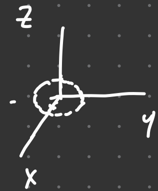
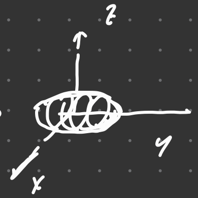
解析几何 补充内容

回忆: $\Gamma: \begin{cases} \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ x = 0 \end{cases}$ 绕 y-轴旋转



$$\xrightarrow[z \mapsto \sqrt{z^2+x^2}]{y\text{值不变}}$$



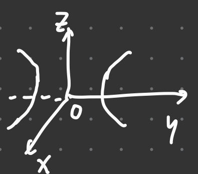
$$\frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{x^2}{c^2} = 1$$

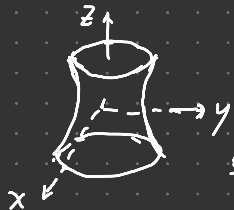
标度改变 Scaling: $x \mapsto \dfrac{cx}{a}$

椭球: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$\Gamma: \begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 \\ x = 0 \end{cases}$ $\xrightarrow[z\text{不变}.\ y \mapsto \sqrt{y^2+x^2}]{\text{绕}z\text{轴旋转}}$ $\dfrac{x^2}{b^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

$x \mapsto \dfrac{b}{a}x$





单叶双曲面: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

z = 常数 $\rightsquigarrow$ 椭圆截面

x = 常数

y = 常数 $\rightsquigarrow$ 双曲线截面

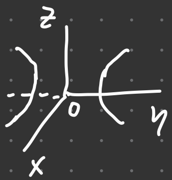
$$\frac{x^2}{a^2} + \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$$

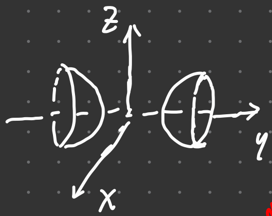
$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$$

都表示单叶双曲面.

$$\Gamma:\begin{cases}\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1\\ x=0\end{cases}\quad \xrightarrow[y\text{不变},\ z\to\sqrt{z^2+x^2}]{\text{绕}y\text{轴旋转}}\quad \frac{y^2}{b^2}-\frac{z^2}{c^2}-\frac{x^2}{c^2}=1$$

$$\Big\downarrow\ x\mapsto \frac{c}{a}x$$





双叶双曲面: $\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}-\dfrac{x^2}{a^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

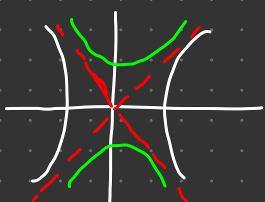
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

渐近锥面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

退化 $\Big\downarrow\ z=0$

类比: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=\pm1$ $\rightsquigarrow$ $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0$

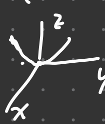


$\Updownarrow$

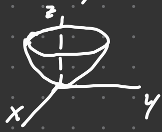
$b^2x^2=a^2y^2$

$bx=\pm ay$ 渐近线

$$\Gamma:\begin{cases}y^2=2pz\\ x=0\end{cases}\quad \overset{\text{绕}z\text{轴}}{\rightsquigarrow}\quad x^2+y^2=2pz,\ \text{即}\quad 2z=\frac{x^2+y^2}{p}$$



$\rightsquigarrow$



一般化

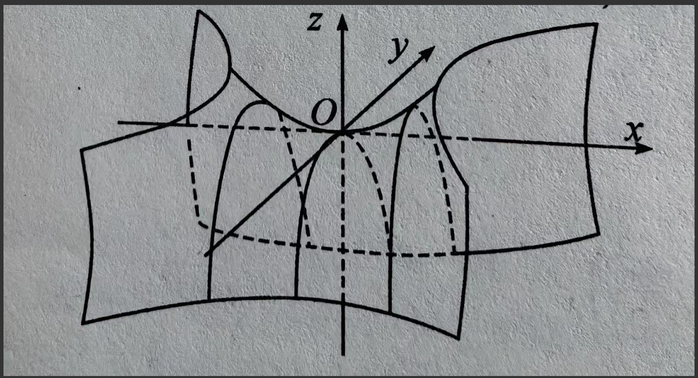
椭圆抛物面: $2z=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$

