SUBJECT : DISCRETE MATHEMATICS

BRANCH : COMPUTER SCIENCE AND ENGINEERING

SEMESTER : III

UNIT II : COMBINATORICS

TOPIC : PERMUTATION AND COMBINATION

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MODULE II
COMBINATORICS

Permutations and combinations – Inclusion and exclusion principle - Pigeonhole principle - Mathematical induction - Recurrence relations - Solving linear recurrence relations using generating function.

CO2: Use induction techniques, generating functions and basics of counting to solve mathematical statements.

An Arrangement is an ordered sequence of items where not every item is used.

Eg 1. How many ways can 8 athletes receive gold, silver and bronze medals?

$$\boxed{8}\ \boxed{7}\ \boxed{6} = 336$$

Eg. 2 How many 4 letter sequences can be made with the vowels a,e,i,o,u & y without repeating?

$$\boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3} = 360$$

A Permutation is an <u>ordered</u> arrangement where ALL of the items in a set are used.

Eg.3 List the permutations of students at the front

2 students	3 students	4 students	
AB	ABC	ABCD	BACD
BA	ACB	ABDC	BADC
	BAC	ACBD	BCAD
	BCA	ACDB	BCDA
	CAB	ADBC	BDAC
	CBA	ADCB	BDCA
		CABD	DABC
		CADB	DACB
		CBAD	DBAC
		CBDA	DBCA
		CDAC	DCAB
		CDBA	DCBA
$2! = 2$	$3 \times 2 \times 1 = 3! = 6$	$4 \times 3 \times 2 \times 1 = 4! = 24$	

A Combination is an arrangement of some of the items are chosen. The order does NOT matter.

It should really be called a permutation lock!

Eg. 4 Choose 3 students from a set of 5 to go to the office

$$\frac{\boxed{5}\ \boxed{4}\ \boxed{3}}{\boxed{3!}} = 10$$

← Because order does not matter

Eg. 5 Choose 5 movies from 20 on the shelf

$$\frac{\boxed{20}\ \boxed{19}\ \boxed{18}\ \boxed{17}\ \boxed{16}}{\boxed{5!}} = 15\,504$$

Permutation

- A permutation is an arrangement of all or part of a set of objects.
- Number of permutations of n objects is n!
- Number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

- Number of permutations of n objects arranged is a circle is (n-1)!

Permutations

- Ordered arrangement of objects selected from a set

$$P(n,r) = {}_nP_r = P_r^n = \frac{n!}{(n-r)!}$$

- Ordered arrangement containing a identical objects of one kind is

$$\frac{n!}{a!b!c!\ldots}$$

Combinations

- The number of combinations of n distinct objects taken r at a time is

With Replacement : $$_nC_r = \frac{n!}{r!\ (n-r)!}$$

Without Replacement : $$_{n+r-1}C_r = \frac{(n+r-1)!}{r!\ (n-1)!}$$

Combination formula proof

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Corollary 1

- Let n and r be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$

- Proof:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

Corollary example

- There are C(52,5) ways to pick a 5-card poker hand
- There are C(52,47) ways to pick a 47-card hand
- P(52,5) = 2,598,960 = P(52,47)

- When dealing 47 cards, you are picking 5 cards to not deal
  - As opposed to picking 5 card to deal
  - Again, the order the cards are dealt in does matter

PROBLEMS ON PERMUTATION

Ordered arrangement containing a identical objects of one kind is

$$\frac{n!}{a!b!c!\ldots}$$

Find the number of distinct permutations that can be formed from all the letters of each word

(i) RADAR (ii) UNUSUAL

Solution:

(i) The word RADAR contains five letters of which two A's and two R's are there.

Therefore the number of possible words = $\frac{5!}{2!\ 2!}$ = 30

Number of distinct permutations = 30

(ii) The word UNUSUAL contains 7 letters of which 3 U's are there.

The number of possible words = $\frac{7!}{3!}$ = 840.

