# hw_10

## 1. 求下列定积分

**a)** $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$**;**

**解:**

原式 $= \arcsin x|_0^{\frac{1}{2}}$
$= \arcsin\frac{1}{2} = \frac{\pi}{6}$

**b)** $\displaystyle\int_0^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$**;**

**解:**

原式 $= \displaystyle\int_0^{\frac{\pi}{2}} \left|\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)\right| dx$

$= -\sqrt{2}\displaystyle\int_0^{\frac{\pi}{4}} \sin\left(x-\frac{\pi}{4}\right)dx + \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(x-\frac{\pi}{4}\right)dx$

$= -\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\Big|_0^{\frac{\pi}{4}} - \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\sqrt{2}\left(\cos 0 - \cos\left(-\frac{\pi}{4}\right)\right) - \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - \cos 0\right)$

$= -\sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right) - \sqrt{2}\left(\frac{\sqrt{2}}{2}-1\right)$

$= -\sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right) + \sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right)$
$= \sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right) + \sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right)$

$= 2\sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}-2$

**c)** $\displaystyle\int_0^{2\pi} \sqrt{1+\cos x}\, dx$**;　（提示：倍角公式）**

**解:**

原式 $= \displaystyle\int_0^{2\pi} \sqrt{2\cos^2\frac{x}{2}}\, dx$

$= \sqrt{2}\displaystyle\int_0^{2\pi} \left|\cos\frac{x}{2}\right| dx$

$= \sqrt{2}\left(\displaystyle\int_0^{\pi} \cos\frac{x}{2}dx - \int_{\pi}^{2\pi} \cos\frac{x}{2}dx\right)$

$= \sqrt{2}\left(2\sin\frac{x}{2}\Big|_0^{\pi} - 2\sin\frac{x}{2}\Big|_{\pi}^{2\pi}\right)$

$= \sqrt{2}\left(2\sin\frac{\pi}{2} - 0 - \left(2\sin\pi - 2\sin\frac{\pi}{2}\right)\right) = 4\sqrt{2}$

### d) $\int_0^3 x^2[x]\,dx$

**解:**

原式 $= \int_1^2 x^2dx + \int_2^3 2x^2dx$

$= \frac{1}{3}x^3\Big|_1^2 + \frac{2}{3}x^3\Big|_2^3$

$= \frac{1}{3}(8-1) + \frac{2}{3}(27-8)$

$= \frac{1}{3}\cdot 7 + \frac{2}{3}\cdot 19$

$= 15$

### e) $\int_{-5}^3 \frac{dx}{\sqrt[3]{(x-3)^2}}$

**解:**

原式 $= \int_{-5}^3 (x-3)^{-\frac{2}{3}}dx$

令 $u = x - 3$，则 $du = dx$，当 $x = -5$ 时 $u = -8$，当 $x = 3$ 时 $u = 0$

原式$= \int_{-8}^0 u^{-\frac{2}{3}}du = 3u^{\frac{1}{3}}\Big|_{-8}^0 = 3\left(0 - (-8)^{\frac{1}{3}}\right) = 3\left(-(-2)\right) = 6$

### f) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x\,dx$

**解:**

原式 $= \int_0^{\frac{\pi}{2}} 2\sin x\cos^6 x\,dx$

令 $u = \cos x$，则 $du = -\sin x\,dx$

当 $x = 0$，$u = 1$；当 $x = \frac{\pi}{2}$，$u = 0$

原式$= -2\int_1^0 u^6du = 2\int_0^1 u^6du = 2\cdot\frac{u^7}{7}\Big|_0^1 = \frac{2}{7}$

## 2. 求下列定积分

### a) $\int_0^{\frac{1}{2}} \arcsin x\, dx$;

**解:**

$$
\begin{aligned}
&原式 = x\arcsin x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x\, d(\arcsin x) \\
&= \tfrac{1}{2}\arcsin\tfrac{1}{2} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx \\
&= \tfrac{\pi}{12} + \sqrt{1-x^2}\Big|_0^{\frac{1}{2}} \\
&= \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1
\end{aligned}
$$

