hw_9

1. 求下列不定积分（可考虑利用三角恒等式化简后凑微分）

a) $\int \cos^2 \frac{x}{2} dx$

解:

$$\begin{aligned} 原式 &= \int \frac{1+\cos x}{2}\, dx \\ &= \frac{1}{2}x + \frac{1}{2}\sin x + C \end{aligned}$$

b) $\int \frac{\cos 2x}{\cos x + \sin x} dx$

解:

$$\begin{aligned} 原式 &= \int \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}\, dx \\ &= \int (\cos x - \sin x)\, dx \\ &= \sin x + \cos x + C \end{aligned}$$

c) $\int \frac{1+\cos^2 x}{1+\cos 2x} dx$

解:

$$\begin{aligned} 原式 &= \int \frac{1+\cos^2 x}{2\cos^2 x}\, dx \\ &= \int \left(\frac{1}{2} + \frac{1}{2\cos^2 x}\right) dx \\ &= \frac{1}{2}x + \frac{1}{2}\tan x + C \end{aligned}$$

d) $\int \frac{dx}{\cos^2 x \sin^2 x}$

解:

$$\begin{aligned} 原式 &= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx \\ &= \tan x - \cot x + C \end{aligned}$$

2. 求下列不定积分（可化简后（利用换元）凑微分）

a) $\int \frac{dx}{e^x - e^{-x}}$

解:

令 $u = e^x$

则 $x = \ln u$，$dx = \frac{du}{u}$

$$\begin{aligned}
\text{原式} &= \int \frac{du}{u^2 - 1} \\
&= \int \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\
&= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C
\end{aligned}$$

b) $\int \frac{2x-5}{(x^2-5x+8)^2} dx$

解:

令 $u = x^2 - 5x + 8$

则 $du = (2x-5)\,dx$

$$\begin{aligned}
\text{原式} &= \int \frac{du}{u^2} \\
&= -\frac{1}{u} + C \\
&= -\frac{1}{x^2-5x+8} + C
\end{aligned}$$

c) $\int \frac{x^2}{\sqrt[3]{1-2x^3}} dx$

解:

令 $u = \sqrt[3]{1-2x^3}$

则 $u^3 = 1 - 2x^3$

$3u^2\,du = -6x^2\,dx$

$x^2\,dx = -\frac{1}{2}u^2\,du$

$$\begin{aligned}
\text{原式} &= \int \frac{-u^2}{2u}\,du \\
&= \int -\frac{1}{2}u\,du \\
&= -\frac{1}{4}u^2 + C \\
&= -\frac{1}{4}(1-2x^3)^{\frac{2}{3}} + C
\end{aligned}$$

d) $\int e^x \sin(e^x)dx$

解:
令 $u = e^x$
则 $du = e^x\,dx$

$$\begin{aligned}\text{原式} &= \int \sin u\,du \\ &= -\cos u + C \\ &= -\cos(e^x) + C\end{aligned}$$

e) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}dx$

解:
令 $u = \sin x - \cos x$
则 $du = (\cos x + \sin x)\,dx$

$$\begin{aligned}\text{原式} &= \int \frac{du}{u^{\frac{1}{3}}} \\ &= \frac{3}{2}u^{\frac{2}{3}} + C \\ &= \frac{3}{2}(\sin x - \cos x)^{\frac{2}{3}} + C\end{aligned}$$

f) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx$

解:
令 $u = \sqrt{x}$
则 $x = u^2$，$dx = 2u\,du$

$$\begin{aligned}\text{原式} &= \int \frac{2\arctan u}{1+u^2}\,du \\ \text{令 } t &= \arctan u \\ dt &= \frac{1}{1+u^2}\,du\end{aligned}$$

$$\begin{aligned}\text{原式} &= \int 2t\,dt \\ &= t^2 + C \\ &= (\arctan\sqrt{x})^2 + C\end{aligned}$$

g) $\int \frac{dx}{x\ln x}$

解:
令 $u = \ln x$
则 $du = \frac{dx}{x}$

$$\begin{aligned} 原式 &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\ln x| + C \end{aligned}$$

h) $\int \frac{x^2}{4+x^6}dx$

解:
令 $u = x^3$
则 $du = 3x^2\,dx$

$$\begin{aligned} 原式 &= \int \frac{1}{3}\cdot\frac{du}{4+u^2} \\ &= \frac{1}{6}\arctan\frac{u}{2} + C \\ &= \frac{1}{6}\arctan\frac{x^3}{2} + C \end{aligned}$$