双曲抛物面（马鞍面）

$$2z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

z-截 = 双曲线 $\begin{cases} z>0 & \text{开口朝向 } x \\ z<0 & \text{开口朝向 } y \\ z=0 & \text{退化为渐近线} \end{cases}$

x-截
y-截 = 抛物线



$$2z = \frac{x^2}{a^2} - \frac{y^2}{b^2} =$$

$$= \underbrace{\left(\frac{x}{a} + \frac{y}{b}\right)}_{2\lambda} \underbrace{\left(\frac{x}{a} - \frac{y}{b}\right)}_{2\mu}$$

$$\begin{cases} \frac{x}{a} + \frac{y}{b} = 2\lambda \\ \frac{x}{a} - \frac{y}{b} = 2\mu \end{cases} \Rightarrow \begin{cases} x = a(\lambda + \mu) \\ y = b(\lambda - \mu) \\ z = 2\lambda\mu \end{cases}$$ 参数方程

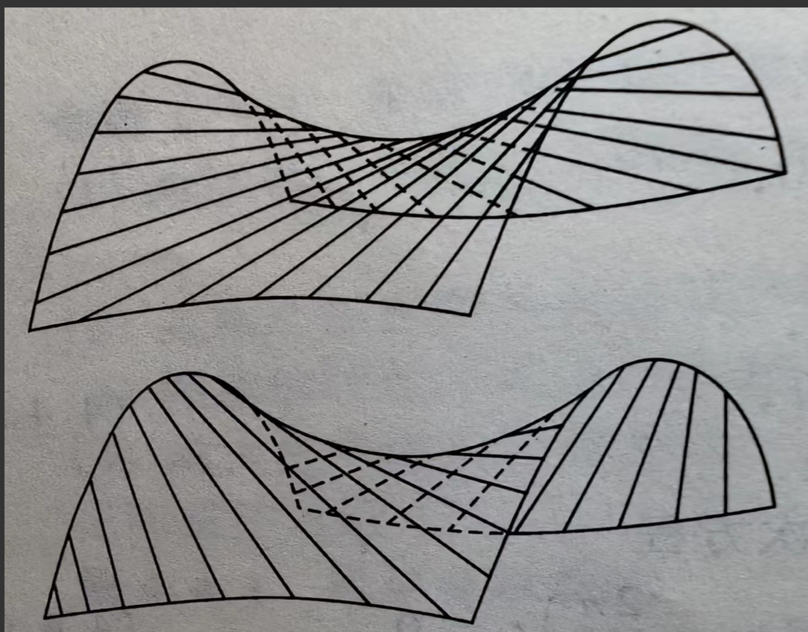
μ = 常数 $\rightsquigarrow$ λ-直线族

λ = 常数 $\rightsquigarrow$ μ-直线族

马鞍面上两族直线的等价刻画

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z \iff \left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z$$

$$\Rightarrow \begin{cases} \frac{x}{a}+\frac{y}{b}=2\lambda \\ \lambda\left(\frac{x}{a}-\frac{y}{b}\right)=z \end{cases} \quad 或 \quad \begin{cases} \frac{x}{a}-\frac{y}{b}=2u \\ u\left(\frac{x}{a}+\frac{y}{b}\right)=z \end{cases}$$



二次曲面中，类似上 双叶双曲面那样，可由两族直线生成的曲面（直纹面）还有单叶双曲面.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \iff \frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$$

$$\iff \left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right)$$

$$\implies \begin{cases} \lambda_1\left(\frac{x}{a}+\frac{z}{c}\right)=\lambda_2\left(1+\frac{y}{b}\right) \\ \lambda_2\left(\frac{x}{a}-\frac{z}{c}\right)=\lambda_1\left(1-\frac{y}{b}\right) \end{cases}$$

