作业　九

1. 求下列不定积分（可考虑利用三角恒等式化简后凑微分）

$$a)\ \int \cos^2 \frac{x}{2} dx; \qquad b)\ \int \frac{\cos 2x}{\cos x + \sin x} dx$$

$$c)\ \int \frac{1+\cos^2 x}{1+\cos 2x} dx; \qquad d)\ \int \frac{dx}{\cos^2 x \sin^2 x}$$

解： a) $\int \cos^2 \frac{x}{2} dx = \int \frac{\cos x + 1}{2} dx = \frac{\sin x + x}{2} + C$

$$b)\ \int \frac{\cos 2x}{\cos x + \sin x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} dx = \int (\cos x - \sin x) dx = \sin x + \cos x + C$$

$$c)\ \int \frac{1+\cos^2 x}{1+\cos 2x} dx = \int \frac{1+\cos^2 x}{2\cos^2 x} dx = \frac{1}{2} \int \sec^2 x dx + \frac{1}{2} dx = \frac{\tan x + x}{2} + C$$

$$d)\ \int \frac{dx}{\cos^2 x \sin^2 x} = \int \frac{4dx}{\sin^2 2x} = 2 \int \sec^2 2x \, d(2x) = 2 \tan 2x + C$$

2. 求下列不定积分（可化简后（利用换元）凑微分）

$$a)\ \int \frac{dx}{e^x - e^{-x}}; \qquad b)\ \int \frac{2x-5}{(x^2-5x+8)^2} dx$$

$$c)\ \int \frac{x^2}{\sqrt[4]{1-2x^3}} dx; \qquad d)\ \int e^x \sin(e^x) dx$$

$$e)\ \int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx; \qquad f)\ \int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx$$

$$g)\ \int \frac{dx}{x \ln x}; \qquad h)\ \int \frac{x^2}{4+x^6} dx$$

解： a) $\int \frac{dx}{e^x - e^{-x}} = \int \frac{de^x}{(e^x)^2 - 1} \overset{u=e^x}{=\!=\!=}$

$$\int \frac{du}{u^2-1} = \frac{1}{2} \int \frac{du}{u-1} - \frac{1}{2} \int \frac{du}{u+1} = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C = \frac{1}{2} \ln \left| \frac{e^x-1}{e^x+1} \right| + C$$

$$b)\ \int \frac{2x-5}{(x^2-5x+8)^2}dx = \int \frac{d(x^2-5x+8)}{(x^2-5x+8)^2} = -\frac{1}{x^2-5x+8} + C$$

$$c)\ \int \frac{x^2dx}{\sqrt[4]{1-2x^3}} = \frac{-1}{6}\int \frac{d(1-2x^3)}{\sqrt[4]{1-2x^3}} \xlongequal{t=1-2x^3} -\frac{1}{6}\int t^{-\frac{1}{4}}dt$$

$$= -\frac{1}{6}\frac{t^{-\frac{1}{4}+1}}{1-\frac{1}{4}} + C = -\frac{2}{9}t^{\frac{3}{4}} + C = -\frac{2}{9}(1-2x^3)^{\frac{3}{4}} + C$$

$$d)\ \int e^x \sin(e^x)dx = \int \sin e^x de^{x=-\cos e^x} + C$$

$$e)\ \int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}dx = \int \frac{d(\sin x - \cos x)}{\sqrt[3]{\sin x - \cos x}} = \frac{1}{-\frac{1}{3}+1}(\sin x - \cos x)^{-\frac{1}{3}+1} + C$$

$$= \frac{3}{2}(\sin x - \cos x)^{\frac{2}{3}} + C$$

$$f)\ \int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx \xlongequal[x=\tan^2 t]{t:=\arctan\sqrt{x}} \int \frac{2t\tan t\sec^2 t\, dt}{\tan t\,(1+\tan^2 t)} = 2\int tdt$$

$$= t^2 + C = \arctan^2\sqrt{x} + C$$

$$g)\ \int \frac{dx}{x\ln x} = \int \frac{d\ln x}{\ln x} = \ln|\ln x| + C$$

$$h)\ \int \frac{x^2}{4+x^6}dx = \frac{1}{3}\int \frac{dx^3}{4+(x^3)^2} \xlongequal{t=x^3} \frac{1}{3}\int \frac{dt}{4+t^2} = \frac{1}{6}\int \frac{d\left(\frac{t}{2}\right)}{1+\left(\frac{t}{2}\right)^2}$$

$$= \frac{1}{6}\arctan\frac{t}{2} + C = \frac{1}{6}\arctan\frac{x^3}{2} + C$$

3. 求下列不定积分（最好直接变量替换后化简整理成简单积分）

$$a)\ \int e^x\sqrt{1-e^{2x}}dx; \qquad b)\ \int \frac{dx}{\sqrt{1+e^x}}$$

$$c)\ \int \frac{dx}{1+e^x}; \qquad d)\ \int \frac{1+\ln x}{(x\ln x)^2}dx$$

$$e)\ \int \frac{x^2}{(x+1)^{100}}dx; \qquad f)\ \int \frac{\sqrt{3+2x}}{x}dx$$

解： $a)$ $$\int e^x\sqrt{1-e^{2x}}dx \xlongequal{t=e^x} \int \sqrt{1-t^2}\,dt \xlongequal{t=\sin\theta} \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\arcsin t + \frac{1}{2}t\sqrt{1-t^2} + C$$

$$= \frac{1}{2}\arcsin e^x + \frac{e^x\sqrt{1-e^{2x}}}{2} + C$$

$b)$ $$\int \frac{dx}{\sqrt{1+e^x}} \xlongequal[x=\ln(t^2-1)]{1+e^x=t^2} \int \frac{2t\,dt}{t(t^2-1)} = \int \frac{2dt}{t^2-1} = \ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$$

