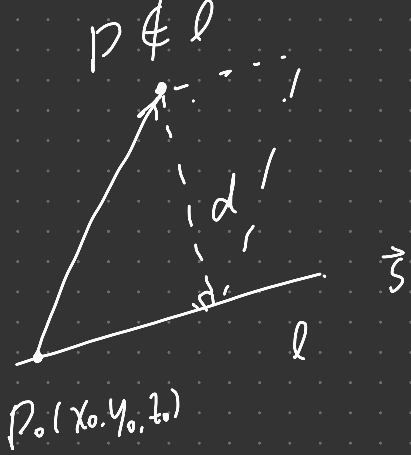
直线,平面的度量关系及位置关系

1. 点到直线的距离



$$\boxed{d = \frac{|\overrightarrow{P_0P} \times \vec{s}|}{|\vec{s}|}}$$

$$d = d(P, l)$$

例: $P(2,-1,3)$ $\quad l: \dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z}{5}$

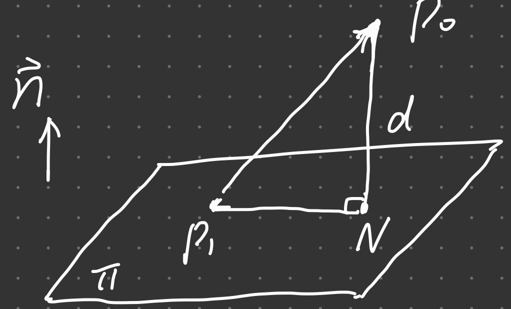
$P_0(2,-1,0)$ $\quad \vec{s} = (3,4,5)$

$\overrightarrow{P_0P} = (0,0,3)$ $\quad \vec{s} \times \overrightarrow{P_0P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ 0 & 0 & 3 \end{vmatrix}$

$$= \left(\begin{vmatrix} 4 & 5 \\ 0 & 3 \end{vmatrix}, \begin{vmatrix} 5 & 3 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 0 & 0 \end{vmatrix} \right) = (12,-9,0).$$

$$d = \frac{|\vec{s} \times \overrightarrow{P_0P}|}{|\vec{s}|} = \frac{\sqrt{12^2+(-9)^2+0^2}}{\sqrt{3^2+4^2+5^2}} = \frac{3}{2}\sqrt{2}.$$

2. 点到平面的距离



$\pi: Ax+By+Cz+D=0$

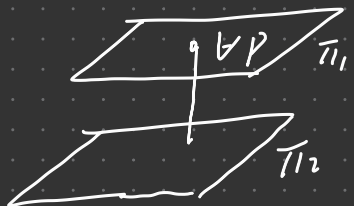
$$d(P_0,\pi)=|NP_0|$$

$$=\left|(\overrightarrow{P_1P_0})_{\vec{n}}\right|=\frac{|\vec{n}\cdot\overrightarrow{P_1P_0}|}{|\vec{n}|}$$

$$=\frac{|A(x_0-x_1)+B(y_0-y_1)+C(z_0-z_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

两平行平面间的距离



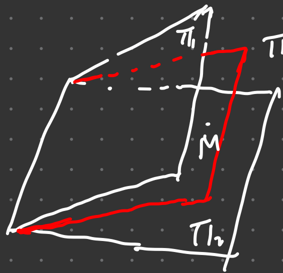
$$d(\pi_1,\pi_2)=d(P,\pi_2)$$

$$\forall P\in\pi_1$$

例：π_1: $2x-y+2z-3=0$ 在此二面角内

π_2: $3x+2y-6z-1=0$ 含有点 $P_0(1,2,-3)$

求 π_1,π_2 构成的二面角的角平分面方程.