对$\lambda_1:\lambda_2$的每个值，对应曲面上一直线

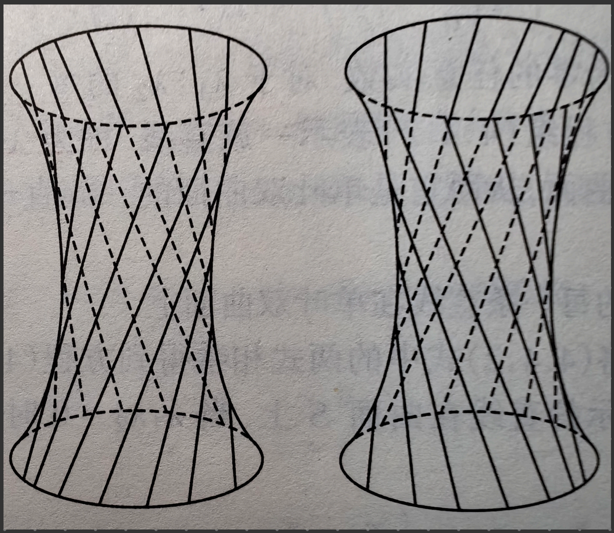
称为曲面上的λ-族直线

另，$\begin{cases} u_1\left(\frac{x}{a}+\frac{z}{c}\right)=u_2\left(1-\frac{y}{b}\right) \\ u_2\left(\frac{x}{a}-\frac{z}{c}\right)=u_1\left(1+\frac{y}{b}\right) \end{cases}$

对$u_1:u_2$的每个值，对应曲面上一条直线，称为曲面上的u-族直线

- 对曲面上任一点，λ和u-族中各有唯一一条直线过此点.

- 同族直线中任两条都异面
- 两族直线中无公共直线



例. 求单叶双曲面 $\frac{x^2}{4}+\frac{y^2}{1}-\frac{z^2}{9}=1$ 上过 $P(2,1,3)$ 的两条直母线.

法一: $\begin{cases}\lambda_1\left(\frac{x}{2}+\frac{z}{3}\right)=\lambda_2(1+y)\\ \lambda_2\left(\frac{x}{2}-\frac{z}{3}\right)=\lambda_1(1-y)\end{cases}$; $\begin{cases}\mu_1\left(\frac{x}{2}+\frac{z}{3}\right)=\mu_2(1-y)\\ \mu_2\left(\frac{x}{2}-\frac{z}{3}\right)=\mu_1(1+y)\end{cases}$

将 $P(2,1,3)$ 代入, 得 $\lambda_1:\lambda_2=1$, $\mu_1=0$

故 $\begin{cases}\frac{x}{2}+\frac{z}{3}=1+y\\ \frac{x}{2}-\frac{z}{3}=1-y\end{cases}$, $\begin{cases}1-y=0\\ \frac{x}{2}-\frac{z}{3}=0\end{cases}$ 为所求.

法二. 过 $P(2,1,3)$ 直线的参数式为

$$\begin{cases} x = 2+lt \\ y = 1+mt \\ z = 3+nt \end{cases}$$ 代入曲面方程,得

$$\left(\frac{l^2}{4}+m^2-\frac{n^2}{9}\right)t^2+\left(l+2m-\frac{2n}{3}\right)t=0$$

若该直线位于曲面上. 则上二次方程应对任意 t 都成立, 从而有:

$$\begin{cases} \frac{l^2}{4}+m^2-\frac{n^2}{9}=0 \\ l+2m-\frac{2n}{3}=0 \end{cases} \Rightarrow \begin{matrix} l:m:n=2:0:3 \\ \text{或 } l:m:n=0:1:3 \end{matrix}$$

从而所求直线为:

$$\frac{x-2}{2}=\frac{y-1}{0}=\frac{z-3}{3} \quad \text{或} \quad \frac{x-2}{0}=\frac{y-1}{1}=\frac{z-3}{3}.$$