$c)$ $$\int \frac{dx}{1+e^x} \xlongequal[x=\ln(t-1)]{t=e^x+1} \int \frac{dt}{t(t-1)} = \ln\left|\frac{t-1}{t}\right| + C = \ln\frac{e^x}{e^x+1} + C$$

$d)$ $$\int \frac{1+\ln x}{(x\ln x)^2}dx = \int \frac{d(x\ln x)}{(x\ln x)^2} = -\frac{1}{x\ln x} + C$$

$e)$ $$\int \frac{x^2}{(x+1)^{100}}dx \xlongequal{t=x+1} \int \frac{(t-1)^2}{t^{100}}dt = \int t^{-98}dt - 2\int t^{-99}dt + \int t^{-100}dt$$

$$-\frac{1}{97}t^{-97} + \frac{1}{49}t^{-98} - \frac{1}{99}t^{-99} + C = -\frac{1}{97}(x+1)^{-97} + \frac{1}{49}(x+1)^{-98} - \frac{1}{99}(x+1)^{-98} + C$$

$f)$ $$\int \frac{\sqrt{3+2x}}{x}dx \xlongequal[x=\frac{t^2-3}{2}]{\sqrt{3+2x}=:t} \int \frac{2t}{t^2-3}\cdot\frac{2t\,dt}{t} = 2\int \frac{t^2-3+3}{t^2-3}dt$$

$$= 2\int dt + 6\int \frac{dt}{t^2-3} \xlongequal{t=\sqrt{3}\sec\theta} 2t + 6\int \frac{\sqrt{3}\sec\theta\tan\theta}{3\tan^2\theta}d\theta$$

$$= 2\sqrt{3+2x} + 2\sqrt{3}\int \csc\theta\,d\theta = 2\sqrt{3+2x} - 2\sqrt{3}\ln|\csc\theta + \cot\theta| + C$$

$$= 2\sqrt{3+2x} - 2\sqrt{3}\ln\left|\frac{t}{\sqrt{t^2-3}} + \frac{\sqrt{3}}{\sqrt{t^2-3}}\right| + C$$

$$= 2\sqrt{3+2x} - 2\sqrt{3}\ln\left|\frac{\sqrt{3+2x}+\sqrt{3}}{\sqrt{2x}}\right| + C$$

4. 求下列定积分

$$a)\ \int_1^4 \frac{dx}{x(1+\sqrt{x})}; \qquad b)\ \int_0^{\frac{\pi}{4}} \frac{\sin x}{1+\sin x}dx$$

解： a) $\displaystyle\int_1^4 \frac{dx}{x(1+\sqrt{x})} \xlongequal{x:=t^2} \int_1^2 \frac{2tdt}{t^2(1+t)} = \int_1^2 \frac{2dt}{t(1+t)} =$

$$2\int_1^2 \frac{dt}{t} - 2\int_1^2 \frac{dt}{1+t} = 2\ln t\Big|_1^2 - 2\ln(1+t)\Big|_1^2 = 2\ln 2 - 2\ln\frac{3}{2} = 2\ln\frac{4}{3}$$

5. 求下列不定积分（用三角替换处理比较适宜）

$$a)\ \int \frac{dx}{\sqrt{(1-x^2)^3}}; \qquad b)\ \int \frac{dx}{x\sqrt{a^2-x^2}}\ (a>0)$$

$$c)\ \int \frac{dx}{x^2\sqrt{1+x^2}}; \qquad d)\ \int \frac{\sqrt{x^2-9}}{x}dx$$

$$e)\ \int \frac{x^2}{\sqrt{a^2-x^2}}dx\ (a>0); \qquad f)\ \int \frac{dx}{1+\sqrt{1-x^2}}$$

解： $a)$ $\displaystyle\int \frac{dx}{\sqrt{(1-x^2)^3}} \xlongequal{x=\sin t} \int \frac{\cos t\, dt}{\cos^3 t} =$

$$\int \sec^2 t\, dt = \tan t + C = \tan\arcsin x + C$$

$$b)\ \int \frac{dx}{x\sqrt{a^2-x^2}} \xlongequal[dx=a\cos t dt]{x=a\sin t} \int \frac{a\cos t\, dt}{a\sin t\cos t} = \int \sec t\, dt$$

$$= \ln|\sec t + \tan t| + C = \ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C$$

$$c)\ \int \frac{dx}{x^2\sqrt{1+x^2}} \xlongequal{x=\tan t} \int \frac{\sec t\tan t\, dt}{\tan^2 t\sec t} = \int \frac{\cos t}{\sin t}dt = \ln|\sin t| + C$$

$$= \ln\left|\frac{1}{\sqrt{1+x^2}}\right| + C$$

$$d)\ \int \frac{\sqrt{x^2-9}}{x}dx \xlongequal{x:=3\sec\theta} \int \frac{3\tan\theta}{3\sec\theta}3\sec\theta\tan\theta d\theta = 3\int \tan^2\theta d\theta$$