$M \in \pi$ 当且仅当

$$d(M, \pi_1) = d(M, \pi_2)$$

$$2\times 1 - 2 + 2\times(-3) - 3 = -9 < 0$$

$$3\times 1 + 2\times 2 - 6\times(-3) - 1 = 24 > 0$$

M 满足：

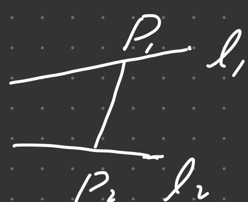
$$\frac{|2x-y+2z-3|}{\sqrt{2^2+(-1)^2+2^2}} = \frac{|3x+2y-6z-1|}{\sqrt{3^2+2^2+(-6)^2}}$$

且适合 $\begin{cases} 2x-y+2z-3 \le 0 \\ 3x+2y-6z-1 \ge 0 \end{cases}$ 或 $\begin{cases} 2x-y+2z-3 \ge 0 \\ 3x+2y-6z-1 \le 0 \end{cases}$

$$\Rightarrow \quad 23x - y - 4z - 24 = 0.$$

3. 两直线之间的距离

$$d(l_1, l_2) := \min_{\substack{P_i \in l_i \\ i=1,2}} d(P_1, P_2)$$



异面直线

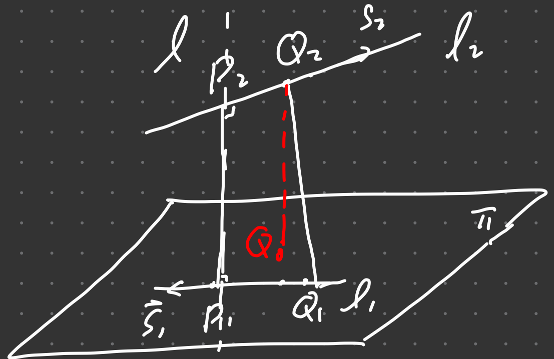
$M_i(x_i, y_i, z_i) \in l_i$, $\quad i = 1, 2.$

方向向量为 $\vec{s}_i = (X_i, Y_i, Z_i)$ $\quad i = 1, 2.$

即 $l_i: \dfrac{x - x_i}{X_i} = \dfrac{y - y_i}{Y_i} = \dfrac{z - z_i}{Z_i}$, $\quad i = 1, 2.$

命题：两条异面直线 l_1 与 l_2 的公垂线段的长就是 l_1 与 l_2 之间的距离.

证明： 设 P_1P_2 是 l_1 和 l_2 的公垂线段



任取点 $Q_i \in l_i$, $i=1,2$.

π 由 P_1, $\vec{s}_1$, $\vec{s}_2$ 决定.

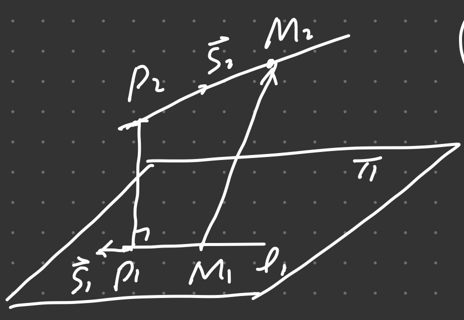
Q_0 为 Q_2 向 π 的垂足.

$l_2 // \pi \Rightarrow |\overrightarrow{P_1P_2}| = |\overrightarrow{Q_0Q_2}|$

$$|\overrightarrow{Q_1Q_2}| \geq |\overrightarrow{Q_0Q_2}| = |\overrightarrow{P_1P_2}| \qquad \square$$

公垂线 l 的方向向量为 $\vec{s}_1 \times \vec{s}_2$.

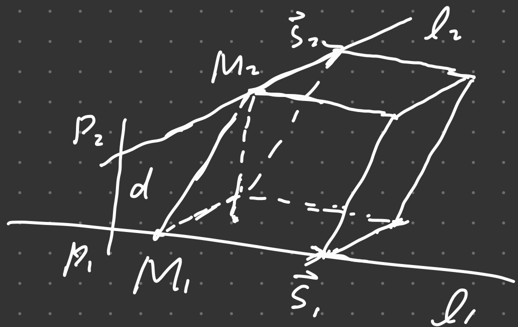
$$d = d(l_1, l_2) = |\overrightarrow{P_1P_2}| = \mathrm{Prj}_{\vec{e}}\, \overrightarrow{M_1M_2}$$