### b) $\int_0^{\frac{\pi}{4}} \sec^3 x\, dx$

**解:**

$$
\begin{aligned}
&原式 = \int_0^{\frac{\pi}{4}} \sec x\, d(\tan x) \\
&= \tan x \sec x\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\, d(\sec x) \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \tan x \sec x\, dx \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} (\sec^3 x - \sec x)\, dx \\
&= \sqrt{2} - \left(\int_0^{\frac{\pi}{4}} \sec^3 x\, dx - \int_0^{\frac{\pi}{4}} \sec x\, dx\right)
\end{aligned}
$$

令原式为 $I$

$$
I = \sqrt{2} - \left(\int_0^{\frac{\pi}{4}} \sec^3 x\, dx - \int_0^{\frac{\pi}{4}} \sec x\, dx\right)
$$

$$
I = \sqrt{2} - I + \int_0^{\frac{\pi}{4}} \sec x\, dx \quad \Rightarrow \quad 2I = \sqrt{2} + \int_0^{\frac{\pi}{4}} \sec x\, dx
$$

$$
\int_0^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}} = \ln(1+\sqrt{2}) - \ln(1) = \ln(1+\sqrt{2})
$$

所以:

$$
2I = \sqrt{2} + \ln(1+\sqrt{2}) \quad \Rightarrow \quad I = \tfrac{\sqrt{2}}{2} + \tfrac{1}{2}\ln(1+\sqrt{2})
$$

**c)** $\displaystyle\int_0^{\frac{\pi}{2}} e^x \sin^2 x\,dx$;

**解:**

原式 = $\displaystyle\int_0^{\frac{\pi}{2}} e^x \cdot \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} e^x\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \cos 2x\,dx$

$= \frac{1}{2}(e^{\frac{\pi}{2}} - 1) - \frac{1}{10}e^x(\cos 2x + 2\sin 2x)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}e^{\frac{\pi}{2}} - \frac{1}{2} + \frac{1}{10}e^{\frac{\pi}{2}} + \frac{1}{10}$

$= \frac{3}{5}e^{\frac{\pi}{2}} - \frac{2}{5}$

**d)** $\displaystyle\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2}\,dx$

**解:**

原式 = $\frac{1}{2}\displaystyle\int_0^{\sqrt{\ln 2}} x^2 e^{-x^2}\,d(x^2)$

令 $u = x^2$，则

原式$=$ $\frac{1}{2}\displaystyle\int_0^{\ln 2} u e^{-u}\,du$

$= -\frac{1}{2}\displaystyle\int_0^{\ln 2} u\,d(e^{-u})$

$= -\frac{1}{2}ue^{-u}\Big|_0^{\ln 2} + \frac{1}{2}\displaystyle\int_0^{\ln 2} e^{-u}\,du$

$= -\frac{1}{2}\ln 2 \cdot \frac{1}{2} - \frac{1}{2}e^{-u}\Big|_0^{\ln 2}$

$= -\frac{1}{4}\ln 2 - \frac{1}{2}\left(\frac{1}{2} - 1\right)$

$= \frac{1}{4} - \frac{1}{4}\ln 2$

**e)** $\displaystyle\int_0^1 x\sqrt{(1-x^4)^3}\,dx$;

**解:**

令 $u = x^2$，则 $du = 2x\,dx$

原式 = $\frac{1}{2}\displaystyle\int_0^1 \sqrt{(1-u^2)^3}\,du$

令 $u = \sin\theta$，则当 $u = 0$ 时 $\theta = 0$；当 $u = 1$ 时 $\theta = \frac{\pi}{2}$

原式 $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$= \frac{1}{8}\int_0^{\frac{\pi}{2}} (1+2\cos 2\theta+\cos^2 2\theta)\, d\theta$

$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \left(1+2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right) d\theta$

$= \frac{1}{16}\int_0^{\frac{\pi}{2}} (3+4\cos 2\theta+\cos 4\theta) d\theta$

$= \frac{1}{16}\left(3\theta+2\sin 2\theta+\frac{1}{4}\sin 4\theta\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{3\pi}{32}$

### f) $\int_0^{\frac{\pi}{4}} \frac{x\, dx}{1+\cos 2x}$

**解:**

原式 = $\int_0^{\frac{\pi}{4}} \frac{x}{2\cos^2 x}\, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} x\, d(\tan x)$

$= \frac{1}{2}x\tan x\Big|_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x\, dx$