3. 求下列不定积分（最好直接变量替换后化简整理成简单积分）

a) $\int e^x\sqrt{1-e^{2x}}dx$

解:
令 $u = e^x$
则 $du = e^x\,dx$

$$\begin{aligned} 原式 &= \int \sqrt{1-u^2}\,du \\ 令\ u &= \sin t \\ du &= \cos t\,dt \end{aligned}$$

$$
\begin{aligned}
\text{原式} &= \int \cos^2 t \, dt \\
&= \int \frac{1+\cos 2t}{2} \, dt \\
&= \frac{1}{2} t + \frac{\sin 2t}{4} + C \\
&= \frac{1}{2} \arcsin(e^x) + \frac{1}{4} \cdot 2 \sin t \cos t + C \\
&= \frac{1}{2} (\arcsin(e^x) + e^x \sqrt{1-e^{2x}}) + C
\end{aligned}
$$

b) $\int \frac{dx}{\sqrt{1+e^x}}$

解:

令 $\sqrt{1+e^x} = u$
则 $1 + e^x = u^2$
$e^x \, dx = 2u \, du$
又 $e^x = u^2 - 1$，故 $dx = \frac{2u}{u^2-1} \, du$

$$
\begin{aligned}
\text{原式} &= \int \frac{2 \, du}{u^2 - 1} \\
&= \ln \left| \frac{u-1}{u+1} \right| + C \\
&= \ln \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} + C
\end{aligned}
$$

c) $\int \frac{dx}{1+e^x}$

解:

$$
\text{原式} = \int \frac{e^{-x}}{e^{-x}+1} \, dx
$$

令 $u = 1 + e^{-x}$
则 $du = -e^{-x} \, dx$

$$
\begin{aligned}
\text{原式} &= \int \frac{-du}{u} \\
&= -\ln |u| + C \\
&= -\ln(1 + e^{-x}) + C
\end{aligned}
$$

d) $\displaystyle\int \frac{1+\ln x}{(x\ln x)^2}dx$

解:
令 $u = x\ln x$
则 $du = (\ln x + 1)\,dx$

$$\begin{aligned}\text{原式} &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{x\ln x} + C\end{aligned}$$

e) $\displaystyle\int \frac{x^2}{(x+1)^{100}}dx$

解:
令 $u = x + 1$
则 $du = dx$

$$\begin{aligned}\text{原式} &= \int \frac{u^2 - 2u + 1}{u^{100}}\,du \\ &= \int \left(u^{-98} - 2u^{-99} + u^{-100}\right)du \\ &= -\frac{u^{-97}}{97} + \frac{u^{-98}}{49} - \frac{u^{-99}}{99} + C \\ &= -\frac{1}{97(x+1)^{97}} + \frac{1}{49(x+1)^{98}} - \frac{1}{99(x+1)^{99}} + C\end{aligned}$$

f) $\displaystyle\int \frac{\sqrt{3+2x}}{x}dx$

解:
令 $u = \sqrt{3+2x}$
则 $u^2 = 3 + 2x$，$dx = u\,du$
解得 $x = \frac{u^2-3}{2}$

$$\begin{aligned}\text{原式} &= \int \frac{2u^2}{u^2-3}\,du \\ &= \int \left(2 + \frac{6}{u^2-3}\right)du \\ &= \int \left(2 + \frac{6}{(u-\sqrt{3})(u+\sqrt{3})}\right)du\end{aligned}$$

$$
\begin{aligned}
&= \int \left(2 + \frac{\sqrt{3}}{u-\sqrt{3}} - \frac{\sqrt{3}}{u+\sqrt{3}}\right) du \\
&= 2u + \sqrt{3}\ln\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right| + C \\
&= 2\sqrt{3+2x} + \sqrt{3}\ln\frac{\sqrt{3+2x}-\sqrt{3}}{\sqrt{3+2x}+\sqrt{3}} + C
\end{aligned}
$$

4. 求下列定积分

a) $\int_1^4 \frac{dx}{x(1+\sqrt{x})}$

解:

令 $u = \sqrt{x}$
则 $u^2 = x$，$dx = 2u\,du$

$$
\begin{aligned}
\int \frac{dx}{x(1+\sqrt{x})} &= \int \frac{2u\,du}{u^2(1+u)} \\
&= \int \frac{2\,du}{u(1+u)} \\
&= 2\ln\left|\frac{u}{1+u}\right| + C \\
&= 2\ln\frac{\sqrt{x}}{1+\sqrt{x}} + C
\end{aligned}
$$

$$
\begin{aligned}
\text{原式} &= 2\left(\ln 2 - \ln 3 - \ln 1 + \ln 2\right) \\
&= 2\left(2\ln 2 - \ln 3\right) \\
&= 2\ln\frac{4}{3}
\end{aligned}
$$

b) $\int_0^{\frac{\pi}{4}} \frac{\sin x}{1+\sin x} dx$

解:

$$
\begin{aligned}
&\int \frac{\sin x}{1+\sin x}\,dx \\
&= \int \left(1 - \frac{1}{1+\sin x}\right) dx \\
&= \int \left(1 - \sec^2 x + \tan x \sec x\right) dx \\
&= x - \tan x + \sec x + C
\end{aligned}
$$

$$
\begin{aligned}
\text{原式} &= [x - \tan x + \sec x]_0^{\frac{\pi}{4}} \\
&= \left(\frac{\pi}{4} - 1 + \sqrt{2}\right) - (0 - 0 + 1) \\
&= \frac{\pi}{4} - 1 + \sqrt{2} - 1 \\
&= \frac{\pi}{4} - 2 + \sqrt{2}
\end{aligned}
$$

5. 求下列不定积分（用三角替换处理比较适宜）

a) $\displaystyle\int \frac{dx}{\sqrt{(1-x^2)^3}}$

解:

令 $x = \sin t$

则 $dx = \cos t\, dt$

$$
\begin{aligned}
\text{原式} &= \int \frac{\cos t\, dt}{\cos^3 t} \\
&= \int \frac{1}{\cos^2 t}\, dt \\
&= \int \sec^2 t\, dt \\
&= \tan t + C \\
&= \frac{\sin t}{\cos t} + C \\
&= \frac{x}{\sqrt{1-x^2}} + C
\end{aligned}
$$

b) $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} \quad (a > 0)$

解:

令 $x = a\sin t$

则 $dx = a\cos t\, dt$

$$
\begin{aligned}
\text{原式} &= \int \frac{a\cos t\, dt}{a\sin t \cdot a\cos t} \\
&= \int \frac{dt}{a\sin t} \\
&= \frac{1}{a}\int \csc t\, dt \\
&= \frac{1}{a}\ln\left|\tan\frac{t}{2}\right| + C \\
&= \frac{1}{a}\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right| + C
\end{aligned}
$$

c) $\int \frac{dx}{x^2\sqrt{1+x^2}}$

解:

令 $x = \tan t$
则 $dx = \frac{dt}{\cos^2 t}$

$$\begin{aligned}
\text{原式} &= \int \frac{\cos t}{\sin^2 t}\, dt \\
&= -\frac{1}{\sin t} + C \\
&= -\frac{1}{\sin t} + C \\
&= -\frac{\sqrt{1+x^2}}{x} + C
\end{aligned}$$

d) $\int \frac{\sqrt{x^2-9}}{x} dx$

解:

令 $x = 3\sec t$
则 $dx = 3\sec t \tan t\, dt$

$$\begin{aligned}
\text{原式} &= \int 3\tan^2 t\, dt \\
&= 3\int (\sec^2 t - 1)\, dt \\
&= 3(\tan t - t) + C \\
&= 3\left(\frac{\sqrt{x^2-9}}{3} - \arccos\left(\frac{3}{x}\right)\right) + C \\
&= \sqrt{x^2-9} - 3\arccos\left(\frac{3}{x}\right) + C
\end{aligned}$$

e) $\int \frac{x^2}{\sqrt{a^2-x^2}} dx \quad (a > 0)$

解:

令 $x = a\sin t$
则 $dx = a\cos t\, dt$

$$
\begin{aligned}
\text{原式} &= \int a^2 \sin^2 t \, dt \\
&= a^2 \int \sin^2 t \, dt \\
&= a^2 \int \frac{1-\cos 2t}{2} \, dt \\
&= \frac{a^2}{2}\left(t - \frac{\sin 2t}{2}\right) + C \\
&= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) - \frac{1}{2} \cdot \frac{2x\sqrt{a^2-x^2}}{a^2}\right) + C \\
&= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) - \frac{x}{2}\sqrt{a^2-x^2} + C
\end{aligned}
$$

f) $\int \frac{dx}{1+\sqrt{1-x^2}}$

解:

$$
\begin{aligned}
\text{原式} &= \int \frac{1-\sqrt{1-x^2}}{x^2} \, dx \\
&= -\frac{1}{x} + \frac{\sqrt{1-x^2}}{x} + C
\end{aligned}
$$