$$= 3\int \sec^2\theta d\theta - 3\int d\theta = 3\tan\theta - 3\theta + C = 3\frac{\sqrt{x^2-9}}{3} - 3\arccos\frac{3}{x} + C$$

$$e)\ \int \frac{x^2dx}{\sqrt{a^2-x^2}} \xlongequal{x=a\sin\theta} \int \frac{a^2\sin^2\theta a\cos\theta d\theta}{a\cos\theta} = a^2\int \sin^2\theta d\theta$$

$$= a^2\int \frac{1-\cos 2\theta}{2}d\theta = \frac{a^2}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} - \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) + C$$

$$f)\ \int \frac{dx}{1+\sqrt{1-x^2}} \xlongequal{x=\sin\theta} \int \frac{\cos\theta d\theta}{1+\cos\theta} = \int d\theta - \int \frac{d\theta}{1+\cos\theta}$$

$$= \theta - \int \frac{d\theta}{2\cos^2\frac{\theta}{2}} = \theta - \int \sec^2\frac{\theta}{2}d\left(\frac{\theta}{2}\right) = \theta - \tan\frac{\theta}{2} + C$$

$$= \arcsin x - \frac{x}{1+\sqrt{1-x^2}} + C$$

6. 求下列不定积分（分部积分法比较适合）

$$a)\ \int xe^{2x}dx; \qquad b)\ \int \arctan x dx$$

$$c)\ \int x^2\arctan x dx; \qquad d)\ \int x^2\ln x dx$$

$$e)\ \int e^{-2x}\sin\frac{x}{2}dx; \qquad f)\ \int \frac{\arcsin x}{\sqrt{1-x}}dx$$

$$g)\ \int \ln(x+\sqrt{1+x^2})dx \qquad h)\ \int \frac{\ln(\cos x)}{\cos^2 x}dx$$

$$i)\ \int \sin x\ln(\tan x)dx; \qquad g)\ \int \frac{\arcsin x}{x^2}dx$$

解： a) $\displaystyle\int xe^{2x}dx = \frac{1}{2}\int xde^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

$$b)\ \int \arctan x dx = x\arctan x - \int \frac{xdx}{1+x^2} = x\arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

$c)$ $\displaystyle\int x^2 \arctan x dx = \frac{1}{3}\int \arctan x dx^3 = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \frac{x^3 dx}{1+x^2}$

$$= \frac{1}{3}x^3\arctan x - \frac{1}{12}\int \frac{dx^4}{1+x^2} \overset{t=x^2}{=\!=} \frac{1}{3}x^3\arctan x - \frac{1}{12}\int \frac{dt^2}{1+t}$$

$$= \frac{x^3\arctan x}{3} - \frac{1}{6}\int \frac{1+t-1}{1+t}dt = \frac{x^3\arctan x}{3} - \frac{t}{6} + \frac{1}{6}\ln|1+t| + C$$

$$= \frac{x^3\arctan x}{3} - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$$

$d)$ $\displaystyle\int x^2\ln x dx = \frac{1}{3}\int \ln x dx^3 = \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2 dx = \frac{x^3\ln x}{3} - \frac{x^3}{9} + C$

$e)$ $\displaystyle\int e^{-2x}\sin\frac{x}{2}dx = -\frac{1}{2}\int \sin\frac{x}{2}de^{-2x} = \frac{1}{2}e^{-2x}\sin\frac{x}{2} - \frac{1}{4}\int e^{-2x}\cos\frac{x}{2}dx$

$$= \frac{1}{2}e^{-2x}\sin\frac{x}{2} + \frac{1}{8}\int \cos\frac{x}{2}de^{-2x} = \frac{1}{2}e^{-2x}\sin\frac{x}{2} + \frac{1}{16}\left(e^{-2x}\cos\frac{x}{2} + \int e^{-2x}\sin\frac{x}{2}dx\right)$$

令 $I = \displaystyle\int e^{-2x}\sin\frac{x}{2}dx$，则上计算表明:

$$I = \frac{1}{2}e^{-2x}\sin\frac{x}{2} + \frac{1}{16}e^{-2x}\cos\frac{x}{2} + \frac{1}{16}I$$

从中解得 $I = \dfrac{8}{15}e^{-2x}\sin\dfrac{x}{2} + 15e^{-2x}\cos\dfrac{x}{2}$

$f)$ $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x}}dx \underset{x=1-t^2}{\overset{t:=\sqrt{1-x}}{=\!=\!=}} \int \frac{\arcsin(1-t^2)}{t}(-2t)dt = -2\int \arcsin(1-t^2)dt$

$$= 2\int td\arcsin(1-t^2) - 2t\arcsin(1-t^2) = 2\int \frac{(-2)t^2dt}{\sqrt{1-(1-t^2)^2}} - 2t\arcsin(1-t^2)$$

$$= -4\int \frac{tdt}{\sqrt{2-t^2}} - 2t\arcsin(1-t^2) + C = 2\int \frac{d(2-t^2)}{\sqrt{2-t^2}} - 2t\arcsin(1-t^2) + C$$

$$= 2\frac{1}{\frac{1}{2}}(2-t^2)^{\frac{1}{2}} - 2t\arcsin(1-t^2) + C$$

$$= 4\left(2-(1-x)\right)^{\frac{1}{2}} - 2\sqrt{1-x}\arcsin\left(1-(1-x)\right) + C$$

