作业 十 答案

1. 求下列定积分

$$a)\ \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}; \qquad b)\ \int_0^{\frac{\pi}{2}} |\sin x - \cos x| dx;$$

$$c)\ \int_0^{2\pi} \sqrt{1+\cos x}\, dx;\ (\text{提示：倍角公式}) \qquad d)\ \int_0^3 x^2 [x]\, dx$$

$$e)\ \int_{-5}^{2} \frac{dx}{\sqrt[3]{(x-3)^2}} \qquad f)\ \int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x\, dx$$

解： a) $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_0^{\frac{1}{2}} = \frac{\pi}{6}.$

$$b)\ \int_0^{\frac{\pi}{2}} |\sin x - \cos x| dx = \int_0^{\frac{\pi}{2}} \left|\sqrt{2} \sin\left(x - \frac{\pi}{4}\right)\right| dx = 2\sqrt{2} \int_0^{\frac{\pi}{4}} \sin y dy$$

$$= 4\sqrt{2} \sin y \Big|_0^{\frac{\pi}{0}} = 4\sqrt{2}$$

$$c)\ \int_0^{2\pi} \sqrt{1+\cos x} dx = \int_0^{2\pi} \sqrt{2\cos^2 \frac{x}{2}} dx = \int_0^{\pi} \sqrt{2} \left|\cos \frac{x}{2}\right| 2d\left(\frac{x}{2}\right)$$

$$= 2\sqrt{2} \cdot 2 \cdot \sin y \Big|_0^{\frac{\pi}{2}} = 4\sqrt{2}$$

$$d)\ \int_0^3 x^2 [x] dx = \int_0^1 0 dx + \int_1^2 x^2 dx + \int_2^3 2x^2 dx = 15$$

$$e)\ \int_{-5}^{2} \frac{dx}{\sqrt[3]{(x-3)^2}} = \int_{-5}^{2} (x-3)^{-\frac{2}{3}} dx \overset{y=x-3}{=\!=\!=} \int_{-8}^{1} y^{-\frac{2}{3}} dy = -3y^{\frac{1}{3}} \Big|_{-8}^{1} = 3$$

$$f)\ \int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x dx = \int_0^{\frac{\pi}{2}} 2 \sin x \cos^6 x dx = -\int_0^{\frac{\pi}{2}} \cos^6 x (-\sin x dx)$$

$$= -\int_1^0 2y^6 dy = \frac{2}{7}$$

2. 求下列定积分

$$a)\ \int_0^{\frac{1}{2}} \arcsin x dx; \qquad b)\ \int_0^{\frac{\pi}{4}} \sec^3 x dx$$

$$c)\ \int_0^{\frac{\pi}{2}} e^x \sin^2 x dx; \qquad d)\ \int_0^{\sqrt{\ln 2}} x^3 e^{-x^2} dx$$

$$e)\ \int_0^1 x\sqrt{(1-x^4)^3} dx; \qquad f)\ \int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x}$$

解： a) $\displaystyle\int_0^{\frac{1}{2}} \arcsin x\, dx = x \arcsin x\Big|_0^{\frac{1}{2}} - \int_0^1 \frac{xdx}{\sqrt{1-x^2}}$

$$= \frac{\arcsin 1/2}{2} + \frac{1}{2}\int_0^1 \frac{d(1-x^2)}{\sqrt{1-x^2}} = \frac{\pi}{8} + 2\sqrt{1-x^2}\Big|_0^1 = \frac{\pi}{8} - 2$$

$$b)\ \int_0^{\frac{\pi}{4}} \sec^3 x\, dx = \int_0^{\frac{\pi}{4}} \sec x d\tan x = \sec x \tan x\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2 x \sec x dx$$

$$= \sec\frac{\pi}{4}\tan\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\sec x\, dx = \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 x dx - \int_0^{\frac{\pi}{4}} \sec x dx$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 x dx - \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}}$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 x dx - \ln(\sqrt{2}+1) \quad \Longrightarrow \quad \int_0^{\frac{\pi}{4}} \sec^3 x dx = \frac{\sqrt{2}}{2} - \frac{1}{2}\ln(\sqrt{2}+1)$$

$$c)\ \int_0^{\frac{\pi}{2}} e^x \sin^2 x\, dx = e^x \sin^2 x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x 2\sin x \cos x dx$$