其中 $\vec{e} = \dfrac{\vec{s}_1 \times \vec{s}_2}{|\vec{s}_1 \times \vec{s}_2|}$

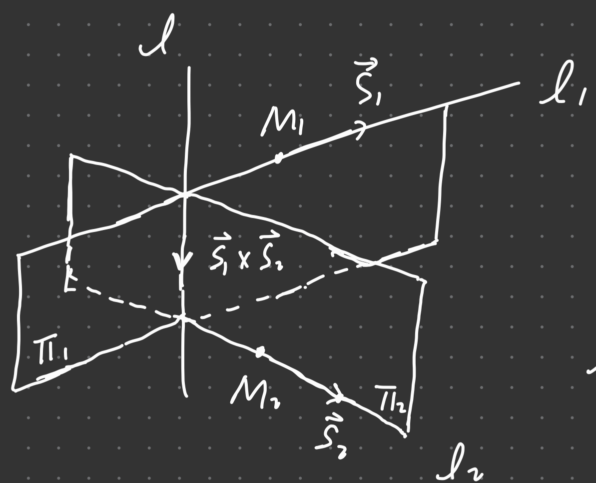
$$= |\vec{e} \cdot \overrightarrow{M_1M_2}| = \frac{|\overrightarrow{M_1M_2} \cdot (\vec{s}_1 \times \vec{s}_2)|}{|\vec{s}_1 \times \vec{s}_2|}$$

$$= \frac{|(\overrightarrow{M_1M_2}, \vec{s_1}, \vec{s_2})|}{|\vec{s_1} \times \vec{s_2}|}$$



= 以 $\overrightarrow{M_1M_2}$, $\vec{s_1}$, $\vec{s_2}$ 为棱的平行六面体的体积除以 $\vec{s_1}$, $\vec{s_2}$ 为邻边的平行四边形的面积.

公垂线 l 方程:



$l = \pi_1 \cap \pi_2$

$l \subseteq \pi_1$ (由 M_1, $\vec{s_1}$, $\vec{s_1} \times \vec{s_2}$ 决定)

$l \subseteq \pi_2$ (由 M_2, $\vec{s_2}$, $\vec{s_1} \times \vec{s_2}$ 决定)

设 $\vec{s_1} \times \vec{s_2} = (X, Y, Z)$

⑥

$$l = \pi_1 \cap \pi_2 \Longrightarrow \begin{cases} (\vec{r}-\vec{r}_1, \vec{s}_1, \vec{s}_1 \times \vec{s}_2) = 0 \\ (\vec{r}-\vec{r}_2, \vec{s}_2, \vec{s}_1 \times \vec{s}_2) = 0 \end{cases}$$

即
$$\begin{cases} \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ X_1 & Y_1 & Z_1 \\ X & Y & Z \end{vmatrix} = 0 \\ \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ X_2 & Y_2 & Z_2 \\ X & Y & Z \end{vmatrix} = 0 \end{cases}$$

例: $l_1: \dfrac{x-1}{3} = \dfrac{y-7}{-1} = \dfrac{z+4}{2}$

$l_2: \dfrac{x-1}{1} = \dfrac{y+2}{-2} = \dfrac{z}{2}$.

l_1 过 $M_1(1,7,-4)$, $\vec{s}_1 = (3,-1,2)$

l_2 过 $M_2(1,-2,0)$, $\vec{s}_2 = (1,-2,2)$

混合积 $(\overrightarrow{M_1M_2}, \vec{s}_1, \vec{s}_2) = \begin{vmatrix} 0 & -9 & 4 \\ 3 & -1 & 2 \\ 1 & -2 & 2 \end{vmatrix} = 16 \neq 0$

故 l_1 与 l_2 是异面直线, 它们的公垂线的方向向量可取为 $\vec{s}_1 \times \vec{s}_2 = (2,-4,-5)$

$$d(l_1, l_2) = \frac{|(\overrightarrow{M_1M_2}, \vec{v}_1, \vec{v}_2)|}{|\vec{v}_1 \times \vec{v}_2|} = \frac{16}{\sqrt{45}} = \frac{16}{3\sqrt{2}}$$