$$= 4\sqrt{1+x} - 2\sqrt{1-x}\arcsin x + C$$

$$g)\quad \int \ln\left(x+\sqrt{1+x^2}\right)dx = x\ln\left(x+\sqrt{1+x^2}\right) - \int x\frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}dx$$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - \int\frac{xdx}{\sqrt{1+x^2}} = x\ln\left(x+\sqrt{1+x^2}\right) - \frac{1}{2}\int\frac{d(1+x^2)}{\sqrt{1+x^2}}$$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - 2\sqrt{1+x^2} + C$$

$$h)\quad \int\frac{\ln(\cos x)}{\cos^2 x}dx = \int\ln(\cos x)d\tan x = \tan x\ln(\cos x) + \int\tan^2 xdx$$

$$= \tan x\ln(\cos x) + \int\left(\sec^2 x - 1\right)dx = \tan x\ln(\cos x) + \tan x - x + C$$

$$i)\quad \int\sin x\ln(\tan x)dx = \int\ln(\tan x)d(-\cos x) = \int\cos x\frac{\sec^2 x}{\tan x}dx - \cos x\ln(\tan x)$$

$$= \int\csc xdx - \cos x\ln(\tan x) = -\ln|\csc x + \cot x| - \cos x\ln(\tan x) + C$$

$$g)\quad \int\frac{\arcsin x}{x^2}dx = -\int\arcsin x\, d\left(\frac{1}{x}\right) = \int\frac{dx}{x\sqrt{1-x^2}} - \frac{\arcsin x}{x}$$

$$\overset{x=\sin t}{=\!=\!=} \int\frac{\cos t\, dt}{\sin t\cos t} - \frac{\arcsin x}{x} = \int\csc tdt - \frac{\arcsin x}{x}$$

$$= -\ln|\cos t + \cot t| - \frac{\arcsin x}{x} + C = -\ln|\cos\arcsin x + \cot\arcsin x| - \frac{\arcsin x}{x} + C$$

7. 求下列不定积分（有理分式展开或其它方法）

$$a)\ \int\frac{dx}{x^2-2x+2};\qquad b)\ \int\frac{x+1}{x^2-3x+2}dx$$

$$c)\ \int\frac{2x+3}{(x^2-1)(x^2+1)}dx;\qquad d)\ \int\frac{x^5+x^4-8}{x^3+x}dx$$

$$e)\ \int\frac{x^4+1}{(x-1)(x^2+1)}dx;\qquad f)\ \int\frac{dx}{(x-1)(x+1)^2}$$

解： a) $\displaystyle\int\frac{dx}{x^2-2x+2} = \int\frac{dx}{1+(x-1)^2} = \arctan(x-1) + C.$

b) 设 $\dfrac{x+1}{x^2-3x+2}=\dfrac{x-1}{(x-1)(x-2)}=\dfrac{A}{x-1}-\dfrac{B}{x-2}$，则 $A(x-2)-B(x-1)=x-1$，从而 $A-B=1$，$B-2A=1$，解得 $A=-2$，$B=-3$

$$\int\frac{x+1}{x^2-3x+2}dx=-2\int\frac{dx}{x-1}+3\int\frac{dx}{x-2}=-2\ln|x-1|+3\ln|x-2|+C$$

c) $\dfrac{2x+3}{(x^2-1)(x^2+1)}=\dfrac{2x+3}{(x-1)(x+1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+1}$，即

$$A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)=2x+3$$

- x^3 前系数：$A+B+C=0$；
- x^2 前系数：$A-B+D=0$；
- x 前系数：$A+B-C=2$；
- 1 前系数：$A-B-D=3$

解得 $A=\dfrac{5}{4}$，$B=-\dfrac{1}{4}$，$C=-1$，$D=-\dfrac{3}{2}$. 故所求积分为

$$\frac{5}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\int\frac{-x+\frac{3}{2}}{x^2+1}dx=\frac{5}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-$$

$$-\quad\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\frac{3}{2}\int\frac{dx}{x^2+1}=\frac{5}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-$$

$$-\frac{1}{2}\ln 1+x^2+\frac{3}{2}\arctan x+C$$

d) 由带余除法，知 $x^5+x^4-8=(x^2+x-1)(x^3+x)-x^2+x-8$，从而被积函数

$$\frac{x^5+x^4-8}{x^3+x}=x^2+x-1-\frac{x^2-x+8}{x^3+x}$$

设 $\dfrac{x^2-x+8}{x(x^2+1)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}=\dfrac{8}{x}-\dfrac{7x+1}{x^2+1}=\dfrac{8}{x}-7\dfrac{x}{x^2+1}-\dfrac{1}{x^2+1}$，综上，所求积分为

$$\int(x^2+x-1)dx-\int\frac{8dx}{x}+\frac{7}{2}\int\frac{d(x^2+1)}{x^2+1}+\int\frac{dx}{x^2+1}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - x + 8\ln|x| + \frac{7}{2}\ln(x^2+1) + \arctan x + C$$

e) 由带余除法，得 $x^4+1=(x+1)(x-1)(x^2+1)+2$，故

$$\frac{x^4+1}{(x-1)(x^2+1)} = x+1+\frac{2}{(x-1)(x^2+1)} = x+1+\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$= x+1+\frac{1}{x+1}+\frac{-x-1}{x^2+1}$$