$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin 2x\, dx = e^{\frac{\pi}{2}} - \left(e^x \sin 2x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^2 2\cos 2x dx\right)$$

$$= e^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^x(1 - 2\sin^2 x) dx = e^{\frac{\pi}{2}} + 2e^x\Big|_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} e^x \sin^2 x dx$$

$$\Longrightarrow \quad \int_0^{\frac{\pi}{2}} e^x \sin^2 x dx = \frac{3}{5}e^{\frac{\pi}{2}} - \frac{2}{5}$$

$$d)\ \int_0^{\sqrt{\ln 2}} x^3 e^{-x^2}dx = \int_0^{\sqrt{\ln 2}} \left(-\frac{x^2}{2}\right)\left(-2xe^{-x^2}\right)dx = -\frac{x^2}{2}e^{-x^2}\Big|_0^{\sqrt{\ln 2}} - \int_0^{\sqrt{\ln 2}} -xe^{-x^2}dx$$

$$= \left(-\frac{x^2}{2}-\frac{1}{2}\right)e^{-x^2}\Big|_0^{\sqrt{\ln 2}} = -\frac{\ln 2}{4}+\frac{1}{4}$$

$$e)\ \int_0^1 x\sqrt{(1-x^4)^3}dx \xlongequal{x=\sqrt{\sin t}} \int_0^{\frac{\pi}{2}} \sqrt{\sin t}\cos^3 t\, d\sqrt{\sin t} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 t dt = \frac{3\pi}{32}$$

$$f)\ \int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x} = \int_0^{\frac{\pi}{4}} \frac{x}{2\cos^2 x} = \frac{x}{2}\tan x\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\tan x}{2}dx = \frac{\pi}{8}-\frac{\ln 2}{4}$$

3. 求下列定积分

$$a)\ \int_0^1 (e^x-1)^4 e^x dx; \qquad b)\ \int_1^e \frac{1+\ln x}{x}dx$$

$$c)\ \int_0^1 \frac{dx}{\sqrt{1+e^{2x}}}; \qquad d)\ \int_0^1 \frac{xdx}{1+\sqrt{x}}$$

$$e)\ \int_0^1 \frac{x^2dx}{\sqrt{2x-x^2}}; \qquad f)\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}dx$$

解： a) $\displaystyle\int_0^1 (e^x-1)^4 e^x dx \xlongequal{y=e^x} \int_1^e (y-1)^4 dy = \frac{1}{5}(y-1)^5\Big|_1^e = \frac{(e-1)^5}{5}.$

$$b)\ \int_1^e \frac{1+\ln x}{x}dx = \int_1^e \frac{1}{x}dx + \int_1^e \frac{\ln x}{x}dx = \ln|x|\Big|_1^e + \frac{y^2}{2}\Big|_0^1 = \frac{3}{2}$$

$$c)\ \int_0^1 \frac{dx}{\sqrt{1+e^{2x}}} = \int_0^1 \frac{e^{-x}dx}{\sqrt{1+e^{-2x}}} \xlongequal{y=e^{-x}} -\int_1^{\frac{1}{e}} \frac{dy}{\sqrt{1+y^2}}$$

$$= \ln\left(y+\sqrt{y^2+1}\right)\Big|_{\frac{1}{e}}^1 = \ln(1+\sqrt{2})+\ln(\sqrt{1+e^2}-1)-1$$

$$d)\ \int_0^1 \frac{xdx}{1+\sqrt{x}} \xlongequal{y=\sqrt{x}} \int_0^1 \frac{y^2\cdot 2y\,dy}{1+y} = 2\int_0^1 \frac{y^3+y^2-y^2-y+y+1-1}{1+y}$$

$$= 2\int_0^1 \left(y^2-y+1-\frac{1}{1+y}\right)dy = 2\left(\frac{1}{3}-\frac{1}{2}+1-\ln 2\right) = \frac{5}{3}-2\ln 2$$

$$e)\ \int_0^1 \frac{x^2dx}{\sqrt{2x-x^2}} = \int_0^1 \frac{x^2}{\sqrt{1-(x-1)^2}} \xlongequal{x-1=\cos t} \int_{-\pi}^{-\frac{\pi}{2}} \frac{(\cos t+1)^2(-\sin t)dt}{|\sin t|}$$

$$= \int_{-\pi}^{-\frac{\pi}{2}} (\cos^2 t + 2\cos t + 1)dt = \left(\frac{1}{4}\sin 2t + 2\sin t + \frac{3}{2}t\right)\Bigg|_{-\pi}^{-\frac{\pi}{2}} = \frac{3\pi}{4} - 2$$

$$f)\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x \sin^2 x}dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\sin x dx$$

$$= 2\int_1^0 -\sqrt{y}dy = 2\cdot -\frac{2}{3}y^{\frac{3}{2}}\Big|_1^0 = \frac{4}{3}$$