Number of distinct permutations = 840

There are 3 red balls , 4 green balls and 5 blue balls in a bag. They are arrange one by one and arranged in a row. Assuming that all the 12 balls are drawn, determine the number of different arrangements.

Solution:

Total number of balls = 3+4+5 = 12

Required number of arrangements = $\dfrac{12!}{3!\ 4!\ 5!}$

$= 27720$

Permutations with Repetition

The number of permutation of 'n' objects taken 'r' at a time with repetitions allowed is $\boldsymbol{n^r}$.

For example, the number of string of length 'n' can be formed from the English alphabet = 26^n

Problem 1:

How many bit string of length 10 that begin and end with 1 ?

Solution:

The bits in the remaining 8 places can be filled in 2^8 ways, after fixing 1 in the first and last places.

∴ No. of bit strings of length 10,start end with 1 = 2^8 = 256.

Problem 2:

How many 4 digit numbers can be formed using the digits 2,4,6,8 when repetition of digits is allowed?

Solution:

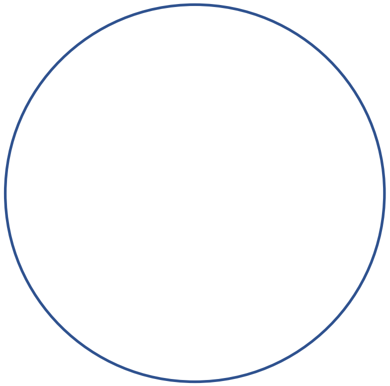
There are 4 ways of filling each of the digit positions.

$\therefore$ Total No. of 4-digits formed = $4\times4\times4\times4=256$

Circular Permutation

The arrangement of objects in a circle is called Circular Permutation.

The number of arrangements of n objects in a circle = $(n-1)!$



Problem 3:

Determine the number of ways in which 5 software engineers and 6 electronics engineers can be seated around a table so that no two software engineers can sit together.

Solution:

The 6 electronics engineers sit in a circular table in (6-1)!=5! Ways.

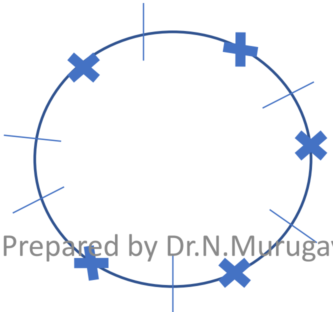
The 5 software engineers must be seated in between two electronics engineers so that no two software engineers can sit together.

There are six places for software engineers.

Therefore, they can be placed in 6! different ways.

Total no.of ways to arrange the engineers to sit on a round table = (5!)(6!)

= 86400.



Problem 4:

How many bit strings of length 10 contain (a) exactly four 1's (b) atmost four 1's (c) atleast four 1's (d)an equal number of 0's and 1's.

Solution:

(a) A bit string of length 10 can be considered to have 10 positions.These 10 positions should be filled with four 1's and six 0's.

∴ No. of required bit strings = $\frac{10!}{4!6!}$ = 210.

(b) The 10 positions should be filled up with no 1 and ten 0's (or) one 1 and 9 0's(or)two 1's and eight 0's (or) three 1's and seven 0's (or) four 1's and six 0's.

Required no. of bit strings = $\frac{\mathbf{10!}}{\mathbf{0!10!}} + \frac{\mathbf{10!}}{\mathbf{1!9!}} + \frac{\mathbf{10!}}{\mathbf{2!8!}} + \frac{\mathbf{10!}}{\mathbf{3!7!}} + \frac{\mathbf{10!}}{\mathbf{4!6!}} = \mathbf{386}$

(c) The ten positions are to be filled up with four 1's and six 0's (or) five 1's and five 0's etc (or) ten 1's and no 0's.