6. 求下列不定积分（分部积分法比较适合）

a) $\int xe^{2x} dx$

解:

令 $u = x, dv = e^{2x}dx$

则 $du = dx, v = \frac{1}{2}e^{2x}$

$$
\begin{aligned}
\text{原式} &= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{2} \cdot \frac{1}{2}e^{2x} + C \\
&= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \\
&= \frac{1}{4}e^{2x}(2x-1) + C
\end{aligned}
$$

b) $\int \arctan x dx$

解:
令 $u = \arctan x, dv = dx$
则 $du = \frac{1}{1+x^2} dx, v = x$

$$\begin{aligned} \text{原式} &= x \arctan x - \int \frac{x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

c) $\int x^2 \arctan x dx$

解:
令 $u = \arctan x, dv = x^2 dx$
则 $du = \frac{1}{1+x^2} dx, v = \frac{1}{3} x^3$

$$\begin{aligned} \text{原式} &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \left(\frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) \right) + C \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1+x^2) + C \end{aligned}$$

d) $\int x^2 \ln x dx$

解:
令 $u = \ln x, dv = x^2 dx$
则 $du = \frac{1}{x} dx, v = \frac{1}{3} x^3$

$$\begin{aligned} \text{原式} &= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx \\ &= \frac{1}{3} x^3 \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C \\ &= \frac{1}{3} x^3 \ln x - \frac{x^3}{9} + C \end{aligned}$$

e) $\int e^{-2x}\sin\frac{x}{2}dx$

解:

令 $u=\sin\frac{x}{2}, dv=e^{-2x}dx$
则 $du=\frac{1}{2}\cos\frac{x}{2}dx, v=-\frac{1}{2}e^{-2x}$

$$\text{原式}=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}+\frac{1}{4}\int e^{-2x}\cos\frac{x}{2}dx$$

对 $\int e^{-2x}\cos\frac{x}{2}dx$ 再次积分:
令 $u=\cos\frac{x}{2}, dv=e^{-2x}dx$
则 $du=-\frac{1}{2}\sin\frac{x}{2}dx, v=-\frac{1}{2}e^{-2x}$

$$\begin{aligned}\int e^{-2x}\cos\frac{x}{2}dx&=-\frac{1}{2}e^{-2x}\cos\frac{x}{2}-\frac{1}{4}\int e^{-2x}\sin\frac{x}{2}dx\\&=-\frac{1}{2}e^{-2x}\cos\frac{x}{2}-\frac{1}{4}\cdot\text{原式}\end{aligned}$$

代入原式:

$$\begin{aligned}\text{原式}&=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}+\frac{1}{4}\left(-\frac{1}{2}e^{-2x}\cos\frac{x}{2}-\frac{1}{4}\text{原式}\right)\\&=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}-\frac{1}{8}e^{-2x}\cos\frac{x}{2}-\frac{1}{16}\text{原式}\end{aligned}$$

移项整理:

$$\begin{aligned}\text{原式}+\frac{1}{16}\text{原式}&=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}-\frac{1}{8}e^{-2x}\cos\frac{x}{2}\\\frac{17}{16}\text{原式}&=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}-\frac{1}{8}e^{-2x}\cos\frac{x}{2}\\\text{原式}&=\frac{16}{17}\left(-\frac{1}{2}e^{-2x}\sin\frac{x}{2}-\frac{1}{8}e^{-2x}\cos\frac{x}{2}\right)\\&=-\frac{8}{17}e^{-2x}\sin\frac{x}{2}-\frac{2}{17}e^{-2x}\cos\frac{x}{2}+C\\&=-\frac{2}{17}e^{-2x}\left(4\sin\frac{x}{2}+\cos\frac{x}{2}\right)+C\end{aligned}$$

f)

解:

令 $u=\arcsin x, dv=\frac{dx}{\sqrt{1-x}}$
则 $du=\frac{1}{\sqrt{1-x^2}}dx, v=-2\sqrt{1-x}$

$$
\begin{aligned}
\text{原式} &= -2\arcsin x \cdot \sqrt{1-x} + 2\int \frac{\sqrt{1-x}}{\sqrt{1-x^2}} dx \\
&= -2\arcsin x \cdot \sqrt{1-x} + 2\int \frac{\sqrt{1-x}}{\sqrt{(1-x)(1+x)}} dx \\
&= -2\arcsin x \cdot \sqrt{1-x} + 2\int \frac{1}{\sqrt{1+x}} dx \\
&= -2\arcsin x \cdot \sqrt{1-x} + 2 \cdot 2\sqrt{1+x} + C \\
&= -2\arcsin x \cdot \sqrt{1-x} + 4\sqrt{1+x} + C
\end{aligned}
$$

g) $\int \ln(x+\sqrt{1+x^2}) dx$

解:

令 $u = \ln(x+\sqrt{1+x^2}), dv = dx$
则 $du = \frac{dx}{\sqrt{1+x^2}}, v = x$

$$
\begin{aligned}
\text{原式} &= x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx \\
&= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C
\end{aligned}
$$

h) $\int \frac{\ln(\cos x)}{\cos^2 x} dx$

解:
令 $u = \ln(\cos x), dv = \sec^2 x\, dx$
则 $du = -\tan x\, dx, v = \tan x$

$$
\begin{aligned}
\text{原式} &= \tan x \cdot \ln(\cos x) + \int \tan^2 x\, dx \\
&= \tan x \cdot \ln(\cos x) + \int (\sec^2 x - 1)\, dx \\
&= \tan x \cdot \ln(\cos x) + \tan x - x + C
\end{aligned}
$$

i) $\int \sin x \ln(\tan x) dx$

解:
令 $u = \ln(\tan x), dv = \sin x\, dx$
则 $du = \frac{dx}{\sin x \cos x}, v = -\cos x$

$$
\begin{aligned}
\text{原式} &= -\cos x \cdot \ln(\tan x) + \int \frac{\cos x}{\sin x \cos x}\, dx \\
&= -\cos x \cdot \ln(\tan x) + \int \frac{1}{\sin x}\, dx \\
&= -\cos x \cdot \ln(\tan x) + \int \csc x\, dx \\
&= -\cos x \cdot \ln(\tan x) + \ln\left|\tan \frac{x}{2}\right| + C
\end{aligned}
$$

j) $\int \frac{\arcsin x}{x^2} dx$

解:

令 $u = \arcsin x, dv = x^{-2} dx$

则 $du = \frac{1}{\sqrt{1-x^2}} dx, v = -\frac{1}{x}$

$$
\begin{aligned}
\text{原式} &= -\frac{\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx \\
&= -\frac{\arcsin x}{x} + \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + C
\end{aligned}
$$

7. 求下列不定积分（有理分式展开或其它方法）

a) $\int \frac{dx}{x^2 - 2x + 2}$

解:

$$
\begin{aligned}
\text{原式} &= \int \frac{dx}{(x-1)^2 + 1} \\
&= \arctan(x-1) + C
\end{aligned}
$$

b) $\int \frac{x+1}{x^2 - 3x + 2} dx$

解:

$$
\begin{aligned}
\text{原式} &= \int \frac{x+1}{(x-1)(x-2)} dx \\
&= \int \left(\frac{-2}{x-1} + \frac{3}{x-2}\right) dx \\
&= -2\ln|x-1| + 3\ln|x-2| + C
\end{aligned}
$$

c) $\int \frac{2x+3}{(x^2-1)(x^2+1)}dx$

解:

$$\begin{aligned} \text{原式} &= \int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{x^2+1} \right) dx \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \arctan x + C \end{aligned}$$

d) $\int \frac{x^5+x^4-8}{x^3+x}dx$

解:

$$\begin{aligned} \text{原式} &= \int \left(x^2+x-1-\frac{8}{x}+\frac{x}{x^2+1} \right) dx \\ &= \frac{1}{3}x^3+\frac{1}{2}x^2-x-8\ln|x|+\frac{1}{2}\ln(1+x^2)+C \end{aligned}$$

e) $\int \frac{x^4+1}{(x-1)(x^2+1)}dx$

解:

$$\begin{aligned} \text{原式} &= \int \left(x+1+\frac{1}{x-1}-\frac{x-1}{x^2+1} \right) dx \\ &= \frac{1}{2}x^2+x+\ln|x-1|-\frac{1}{2}\ln(1+x^2)+C \end{aligned}$$

f) $\int \frac{dx}{(x-1)(x+1)^2}$

解:

$$\begin{aligned} \text{原式} &= \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2} \right) dx \\ &= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2(x+1)} + C \end{aligned}$$

8. 利用定积分的几何意义计算下列积分

a) $\int_{-1}^{1} |x| dx$

解:

$$\text{原式} = 2 \times 1 \times \frac{1}{2} = 1$$

b) $\int_{-\pi}^{\pi} \sin x dx$

解:

$$\text{原式} = 0$$

9. 已知 $\int_0^{\pi} \sin x dx = 2$，计算极限

$$\lim_{n\to\infty} \frac{1}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n}\right]$$