公垂线 l 的方程为
$$\begin{cases} \begin{vmatrix} x-1 & y-7 & z+4 \\ 3 & -1 & 2 \\ 2 & -4 & -5 \end{vmatrix} = 0 \\ \begin{vmatrix} x-1 & y+2 & z \\ 1 & -2 & 2 \\ 2 & -4 & -5 \end{vmatrix} = 0. \end{cases}$$

即
$$\begin{cases} 13x + 19y - 10z - 186 = 0 \\ 2x + y = 0. \end{cases}$$

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1. 两平面的夹角 $\quad \pi_i: A_i x + B_i y + C_i z + D_i = 0$

$$\vec{n}_i = (A_i, B_i, C_i) \qquad i = 1, 2.$$

$$\theta = \min\{\widehat{(\vec{n}_1, \vec{n}_2)}, \pi - \widehat{(\vec{n}_1, \vec{n}_2)}\} \in [0, \tfrac{\pi}{2}]$$

$$\cos\theta = |\cos\widehat{(\vec{n}_1, \vec{n}_2)}| = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}.$$

$$\theta = \arccos\left(\frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}\right)$$

- $\pi_1 \perp \pi_2 \Longleftrightarrow \vec{n}_1 \perp \vec{n}_2 \Longleftrightarrow A_1A_2+B_1B_2+C_1C_2=0$
- $\pi_1 \parallel \pi_2 \Longleftrightarrow \vec{n}_1 \parallel \vec{n}_2 \Longleftrightarrow \dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}$.

例：求过 $L:\begin{cases} x+2z+1=0 \\ x-y-z+1=0 \end{cases}$ 且与平面

$\pi: x+y+2z-4=0$ 成 $\frac{\pi}{3}$ 夹角的平面.

解：考虑平面束：$x+2z+1+\lambda(x-y-z+1)=0$

即 $(\lambda+1)x-\lambda y+(-\lambda+2)z+\lambda+1=0$.

$$\frac{|(\lambda+1)-\lambda+2(-\lambda+2)|}{\sqrt{(\lambda+1)^2+(-\lambda)^2+(-\lambda+2)^2}\sqrt{1^2+1^2+2^2}}=\cos\frac{\pi}{3}$$

$\Rightarrow \lambda^2+34\lambda-35=0 \Rightarrow \lambda=1,\ -35$

$2x-y+z+2=0$ 或 $-34x+35y+27z-34=0$

2. 两直线的夹角.

l_i 过 $M_i(x_i, y_i, z_i)$, 方向 $\vec{s}_i=(m_i, n_i, p_i)$

$i=1, 2$

$$\varphi := \min\{(\widehat{\vec{s}_1, \vec{s}_2}),\ \pi - (\widehat{\vec{s}_1, \vec{s}_2})\} \in [0, \tfrac{\pi}{2}]$$

$$= \arccos \frac{|m_1m_2 + n_1n_2 + p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2} \cdot \sqrt{m_2^2+n_2^2+p_2^2}}$$

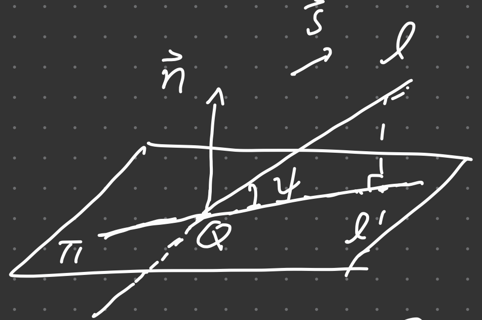
- $l_1 \perp l_2 \Longleftrightarrow \vec{s}_1 \perp \vec{s}_2 \Longleftrightarrow m_1m_2 + n_1n_2 + p_1p_2 = 0$
- $l_1 \parallel l_2 \Longleftrightarrow \vec{s}_1 \parallel \vec{s}_2 \Longleftrightarrow \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = \dfrac{p_1}{p_2}$.
- l_1, l_2 共面 $\Longleftrightarrow \overrightarrow{M_1M_2}, \vec{s}_1, \vec{s}_2$ 共面