Required no.of bit strings = $\left[\frac{\mathbf{10!}}{\mathbf{4!6!}} + \frac{\mathbf{10!}}{\mathbf{5!5!}} + \frac{\mathbf{10!}}{\mathbf{6!4!}} + \frac{\mathbf{10!}}{\mathbf{7!3!}} + \frac{\mathbf{10!}}{\mathbf{8!2!}} + \frac{\mathbf{10!}}{\mathbf{9!1!}} + \frac{\mathbf{10!}}{\mathbf{10!0!}}\right] = \mathbf{848.}$

(d) The ten positions are to be filled up with five 1's and five 0's = $\frac{\mathbf{10!}}{\mathbf{5!5!}} = \mathbf{252}$.

Problem 5:

In how many of the permutations of 10 things taken 4 at a time will (a) one thing always occur (b) never occur

Solution:

We can keep aside the particular thing which will always occur, the number of permutations of 9 things taken at a time is $9P_3$.

Now this particular thing can take up any one of the four places and can be arranged in 4 ways.

$\therefore$ The no. of permutations $= 9P_3 \times 4$

$= 9\times8\times7\times4$

$= 2016.$

If we are keeping the particular thing aside which has never to occur, the number of 9 things

(10-1= 9) taken 4 at a time is $9p_4 = 9\times8\times7\times4 = 3026.$

Combinations

- The number of combinations of n distinct objects taken r at a time is

With Replacement : $${}_{n}C_{r} = \frac{n!}{r!\ (n-r)!}$$

Without Replacement : $${}_{n+r-1}C_{r} = \frac{(n+r-1)!}{r!\ (n-1)!}$$

Problem 1:

If nC_5=20 $nC_{4,}$ find 'n'

Solution:

Given nC_5=20 nc_4

$\Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}$ =20 $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$

$\Rightarrow \frac{n-4}{5}$ = 20

$\Rightarrow n-4$ = 20 . 5

$\Rightarrow n$ = 100 + 4

$\Rightarrow n$ = 104

Problem 2:

In how many ways can 5 persons be selected from amongst 10 persons?

Solution:

The selection can be done in 10_{C_5} ways.

$$\mathbf{nCr} = \frac{n!}{r!\,(n-r)!}$$

$$\mathbf{10_{C_5}} = \frac{10!}{5!(10-5)!}$$

$$= \frac{10\times9\times8\times7\times6\times5!}{5!\ \ 5!}$$

$$= \frac{10\times9\times8\times7\times6}{120} = 9\times28$$

= 252 ways.

Problem 3:

A committee of 5 is to be selected from 6 boys and 5 girls. Determine the number of ways of selecting the committee if it is to consist of atleast 1 boy and 1 girl.

Solution:

The committee may consist of

(i) 1 boy, 4 girls
(ii) 2 boys, 3 girls
(iii) 3 boys, 2 girls
(iv) 4 boys, 1 girl

The number of committees of type (i)

$$= 6c_1 \times 5c_4 = 6\times5 = 30$$

The number of committees of type (ii)

$$= 6c_2 \times 5c_3 = 15\times10 = 150$$

The number of committees of type (iii)

$$= 6c_3 \times 5c_2 = \frac{6*5*4}{1*2*3} = 200$$

The number of committees of type (iv)

$$= 6c_4 \times 5c_1 = 15\times5 = 75$$

The total number of ways of forming the committee

$$= 50+150+200+75 = 455.$$

Problem 4:

A committee of 3 people is to be chosen from 3 women,5 men and 4 students. (i) How many ways can the committee be chosen?(ii)How many ways can the committee be chosen if it cannot include both men and women?

Solution :

Total No. of people **= 5+4+3 = 12**

(i) Number of ways of selecting the committee of 3 people $= 12c_3 = \frac{12*11*10}{1*2*3} = 220$

(ii) Number of ways of selecting committees of 3 people with no women $= 9c_3$

$$= \frac{9*8*7}{1*2*3} = 84$$

Number of ways of selecting committees of 3 people with no men $= 7c_3 = \frac{7*6*5}{1*2*3} = 35$

Number of ways of selecting committees of 3 people with students alone $= 4c_3$

$$= \frac{4*3*2}{1*2*3} = 4$$

∴ **Number of ways choosing the committee that include both men and women**

= 84+35-4 = 115.

