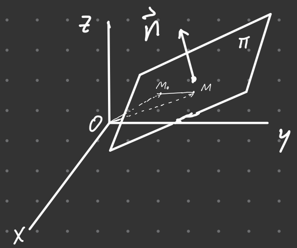
空间的平面和直线

平面：



$\vec{n}=(A,B,C)$

$M_0(x_0,y_0,z_0)$ π上给定点

$\forall M(x,y,z)$: $M\in\pi \iff \overrightarrow{M_0M}\perp\vec{n} \iff \overrightarrow{M_0M}\cdot\vec{n}=0$

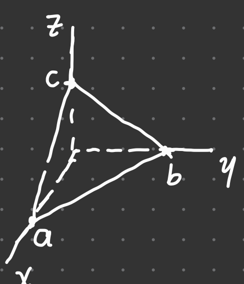
$(x-x_0,\ y-y_0,\ z-z_0)\cdot(A,B,C)=0$

点-法式：$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$$Ax+By+Cz\underbrace{-(Ax_0+By_0+Cz_0)}_{D}=0$$

$Ax+By+Cz+D=0$ （一般式）

- $D=0$，过原点
- $A=0$，平行于 x-轴
- $A=B=0$ 平行于 x-y 坐标面



设方程为：$Ax+By+Cz+D=0$

代入 $(x,y,z)=(a,0,0),(0,b,0),(0,0,c)$，得

$$\begin{cases}Aa+D=0\\Bb+D=0\\Cc+D=0\end{cases}\Rightarrow\begin{cases}A=-D/a\\B=-D/b\\C=-D/c\end{cases}$$

$\Rightarrow \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. 截距式方程.

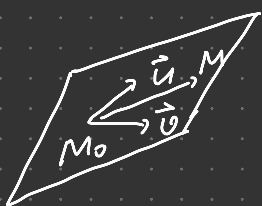
或 $A(a,0,0)$ $B(0,b,0)$ $C(0,0,c)$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & b & 0 \\ 0 & -b & c \end{vmatrix} = (bc, ac, ab)$$

$$bc(x-a) + acy + abz = 0$$

$$bcx + acy + abz = abc \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1.$$

• 若 π 过点 $M_0(x_0, y_0, z_0)$ 且 // 于两不共线向量 $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$



$\forall M(x,y,z)$, $\exists!\ \lambda, \mu$, 使得

$$\overrightarrow{M_0M} = \lambda \vec{u} + \mu \vec{v}$$

$$(x-x_0, y-y_0, z-z_0) = \lambda(u_1, u_2, u_3) + \mu(v_1, v_2, v_3)$$

即 $\begin{cases} x = x_0 + \lambda u_1 + \mu v_1 \\ y = y_0 + \lambda u_2 + \mu v_2 \\ z = z_0 + \lambda u_3 + \mu v_3 \end{cases}$ 平面的参数式方程.

或 $\vec{n} = \vec{u} \times \vec{v}$, $\overrightarrow{M_0M} \cdot \vec{n} = \overrightarrow{M_0M} \cdot \vec{u} \times \vec{v} = 0$

即 $[\overrightarrow{M_0M}, \vec{u}, \vec{v}] = 0$.

③

标准式方程: $\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$

例: $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ 不共线

$\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

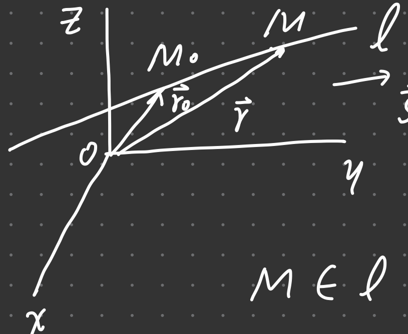
$\Longleftrightarrow \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$ 三点式方程

特别地, 取 $P_1(a, 0, 0)$ $P_2(0, b, 0)$ $P_3(0, 0, c)$

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$$(x-a)bc + yac + zab = 0 \Longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

空间直线方程



$\vec{s}$ (方向向量) $\qquad \vec{s} = (m, n, p)$

$M \in l \iff \overrightarrow{M_0M} \parallel \vec{s} \iff \exists t \in \mathbb{R}$, s.t. $\overrightarrow{M_0M} = t\vec{s}$

$(x-x_0, y-y_0, z-z_0) = t(m, n, p)$

$$\begin{cases} x = x_0 + tm \\ y = y_0 + tn \\ z = z_0 + tp \end{cases} \quad t \in \mathbb{R} \quad \text{参数式}$$

$\iff \vec{r} = \vec{r}_0 + t\vec{s}$ 向量式.

