cos x + \sin x$$

5. 计算导数

a) $y = \frac{1-\sqrt[3]{2x-1}}{1+\sqrt[3]{2x-1}}$;

解:

$$y = -1 + \frac{2}{1+\sqrt[3]{2x-1}}$$

$$y' = -\frac{2}{(1+\sqrt[3]{2x-1})^2} \cdot \frac{2}{3}(2x-1)^{-\frac{2}{3}}$$

$$= -\frac{4}{3(2x-1)^{\frac{2}{3}}(1+\sqrt[3]{2x-1})^2}$$

b) $y=\sin(\cos^2 x)\cdot\cos(\sin^2 x)$

解:

$$\begin{aligned} y' &= \cos(\cos^2 x)\cdot 2\cos x(-\sin x)\cdot\cos(\sin^2 x)+\sin(\cos^2 x)\cdot[-\sin(\sin^2 x)\cdot 2\sin x\cos x] \\ &= -2\sin x\cos x[\cos(\cos^2 x)\cos(\sin^2 x)+\sin(\cos^2 x)\sin(\sin^2 x)] \\ &= -2\sin x\cos x\cos(\cos^2 x-\sin^2 x) \end{aligned}$$

c) $y=\frac{x}{2}\sqrt{x^2+a}+\frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$**;**

解:

$$\begin{aligned} y' &= \sqrt{x^2+a}+\frac{x}{2}\cdot\frac{1}{2\sqrt{x^2+a}}\cdot x+\frac{a^2}{2}\cdot\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{x}{\sqrt{x^2+a^2}}\right) \\ &= \frac{2x^2+a}{2\sqrt{x^2+a}}+\frac{a^2}{2\sqrt{x^2+a^2}} \end{aligned}$$

d) $y=x\arcsin(\ln x)$

解:

$$\begin{aligned} y' &= \arcsin(\ln x)+x\cdot\frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{1}{x} \\ &= \arcsin(\ln x)+\frac{1}{\sqrt{1-(\ln x)^2}} \end{aligned}$$

e) $y=e^{\arctan\sqrt{x}}$**;**

解:

$$y'=e^{\arctan\sqrt{x}}\cdot\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}$$

f) $y=10^{x\tan x^2}$

解:

$$\begin{aligned} y' &= 10^{x\tan x^2}\ln 10\cdot(\tan x^2+x\cdot\frac{1}{\cos^2 x^2}\cdot 2x) \\ &= 10^{x\tan x^2}\ln 10\cdot(\tan x^2+\frac{2x^2}{\cos^2 x^2}) \end{aligned}$$

6. 用对数求导法求导数

a) $y=\frac{(x+1)^2\sqrt[3]{4x+3}}{\sqrt[3]{2x^2+2x+1}}$;

解:

$$\ln y=2\ln(x+1)+\frac{1}{3}\ln(4x+3)-\frac{1}{3}\ln(2x^2+2x+1)$$

$$\frac{y'}{y}=\frac{2}{x+1}+\frac{4}{3(4x+3)}-\frac{4x+2}{3(2x^2+2x+1)}$$

$$y'=\frac{(x+1)^2\sqrt[3]{4x+3}}{\sqrt[3]{2x^2+2x+1}}\left[\frac{2}{x+1}+\frac{4}{3(4x+3)}-\frac{4x+2}{3(2x^2+2x+1)}\right]$$

b) $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$

解:
$y_1=(\sin x)^{\cos x}$, $y_2=(\cos x)^{\sin x}$

$$\frac{y_1'}{y_1}=-\sin x\ln\sin x+\cos x\cdot\frac{\cos x}{\sin x}$$

$$\frac{y_2'}{y_2}=\sin x\ln\cos x+\cos x\cdot\left(-\frac{\sin x}{\cos x}\right)$$

$$y'=y_1'+y_2'=(\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x}-\sin x\ln\sin x\right)+(\cos x)^{\sin x}\left(\sin x\ln\cos x-\frac{\sin^2 x}{\cos x})\right)$$

c) $y=\left(\frac{x}{1+x}\right)^x$;