Problem 5:

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways there are to select a committee to develop a discrete maths course at a school in the committee is to consist of three faculty members from the maths department and four from the computer science dept.

Solution:

3 educationalists can be chosen from 9 educationalists in ${}_9C_3$ ways.

4 socialist can be chosen from 11 in ${}_9C_3$ ways.

Therefore, by product rule, the number of ways to select the committee is

${}_9C_3 \times {}_{11}C_4 = 84 \times 330 = 27720$ ways.

Problem 6:

A box contains 6 white balls and 5 red balls . Find the number of ways 4 balls can be drawn from the box. If

(i) **They can be any colour.**

(ii) **Two must be white and 2 red.**

(iii) **They all must be of the same colour.**

Solution:

(i) 4 balls of any colour can be chosen from 11 balls in ${}_{11}C_4$ ways = 330 ways.

(ii) 2 white balls can be chosen in ${}_{6}C_2$ ways and 2 red balls can be chosen in ${}_{5}C_2$ ways.

$\therefore$ No. of ways to choose 2 white and 2 red balls = ${}_{6}C_2 + {}_{5}C_2$

= 15+10 = 25 ways.

(iii) No. of ways selecting 4 balls all of the same colour is ${}_{6}C_4 + {}_{5}C_4$

= 15+5 = 20 ways.

Find the number of non-negative integer solution of the equation $x_1 + x_2 + x_3 = 11$.

Solution:

Let P_1 be the property that $x_1 > 3$.

Let P_2 be the property that $x_2 > 4$

Let P_3 be the property that $x_3 > 6$.

Now the number of solutions satisfying the inequalities

$x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$ is $N(P_1' P_2' P_3')$.

By the principle of inclusion – exclusion we have

$N(P_1' P_2' P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1 P_2) + N(P_1 P_3) + N(P_2 P_3) - N(P_1 P_2 P_3)$

N = Total number of solutions = $C(3+11-1,11) = 78$.

$N(P_1)$ = No. of solutions with $x_1 \geq 3$.

$= c_1(3+7-1,7) = c(9,7) = 36.$

$N(P_2)$ = No. of solutions with $x_2 \geq 5$.

$= c_1(3+6-1,6) = c(8,6) = 28.$

$N(P_3)$ = No. of solutions with $x_3 \geq 7$.

$= c_1(3+4-1,4) = c(6,4) = 15.$

$N(P_1P_2)$ = No. of solutions with $x_1 \geq 4$ and $x_2 \geq 5$.

$= c_1(3+2-1,2) = c(4,2) = 6.$

$N(P_1P_3)$ = No. of solutions with $x_1 \geq 4$ and $x_3 \geq 7 = 0$.

$N(P_1P_2\ P_3)$ = No. of solutions with $x_1 \geq 4$ and $x_2 \geq 5$ and $x_3 \geq 7 = 0$.

$N(P_1'\ P_2'\ P_3') = 78 - 36 - 28 - 15 + 6 + 1 + 0 + 0 = 6.$

$\therefore$ The equation $x_1 + x_2 + x_3 = 11$ with respect to the given constraints has 6 solutions

Combinations with Repetitions

The number of r-combinations (selection of r objects) of n kinds of objects, if repetitions of the objects is allowed = $C(n+r-1, r)$.

Problems :

1. Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be choosen?

Solution :