$= \frac{\pi}{8} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x}\, d(\cos x)$

$= \frac{\pi}{8} + \frac{1}{2}\ln|\cos x|\Big|_0^{\frac{\pi}{4}}$

$= \frac{\pi}{8} + \frac{1}{2}\ln\frac{\sqrt{2}}{2}$

$= \frac{\pi}{8} - \frac{1}{4}\ln 2$

## 3. 求下列定积分

### a) $\int_0^1 (e^x-1)^4 e^x\, dx$;

**解:**

令 $u = e^x$，则 $du = e^x dx$

原式 = $\int_1^e (u-1)^4\, du$

$= \frac{1}{5}(u-1)^5\Big|_1^e$

$= \frac{1}{5}(e-1)^5$

**b)** $\displaystyle\int_1^e \frac{1+\ln x}{x}\,dx$

**解:**

令 $u = 1 + \ln x$，则 $du = \frac{dx}{x}$

原式 = $\displaystyle\int_1^2 u\,du$

$= \frac{1}{2}u^2\Big|_1^2 = \frac{3}{2}$

**c)** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+e^{2x}}}$;

**解:**

令 $u = \sqrt{1+e^{2x}}$

则:

$u^2 = 1 + e^{2x}$
$e^{2x} = u^2 - 1$
$2x = \ln(u^2 - 1)$
$x = \frac{1}{2}\ln(u^2 - 1)$

$dx = \frac{u}{u^2-1}\,du$

原式 = $\displaystyle\int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{1}{u^2-1}\,du$

$= \frac{1}{2}\displaystyle\int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$

$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\Big|_{\sqrt{2}}^{\sqrt{1+e^2}}$

$= \frac{1}{2}\ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

**d)** $\displaystyle\int_0^1 \frac{x\,dx}{1+\sqrt{x}}$

**解:**

令 $u = 1 + \sqrt{x}$

则:

$x = u^2 - 2u + 1$
$dx = (2u - 2)\,du$

原式 = $\displaystyle\int_1^2 \frac{(u^2-2u+1)(2u-2)}{u}\,du$

$$= \int_1^2 \frac{2(u-1)^3}{u}\,du$$

$$= 2\int_1^2 \frac{u^3-3u^2+3u-1}{u}\,du$$

$$= 2\int_1^2 \left(u^2-3u+3-\frac{1}{u}\right)du$$

$$= 2\left(\tfrac{1}{3}u^3-\tfrac{3}{2}u^2+3u-\ln u\right)\Big|_1^2$$

$$= \tfrac{5}{3}-2\ln 2$$

### e) $\int_0^1 \frac{x^2\,dx}{\sqrt{2x-x^2}}$;

**解:**

令 $x-1=\sin t$，则 $dx=\cos t\,dt$

当 $x=0$ 时，$t=-\frac{\pi}{2}$；当 $x=1$ 时，$t=0$

原式 = $\int_{-\frac{\pi}{2}}^0 \frac{(1+\sin t)^2}{\cos t}\cdot\cos t\,dt$

$$= \int_{-\frac{\pi}{2}}^0 (1+\sin t)^2\,dt$$

$$= \int_{-\frac{\pi}{2}}^0 (\sin^2 t+2\sin t+1)\,dt$$

$$= \int_{-\frac{\pi}{2}}^0 \left(\frac{1-\cos 2t}{2}+2\sin t+1\right)dt$$

$$= \int_{-\frac{\pi}{2}}^0 \left(\frac{3}{2}-\frac{1}{2}\cos 2t+2\sin t\right)dt$$

$$= \left(\tfrac{3}{2}t-\tfrac{1}{4}\sin 2t-2\cos t\right)\Big|_{-\frac{\pi}{2}}^0$$

$$= \tfrac{3\pi}{4}-2$$

### f) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\,dx$

**解:**

原式 = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x}\cdot|\sin x|\,dx$

由于被积函数是偶函数（因为 $\cos(-x)=\cos x$，$|\sin(-x)|=|\sin x|$），可化为:

原式 $= 2\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\cdot\sin x\,dx$

令 $u=\cos x$，则 $du=-\sin x\,dx$

当 $x=0$ 时，$u=1$；当 $x=\frac{\pi}{2}$ 时，$u=0$

原式 $=2\int_1^0 \sqrt{u}\,(-du)$

$=2\int_0^1 u^{1/2}\,du$

$=2\cdot\frac{2}{3}u^{3/2}\Big|_0^1=\frac{4}{3}$

## 4. 求下列定积分

### a) $\int_{\sqrt{2}}^{2} \frac{dx}{2+\sqrt{4+x^2}}$;

**解:**

原式 = $\int_{\sqrt{2}}^{2} \frac{1}{1+\sqrt{1+\left(\frac{x}{2}\right)^2}}\,dx$

令 $x=2\tan t$，则:
$\tan t=\frac{x}{2},\quad dx=2\sec^2 t\,dt$

当 $x=\sqrt{2}$ 时，$t=\arctan\left(\frac{\sqrt{2}}{2}\right)$;
当 $x=2$ 时，$t=\frac{\pi}{4}$

原式$=\int_{\arctan(\frac{\sqrt{2}}{2})}^{\frac{\pi}{4}} \frac{\sec^2 t}{1+\sec t}\,dt$

$=\int_{\arctan(\frac{\sqrt{2}}{2})}^{\frac{\pi}{4}} \frac{1}{\cos^2 t+\cos t}$

$=\int_{\arctan(\frac{\sqrt{2}}{2})}^{\frac{\pi}{4}} \left(\frac{1}{\cos t}-\frac{1}{\cos t+1}\right)dt$

$=\left(\ln|\sec t+\tan t|-\tan\frac{t}{2}\right)\Big|_{\arctan(\frac{\sqrt{2}}{2})}^{\frac{\pi}{4}}$

$=\sqrt{3}+1-2\sqrt{2}+\ln\left(\frac{\sqrt{2}+2}{\sqrt{3}+1}\right)$

### b) $\int_0^2 \frac{dx}{2+\sqrt{4+x^2}}$

**解:**

原式 = $\left(\ln|\sec t+\tan t|-\tan\frac{t}{2}\right)\Big|_0^{\frac{\pi}{4}}$
$=\ln(\sqrt{2}+1)-\sqrt{2}+1$

**c)** $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} \quad (a>0)$**;**

**解:**

令 $x = a\sin t$，则 $dx = a\cos t\,dt$

当 $x=0$ 时，$t=0$；当 $x=a$ 时，$t=\frac{\pi}{2}$

原式 = $\int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t + a\cos t}\,dt$

$= \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}\,dt$

利用对称性：令 $u=\frac{\pi}{2}-t$，则：

$= \int_0^{\frac{\pi}{2}} \frac{\sin u}{\cos u+\sin u}\,du = \int_0^{\frac{\pi}{2}} \frac{\sin t}{\cos t+\sin t}\,dt$

将两个表达式相加：

$2I = \int_0^{\frac{\pi}{2}} \frac{\cos t+\sin t}{\sin t+\cos t}\,dt = \int_0^{\frac{\pi}{2}} 1\,dt = \frac{\pi}{2}$

故：

$I = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

**d)** $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$

**解:**

d) 令 $x=\sin t$，则 $dx=\cos t\,dt$

当 $x=0$ 时，$t=0$；当 $x=\frac{1}{2}$ 时，$t=\frac{\pi}{6}$

原式 = $2\int_0^{\frac{\pi}{6}} \frac{t\sin t}{\cos t}\cdot\cos t\,dt = 2\int_0^{\frac{\pi}{6}} t\sin t\,dt$

$= 2(-t\cos t+\sin t)|_0^{\frac{\pi}{6}}$
$= 1-\frac{\sqrt{3}}{6}\pi$

## 5. 设 $f\in C[0,+\infty)$，且 $\forall a,b>0$ 满足不等式 $f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}$.

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## 证明：$F(x) := \frac{1}{x}\int_0^x f(t)\,dt$ 满足不等式

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$$F\left(\frac{a+b}{2}\right) \le \frac{F(a)+F(b)}{2},$$