从而所求积分为 $\int(x+1)dx + \int\frac{dx}{x+1} - \frac{1}{2}\int\frac{d(x^2+1)}{x^2+1} - \int\frac{dx}{x^2+1}$

$$= \frac{x^2}{2} + x + \ln|x+1| - \frac{1}{2}\ln(1+x^2) - \arctan x + C$$

$$f)\quad \frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{(x+1)^2}$$

故所求积分为 $\frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1} - \int\frac{dx}{(x+1)^2}$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{2}\frac{1}{x+1} + C$$

8. 利用定积分的几何意义计算下列积分

$$a)\ \int_{-1}^{1}|x|dx; \qquad b)\ \int_{-\pi}^{\pi}\sin x dx$$

解： a) 转化为关于 y 轴对称的两三角形的面积，得 1；b) x 轴上方面积和 x 轴下方面积相互抵消，得 0.

9. 已知 $\int_0^{\pi}\sin x dx = 2$，求 $\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n}\right]$

解： 原极限 $= \lim_{n\to\infty}\frac{1}{\pi}\sum_{i=1}^{n}\left(\frac{i}{n}-\frac{i-1}{n}\right)\cdot\sin\frac{i\pi}{n} = \frac{1}{\pi}\int_0^{\pi}\sin x dx = \frac{2}{\pi}.$

10. 已知 $\int_1^2 \ln x dx = 2\ln 2 - 1$，求极限 $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}$

解： 原极限 $= exp\left\{\lim\limits_{n\to\infty} \dfrac{1}{n}\left(\ln\left(1+\dfrac{1}{n}\right)+\cdots+\ln\left(1+\dfrac{n}{n}\right)\right)\right\}$

$$= exp\left\{\int_1^2 \ln x dx\right\} = e^{2\ln 2-1} = \frac{4}{e}$$

11. 设 $f, g \in C[a,b]$，证明：若 $f(x) \geq g(x)\,(x\in[a,b])$，且 $f(x)$ 不恒为 $g(x)$，则 $\int_a^b f(x) > \int_a^b g(x)dx.$

证明： 只需证明 $\int_a^b (f(x)-g(x))\,dx > 0$. 由于 f 不恒为 g，故 $\exists x_0 \in [a,b]$，使得 $f(x_0)-g(x_0)>0$. 又 f, g 连续，则由局部保号性知存在 x_0 的一个领域 $[x_1,x_2]\subseteq[a,b]$，使得在该领域内 $h(x) := f(x)-g(x)>0$，故 $\int_{x_1}^{x_2} h(x)dx > m(x_2-x_1)>0$，其中 $m := \min\limits_{x_1\le x\le x_2} h(x)$，于是

$$\int_a^b h(x)dx = \int_a^{x_1} h(x)dx + \int_{x_1}^{x_2} h(x)dx + \int_{x_2}^{b} h(x)dx \geq \int_{x_1}^{x_2} h(x)dx > 0$$

12. 比较下列各组中积分的大小.

$a)$ $\int_0^{\frac{\pi}{2}} \sin^2 x$ 与 $\int_0^{\frac{\pi}{2}} \sin^4 x dx$; $b)$ $\int_1^e \ln x dx$, $\int_1^e (\ln x)^2 dx$ $\int_1^e \ln(x^2)dx$

解： a) 由于 $\int_0^{\frac{\pi}{2}} \sin^2 x dx - \int_0^{\frac{\pi}{2}} \sin^4 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x\left(1-\sin^2 x\right)dx$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x dx > 0$$

b) 记 $I_1 = \int_1^e \ln x dx$，$I_2 = \int_1^e (\ln x)^2 dx$，$I_3 = \int_1^e \ln x^2 dx$. 则有

$$I_1 - I_2 = \int_1^e \ln x(1-\ln x)dx > 0$$

$$I_1 - I_3 = \int_1^e (\ln x - \ln x^2)dx = \int_1^e \ln \frac{1}{x} dx = -\int_1^e \ln x dx < 0$$

从而 $I_3 > I_1 > I_2$.

13. 证明不等式 $\frac{1}{2} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{2}$

证明： 当 $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ 时，$\left(\frac{\sin x}{x}\right)' = \frac{\cos x(x - \tan x)}{x^2} < 0$，从而有估计

$$\frac{2}{\pi} < \frac{\sin x}{x} < \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}$$

于是 $\frac{2}{\pi} \cdot \frac{\pi}{4} = \frac{1}{2} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx < \frac{2\sqrt{2}}{\pi} \cdot \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \square$.

14. 求下列极限

$$a)\ \lim_{n\to\infty} \int_0^{1/2} \frac{x^n}{1+x} dx; \qquad b)\ \lim_{n\to\infty} \int_{n^2}^{n^2+n} \frac{1}{\sqrt{x}} e^{-\frac{1}{x}} dx$$

解： a) $0 < \int_0^{1/2} \frac{x^n}{1+x} dx < \int_0^{\frac{1}{2}} x^n dx = \frac{1}{n+1} x^{n+1} \Big|_0^{\frac{1}{2}} = \frac{1}{(n+1)2^{n+1}} \to 0$.