No. of available cookies $n = 4$

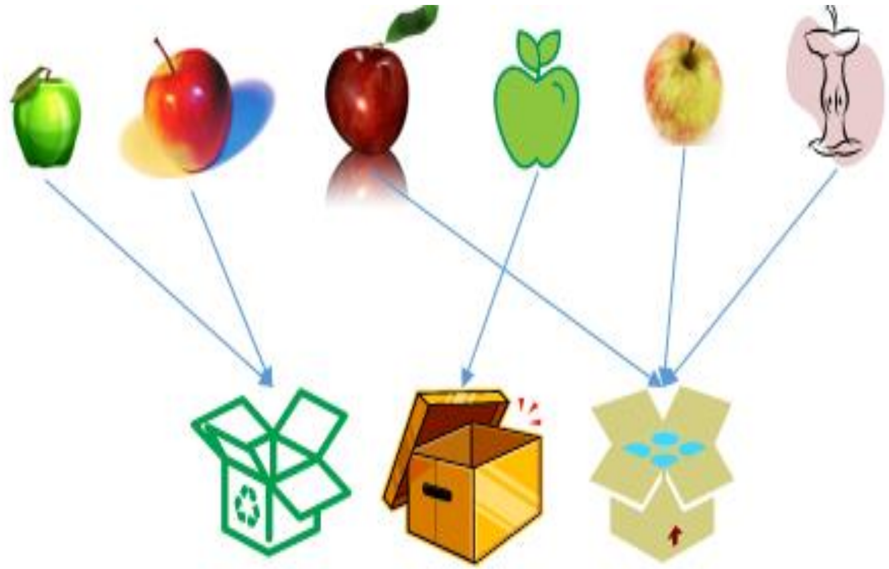
No .of cookies to be selected $r = 6$

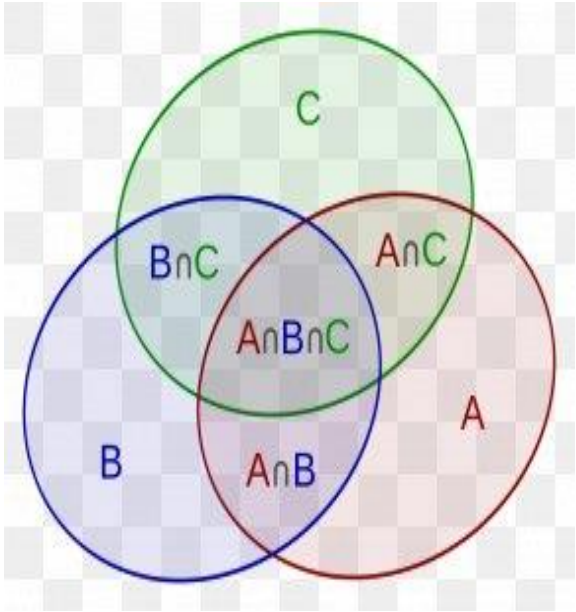
The number of ways to choose 6 cookies is the number of 6-combinations of a set with four elements.

$\therefore$ No. of ways of selection = $C(4+6-1,6) = C(9,6) = C(9,3) = 84$.

MODULE-II

PRINCIPLE OF INCLUSION AND EXCLUSION



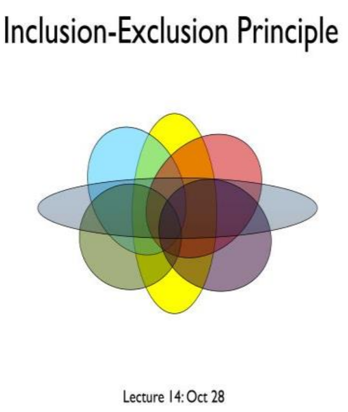


This Principle is used to count the number of elements in the union of two or more sets.

(i) If A and B are two sets, then the number of elements in their union set $(A \cup B)$ is given by

$|A \cup B|=|A|+|B|-|A \cap$

$n(A \cup B)=n(A)+n(B)-$



(ii) If A,B,C are any three sets then

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ (or)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

(iii) If A,B,C,D are any four sets then,

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| -$

$|B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap D| + |B \cap C \cap D| -$

$|A \cap B \cap C \cap D|$.

The Principle of Inclusion-Exclusion

$| A_1 \cup A_2 \cup \ldots \cup A_n | = \Sigma | A_i |$

$- \Sigma | A_i \cap A_j |$

$+ \Sigma | A_i \cap A_j \cap A_k |$

$- \ldots$

$+ (-1)^{n-1} \Sigma | A_1 \cap A_2 \cap \ldots \cap$

$A_n |$

EXAMPLE: 1

THE SURVEY WAS CONDUCTED AMONG 1000 STUDENTS. 595 LIKE DM, 565 LIKE POM, 550 LIKE DSP, 395 LIKE DM AND POM, 350 LIKE POM AND DSP, 400 LIKE DSP AND DM, & 250 LIKE ALL THE THREE SUBJECTS.

