Definition 1 (Supremum / Least Upper Bound)**.** Let $S \subset \mathbb{R}$ be a non-empty set that is bounded above. A real number u is called the *supremum* (least upper bound) of S, denoted by $\sup S$, if

1. u is an upper bound of S, i.e., $s \leq u$ for all $s \in S$;
2. for any $\epsilon > 0$, there exists $s \in S$ such that $s > u - \epsilon$.

Definition 2 (Infimum / Greatest Lower Bound)**.** Let $S \subset \mathbb{R}$ be a non-empty set that is bounded below. A real number l is called the *infimum* (greatest lower bound) of S, denoted by $\inf S$, if

1. l is a lower bound of S, i.e., $s \geq l$ for all $s \in S$;
2. for any $\epsilon > 0$, there exists $s \in S$ such that $s < l + \epsilon$.

By convention,

1. $\sup S = \infty$ (resp. $\inf S = -\infty$) whenever S has no upper (resp. lower) bound.
2. S is bounded above (resp. below) $\iff$ S has a upper (resp. lower) bound.

Remark 1. Every $r \in \mathbb{R}$ is a upper bound and lower bound of $\varnothing$.

Definition 3 (Dedekind cut)**.** Let $A, B \subseteq \mathbb{R}$. We say that (A, B) is a *Dedekind cut* of $\mathbb{R}$ if following conditions hold:

1. $A \neq \varnothing \neq B$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, \forall b \in B,\ a < b$.

Property 1 (Dedekind's gapless property)**.** If (A,B) is a Dedekind cut, then exactly one of the following happens:

1. $\max A$ exists but $\min B$ doesn't;
2. $\min B$ exists but $\max A$ doesn't.

Exercise 1. We may define Dedekind cuts of $\mathbb{Q}$ and $\mathbb{Z}$ similarly. Does the **Property 1** hold for $\mathbb{Q}$ or $\mathbb{Z}$?

Property 2 (Least upper bound property)**.** If S has a upper bound, then $\sup S$ exists.

Proof. Let $\varnothing \neq S \subseteq \mathbb{R}$, $B \coloneqq \{b \in \mathbb{R} \mid b \text{ is upper bound of } S\}$ and $A \coloneqq \mathbb{R} \setminus B$. We need to show that $\min B$ exists. First, we prove that (A, B) forms a *Dedekind cut* of $\mathbb{R}$.

1. $S \neq \varnothing \implies A \neq \varnothing$, and S has an upper bound $\iff B \neq \varnothing$
2. $A = \mathbb{R} \setminus B \implies A \cup B = \mathbb{R}$

3. For $a \in A$ and $b \in B$, we need to show that $a < b$. Assume the contrary, that $a \geq b$. Hence a is an upper bound of S, i.e., $a \in B$ and $a \in A \cap B = \varnothing$. Therefore, our assumption is false, and $a < b$.

Hence (A, B) is a *Dedekind cut* of $\mathbb{R}$.
Next we prove that $\min B$ exists. Again, we assume the contrary, that $\max A$ exists, denoted by a_0.

$$\begin{aligned} a_0 \in A \iff a_0 \notin B &\iff a_0 \text{ is not an upper bound of } S \\ &\iff \exists s_0 \in S \text{ such that } a_0 < s_0 \end{aligned}$$

Choose x such that $a_0 < x < s_0$. Then $a_0 < x \iff x \in B$, hence x is an upper bound of S, but $x < s_0$. Therefore, our assumption is false, and $\min B$ exists.(Property 1) $\square$

Property 3 (Greatest lower bound property)**.** If S has a lower bound, then $\inf S$ exists.

Exercise 2 (Archimedean property)**.** Prove that

$$\forall r \in \mathbb{R}, r > 0 \implies \exists n \in \mathbb{N} \text{ such that } \frac{1}{n} < r$$

Hint: Rephrase this statement in terms of the set $S = \mathbb{N} \subseteq \mathbb{R}$ and consider its upper bounds.

Definition 4 (Limit of a Sequence)**.** Let (a_n) be a sequence. We say that (a_n) *converges* to a real number L, or that L is the *limit* of (a_n), if $\forall \varepsilon > 0$, there exists a positive integer N such that $\forall n > N$,

$$|a_n - L| < \varepsilon.$$

In this case, we write

$$\lim_{n \to \infty} a_n = L.$$

Exercise 3. Prove that

1. $\lim\limits_{n \to \infty} a_n = L, \lim\limits_{n \to \infty} a_n = M \implies L = M$
2. a_n $(n \in \mathbb{N})$ is convergent $\implies$ $\{a_n \mid n \in \mathbb{N}\}$ is bounded.
3. If $a_n \leq b_n$ for all $n \in \mathbb{N}$, $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} b_n = M$, then $L \leq M$. What if "$\leq$" is replaced by "$<$"?

Remark 2. Modifying or removing *finitely many terms* of a sequence (a_n) does not affect its convergence or divergence, nor the value of its limit if it exists.

Proposition 1. If $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} b_n = M$, then

1. $\lim\limits_{n \to \infty} (a_n \pm b_n) = L \pm M$;

2. $\lim_{n\to\infty} a_n b_n = L \cdot M$;

3. If $M \neq 0$, then $b_n \neq 0$ for all but finitely many n, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

Proof.
1. Consider $|(a_n \pm b_n) - (L \pm M)|$.

$$|(a_n \pm b_n) - (L \pm M)| = |(a_n - L) \pm (b_n - M) \leq |a_n - L| + |b_n - M|$$

$$\begin{gathered}\forall \varepsilon > 0,\ \exists N_1, N_2 \in \mathbb{N}[n \geq N_1 \implies |a_n - L| < \frac{\varepsilon}{2}] \\ \text{and } [n \geq N_2 \implies |b_n - M| < \frac{\varepsilon}{2}]\end{gathered}$$

Let $N = \max\{N_1, N_2\}$. Then

$$n \geq N \implies |(a_n \pm b_n) - (L \pm M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$

2. Consider $|a_n b_n - LM|$.

$$\begin{aligned}|a_n b_n - LM| &= |a_n b_n - L b_n + L b_n - LM| \\ &\leq |a_n - L|\,|b_n| + |L|\,|b_n - M|.\end{aligned}$$

Since $b_n \to M$, the sequence (b_n) is bounded (By **Exercise 3**). Thus, there exists $c > 0$ such that $|b_n| \leq c$ and $|L| \leq c$ for all $n \in \mathbb{N}$. Therefore,

$$|a_n b_n - LM| \leq c|a_n - L| + c\,|b_n - M|.$$

$$\begin{gathered}\forall \varepsilon > 0,\ \exists N \in \mathbb{N}[n \geq N \implies |a_n - L| < \frac{\varepsilon}{2c} \text{ and } |b_n - M| < \frac{\varepsilon}{2c}] \\ \implies |a_n b_n - LM| < c\frac{\varepsilon}{2c} + c\frac{\varepsilon}{2c} = \varepsilon\end{gathered}$$

$\square$

Exercise 4. Prove 3. in **Proposition 1**.