解:

$$\begin{aligned}
\text{原式} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \sin\frac{k\pi}{n} \\
&= \frac{1}{\pi} \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) \\
&= \frac{1}{\pi} \int_0^{\pi} \sin x \, dx \\
&= \frac{2}{\pi}
\end{aligned}$$

10. 已知 $\int_1^2 \ln x dx = 2\ln 2 - 1$，求极限

$$\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}$$

解:

令原式 $= t$

$$\begin{aligned}\ln t &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=n+1}^{2n}\ln\frac{k}{n}\\ &= \int_1^2 \ln x\,dx\\ &= [x\ln x - x]_1^2\\ &= (2\ln 2-2)-(0-1)\\ &= 2\ln 2-1\end{aligned}$$

$$\text{原式} = t = \frac{4}{e}$$

11. 设 $f, g \in C[a,b]$，证明：若 $f(x) \geq g(x)$($x \in [a,b]$)，且$f(x)$ 不恒为 $g(x)$，则$\int_a^b f(x)dx > \int_a^b g(x)dx$

解:

令 $h(x) = f(x) - g(x) \geq 0$，$x \in [a,b]$

存在 x_0，使得 $h(x_0) > 0$

又 $h(x) \in C[a,b]$

则存在区间 $[x_0-\delta, x_0+\delta] \subset [a,b]$，
使得 $h(x) > 0$

$$\int_a^b h(x)\,dx = \int_a^b [f(x)-g(x)]\,dx > 0$$

$$\Rightarrow \int_a^b f(x)\,dx > \int_a^b g(x)\,dx$$

12. 比较下列各组中积分的大小.

a) $\int_0^{\frac{\pi}{2}} \sin^2 x dx$与 $\int_0^{\frac{\pi}{2}} \sin^4 x dx$

解:

$$0 \leq \sin x \leq 1$$

$$\sin^4 x \leq \sin^2 x$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx < \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

b) $\int_1^e \ln x dx$，$\int_1^e (\ln x)^2 dx$，$\int_1^e \ln(x^2) dx$

解:

$$\ln x \in [0,1] \quad 故 \quad (\ln x)^2 \le \ln x$$

$$又 \quad \ln(x^2) = 2\ln x$$

$$(\ln x)^2 \le \ln x \le \ln(x^2)$$

$$\int_1^e (\ln x)^2 dx < \int_1^e \ln x\, dx < \int_1^e \ln(x^2)\, dx$$

13. 证明不等式 $\frac{1}{2} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{2}$

解:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{x} = \ln 2 < \frac{\sqrt{2}}{2}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx > \frac{2}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{2}$$

故成立

14. 求下列极限

a) $\lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{x^n}{1+x} dx$

解:

$$0 < \lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{x^n}{1+x}\, dx < \lim_{n\to\infty} \int_0^{\frac{1}{2}} x^n\, dx = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^{n+1}}{n+1} = 0$$

故原式 $= 0$

b) $\lim_{n\to\infty} \int_{n^2}^{n^2+n} \frac{1}{\sqrt{x}} e^{-\frac{1}{x}} dx$

解:

$$原式 = \lim_{n\to\infty} \int_{n^2}^{n^2+n} \frac{dx}{\sqrt{x}}$$

$$= \lim_{n\to\infty} 2\left(\sqrt{n^2+n}-n\right)$$

$$= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+n}+n} = 1$$

15. 求下列函数 $y = y(x)$ 的导数 $\frac{dy}{dx}$

a) $y = \int_{\cos x}^{\sin x} e^{-t^2} dt$

解:

$$y' = e^{-\sin^2 x} \cos x + e^{-\cos^2 x} \sin x$$

b) $y = \left(\int_0^{\sqrt{x}} \ln(1+t^2) dt\right)^2$

解:

$$y' = 2\left(\int_0^{\sqrt{x}} \ln(1+t^2)\, dt\right) \cdot \frac{\ln(1+x)}{2\sqrt{x}}$$

$$= \frac{\ln(1+x)}{\sqrt{x}} \int_0^{\sqrt{x}} \ln(1+t^2)\, dt$$

c) $\int_0^{xy} e^t dt + \int_0^y \sin t dt = 0$

解:

$$e^{xy}(y+xy') + \sin y + y' = 0$$

$$y' = -\frac{ye^{xy}}{xe^{xy} + \sin y}$$

d)

$$\begin{cases} x = \int_1^t \ln u du \\ y = \int_1^t u \ln u du \end{cases}$$

解:

$$\frac{dy}{dx} = \frac{t \ln t}{\ln t} = t$$

16. 求下列极限

a) $\lim\limits_{x\to 0}\dfrac{\displaystyle\int_{cosx}^{1}e^{-t^2}dt}{x^2}$

解:

$$
\begin{aligned}
\text{原式} &= \lim_{x\to 0}\frac{e^{-\cos^2 x}\sin x}{2x} \\
&= \lim_{x\to 0}\frac{e^{-\cos^2 x}}{2} = \frac{1}{2e}
\end{aligned}
$$

b) $\lim\limits_{x\to 0}\dfrac{\displaystyle\int_{0}^{x}\ln(1+t^2)dt}{\ln\frac{\sin x}{x}}$

解:

$$
\begin{aligned}
\text{原式} &= \lim_{x\to 0}\frac{\ln(1+x^2)}{\frac{\cos x}{\sin x}-\frac{1}{x}} \\
&= \lim_{x\to 0}\frac{x^2+o(x^2)}{-\frac{x}{3}+o(x)} = 0
\end{aligned}
$$

17. 设函数

$$
f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ x-2, & 1 \le x \le 2 \end{cases}
$$

求 $\Phi(x) = \displaystyle\int_0^x f(t)dt$在$[0,2]$ 上的表达式.

解:
当 $x\in[0,1)$ 时,

$$
\phi(x) = \frac{1}{3}x^3
$$

当 $x\in[1,2]$ 时,

$$
\phi(x) = \frac{1}{3} + \int_1^x (t-2)\,dt = \frac{1}{2}x^2 - 2x + \frac{11}{6}
$$

$$\phi(x)=\begin{cases}\frac{1}{3}x^3, & x\in[0,1)\\ \frac{1}{2}x^2-2x+\frac{11}{6}, & x\in[1,2]\end{cases}$$

18. 设函数 $f\in C[a,b]$，且 $f(x)>0$ ($x\in[a,b]$)，记 $F(x)=\int_a^x f(t)dt+\int_b^x \frac{dt}{f(t)}\quad(x\in[a,b])$

证明:

a) $F'(x)\geq 2$

解:

$$F'(x)=f(x)+\frac{1}{f(x)}\geq 2$$

b) 方程$F(x)=0$ 在区间 (a,b) 内有且仅有一根.

解:

$F(x)$ 连续，$F'(x)>0$，故 $F(x)$ 单调增加。

又 $F(a)<0$，$F(b)>0$，
故方程 $F(x)=0$ 在区间 (a,b) 内有且仅有一根.

19. 设函数 $f\in C[a,b]$，且 $f(x)>0$ ($x\in[a,b]$). 证明：至少存在一个点 $\xi\in[a,b]$，使得 $\int_a^\xi f(x)dx=\int_\xi^b f(x)dx=\frac{1}{2}\int_a^b f(x)dx$

解:

令 $F(x)=\int_a^x f(t)\,dt$

则 $F(a)=0$，$F(b)=\int_a^b f(t)\,dt>0$

又 $f(x)>0$，故 $F(x)$ 单调增加

令 $k=\frac{1}{2}\int_a^b f(t)\,dt$

存在 $\xi\in[a,b]$，使得 $F(\xi)=\int_a^\xi f(t)\,dt=k$

$$\int_\xi^b f(t)\,dt=\int_a^b f(t)\,dt-\int_a^\xi f(t)\,dt=k$$

故成立

20. 设函数 $f \in C[0,1] \cap D(0,1)$，且 $3\int_0^{\frac{1}{3}} e^{1-x^2} f(x)dx = f(1)$

证明：至少存在一点 $\xi \in (0,1)$，使得$f'(\xi) = 2\xi f(\xi)$.

解:

令 $F(x) = e^{-x^2} f(x)$

则 $F(1) = e^{-1} f(1)$

故

$$3\int_0^{\frac{1}{3}} f(x)\,dx = e^{-1} f(1) = F(1)$$

又存在 $\eta \in \left[0, \frac{1}{3}\right]$，使得

$$\int_0^{\frac{1}{3}} f(x)\,dx = \frac{1}{3} F(\eta)$$

故

$$F(\eta) = F(1)$$

在区间 $[\eta, 1]$ 上，$F(\eta) = F(1)$

故存在 $\xi \in (\eta, 1) \subset (0,1)$，使得

$$F'(\xi) = 0$$

代入，得

$$f'(\xi) = 2\xi f(\xi)$$

21. 设函数 $S(x) = \int_0^x |\cos t| dt$

(a) 当 $(n \in \mathbb{N}_+$，且 $(n\pi \le x < (n+1)\pi$ 时，证明：$2n \le S(x) < 2(n+1)$.