b) $f(x) = \frac{1}{\sqrt{x}} e^{-\frac{1}{x}}$，$f'(x) = x^{-\frac{3}{2}} e^{-\frac{1}{x}} \left(-\frac{1}{2} - x\right) < 0$，$\forall x > -\frac{1}{2}$. 故函数 $f(x)$ 在 $[n^2, n^2+n]$ 上单调减少. 从而 $\frac{e^{-\frac{1}{n^2+n}}}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{x}} e^{-\frac{1}{x}} \le \frac{e^{-\frac{1}{n^2}}}{\sqrt{n^2}}$

$$\frac{ne^{-\frac{1}{n^2+n}}}{\sqrt{n^2+n}} \le \int_{n^2}^{n^2+n} \frac{1}{\sqrt{x}} e^{-\frac{1}{x}} dx \le \frac{ne^{-\frac{1}{n^2}}}{\sqrt{n^2}}$$

$$\lim_{n\to\infty} \frac{ne^{-\frac{1}{n^2}}}{\sqrt{n^2}} = \lim_{n\to\infty} e^{-\frac{1}{n^2}} = 1 \qquad \lim_{n\to\infty} \frac{ne^{-\frac{1}{n^2+n}}}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} \lim_{n\to\infty} e^{-\frac{1}{n^2+n}} = 1$$

从而由夹逼定理，知 $\lim_{n\to\infty} \int_{n^2}^{n^2+n} \frac{1}{\sqrt{x}} e^{-\frac{1}{x}} dx = 1 \quad \square$.

15. 求下列函数 $y = y(x)$ 的导数 $\dfrac{dy}{dx}$

$$a)\ y = \int_{\cos x}^{\sin x} e^{-t^2} dt; \qquad b)\ y = \left(\int_0^{\sqrt{x}} \ln(1+t^2) dt \right)^2$$

$$c)\ \int_0^{xy} e^t dt + \int_0^y \sin t dt = 0; \qquad d)\ \begin{cases} x = \int_1^t \ln u du \\ y = \int_1^t u \ln u du \end{cases}$$

解： a) $\dfrac{dy}{dx} = e^{-\sin^2 x} \cos x - e^{-\cos^2 x} (\cos x)' = e^{-\sin^2 x} \cos x + e^{-\cos^2 x} \sin x$

b) $y' = 2\int_0^{\sqrt{x}} \ln(1+t^2) dt \cdot \ln(1+x) \dfrac{1}{2\sqrt{x}} = \dfrac{\ln(1+x)}{\sqrt{x}} \int_0^{\sqrt{x}} \ln(1+t^2) dt$

c) 方程 $\int_0^{xy} e^t dt + \int_0^y \sin t dt = 0$ 两边同时对 x 求导，得

$$e^{xy}(y + xy') + \sin y \cdot y' = 0 \quad \Longrightarrow \quad y' = \frac{-ye^{xy}}{\sin y + xe^{xy}}$$

d) $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{t \ln t}{\ln t} = t$

16. 求极限 $a)\ \lim\limits_{x \to 0} \dfrac{\int_{\cos x}^1 e^{-t^2} dt}{x^2}$; $\quad b)\ \lim\limits_{x \to 0} \dfrac{\int_0^x \ln(1+t^2) dt}{\ln \dfrac{\sin x}{x}}$.

解： a) $\lim\limits_{x \to 0} \dfrac{\int_{\cos x}^1 e^{-t^2} dt}{x^2} \xlongequal{\text{洛必达}} \lim\limits_{x \to 0} \dfrac{-e^{-\cos^2 x} (\cos x)'}{2x} = \lim\limits_{x \to 0} \dfrac{\sin x e^{-\cos^2 x}}{2x} = \dfrac{1}{2e}$

b) $\lim\limits_{x \to 0} \dfrac{\int_0^x \ln(1+t^2) dt}{\ln \dfrac{\sin x}{x}} = \lim\limits_{x \to 0} \dfrac{\ln(1+x^2)}{\frac{x}{\sin x} \cdot \frac{\cos x \cdot x - \sin x}{x^2}} = \lim\limits_{x \to 0} \dfrac{x^2 \ln(1+x^2)}{x \cos x - \sin x} = 0$

17. 设函数 $f(x)=\begin{cases} x^2, & 0\le x<1 \\ x-2, & 1\le x\le 2 \end{cases}$ 求 $\Phi(x)=\int_0^x f(t)dt$ 在 $[0,2]$ 上的表达式.

解： 当 $0<x<1$ 时，$\Phi(x)=\frac{1}{3}\Big|_0^x=\frac{x^3}{3}$；而当 $1<x<2$ 时，$\Phi(x)=\int_0^1 f(t)dt+\int_1^x f(t)dt=\frac{1}{3}+\int_1^x(t-2)dt=\frac{1}{3}+\frac{1}{2}(x-2)^2-\frac{1}{2}$，故

$$\Phi(x)=\begin{cases} \frac{1}{3}x^3, & 0\le x\le 1 \\ \frac{1}{2}(x-1)^2-\frac{1}{6}, & 1\le x\le 2. \end{cases}$$

18. 设函数 $f\in C[a,b]$，且 $f(x)>0$（$x\in[a,b]$），记 $F(x)=\int_a^x f(t)dt+\int_b^x \frac{dt}{f(t)}$ $(x\in[a,b])$. 证明:

a) $F'(x)\ge 2$; *b*) 方程 $F(x)=0$ 在区间 (a,b) 内有且仅有一根.

