Example 1. If $a > 1$, then $\lim\limits_{n\to\infty} \frac{1}{a^n} = 0$.

Solution. $\frac{1}{a^n} = \frac{1}{(1+(a-1))^n} \leq \frac{1}{1+n(a-1)}$ □

Exercise 1 (Squeeze theorem)**.** If $n \in \mathbb{N}$, $a_n \leq c_n \leq b_n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \implies \lim_{n\to\infty} c_n = L$$

Theorem 1. If a_n is increasing and $\{a_n \mid n \in \mathbb{N}\}$ has an upper bound, then a_n converges.

Proof. $\{a_n \mid n \in \mathbb{N}\}$ has an upper bound $\implies \sup\{a_n \mid n \in \mathbb{N}\}$ exists.
For convenience, we denote $\sup\{a_n \mid n \in \mathbb{N}\}$ by L.
$\forall \varepsilon > 0, L - \varepsilon < L$, and hence $\exists N \in \mathbb{N}\ [\![L - \varepsilon < a_N]\!](\implies L - \varepsilon$ is not an upper bound.)
$\forall n \geq \mathbb{N}\ [\![L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon]\!] \implies |a_n - L| < \varepsilon$ □

Definition 1. A sequence of intervals I_n $(n \in \mathbb{N})$ is nested if $I_n \neq \varnothing$ and $I_{n+1} \subseteq I_n$ for all n $\in \mathbb{N}$.

Theorem 2 (Nested intervals theorem)**.** If I_n $(n \in \mathbb{N})$ is a sequence of bounded closed nested intervals, then

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

Proof. Write $I_n = [a_n, b_n]$ $(n \in \mathbb{N})$
I_n is nested $\iff a_n \leq b_n$, $a_n \nearrow$ and $b_n \searrow$
$\forall n, m \in \mathbb{N} \quad a_n \leq a_{\max(n,m)} \leq b_{\max(n,m)} \leq b_m$
In other words, for every $m \in \mathbb{N}$, b_m is an upper bound of $\{a_n \mid n \in \mathbb{N}\}$.
Let $c = \lim_{n\to\infty} a_n$. Then $a_n \leq c \leq b_m m$ for all $m \in \mathbb{N} \implies$

$$c \in \bigcap_{n\in\mathbb{N}} I_n$$

□

Exercise 2. What if

1. $I_n = (a_n, b_n)$, nested, but $a_n \nearrow\!\nearrow$ and $b_n \searrow\!\searrow$.
2. $I_n = (a_n, \infty)$, nested and $\{a_n \mid n \in \mathbb{N}\}$ is bounded from above.

Exercise 3. Prove the **Dedekind gapless property** using **Archimedean property** and the **nested intervals theorem**.

Definition 2. A sequence a_n $(n \in \mathbb{N})$ in $\mathbb{R}$ is a Cauchy sequence if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \; [\![n, m \geq \mathbb{N} \implies |a_n - a_m| < \varepsilon]\!]$$

Obviously,

- a_n is convergent $\implies$ a_n is Cauchy sequence
- a_n is Cauchy sequence $\implies$ a_n is bounded

Definition 3. Let a_n $(n \in \mathbb{N})$ be a *bounded* sequence in $\mathbb{R}$.
$u_n := \sup\{a_m \mid m \geq n\}$, $l_n := \inf\{a_m \mid m \geq n\}$
$\forall n \in \mathbb{N} \; [\![l_n \leq a_n \leq u_n]\!]$
$\overline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} u_n$, $\underline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} l_n$

Exercise 4. a_n converges $\iff$ $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$, and if any of both sides holds, then $\lim_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$.

Theorem 3. Let a_n $(n \in \mathbb{N})$ be a sequence in $\mathbb{R}$.

$$a_n \text{ is convergent } \iff a_n \text{ is a Cauchy sequence}$$