(I) FIND THE NUMBER OF STUDENTS WHO LIKE ATLEAST ONE OF THE SUBJECTS,

(II) HOW MANY OF THEM LIKE POM AND DO NOT LIKE DM AND DSP.

(III) HOW MANY OF THEM WHO DO NOT LIKE DM, POM, DSP.

SOLUTION:

Let D : Set of Students like DM,

P : Set of Students like POM

S : Set of Students like DSP

$|U|=1000$, $|D|=595$, $|P|=565$, $|S|=550$,

$|D\cap P|=395$, $|P\cap S|=350$, $|D\cap S|=400$, $|D\cap P\cap S|=250$

i) Number of students who like atleast one of the subjects

$=|D\cup P\cup S|$.

$$|D\cup P\cup S|= |D|+|P|+|S|-|D\cap P|-|P\cap S|-|D\cap S|+|D\cap P\cap S|$$

$$= 595+565+550-395-400-350+250 =815.$$

$|D\cup P\cup S|= 815$.

ii) n(P alone)= 565-145-250-100=70

Number of students who like POM but not DM & DSP

$=|P \cap \bar{D} \cap \bar{S}|$

$= |P| - |D \cap P| - |P \cap S| + |D \cap P \cap S|$

=565-395-350+250

=70

(iii) Number of students don’t like all

$=|\bar{P} \cap \bar{D} \cap \bar{S}|$

$= |U|- |P \cup D \cup S|$

=1000-815=185.

EXAMPLE : 2

DETERMINE THE NUMBER OF INTEGERS BETWEEN 1 AND 250 THAT ARE NOT DIVISIBLE BY 2 , 3 OR 5.

Solution:

Let A : Set of Integers between 1 and 250 are divisible by 2.

B : Set of Integers between 1 and 250 are divisible by 3.

C : Set of Integers between 1 and 250 are divisible by 5

$|U| = 250$

Therefore, $|A| = \left[\frac{250}{2}\right] = 125$, $|B| = \left[\frac{250}{3}\right] = 83$, $|C| = \left[\frac{250}{5}\right] = 50$

$|A \cap B| = \left\lfloor \frac{250}{2\times 3} \right\rfloor = \left\lfloor \frac{250}{6} \right\rfloor = 41$, $|B \cap C| = \left\lfloor \frac{250}{3\times 5} \right\rfloor = \left\lfloor \frac{250}{15} \right\rfloor = 16$

$|A \cap C| = \left\lfloor \frac{250}{2\times 5} \right\rfloor = \left\lfloor \frac{250}{10} \right\rfloor = 25$, $|A \cap B \cap C| = \left\lfloor \frac{250}{2\times 3\times 5} \right\rfloor = \left\lfloor \frac{250}{30} \right\rfloor = 8.$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 125+83+50+41-16-25+8 \quad = 184$

Therefore , The number of integers that are not divisible by 2 , 3 or 5 is $|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C| = 250 - 184 = 66.$

EXAMPLE : 3

A survey of 150 college students reveals that 83 own automobiles , 97 own bikes , 28 own motocycles , 53 own a car and a bike ,14 own a car and motorcycles , 7 own a bike and a motorcycle and 2 all three.

(i) How many students own a bike and nothing else?

(ii) How many students do not own any of the three?

Solution:

Let A : Set of students own a car or automobile.

B : Set of students own a bike .

C : Set of students own a motorcycle.

From the problem, we have the data

$n(A) = 83$, $n(B) = 97$, $n(C) = 28$, $n(A\cap B) = 53$, $n(A\cap C) = 14$,

$n(B\cap C) = 7$, $n(A\cap B\cap C) = 2$

(i) Number of students who own a bike and nothing else:

$= |B - (A \cup C)|$

$= |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$

$= 97 - 53 - 7 + 2 = 39$

Therefore , 39 Students own only bike.