4. 求下列定积分

$$a)\ \int_{\sqrt{2}}^2 \frac{dx}{2+\sqrt{4+x^2}}; \qquad b)\ \int_0^2 \frac{dx}{2+\sqrt{4+x^2}}$$

$$c)\ \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}\ (a>0); \qquad d)\ \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}dx$$

解： a) $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{2+\sqrt{4+x^2}} \xlongequal{x=2\tan t} \int_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} \frac{1}{\cos t}\frac{1}{1+\cos t}dt =$

$$\int_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} \sec t dt - \int_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} \frac{dt}{1+\cos t} = \ln|\sec t + \tan t|\Big|_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} -$$

$$- \int_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} \frac{dt}{2\cos^2\frac{t}{2}} = \ln|\sqrt{2}+1| - \ln\left|\frac{\sqrt{2}+\sqrt{3}}{2}\right| - \int_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} \sec^2\frac{t}{2}d\left(\frac{t}{2}\right)$$

$$= \ln\frac{2(\sqrt{2}+1)}{\sqrt{2}+\sqrt{3}} - \tan\frac{t}{2}\Big|_{\arctan\frac{\sqrt{2}}{2}}^{\frac{\pi}{4}} = \ln\frac{2(\sqrt{2}+1)}{\sqrt{2}+\sqrt{3}} - \tan\frac{\pi}{8} + \tan\frac{\arctan\frac{\sqrt{2}}{2}}{2}$$

$$c)\ \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} \xlongequal{x=a\sin\theta} \int_0^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{\sin\theta+\cos\theta} = \frac{1}{2}\left[x + \ln(\sin\theta+\cos\theta)\right]\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$d\ \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}dx \xlongequal{x=\sin\theta} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\theta\sin\theta}{\cos\theta}\cos\theta\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \theta\sin\theta\, d\theta = -\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \theta\, d\cos\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos\theta \, d\theta - \theta\cos\theta\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \sin\theta\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} - \theta\cos\theta\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = 1 - \frac{\sqrt{3}\pi}{6}$$

5. 设 $f \in C[0, +\infty)$，且 $\forall a, b > 0$ 满足下不等式 $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$.

证明：$F(x) := \frac{1}{x}\int_0^x f(t)dt$ 满足下不等式

$$F\left(\frac{a+b}{2}\right) \leq \frac{F(a)+F(b)}{2}$$

即若 $f(x)$ 下凸，则 $F(x)$ 亦下凸.

证明： $F\left(\frac{a+b}{2}\right) = \frac{2}{a+b}\int_0^{\frac{a+b}{2}} f(t)dt \xlongequal{t=\frac{a+b}{2}u} \int_0^1 f\left(\frac{a+b}{2}u\right)du$

$$\leq \frac{1}{2}\int_0^1 (f(au)+f(bu))\,du \xlongequal{au=v_1, bu=v_2} \frac{1}{2}\left(\frac{1}{a}\int_0^a f(v_1)dv_1 + \frac{1}{b}\int_0^b f(v_2)dv_2\right)$$

$$\text{即} \quad F\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(F(a)+F(b)\right) \qquad \square.$$

6. 设 $f(x)$ 在区间 $[a, b]$ 上有二阶连续导数. 证明：$\exists \xi \in [a, b]$，使得

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24}f''(\xi)$$

证明： 将 $f(x)$ 在点 $x = \frac{a+b}{2}$ 处展开为二阶泰勒公式，即

$$f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{f''(\xi)}{2!}\left(x - \frac{a+b}{2}\right)^2$$

两边积分,得 $\int_a^b f(x)dx = \int_a^b f\left(\frac{a+b}{2}\right)dx + f'\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right)dx +$

$$+\frac{1}{2!}\int_a^b f''(\xi)\left(x - \frac{a+b}{2}\right)^2 dx = f\left(\frac{a+b}{2}\right)(b-a) + 0 + \frac{f''(\xi)}{2!}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

$$= f\left(\frac{a+b}{2}\right)(b-a) + \frac{(b-a)^3}{24} f''(\xi) \quad \square.$$

7. 设 $f'(\sin^2 x) = \cos 2x + \tan^2 x$，求 $f(x)$.