证明： a) $F'(x)=f(x)+\frac{1}{f(x)}\ge 2\sqrt{f\cdot\frac{1}{f}}=2.$

b) 由 *a*) 知 $F(x)$ 单调增加，且由于 $F'(x)\ge 2$，所以

$$F(a)=\int_a^a f(t)dt+\int_b^x \frac{dt}{f(t)}=-\int_a^b \frac{dt}{f(t)}<0$$

且 $F(b)=\int_a^b f(t)dt>0$. 而 $F(x)\in C[a,b]$，故由零点定理知道 $\exists\xi\in(a,b)$，使得 $F(\xi)=0$. 而 F 在 (a,b) 上单调，所以 ξ 是 F 在 (a,b) 上的唯一根. $\square$.

19. 设函数 $f\in C[a,b]$，且 $f(x)>0\,(x\in[a,b])$. 证明：至少存在一个点 $\xi\in[a,b]$，使得 $\int_a^\xi f(x)dx=\int_\xi^b f(x)dx=\frac{1}{2}\int_a^b f(x)dx$

解： 令 $G(x)=\int_a^x f(t)dt-\frac{1}{2}\int_a^b f(x)dx$，则 $G'(t)=f(x)>0$，故 G 在

$[a,b]$ 上单调增加, 且 $G(a)=-\dfrac{1}{2}\displaystyle\int_a^b f(t)dt<0$, $G(b)=\dfrac{1}{2}\displaystyle\int_a^b f(t)dt>0$, 则由闭区间上连续函数的介值定理，知 $\exists!\,\xi\in(a,b)$，使得 $G(\xi)=0$.

同理，令 $F(x)=\displaystyle\int_x^b f(t)dt-\dfrac{1}{2}\displaystyle\int_a^b f(x)dx$，则 $F'(x)=-f(x)<0$，故 F 在 (a,b) 上单调减少，又 $F(a)>0$，$F(b)<0$，结合介值定理，知 $\exists!\eta\in(a,b)$，使得 $F(\eta)=0$，即 $\displaystyle\int_\eta^b f(t)dt=\dfrac{1}{2}\displaystyle\int_a^b f(x)dx$. 由于 ξ 和 η 的唯一性，且 $\displaystyle\int_a^\xi f(x)dx=\displaystyle\int_\eta^b f(x)dx$，故 $\xi=\eta$. $\square$.

20. 设函数 $f\in C[0,1]\cap D(0,1)$，且 $3\displaystyle\int_0^{\frac{1}{3}} e^{1-x^2}f(x)dx=f(1)$. 证明：至少存在一点 $\xi\in(0,1)$，使得 $f'(\xi)=2\xi f(\xi)$.

证明： 由积分中值定理，知 $\exists\eta\in[0,1/3]$，使得 $3\displaystyle\int_0^{\frac{1}{3}} e^{1-x^2}f(x)dx=3e^{1-\eta^2}f(\eta)\dfrac{1}{3}=e^{1-\eta^2}f(\eta)$，但由条件知 $e^{1-\eta^2}f(\eta)=f(1)=e^{1-1^2}f(1)$，则令 $F(x)=e^{1-x^2}f(x)\in C[0,1]\cap D(0,1)$，上条件表明 $F(1)=F(\eta)$，从而由罗尔定理知 $\exists\xi\in(\eta,1)$，使得 $F'(\xi)=0$，即

$$2\xi e^{1-\xi^2}f(\xi)+e^{1-\xi^2}f'(\xi)=0 \quad\Longrightarrow\quad f'(\xi)=2\xi f(\xi) \quad \square.$$

21. 设函数 $S(x)=\displaystyle\int_0^x|\cos t|dt$.

(a) 当 $n\in\mathbb{N}_+$，且 $n\pi\le x<(n+1)\pi$ 时，证明：$2n\le S(x)<2(n+1)$.

(b) 求极限 $\displaystyle\lim_{x\to+\infty}\dfrac{S(x)}{x}$.

证明： a) $\dfrac{dS}{dx}=|\cos x|\ge 0$, 故 $S(x)$ 单调增减, 则由于 $n\pi\le x<(n+1)\pi$, 故

$$S(n\pi)\le S(x)\le S\left((n+1)\pi\right)$$

而 $S(n\pi)=\displaystyle\int_0^{n\pi}|\cos t|dt \xlongequal{t=nx} n\displaystyle\int_0^\pi|\cos nx|dx$. 下证 $\displaystyle\int_0^\pi|\cos nx|dx=$

2. 从而 $2n \leq S(x) \leq 2(n+1)$.

$$\int_0^{\pi} |\cos nx| dx = n \times \left(\int_0^{\frac{\pi}{2n}} \cos nx \, dx - \int_{\frac{\pi}{2n}}^{\frac{3\pi}{2n}} \cos kx \, dx + \int_{\frac{3\pi}{2n}}^{\frac{2\pi}{n}} \cos nx \, dx \right)$$

$$= n \times \left(\frac{1}{n} \sin nx \Big|_0^{\frac{\pi}{2n}} - \frac{1}{n} \sin nx \Big|_{\frac{\pi}{2n}}^{\frac{3\pi}{2n}} + \frac{1}{n} \sin nx \Big|_{\frac{3\pi}{2n}}^{\frac{2\pi}{n}} \right) = n \times \frac{2}{n} = 2$$

b) 由 $a)$ 中结论知 $\dfrac{2n}{x} \leq \dfrac{S(x)}{x} \leq \dfrac{2(n+1)}{x}$，而当 $n\pi \leq x < (n+1)\pi$ 时，$\dfrac{1}{(n+1)\pi} \leq x \leq \dfrac{1}{n\pi}$，故而

$$\frac{2n}{(n+1)\pi} \leq \frac{S(x)}{x} \leq \frac{2(n+1)}{n\pi} \quad \Longrightarrow \quad \lim_{x \to +\infty} \frac{S(x)}{x} = \frac{2}{\pi} \quad \square.$$