$\Longleftrightarrow [\overrightarrow{M_1M_2}, \vec{s}_1, \vec{s}_2] = 0$

3. 直线与平面的夹角

$\pi: Ax + By + Cz + D = 0$

$l: \vec{s} = (m, n, p)$



l' 为投影直线

$$\varphi = \left|\frac{\pi}{2} - (\widehat{\vec{s}, \vec{n}})\right| \in [0, \tfrac{\pi}{2}]$$

$$\sin\varphi = |\cos(\widehat{\vec{s}, \vec{n}})| = \frac{|\vec{s} \cdot \vec{n}|}{|\vec{s}|\,|\vec{n}|}$$

$$\varphi = \arcsin \frac{|Am+Bn+Cp|}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}$$

- $l \parallel \pi$ 或 $l \subset \pi \Leftrightarrow \vec{s} \perp \vec{n} \Leftrightarrow Am+Bn+Cp=0$
- $l \perp \pi \Leftrightarrow \vec{s} \parallel \vec{n} \Leftrightarrow \frac{A}{m}=\frac{B}{n}=\frac{C}{p}$.

例：$l: \frac{x-2}{1}=\frac{y-4}{2}=\frac{z-3}{1}$，$\pi: 2x+y-z-2=0$

1) 求 $l \cap \pi = Q$ 和夹角 φ.

2) 求 l 在 π 上的投影直线 l'.

解：1) l: $x=2+t$，$y=4+2t$，$z=3+t$

代入 π 的方程：$2(2+t)+(4+2t)-(3+t)-2=0$

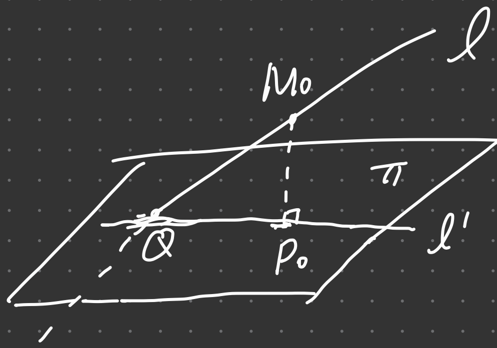
$\Rightarrow t=-1. \rightsquigarrow Q(1,2,2)$

$\vec{s}=(1,2,1)$. $\vec{n}=(2,1,-1)$.

$$\sin\varphi = |\cos(\widehat{\vec{s},\vec{n}})| = \frac{|1\times2+2\times1+1\times(-1)|}{\sqrt{1^2+2^2+1^2}\cdot\sqrt{2^2+1^2+(-1)^2}} = \frac{1}{2}$$

$\Rightarrow \varphi = \frac{\pi}{6}$.

2)



$M_0(2,4,3)$

$\vec{s}=(1,2,1)$

$\vec{n}=(2,1,-1)$

设过 M_0, P_0, Q 的平面为 π'

设 $M \in \pi'$，则 $\overrightarrow{QM}, \vec{s}, \vec{n}$ 共面

即 $[\overrightarrow{QM}, \vec{s}, \vec{n}]=0$ $\quad \begin{vmatrix} x-1 & y-2 & z-2 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}=0$

$x-y+z-1=0$

$l': \begin{cases} 2x+y-z-2=0 \\ x-y+z-1=0 \end{cases}$

π' 亦可由平面束法求得

$2x-y+\lambda(y-2z+2)=0$

$2x+(\lambda-1)y-2\lambda z+2\lambda=0$ $\qquad \pi' \perp \pi$

$(2,\lambda-1,-2\lambda)\cdot(2,1,-1)=0 \Rightarrow \lambda=-1$

$$\pi' : x - y + z - 1 = 0$$

$\square$