## 即若 $f(x)$ 下凸，则 $F(x)$ 亦下凸。

**解：**

令 $t = xu$，则 $dt = x\,du$，其中 $u \in [0,1]$

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt = \int_0^1 f(xu)\,du$$

已知：$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}$

$$f\left(\frac{au+bu}{2}\right) \le \frac{f(au)+f(bu)}{2}$$

$$\int_0^1 f\left(\frac{a+b}{2}u\right)du \le \frac{1}{2}\left(\int_0^1 f(au)\,du + \int_0^1 f(bu)\,du\right)$$

$$F\left(\frac{a+b}{2}\right) \le \frac{F(a)+F(b)}{2}$$

故不等式成立。

## 6. 设 $f(x)$ 在区间 $[a,b]$ 上有二阶连续导数。证明：$\exists \xi \in [a,b]$，使得

$$\int_a^b f(x)\,dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24}f''(\xi).$$

**解：**

令 $x_0 = \frac{a+b}{2}$，由泰勒展开：

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(\eta)(x-x_0)^2$$

对 $f(x)$ 在 $[a,b]$ 上积分：

$$\int_a^b f(x)\,dx = \int_a^b f(x_0)\,dx + \int_a^b f'(x_0)(x-x_0)\,dx + \frac{1}{2}\int_a^b f''(\xi)(x-x_0)^2\,dx$$

$$\int_a^b f(x)\,dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{1}{2}f''(\xi)\cdot\frac{(b-a)^3}{12} = (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(\xi)}{24}(b-a)^3$$

## 7. 设 $f'(\sin^2 x) = \cos 2x + \tan^2 x$，求 $f(x)$。

**提示**：$f(x) = \int f'(x)\,dx + C$，所以先求出 $f'(x)$ 的表达式。

**解：**

令 $u = \sin^2 x$

则：

$$f'(u) = 1 - 2u + \frac{u}{1-u} = -2u + \frac{1}{1-u}$$

$$f(u) = \int f'(u)\,du + C = \int \left(-2u + \frac{1}{1-u}\right) du + C = -u^2 - \ln|1-u| + C$$

$$f(x) = -x^2 - \ln|1-x| + C$$

## 8. 设 $f(\ln x) = \frac{\ln(1+x)}{x}$，求 $\int f(x)\,dx$。

**提示**：同上题提示。

**解：**

令 $u = \ln x$，则 $x = e^u$

$$f(u) = \frac{\ln(1+e^u)}{e^u}$$

$$f(x) = \frac{\ln(1+e^x)}{e^x}$$

$$\int f(x)\,dx = \int \frac{\ln(1+e^x)}{e^x}\,dx$$

$$= -\int \ln(1+e^x)\,d(e^{-x})$$

$$= -e^{-x}\ln(1+e^x) + \int e^{-x} \cdot \frac{e^x}{1+e^x}\,dx + C$$

$$= -\frac{\ln(1+e^x)}{e^x} + \int \frac{1}{1+e^x}\,dx + C$$

$$\int \frac{1}{1+e^x}\,dx = \int \left(1 - \frac{e^x}{1+e^x}\right) dx = \int dx - \int \frac{e^x}{1+e^x}\,dx = x - \ln(1+e^x) + C$$

所以：

$$\int f(x)\,dx = -\frac{\ln(1+e^x)}{e^x} + x - \ln(1+e^x) + C$$

## 9. 已知 $f(x)$ 的一个原函数为 $\frac{\sin x}{1+x\sin x}$，求 $\int f(x) f'(x)\,dx$。

**解：**

$$f(x) = \left(\frac{\sin x}{1+x\sin x}\right)'$$

$$= \frac{\cos x(1+x\sin x) - \sin x(\sin x + x\cos x)}{(1+x\sin x)^2} = \frac{\cos x - \sin^2 x}{(1+x\sin x)^2}$$

$$\int f(x)f'(x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}f^2(x) + C$$

$$= \frac{1}{2}\left(\frac{\cos x - \sin^2 x}{(1+x\sin x)^2}\right)^2 + C = \frac{1}{2}\cdot\frac{(\cos x - \sin^2 x)^2}{(1+x\sin x)^4} + C$$