提示： $f(x) = \int f'(x)dx + C$，所以先求出 $f'(x)$ 的表达式

解： 有 $f'(\sin^2 x) = 1 - 2\sin^2 x + \dfrac{\sin^2 x}{1-\sin^2 x}$，令 $u = \sin x$，得

$$f'(u) = 1 - 2u + \frac{u}{1-u} \quad \Longrightarrow \quad f(u) = \int f'(u)du$$

$$= \int \left(1 - 2u + \frac{u}{1-u}\right) du = \int \left(\frac{1}{1-u} - 2u\right) du = -\ln|1-u| - u^2 + C$$

从而 $f(x) = -\ln|1-x| - x^2 + C$.

8. 设 $f(\ln x) = \dfrac{\ln(1+x)}{x}$，求 $\displaystyle\int f(x)dx$.

提示： 同上题提示

解： 令 $u = \ln x$, 则 $f(u) = \dfrac{\ln(1+e^u)}{e^u}$, 故 $\displaystyle\int f(x)dx = \int \frac{\ln(1+e^x)}{e^x}dx =$

$$-\frac{\ln(1+e^x)}{e^x} + \int \frac{dx}{1+e^x} = -\frac{\ln(1+e^x)}{e^x} + \int \frac{1+e^x-e^x}{1+e^x}dx$$

$$= -\frac{\ln(1+e^x)}{e^x} + x - \ln(1+e^x) + C$$

9. 已知 $f(x)$ 的一个原函数为 $\dfrac{\sin x}{1+x\sin x}$，求 $\displaystyle\int f(x)f'(x)dx$.

解： 由于 $f(x) = \left(\dfrac{\sin x}{1+x\sin x}\right)' = \dfrac{\cos x - \sin^2 x}{(1+x\sin x)^2}$.

$$\int f(x)f'(x)dx = \frac{1}{2}\int d(f(x))^2 = \frac{1}{2}(f(x))^2 + C$$

$$= \frac{1}{2}\frac{(\cos x - \sin^2 x)^2}{2(1+x\sin x)^4} + C$$

10. 曲线 $y=f(x)$ 经过点 $(e,1)$，且在任一点的切线斜率为该点横坐标的倒数，求该曲线的方程.（最简单的建立微分方程然后求解的类型）

解： 根据题意 $y'=\dfrac{1}{x}$，两边积分知 $f(x)=\ln|x|+C$，又由于 $f(e)=1$ 知 $C=0$，故曲线方程为 $f(x)=\ln|x|$.

11. 设 $(0,+\infty)$ 上的连续函数 $f(x)$ 分别满足下列条件，求 $f(x)$ 的表达式:

$$a)\ f(x)=\sin x+\int_0^\pi f(x)dx;\quad b)\ f(x)=2\ln x+x^2\int_1^e \frac{f(x)}{x}dx$$

提示： 题目没说函数可导，所以两边求导不好使，但两边可以求积分啊

解： a) 记 $A:=\displaystyle\int_0^\pi f(x)dx$，则 $f(x)=\sin x+A$，两边积分，得

$$\int_0^\pi f(x)dx=\int_0^\pi \sin x dx+A\int_0^\pi dx=2+A\pi \quad\Longrightarrow\quad A=\frac{2}{1-\pi}$$

从而 $f(x)=\sin x+\dfrac{2}{1-\pi}$.

b) 记 $B=\displaystyle\int_1^e \frac{f(x)}{x}dx$，两边积分，得

$$B=2\int_1^e \frac{\ln x}{x}dx+B\int_1^e xdx \quad\Longrightarrow\quad B=(\ln x)^2\Big|_1^e+B\frac{x^2}{2}\Big|_1^e=1+\frac{e^2-1}{2}B$$

$$\Longrightarrow\quad B=\frac{2}{3-e^2}\quad\Longrightarrow\quad f(x)=2\ln x+\frac{2x^2}{3-e^2}$$

12. 计算 $\displaystyle\int_1^3 f(x-2)dx$，其中 $f(x)=\begin{cases}1+x^2, & x\le 0\\ \frac{1}{e^x}, & x>0\end{cases}$