解:

$$S(x) = \int_0^x |\cos t|\,dt = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} |\cos t|\,dt + \int_{n\pi}^x |\cos t|\,dt = 2n + \int_{n\pi}^x |\cos t|\,dt$$

由于 $\int_{n\pi}^x |\cos t|\,dt \in [0,2)$

故

$$2n \le S(x) < 2(n+1)$$

(b) 求极限 $\lim\limits_{x\to+\infty} \dfrac{S(x)}{x}$

解:

$$2n \le S(x) < 2(n+1)$$

两边同除以 x，注意 $x \in [n\pi, (n+1)\pi)$，所以

$$\frac{2n}{(n+1)\pi} \le \frac{S(x)}{x} \le \frac{2(n+1)}{n\pi}$$

当 $x \to \infty$ 时，$n \to \infty$，夹逼得:

$$\lim_{x\to\infty} \frac{S(x)}{x} = \frac{2}{\pi}$$

22. 证明: $\int_0^x f(t)(x-t)dt = \int_0^x \left(\int_0^t f(x)dx\right)dt$

解:

令 $x - t = \int_t^x 1\,ds$

$$\begin{aligned}\int_0^x f(t)(x-t)\,dt &= \int_0^x \left(\int_t^x f(t)\,ds\right)dt \\ &= \int_0^x \left(\int_0^s f(t)\,dt\right)ds \\ &= \int_0^x \left(\int_0^t f(s)\,ds\right)dt\end{aligned}$$

23. 设非负函数 $f \in R[a,b]$，证明不等式:

$$\left(\int_a^b f(x)\cos x dx\right)^2 + \left(\int_a^b f(x)\sin x dx\right)^2 \le \left(\int_a^b f(x)dx\right)^2$$

提示: 利用柯西-施瓦茨不等式.

解:

左边 $\le \left(\int_a^b f(x)\sqrt{\cos^2 x + \sin^2 x}\,dx\right)^2$

$$= \left(\int_a^b f(x)\,dx\right)^2$$

故成立

24. 设 $f \in C[a,b]$，且$f(a)$ 和 $f(b)$ 分别是 $f(x)$ 在 $[a,b]$ 上最大值和最小值. 证明：至少存在一点$\xi \in [a,b]$，使得 $\int_a^b f(x)dx = f(a)(\xi - a) + f(b)(b - \xi)$

提示：先利用积分中值公式，然后构造函数并利用零点定理之类.

解：

令 $F(\xi) = f(a)(\xi - a) + f(b)(b - \xi) - \int_a^b f(x)\,dx$

则

$$F(a) = f(b)(b-a) - \int_a^b f(x)\,dx \le 0$$

$$F(b) = f(a)(b-a) - \int_a^b f(x)\,dx \ge 0$$

故存在 $\xi \in [a,b]$，使得 $F(\xi) = 0$

25. 设 $f \in C[a,b] \cap D(a,b)$)，且 $f'(x) \ge 0 (x \in (a,b))$，求证 $\int_a^b xf(x)dx \ge \frac{a+b}{2}\int_a^b f(x)dx$

提示：能否让 b 动起来，从而转变为对函数求导以判别单调性的问题？

解：
令 $g(x) = x - \frac{a+b}{2}$

则

$$\int_a^b xf(x)\,dx - \frac{a+b}{2}\int_a^b f(x)\,dx = \int_a^b g(x)f(x)\,dx$$

当 $x \in \left[a, \frac{a+b}{2}\right]$ 时，$g(x) \le 0$，且 $f(x) \le f\left(\frac{a+b}{2}\right)$

当 $x \in \left[\frac{a+b}{2}, b\right]$ 时，$g(x) \ge 0$，且 $f(x) \ge f\left(\frac{a+b}{2}\right)$

故

$$\int_a^b g(x)f(x)\,dx \ge \int_a^b g(x)f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx$$

又

$$\int_a^b g(x)\,dx = \int_a^b \left(x - \frac{a+b}{2}\right) dx = \frac{b^2 - a^2}{2} - \frac{a+b}{2}(b-a) = 0$$

故

$$\int_a^b g(x)f(x)\,dx \geq 0$$

因此

$$\int_a^b xf(x)\,dx \geq \frac{a+b}{2} \int_a^b f(x)\,dx$$