## 10. 曲线 $y = f(x)$ 经过点 $(e, 1)$，且在任一点的切线斜率为该点横坐标的倒数，求该曲线的方程。 （最简单的建立微分方程然后求解的类型）

**解：**
$f(e) = 1$
$f'(x) = \frac{1}{x}$

积分得：

$$f(x) = \int \frac{1}{x}\,dx = \ln|x| + C$$

由初始条件 $f(e) = 1$：

$$\ln e + C = 1 \Rightarrow 1 + C = 1 \Rightarrow C = 0$$

所以：

$$f(x) = \ln x$$

## 11. 设 $(0, +\infty)$ 上的连续函数 $f(x)$ 分别满足下列条件，求 $f(x)$ 的表达式：

### a) $f(x) = \sin x + \int_0^\pi f(x)\,dx$;

**解：**

$$\int_0^\pi f(x)\,dx = \int_0^\pi \sin x\,dx + \int_0^\pi \left(\int_0^x f(t)\,dt\right)dx$$

$$\int_0^\pi f(x)\,dx = (-\cos\pi + \cos 0) + \pi\int_0^\pi f(x)\,dx$$

$$\int_0^\pi f(x)\,dx = \frac{2}{1-\pi}$$

$$f(x) = \sin x + \frac{2}{1-\pi}$$

### b) $f(x) = 2\ln x + x^2 \int_1^e \frac{f(x)}{x}\,dx$

**提示**：题目没说函数可导，所以两边求导不好使，但两边可以求积分啊。

**解：**

$$\frac{f(x)}{x} = \frac{2\ln x}{x} + x\int_1^e \frac{f(x)}{x}\,dx$$

令 $A = \int_1^e \frac{f(x)}{x}\,dx$

$$\frac{f(x)}{x} = \frac{2\ln x}{x} + Ax \Rightarrow f(x) = 2\ln x + Ax^2$$

代入原式求 $A$：

$$\int_1^e \frac{f(x)}{x}\,dx = \int_1^e \left(\frac{2\ln x}{x} + Ax\right)dx = \int_1^e \frac{2\ln x}{x}\,dx + A\int_1^e x\,dx$$

$A = 1 + \frac{A}{2}(e^2-1)$
$A - \frac{A}{2}(e^2-1) = 1$
$A\left(1 - \frac{e^2-1}{2}\right) = 1$
$A\left(\frac{3-e^2}{2}\right) = 1$
$A = \frac{2}{3-e^2}$

$$f(x) = 2\ln x + \frac{2}{3-e^2}x^2$$

## 12. 计算 $\int_1^3 f(x-2)\,dx$，其中 $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ e^x, & x > 0 \end{cases}$

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**解：**

原式 = $\int_1^2 f(x-2)\,dx + \int_2^3 f(x-2)\,dx$
$= \int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx$
$= \int_{-1}^0 (1+x^2)\,dx + \int_0^1 e^x\,dx$
$= e + \frac{1}{3}$

## 13. 求 $\int_0^{\frac{\pi}{2}} \frac{f(x)}{\sqrt{x}}\,dx$，其中

$$f(x)=\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{x}} \frac{dt}{1+\tan t^2}$$

**提示**：直接尝试求出 $f(x)$ 的表达式是不利的，但 $f(x)$ 的导数立马可知，所以。。。

**解**：

原式 = $\int_0^{\frac{\pi}{2}} f(x)\,d(2\sqrt{x})$

$$=\left[2\sqrt{x}\,f(x)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}} 2\sqrt{x}\,f'(x)\,dx$$

$$f'(x)=\frac{1}{2\sqrt{x}}\cdot\frac{1}{1+\tan x}$$

$$\text{原式}=-2\sqrt{\frac{\pi}{2}}\,f\left(\frac{\pi}{2}\right)-\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan x}\,dx$$

$$=-\int_0^{\frac{\pi}{2}}\frac{dx}{1+\tan x}$$

$$=-\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}\,dx$$

$$=-\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}+\frac{\cos x-\sin x}{2(\sin x+\cos x)}\right]dx$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{d(\sin x+\cos x)}{\sin x+\cos x}$$

$$=-\frac{\pi}{4}-\frac{1}{2}\left[\ln(\sin\frac{\pi}{2}+\cos\frac{\pi}{2})-\ln(\sin 0+\cos 0)\right]$$