解： $\displaystyle\int_1^3 f(x-2)dx=\int_1^2 f(x-2)dx+\int_2^3 f(x-2)dx=$

$$\int_1^2(1+(x-2)^2)dx+\int_2^3 e^{-(x-2)}dx=\left(\frac{x^3}{3}-2x^2+5x\right)\Big|_1^2+\left(-e^{-(x-2)}\right)\Big|_2^3$$

$$=\frac{4}{3}+1-\frac{1}{e}=\frac{7}{3}-\frac{1}{e}.$$

13. 求 $\int_0^{\frac{\pi}{2}} \frac{f(x)}{\sqrt{x}}dx$，其中 $f(x) = \int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{x}} \frac{dt}{1+\tan t^2}$

提示： 直接尝试求出 $f(x)$ 的表达式是不利的，但 $f(x)$ 的导数立马可知，所以。。。

解： 由于 $f'(x) = \frac{1}{1+\tan x}\frac{1}{2\sqrt{x}}$，故 $\int_0^{\frac{\pi}{2}} \frac{f(x)}{\sqrt{x}}dx = 2\int_0^{\frac{\pi}{2}} f(x)d\sqrt{x}$

$$= 2\left(\sqrt{x}f(x)\right)\Big|_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sqrt{x}\,df(x) = 0 - 2\int_0^{\frac{\pi}{2}} \sqrt{x}f'(x)\,dx$$

$$= -2\int_0^{\frac{\pi}{2}} \frac{1}{2}\frac{dx}{1+\tan x} = -\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} \xlongequal{x=\frac{\pi}{2}-t} \int_{\frac{\pi}{2}}^0 \frac{-dt}{1+\cot t}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dt}{1+\cot t} = \int_0^{\frac{\pi}{2}} \frac{\tan t}{1+\tan t}dt$$

令 $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}$，则由上可知

$$2I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} + \int_0^{\frac{\pi}{2}} \frac{\tan x}{1+\tan x} = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

可知 $I = \frac{\pi}{4}$，故 $\int_0^{\frac{\pi}{2}} \frac{f(x)}{\sqrt{x}}dx = -I = -\frac{\pi}{4}$.

14. 已知函数 $f(x)$ 在 $[0,+\infty)$ 上具有二阶连续导数，设 $f(0)=2, f(\pi)=1$，求 $\int_0^{\pi} [f(x)+f''(x)]\sin x dx$

解： $I = \int_0^{\pi} f(x)\sin x dx + \int_0^{\pi} f''(x)\sin x dx = I_1 + I_2$，其中

$$I_2 = \int_0^{\pi} f''(x)\sin x dx = f'(x)\sin x\Big|_0^{\pi} - \int_0^{\pi} f'(x)\cos x dx$$

$$= -\int_0^{\pi} f'(x)\cos x dx = -f(x)\cos x\Big|_0^{\pi} + \int_0^{\pi} f(x)(-\sin x)dx$$

$$= 3 - \int_0^{\pi} f(x)\sin x dx \quad \Longrightarrow \quad I = 3$$

15. 设 $f(x)$ 连续，满足 $f(1)=1$，且 $\int_0^x tf(2x-t)dt=\frac{\arctan x^2}{2}$，求 $\int_1^2 f(x)dx$.

提示： 用变量替换先将积分转化为两边可以对 x 求导的形式

解： 设 $u=2x-t$，则 $\int_0^x tf(2x-t)dt=\int_{2x}^x (2x-u)f(u)(-du)=$

$$\int_x^{2x}(2x-u)f(u)du=\frac{\arctan x^2}{2} \implies 2x\int_x^{2x}f(u)du-\int_x^{2x}uf(u)du=\frac{\arctan x^2}{2}$$

两边对 x 求导，得

$$2\int_x^{2x}f(u)du+2x\left[2f(2x)-f(x)\right]-\left[4xf(2x)-xf(x)\right]=\frac{1}{2}\frac{2x}{1+x^4}$$

$$\implies 2\int_x^{2x}f(u)du-xf(x)=\frac{x}{1+x^4} \xRightarrow{x=1} 2\int_1^2 f(u)du-f(1)=\frac{1}{2}$$

从而 $\int_1^2 f(x)dx=\frac{3}{4}$.

