hw_11(2)

习题5.6

1. 找出对角化下面矩阵的一个正交矩阵。

(1)

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

解:

$\det(tE-A)=\begin{vmatrix} t-2 & -1 \\ -1 & t-2 \end{vmatrix}=(t-2)^2-1=(t-1)(t-3)$

特征值:

$t_1=1,\quad t_2=3$

特征向量:

$(A-E)X=0$，得 $X_1=[1,-1]$

$(A-3E)X=0$，得 $X_2=[1,1]$

单位化:

$$b_1=\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right],\quad b_2=\left[\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]$$

正交矩阵:

$$B=\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

(2)

$$\begin{pmatrix} 11 & 2 & -8 \\ 2 & 2 & 10 \\ -8 & 10 & 5 \end{pmatrix}$$

解:

$\det(tE-A)=\begin{vmatrix} t-11 & -2 & 8 \\ -2 & t-2 & -10 \\ 8 & -10 & t-5 \end{vmatrix}=(t-18)(t+9)(t-9)=(t-18)(t+9)(t-9)$

特征值:

$t_1=9,\quad t_2=-9,\quad t_3=18$

特征向量:

$(A-9E)X=0$，得 $X_1=[2,2,1]$

$(A+9E)X=0$，得 $X_2=[1,-2,2]$

$(A-18E)X=0$，得$X_3=[-2,1,2]$

单位化:

$$b_1=\left[\frac{2}{3},\frac{2}{3},\frac{1}{3}\right],\quad b_2=\left[\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right],\quad b_3=\left[-\frac{2}{3},\frac{1}{3},\frac{2}{3}\right]$$

正交矩阵:

$$B=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\ \frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\ \frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{pmatrix}$$

(3)

$$\begin{pmatrix}5&-1&-1\\-1&5&-1\\-1&-1&5\end{pmatrix}$$

解:

$\det(tE-A)=\begin{vmatrix}t-5&1&1\\1&t-5&1\\1&1&t-5\end{vmatrix}=(t-3)(t-6)^2$

特征值:
$t_1=3,\quad t_{2,3}=6$

特征向量:
$(A-3E)X=0$，得 $X_1=[1,1,1]$
$(A-6E)X=0$，得 $X_2=[1,-1,0],\quad X_3=[1,1,-2]$

单位化:
$b_1=\left[\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right],\quad b_2=\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0\right],\quad b_3=\left[\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{3}\right]$

正交矩阵:

$$B=\begin{pmatrix}\frac{\sqrt{3}}{3}&\frac{\sqrt{2}}{2}&\frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3}&-\frac{\sqrt{2}}{2}&\frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3}&0&-\frac{\sqrt{6}}{3}\end{pmatrix}$$

2. 求正交矩阵 T使得变量替换 $x=Ty$ 把下面的二次型化成典范式。（提示：写出二次型的矩阵，找出把这个矩阵对角化的正交矩阵，根据定理5.51的证明，它就是要求的T）

(2) x_1x_2

解:

$$A=\begin{pmatrix}0&\frac{1}{2}\\ \frac{1}{2}&0\end{pmatrix}$$

$$\det(tE-A)=t^2-\frac{1}{4}=\left(t-\frac{1}{2}\right)\left(t+\frac{1}{2}\right)$$

特征值:
$t_1=\frac{1}{2}, \quad t_2=-\frac{1}{2}$

特征向量:
$(A-\frac{1}{2}E)X=0$，得 $X_1=[1,1]$
$(A+\frac{1}{2}E)X=0$，得 $X_2=[1,-1]$

单位化:

$$b_1=\left[\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right], \quad b_2=\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right]$$

正交变换矩阵:

$$T=\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}$$

(4) $34x_1^2-24x_1x_2+41x_2^2$

解:
$\det(tE-A)=(t-41)(t-34)-144=t^2-75t+1250=(t-25)(t-50)$

特征值:
$t_1=25, \quad t_2=50$

特征向量:
$(A-25E)X=0$，得 $X_1=[4,3]$
$(A-50E)X=0$，得$X_2=[-3,4]$

单位化:

$$b_1=\left[\frac{4}{5},\frac{3}{5}\right], \quad b_2=\left[-\frac{3}{5},\frac{4}{5}\right]$$

正交变换矩阵:

$$T=\begin{pmatrix}\frac{4}{5} & -\frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{pmatrix}$$

(6)$3x_1^2+3x_3^2+5x_2^2+6x_1x_2+2x_1x_3+2x_2x_3$

解:

$$\begin{aligned}&\det(tE-A)\\=(t-3)(t-3)(t-5)-2-2-2(t-5)+3+3\\&=t(t-4)(t-7)\end{aligned}$$

特征值:
$t_1=0, \quad t_2=4, \quad t_3=7$

特征向量:
$AX=0$，得 $X_1=[-1,1,0]$
$(A-4E)X=0$，得 $X_2=[1,1,-2]$
$(A-7E)X=0$，得 $X_3=[1,1,1]$

单位化:
$b_1=\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right],\quad b_2=\left[\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{3}\right],\quad b_3=\left[\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right]$

正交变换矩阵:

$$T=\begin{pmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3}\end{pmatrix}$$

3. 确定下面二次曲线的类型并画出其大概图形:

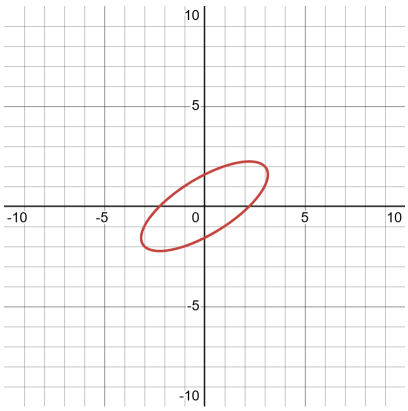
(1) $x_1^2-2x_1x_2+2x_2^2-5=0$

解:

$$A=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$$

$$\chi_A(t)=\det(tI-A)=(t-1)(t-2)-1=t^2-3t+1$$

特征值:
$t=\frac{3\pm\sqrt{5}}{2},\quad t_1t_2=1>0$
故对应的二次曲线为椭圆



(3) $9x_1^2+24x_1x_2+16x_2^2-52x_1+14x_2=6$

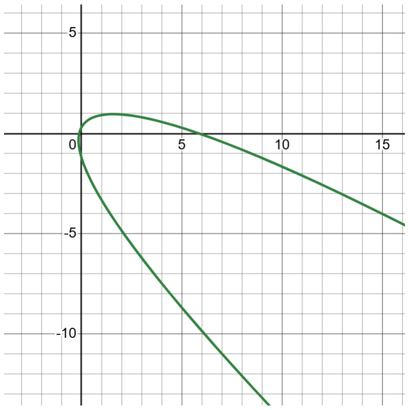
解:

$$A=\begin{pmatrix}9 & 12\\ 12 & 16\end{pmatrix}$$

$$\chi_A(t)=\det(tI-A)=(t-9)(t-16)-144=t(t-25)$$

特征值:
$t_1=0,\quad t_2=25,\quad t_1t_2=0$

故对应的二次曲线为抛物线



(5) $x_1^2 + 2x_1x_2 + x_2^2 - 2x_1 + 2x_2 + 3 = 0$

解:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\chi_A(t) = \det(tI - A) = (t-1)^2 - 1 = t(t-2)$$

特征值:
$t_1 = 0, \quad t_2 = 2, \quad t_1t_2 = 0$

故对应的二次曲线为抛物线