$$=-\frac{\pi}{4}-\frac{1}{2}[\ln(1+0)-\ln(0+1)]$$

$$=-\frac{\pi}{4}$$

## 14. 已知函数 $f(x)$ 在 $[0,+\infty)$ 上具有二阶连续导数，设 $f(0)=2, f(\pi)=1$，求

$$\int_0^{\pi}\left[f(x)+f''(x)\right]\sin x\,dx.$$

**解**：

原式 = $\int_0^{\pi} f(x)\sin x\,dx+\int_0^{\pi} f''(x)\sin x\,dx$

$$=\int_0^{\pi} f(x)\sin x\,dx+\int_0^{\pi}\sin x\,d(f'(x))$$

$$= \int_0^\pi f(x)\sin x\,dx + \left[f'(x)\sin x\right]_0^\pi - \int_0^\pi f'(x)\cos x\,dx$$

$$= \int_0^\pi f(x)\sin x\,dx + f'(\pi)\sin\pi - f'(0)\sin 0 - \int_0^\pi f'(x)\cos x\,dx$$

$$= \int_0^\pi f(x)\sin x\,dx - \int_0^\pi f'(x)\cos x\,dx$$

$$= \int_0^\pi f(x)\sin x\,dx - [f(x)\cos x]_0^\pi - \int_0^\pi f(x)\sin x\,dx$$

$$= \int_0^\pi f(x)\sin x\,dx - (f(\pi)\cos\pi - f(0)\cos 0) - \int_0^\pi f(x)\sin x\,dx$$

$$= -f(\pi)\cos\pi + f(0)\cos 0 = -f(\pi)(-1) + f(0)(1) = f(\pi) + f(0)$$

$$= 2 + 1 = 3$$

## 15. 设 $f(x)$ 连续，满足 $f(1) = 1$，且 $\int_0^x tf(2x-t)\,dt = \frac{1}{2}\arctan x^2$，

## 求 $\int_1^2 f(x)\,dx$.

**提示**：用变量替换先将积分转化为两边可以对 $x$ 求导的形式。

**解**：

令 $u = 2x - t$

$$\int_x^{2x} (2x-u)f(u)\,du = \frac{1}{2}\arctan^2 x$$

对两边求导

$$-xf(x) + 2\int_x^{2x} f(u)\,du = \frac{x}{1+x^2}$$

令 $x = 1$

$$-f(1) + 2\int_1^2 f(u)\,du = \frac{1}{2}$$

$$\int_1^2 f(u)\,du = \frac{3}{4}$$

故原式 = $\frac{3}{4}$

## 16. 设函数 $f(x)$ 在 $U(0)$ 可导，且 $f(0)=0$，求极限

$$\lim_{x\to 0}\frac{\int_0^x t^{n-1}f(x^n-t^n)\,dt}{x^{2n}}\quad (n\in\mathbb{N}^+)$$

**提示**：见上题的提示。

**解**:
令 $u=x^n-t^n$, $du=-nt^{n-1}dt$

原式 = $\lim\limits_{x\to 0}\dfrac{\int_0^{x^n}f(u)\,du}{nx^{2n}}$

= $\lim\limits_{x\to 0}\dfrac{nx^{n-1}f(x^n)}{2n^2x^{2n-1}}$

= $\lim\limits_{x\to 0}\dfrac{f(x^n)}{2nx^n}$

= $\lim\limits_{t\to 0}\dfrac{f(t)}{2nt}$

= $\dfrac{f'(0)}{2n}$

## 17. 设函数 $f(x)$ 连续，且 $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=A$（$A$ 为常数），记 $\varphi(x)=\displaystyle\int_0^1 f(xt)\,dt.$

## 求 $\varphi'(x)$ 并讨论 $\varphi'(x)$ 在 $x=0$ 处的连续性。

**解**：令 $u=xt, du=xdt$

$$\varphi(x)=\frac{1}{x}\int_0^x f(u)\,du$$

$$\varphi'(x)=\frac{xf(x)-\int_0^x f(u)\,du}{x^2}$$

$$\lim_{x\to 0}\varphi'(x)=\lim_{x\to 0}\frac{f(x)+xf'(x)-f(x)}{2x}$$

$$=\lim_{x\to 0}\frac{f'(x)}{2}$$

又 $A=\lim_{x\to 0}\frac{f(x)}{x}=f'(0)$

$$\lim_{x\to 0}\varphi'(x)=\frac{A}{2}$$

$$\varphi'(0)=\lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x}=\lim_{x\to 0}\frac{\int_0^x f(u)\,du}{x^2}=\frac{A}{2}$$