解:

$$\ln y=x\ln\left(\frac{x}{1+x}\right)$$

$$\frac{y'}{y}=\ln\left(\frac{x}{1+x}\right)+x\cdot\frac{1+x}{x}\cdot\frac{1}{(1+x)^2}$$

$$y'=\left(\frac{x}{1+x}\right)^x\left(\ln\frac{x}{1+x}+\frac{1}{1+x}\right)$$

d) $y=\sqrt{x\sin x\sqrt{1-e^x}}$

解:

$$\ln y=\frac{1}{2}\ln x+\frac{1}{2}\ln\sin x+\frac{1}{4}\ln(1-e^x)$$

$$\frac{y'}{y} = \frac{1}{2x} + \frac{\cos x}{2\sin x} - \frac{e^x}{4(1-e^x)}$$

$$y' = \sqrt{x\sin x\sqrt{1-e^x}}\left[\frac{1}{2x} + \frac{\cos x}{2\sin x} - \frac{e^x}{4(1-e^x)}\right]$$

7. 设 $f(x)$ 是定义在 $(-1,1)$ 上的连续正值函数，且 $f(0)=1$，$f'(0)=2$，计算

$$\lim_{x\to 0}(f(x))^{\frac{1}{x}}$$

解:

取对数:

$$\ln\left((f(x))^{\frac{1}{x}}\right) = \frac{\ln f(x)}{x}$$

当 $x\to 0$ 时,

$$f(x) = f(0) + f'(0)x + o(x) = 1 + f'(0)x + o(x)$$

$$\ln f(x) = \ln(1 + f'(0)x + o(x)) \sim f'(0)x + o(x)$$

$$\frac{\ln f(x)}{x} \sim \frac{f'(0)x + o(x)}{x} = f'(0) + o(1)$$

所以:

$$\lim_{x\to 0}\frac{\ln f(x)}{x} = f'(0) = 2$$

$$\lim_{x\to 0}(f(x))^{\frac{1}{x}} = e^2$$

8. 计算函数的微分

a) $y = \ln\tan\frac{x}{2}$;

解:

$$\frac{dy}{dx} = \frac{1}{\tan\frac{x}{2}}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}$$

$$dy = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx = \frac{1}{\sin x}dx$$

b) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$

解:

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)\right)$$

$$dy=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]dx$$

c) $y=\arctan\frac{v}{u}$ **(**v,u **用** du,dv **表示之);**

解:

$$dy=\frac{1}{1+\frac{v^2}{u^2}}d\left(\frac{v}{u}\right)$$

$$dy=\frac{u^2}{u^2+v^2}\cdot\frac{udv-vdu}{u^2}$$

$$=\frac{udv-vdu}{u^2+v^2}$$

d) $y=\ln(\ln(x))$

解:

$$dy=\frac{1}{\ln x}\cdot\frac{1}{x}dx$$

$$dy=\frac{1}{x\ln x}dx$$

9. 证明函数 $y=f(x)$ 的反函数的二阶导数公式

$$\frac{d^2x}{dy^2}=-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

解:

$$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\frac{dx}{dy}=-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{1}{\frac{dy}{dx}}=-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

即

$$\frac{d^2x}{dy^2}=-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

10. 计算下列函数反函数的一阶导数和二阶导数

a) $\theta = r\arctan r$;

解:

$$\frac{dr}{d\theta} = \frac{1}{\arctan r + \frac{r}{1+r^2}}$$

$$\frac{d^2r}{d\theta^2} = -\frac{2}{(1+r^2)^2\left(\arctan r + \frac{r}{1+r^2}\right)^3}$$

b) $y = \frac{1}{2}\ln\frac{1-x}{1+x}$

解:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\frac{1}{1-x^2}} = -1 + x^2$$

$$\frac{d^2x}{dy^2} = -2x(1-x^2)$$

c) $y = e^{\arcsin x}$;

解:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{e^{\arcsin x}}{\sqrt{1-x^2}}} = \frac{\sqrt{1-x^2}}{e^{\arcsin x}}$$

$$\frac{d^2x}{dy^2} = \frac{-x-\sqrt{1-x^2}}{e^{2\arcsin x}}$$

d) $y = 2x - \cos\frac{x}{2}$

解:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2+\frac{1}{2}\sin\frac{x}{2}} = \frac{2}{4+\sin\frac{x}{2}}$$

$$\frac{d^2x}{dy^2} = -\frac{2\cos\frac{x}{2}}{(4+\sin\frac{x}{2})^3}$$

11.

a) 设 $y = e^x\cos x$，求 $y^{(4)}$;