16. 设函数 $f(x)$ 在 $U(0)$ 可导,且 $f(0)=0$,求极限 $\lim\limits_{x\to 0}\frac{\int_0^x t^{n-1}f(x^n-t^n)dt}{x^{2n}}$ $(n\in\mathbb{N}_+)$

提示： 见上题的提示

解： 设 $u=x^n-t^n$，则 $t^{n-1}dt=-\frac{1}{n}du$. 从而

$$\int_0^x t^{n-1}f(x^n-t^n)dt=\int_{x^n}^0 f(u)\left(-\frac{1}{u}\right)du=\frac{1}{n}\int_0^{x^n}f(u)du$$

故 $\lim\limits_{x\to 0}\frac{\int_0^x t^{n-1}f(x^n-t^n)dt}{x^{2n}}=\lim\limits_{x\to 0}\frac{\frac{1}{n}\int_0^{x^n}f(u)du}{x^{2n}}\xlongequal{\text{洛必达}}\frac{1}{n}\lim\limits_{x\to 0}\frac{f(x^n)n\cdot x^{n-1}}{2nx^{2n-1}}$

$$=\frac{1}{n}\lim_{x\to 0}\frac{f(x^n)n\cdot x^{n-1}}{2n\,x^{2n-1}}=\frac{1}{2n}\lim_{x\to 0}\frac{f(x^n)}{x^n}\xlongequal{\text{洛必达}}\frac{1}{2n}\lim_{x\to 0}\frac{f'(x^n)n\cdot x^{n-1}}{n\cdot x^{n-1}}=\frac{1}{2n}f'(0)$$

17. 设函数 $f(x)$ 连续，且 $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=A$（$A$ 为常数），记 $\varphi(x)=\displaystyle\int_0^1 f(xt)dt$. 求 $\varphi'(x)$ 并讨论 $\varphi'(x)$ 在 $x=0$ 处的连续性.

解： 先求 $\varphi'(0)$，因 $\varphi(x)=\displaystyle\int_0^1 f(xt)dt$，故 $\varphi'(0)=\lim\limits_{x\to 0}\dfrac{\varphi(x)-\varphi(0)}{x}=$

$$\lim_{x\to 0}\frac{\displaystyle\int_0^1 f(x-t)dt-f(0)\int_0^1 dt}{x}\xlongequal{f(0)=0}\lim_{x\to 0}\frac{\displaystyle\int_0^1 f(x-t)dt}{x}\xlongequal{xt=u}\lim_{x\to 0}\frac{\displaystyle\int_0^x\frac{f(u)}{x}du}{x}$$

$$=\lim_{x\to 0}\frac{\displaystyle\int_0^x f(u)du}{x^2}\xlongequal{\text{洛必达}}\lim_{x\to 0}\frac{f(x)}{2x}=\frac{A}{2}$$

当 $x\neq 0$ 时，$\varphi(x)=\displaystyle\int_0^1 f(xt)dt=\frac{1}{x}\int_0^x f(u)du$，故

$$\varphi'(x)=-\frac{1}{x^2}\int_0^x f(u)du+\frac{1}{x}f(x)=\frac{xf(x)-\displaystyle\int_0^x f(u)du}{x^2}$$

连续性：$\lim\limits_{x\to 0}\varphi'(x)=\lim\limits_{x\to 0}\dfrac{f(x)}{x}+\lim\limits_{x\to 0}-\dfrac{\displaystyle\int_0^x f(u)du}{x^2}=A-\lim\limits_{x\to 0}\dfrac{\displaystyle\int_0^x f(u)du}{x^2}$

$=A-\dfrac{A}{2}=\dfrac{A}{2}$ 　　故 $\varphi'(x)$ 在 $x=0$ 处连续，且

$$\varphi'(x)=\begin{cases}\dfrac{xf(x)-\displaystyle\int_0^x f(u)du}{x^2}, & x\neq 0\\ \dfrac{A}{2}, & x=0.\end{cases}$$

18. 求不定积分 $I_1=\displaystyle\int\frac{\cos x dx}{\sin x+\cos x}$ 和 $I_2=\displaystyle\int\frac{\sin x dx}{\sin x+\cos x}$

解： $I_1+I_2=\displaystyle\int dx=x+C_1$

$$I_1-I_2=\int\frac{\cos x-\sin x}{\sin x+\cos x}dx=\int\frac{d(\sin x+\cos x)}{\sin x+\cos x}=\ln|\sin x+\cos x|+C_2$$

解得 $I_1 = \dfrac{(I_1 + I_2) + (I_1 - I_2)}{2} = \dfrac{x + \ln|\sin x + \cos x|}{2} + C$

$$I_2 = \frac{(I_1 + I_2) - (I_1 - I_2)}{2} = \frac{x - \ln|\sin x + \cos x|}{2} + C$$