22. 证明：$\displaystyle\int_0^x f(t)(x-t)dt = \int_0^x \left(\int_0^t f(x)dx \right) dt$

证明： $\displaystyle\int_0^x f(t)(x-t)dt = \int_0^x (x-t)\, dF(t) \quad \left(其中\, F(t) := \int_0^t f(x)dx \right)$

$$= F(t)(x-t)\Big|_0^x - \int_0^x F(t)d(x-t) = F(0) \cdot x + \int_0^x F(t)\, dt$$

$$= \int_0^x (F(t) - F(0))\, dt = \int_0^x \left(\int_0^t f(x)dx \right) dt \qquad \square.$$

23. 设非负函数 $f \in R[a,b]$，证明不等式：

$$\left(\int_a^b f(x) \cos x dx \right)^2 + \left(\int_a^b f(x) \sin x dx \right)^2 \leq \left(\int_a^b f(x)dx \right)^2$$

提示： 利用柯西-施瓦茨不等式.

证明： 利用柯西-施瓦茨不等式，得 $\left(\displaystyle\int_a^b f(x) \cos x dx \right)^2 =$

$$\left(\int_a^b \sqrt{f(x)} \left(\sqrt{f(x)} \cos x \right) dx \right)^2 \leq \left(\int_a^b f(x)dx \right) \left(\int_a^b f(x) \cos^2 x dx \right)$$

同理

$$\left(\int_a^b f(x)\sin x dx\right)^2 \leq \left(\int_a^b f(x)dx\right)\left(\int_a^b f(x)\sin^2 x dx\right)$$

两不等式相加，得到

$$\left(\int_a^b f(x)\cos x dx\right)^2 + \left(\int_a^b f(x)\sin x dx\right)^2 \leq$$

$$\left(\int_a^b f(x)dx\right)\left(\int_a^b f(x)\cos^2 x dx + \int_a^b f(x)\sin^2 x dx\right) = \left(\int_a^b f(x)dx\right)^2$$

24. 设 $f \in C[a,b]$，且 $f(a)$ 和 $f(b)$ 分别是 $f(x)$ 在 $[a,b]$ 上最大值和最小值. 证明：至少存在一点 $\xi \in [a,b]$，使得

$$\int_a^b f(x)dx = f(a)(\xi - a) + f(b)(b - \xi)$$

提示： 先利用积分中值公式，然后构造函数并利用零点定理之类.

证明： 令 $g(x) = \int_a^b f(x)dx - f(a)(x-a) - f(b)(b-x)$

$$= \int_a^x (f(t) - f(a))\, dt + \int_x^b (f(t) - f(b))\, dt$$

可见 $g(a) = \int_a^b (f(t) - f(b))dt = \int_a^b f(t)dt - f(b)(b-a) \xlongequal{\text{积分中值定理}}$

$$f(\eta)(b-a) - f(b)(b-a) = (f(\eta) - f(b))(b-a) \leq 0 \quad \eta \in [a,b]$$

同理可知 $g(b) \geq 0$. 如果 $g(a)$ 或 $g(b)$ 为零，则无需再证，否则必有 $g(a)g(b) < 0$，则由于 $g(x) \in C[a,b]$，应用零点定理，知 $\exists \xi \in (a,b)$，使得 $g(\xi) = 0$，即得所证. $\square$.

25. 设 $f \in C[a,b] \cap D(a,b)$，且 $f'(x) \geq 0\,(x \in (a,b)$，求证

$$\int_a^b xf(x)dx \geq \frac{a+b}{2}\int_a^b f(x)dx$$

证明： 只需证明 $\displaystyle\int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \geq 0$. 左边可写为

$$\int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) f(x)dx + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right) f(x)dx \xlongequal[x:=a+b-t]{\text{对第一个积分}}$$

$$\int_{\frac{a+b}{2}}^b \left(a+b-t - \frac{a+b}{2}\right) f(a+b-t)d(a+b-t) + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right) f(x)dx$$

$$= \int_b^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) f(a+b-x)dx + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right) f(x)dx$$

$$= \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right) \left(f(x) - f(a+b-x)\right) dx$$

因 $\forall\, x \in (a,b)$，$f'(x) \geq 0$，从而 f 在 (a,b) 上单调增加

- 若 $x \leq \dfrac{a+b}{2}$，则 $f(x) \leq f(a+b-x)$，此时上面积分的被积函数 ≥ 0.
- 若 $x > \dfrac{a+b}{2}$，则 $f(x) > f(a+b-x)$，此时上面积分的被积函数 > 0.

综上，不论 $x \in (a,b)$ 如何取值，被积函数 ≥ 0，则由保序性知积分 ≥ 0，即

$$\int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \geq 0 \quad \Longrightarrow \quad \int_a^b xf(x)dx \geq \frac{a+b}{2}\int_a^b f(x)dx \quad \square.$$