解:

$$y = e^x\cos x$$

$$y' = e^x\cos x - e^x\sin x$$

$$y'' = (e^x \cos x - e^x \sin x) - (e^x \sin x + e^x \cos x) = -2e^x \sin x$$

$$y''' = -2e^x \sin x - 2e^x \cos x$$

$$y^{(4)} = (-2e^x \sin x - 2e^x \cos x) + (2e^x \sin x - 2e^x \cos x) = -4e^x \cos x$$

b) 设 $y = (x+1)^2 e^{2x}$，求 $y^{(100)}$

解: $y = (x+1)^2 e^{2x}$

$$y^{(n)} = 2^n e^{2x} \left[(x+1)^2 + n(x+1) + \frac{n(n-1)}{4} \right]$$

$$y^{(100)} = 2^{100} e^{2x} \left(x^2 + 102x + 2576 \right)$$

12. 求 n 阶导数

a) $y = \frac{1-x}{1+x}$;

解:

$$y = \frac{1-x}{1+x} = -1 + \frac{2}{1+x}$$

$$y^{(n)} = 2(-1)^n n!(1+x)^{-n-1}$$

b) $y = \frac{1}{x^2-3x+2}$

解:

$$y = \frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$y^{(n)} = (-1)^n n! \left[(x-2)^{-n-1} - (x-1)^{-n-1} \right]$$

c) $y = x \ln x$;

解:

$$y = x \ln x$$

$$y' = \ln x + 1, \quad y'' = \frac{1}{x}$$

$$y^{(n)} = (-1)^n (n-2)! x^{-n+1}, \quad n \geq 2$$

d) $y = \sin^2 x$

解:

$$y = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$y^{(n)} = -2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right)$$

13.

(a) 设函数 $f(x)$ 在 $x = 0$ 处连续，$f^2(x)$ 在 $x = 0$ 处的导数为 A，讨论 $f(x)$ 在 $x = 0$ 处的可导性;

解:

$$A = f^2(0)' = 2f(0)f'(0)$$

当 $f(0) = 0$，$A = 0$ $f(x)$ 不一定可导

当 $f(0) \neq 0$，$f'(0) = \frac{A}{2f(0)}$ 可导

(b) 设 $xf(x)$ 在 $x_0(\neq 0)$ 处可导，证明 $f(x)$ 在 x_0 处可导

解:

$$g(x) = xf(x)$$

$$g'(x_0) = \lim_{x \to x_0} \frac{xf(x) - x_0 f(x_0)}{x - x_0}$$

$$= x_0 \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + f(x_0)$$

$x_0 \neq 0$，$g'(x)$ 存在

故 $f(x)$ 在 x_0 处可导

14. 设 $f(x)$ 是偶函数，且 $f'(0)$ 存在，证明: $f'(0) = 0$

解:

$$f(x) = f(-x)$$

$$f'(-x) = -f'(x)$$

令 $x = 0$:

$$f'(0) = -f'(0)$$

故

$$f'(0) = 0$$

选做题:

1. 若 在 a 处连续，且 $F(x) \neq 0$，试讨论函数 $f(x) = |x-a|F(x)$ 在 $x=a$ 处的可导性

解:

$$f'_-(a) = \lim_{x\to a^-} \frac{|x-a|F(x)-0}{x-a} = \lim_{x\to a^-} \frac{-(x-a)F(x)}{x-a} = -F(a)$$

$$f'_+(a) = \lim_{x\to a^+} \frac{|x-a|F(x)-0}{x-a} = \lim_{x\to a^+} \frac{(x-a)F(x)}{x-a} = F(a)$$

由于 $F(a) \neq 0$, $f'_-(a) \neq f'_+(a)$

故在 $x=a$ 处不可导

2. 若 $\forall x, y \in \mathbb{R}$ 有 $f(x+y) = f(x)+f(y)+2xy$，且 $f'(0)$ 存在，求 $f'(x)$

解:

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{[f(x)+f(\Delta x)+2x\Delta x]-f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{f(\Delta x)}{\Delta x} + 2x \\ &= f'(0)+2x \end{aligned}$$

3. 按定义证明: 可导的偶函数的导函数是奇函数，可导的奇函数的导函数是偶函数；可导的周期函数的导函数仍然是周期函数，且周期不变

解:

① 偶函数：$f(x) = f(-x)$

$$f'(-x) = -f'(x)$$

② 奇函数：$f(-x) = -f(x)$

$$f'(-x) = f'(x)$$

③ 周期函数：$f(x+T) = f(x)$

$$f'(x+T) = f'(x)$$