19. 设 $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx$. 证明当 $n \geq 2$ 时，有

$a)$ $I_n + I_{n-2} = \dfrac{1}{n-1}$; $b)$ $\dfrac{1}{2(n+1)} < I_n < \dfrac{1}{2(n-1)}$; $c)$ $\lim\limits_{n\to\infty}(nI_n) = \dfrac{1}{2}$

证明： $I_n = \displaystyle\int \tan^n x dx = \int \tan^{n-2} x(\sec^2 x - 1)dx$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx = \int \tan^{n-2} x \sec^2 x dx - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2} \quad n \geq 2.$$

且 $I_0 = \displaystyle\int dx = x + C$，$I_1 = \displaystyle\int \tan x dx = -\ln|\cos x| + C$.

a) $I_n + I_{n-2} = \left.\dfrac{\tan^{n-1} x}{n-1}\right|_0^{\frac{\pi}{4}} = \dfrac{1}{n-1}, \quad n > 1.$

b) 当 $0 < x < \dfrac{\pi}{4}$ 时，有 $\tan^n x > \tan^{n+1} x$，由定积分的保号性，知 $I_n > I_{n+1}$，$n = 0, 1, 2, \cdots$ 则当 $n > 1$ 时，成立

$$\frac{1}{n-1} = I_n + I_{n-2} > 2I_n \quad \Longrightarrow \quad I_n < \frac{1}{2(n-1)}$$

又 $\dfrac{1}{n+1} = I_n + I_{n-2} < 2I_n$，故得 $I_n > \dfrac{1}{2(n+1)}$，从而

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}, \quad n > 1$$

c) 由于 $\dfrac{n}{2(n+1)} < nI_n < \dfrac{n}{2(n-1)}$，故由夹逼定理知 $\lim\limits_{n\to\infty} nI_n = \dfrac{1}{2}$.

20. 设函数 $f(x)$ 在 $[0,1]$ 上二阶可导，且 $f''(x) \geq 0\,(x \in [0,1])$，证明: $\int_0^1 f\left(x^2\right)\,dx \geq f\left(\frac{1}{3}\right)$.

证明: 将 $f(x)$ 在 $\frac{1}{3}$ 处展开为二阶泰勒多项式

$$f(x)=f\left(\frac{1}{3}\right)+f'\left(\frac{1}{3}\right)\left(x-\frac{1}{3}\right)+\frac{f''(\xi)}{2}\left(x-\frac{1}{3}\right)^2$$

其中 ξ 介于 x 与 $\frac{1}{3}$ 之间，由题意知 $f''(\xi) \geq 0$.

$$\begin{aligned}\int_0^1 f(x^2)dx &= \int_0^1 \left[f\left(\frac{1}{3}\right)+f'\left(\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)+\frac{f''(\xi)}{2}\left(x^2-\frac{1}{3}\right)^2\right]dx \\ &\geq \int_0^1 \left[f\left(\frac{1}{3}\right)+f'\left(\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)\right]dx \\ &= f\left(\frac{1}{3}\right)+f'\left(\frac{1}{3}\right)\left(\frac{x^3}{3}-\frac{x}{3}\right)\bigg|_0^1 = f\left(\frac{1}{3}\right) \qquad \square.\end{aligned}$$

21. 设 $f \in C^{(1)}[a,b]$，求证 $\max\limits_{a\leq x\leq b}|f(x)| \leq \frac{1}{b-a}\left|\int_a^b f(x)dx\right| + \int_a^b |f'(x)|\,dx$

证明: 设 $\max\limits_{a\leq x\leq b}|f(x)| = f(x_0)$. 由积分中值定理，知 $\exists\xi \in [a,b]$，使得 $\int_a^b f(x)dx = (b-a)f(\xi)$. 则

$$\max_{a\leq x\leq b} f(x) = f(x_0) = f(\xi)+\int_\xi^{x_0} f'(x)dx = \frac{1}{b-a}\int_a^b f(x)dx+\int_\xi^{x_0} f'(x)dx$$

从而

$$\max_{a\leq x\leq b}|f(x)| \leq \frac{1}{b-a}\left|\int_a^b f(x)dx\right| + \int_\xi^{x_0} |f'(x)|\,dx \quad \square.$$