消参，得 $\boxed{\dfrac{x-x_0}{m} = \dfrac{y-y_0}{n} = \dfrac{z-z_0}{p}}$ 标准式 或点向式.

$$\iff \begin{cases} \dfrac{x-x_0}{m} = \dfrac{y-y_0}{n} \\ \dfrac{y-y_0}{n} = \dfrac{z-z_0}{p} \end{cases} \quad \text{两平面之交}$$

一般地 $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

一般式方程

$\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$, i.e. $A_1:B_1:C_1 \neq A_2:B_2:C_2$

例：不同两点 $P_1(x_1, y_1, z_1)$ 和 $P_2(x_2, y_2, z_2)$ 决定的直线　$\overrightarrow{P_1P_2} = (x_2-x_1, y_2-y_1, z_2-z_1)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ 两点式方程

例　$\frac{x-1}{4} = \frac{y+2}{0} = \frac{z-3}{-1} \Longleftrightarrow \begin{cases} y+2=0 \\ \frac{x-1}{4} = \frac{z-3}{-1} \end{cases}$

$\Longleftrightarrow \begin{cases} x = 1+4t \\ y = -2 \\ z = 3-t \end{cases}$.

例　$\begin{cases} 2x-3y+z-5=0 \\ 3x+y-2z-2=0 \end{cases}$

$\vec{n}_1 = (2, -3, 1)$

$\vec{n}_2 = (3, 1, -2)$

$\vec{s} = \vec{n}_1 \times \vec{n}_2 = (5, 7, 11)$

令 $z_0 = 0 \Rightarrow x_0 = 1, y_0 = -1$.

$\begin{cases} x = 1+5t \\ y = -1+7t \\ z = 11t \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 3 & 1 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{11}{2} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 0 & 11 & -7 & -11 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -\frac{10}{11} & 2 \\ 0 & 11 & -7 & -11 \end{array}\right] \longrightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{11} & 1 \\ 0 & 1 & -\frac{7}{11} & -1 \end{array}\right] \qquad \begin{cases} x - \frac{5}{11}z = 1 \\ y - \frac{7}{11}z = -1 \end{cases}$$

$$\begin{cases} x = 1 + \frac{5}{11}t \\ y = -1 + \frac{7}{11}t \\ z = t \end{cases} \qquad \frac{x-1}{5} = \frac{y+1}{7} = \frac{z}{11}$$

平面束： $l: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \rightarrow \pi_1 \\ A_2x + B_2y + C_2z + D_2 = 0 \rightarrow \pi_2 \end{cases}$ $(A_1:B_1:C_1 \neq A_2:B_2:C_2)$

$\mu\pi_1 + \nu\pi_2 = 0$, i.e., $\mu(A_1x + B_1y + C_1z + D_1) + \nu(A_2x + B_2y + C_2z + D_2) = 0$

轴线

$(\mu, \nu) = (1, 0) \rightsquigarrow \pi_1$

$(\mu, \nu) = (0, 1) \rightsquigarrow \pi_2$

$\pi \neq \pi_1, \pi_2$ $\qquad M_0(x_0, y_0, z_0) \in \pi$, $\notin \pi_1, \pi_2$

$$A_1x_0 + B_1y_0 + C_1z_0 + D_1 = a \neq 0$$

$$A_2x_0 + B_2y_0 + C_2z_0 + D_2 = b \neq 0$$

π: $b(A_1x + B_1y + C_1z + D_1) - a(A_2x + B_2y + C_2z + D_2) = 0$

若 $\mu \neq 0$, $\lambda = \frac{\nu}{\mu}$, $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$

例：求过直线 $l_1:\begin{cases} x+y-z+2=0 \\ 4x-3y+z+2=0 \end{cases}$ 且与直线

$l_2:\begin{cases} z=-x+1 \\ y=3 \end{cases}$ 平行的平面方程。

解. 在 l_1 的方程中，令 $y=0$. 解得 $x=-\frac{4}{5}$, $z=\frac{6}{5}$.

l_1 的方向向量为 $\vec{s}_1=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 4 & -3 & 1 \end{vmatrix}=(-2,-5,-7)$

l_2 的方向向量为 $\vec{s}_2=(-1,0,1)$

$$\begin{vmatrix} x+\frac{4}{5} & y & z-\frac{6}{5} \\ -2 & -5 & -7 \\ -1 & 0 & 1 \end{vmatrix}=0$$ 即

$$5x-9y+5z-2=0.$$

另解（利用平面束）设平面方程为：

$$\pi:\ \lambda(x+y-z+2)+\mu(4x-3y+z+2)=0$$

即 $(\lambda+4\mu)x+(\lambda-3\mu)y+(-\lambda+\mu)z+2(\lambda+\mu)=0$

$\pi \parallel (-1,0,1)$ 故 $-1(\lambda+4\mu)+1\cdot(-\lambda+\mu)=0$

即 $2\lambda+3\mu=0$，$\lambda:\mu=-3:2$

$$-3(x+y-z+2)+2(4x-3y+z+2)=0$$

即 $5x-9y+5z-2=0.$