故 $\varphi'(x)$ 在 $x=0$ 处连续

## 18. 求不定积分

$$I_1=\int\frac{\cos x}{\sin x+\cos x}\,dx \quad \text{和} \quad I_2=\int\frac{\sin x}{\sin x+\cos x}\,dx.$$

**解:**

$$I_1+I_2=\int 1\,dx=x+C$$

$$\begin{aligned}I_1-I_2&=\int\frac{\cos x-\sin x}{\sin x+\cos x}\,dx\\&=\int\frac{d(\sin x+\cos x)}{\sin x+\cos x}\\&=\ln|\sin x+\cos x|+C\end{aligned}$$

$$I_1=\frac{1}{2}x+\frac{1}{2}\ln|\sin x+\cos x|+C$$

$$I_2=\frac{1}{2}x-\frac{1}{2}\ln|\sin x+\cos x|+C$$

## 19. 设 $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$. 证明当 $n\geq 2$ 时，有

### a) $I_n+I_{n-2}=\frac{1}{n-1}$;

**解:**

$$\begin{aligned}I_n+I_{n-2}&=\int_0^{\frac{\pi}{4}}(\tan^n x+\tan^{n-2}x)\,dx\\&=\int_0^{\frac{\pi}{4}}\tan^{n-2}x\sec^2 x\,dx\\&=\left.\frac{\tan^{n-1}x}{n-1}\right|_0^{\frac{\pi}{4}}=\frac{1}{n-1}\end{aligned}$$

### b) $\frac{1}{2(n+1)}<I_n<\frac{1}{2(n-1)}$;

**解:**

$$\tan x<1,\quad \tan^n x\ \text{单调减少},\quad I_n>I_{n+1}$$

$$I_n+I_{n-2}>2I_n\Rightarrow I_n<\frac{1}{2(n-1)}$$

$$I_n + I_{n+2} < 2I_n \Rightarrow I_n > \frac{1}{2(n+1)}$$

故

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

### c) $\lim\limits_{n\to\infty}(nI_n) = \frac{1}{2}$.

$$\frac{n}{2(n+1)} < nI_n < \frac{n}{2(n-1)}$$

$$\lim_{n\to\infty}\frac{n}{2n+2} = \lim_{n\to\infty}(nI_n) = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$$

故

$$\lim_{n\to\infty}(nI_n) = \frac{1}{2}$$

## 20. 设函数 $f(x)$ 在 $[0,1]$ 上二阶可导，且 $f''(x) \geq 0$ ( $x \in [0,1]$)，证明：

$$\int_0^1 f(x^2)\,dx \geq f\left(\frac{1}{3}\right).$$

**解:**

$$f\left(\frac{1}{b-a}\int_a^b \varphi(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b f(\varphi(x))\,dx$$

$$f\left(\int_0^1 x^2\,dx\right) \leq \int_0^1 f(x^2)\,dx$$

$$\int_0^1 f(x)\,dx \geq f\left(\frac{1}{3}\right)$$

## 21. 设 $f \in C^{(1)}[a,b]$，求证

$$\max_{a\leq x\leq b}|f(x)| \leq \frac{1}{b-a}\left|\int_a^b f(x)\,dx\right| + \int_a^b |f'(x)|\,dx.$$

**解:**

$\exists\, \xi \in [a,b]$ s.t. $\int_a^b f(x)\,dx = (b-a)f(\xi)$

对 $\forall x \in [a,b]$

$$f(x) = f(\xi) + \int_\xi^x f'(t)\,dt$$

$$|f(x)| \leq |f(\xi)| + \left|\int_\xi^x f'(t)\,dt\right|$$

$$= \frac{1}{b-a}\left|\int_a^b f(x)\,dx\right| + \left|\int_\xi^x f'(t)\,dt\right|$$

$$\leq \frac{1}{b-a}\left|\int_a^b f(x)\,dx\right| + \int_a^b |f'(t)|\,dt$$

故

$$\max_{a\leq x\leq b}|f(x)| \leq \frac{1}{b-a}\left|\int_a^b f(x)\,dx\right| + \int_a^b |f'(x)|\,dx$$