(ii) $n(\bar{A} \cap \bar{B} \cap \bar{C}) = N - n(A \cup B \cup C)$

Now,

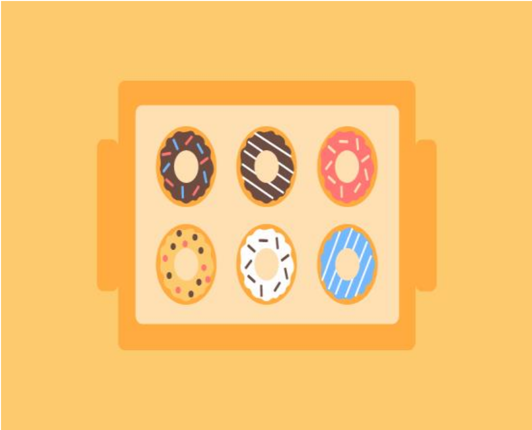
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

$= 83 + 97 + 28 - 53 - 14 - 7 + 2$

$= 136$.

Therefore , Number of Students do not own any of the three vehicles = 150 – 136 =14 Students.

ALTERNATE FORM OF PRINCIPLE OF INCLUSION AND EXCLUSION



Let N be the total number of objects.

Let A, B and C be the three characters possessed by the objects.

Then $n(\bar{A} \cap \bar{B} \cap \bar{C}) = N - n(A \cup B \cup C)$

$$= N - n(A) - n(B) - n(C) + n(A \cap B) + n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)$$

EXAMPLE : 4

USING THE PRINCIPLE OF INCLUSION AND EXCLUSION FIND THE NUMBER OF PRIME NUMBERS NOT EXCEEDING 100.

Solution:

The primes below 10 are 2,3,5,7

Therefore the number of primes not exceeding 100 is

$$4+N(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4})$$

Here, A_1 =Set of integers divisible by 2

A_2 = Set of integers divisible by 3

A_3 = Set of integers divisible by 5

A_4 = Set of integers divisible by 7

Since there are 99 possible integers >1 and not exceeding 100, by principle of inclusion and exclusion.

$n(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4})$ = 99 - $n(A_1 \cup A_2 \cup A_3 \cup A_4)$

$= 99 - n(A_1) - n(A_2) - n(A_3) - n(A_4) + n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_1 \cap A_4) + n(A_2 \cap A_3)$

$+ n(A_2 \cap A_4) + n(A_3 \cap A_4) - n(A_1 \cap A_2 \cap A_3) - n(A_1 \cap A_3 \cap A_4) - n(A_1 \cap A_2 \cap A_4) -$

$n(A_2 \cap A_3 \cap A_4) + n(A_1 \cap A_2 \cap A_3 \cap A_4)$

$| A_1 | = \left[\frac{100}{2}\right] = 50$; $| A_2 | = \left[\frac{100}{3}\right] = 33$

$| A_3 | = \left[\frac{100}{5}\right] = 20$; $| A_4 | = \left[\frac{100}{7}\right] = 14$

$| A_1 \cap A_2 |$ = [100/(2*3)]= [100/LCM(2,3)] = [100/6] =16

$| A_1 \cap A_3 |$ = [100/(2*5)] = [100/10] =10

$| A_1 \cap A_4 |$= [100/(2*7)] = [100/14] =7

$| A_2 \cap A_3 |$ = [100/(3*5)] = [100/15] =6

$| A_2 \cap A_4 |$= [100/(3*7)] = [100/21] =4

$| A_3 \cap A_4 |$ = [100/(5*7)] =[100/35] =3

$| A_1 \cap A_2 \cap A_3 |$ = [100/lcm(2*3*5)]= [100/80] =3

$| A_1 \cap A_2 \cap A_4 |$ = [100/(2*3*7)] =[100/42] =2

$| A_1 \cap A_3 \cap A_4 |$ = [100/(2*5*7)] = [100/70] = 1

$| A_2 \cap A_3 \cap A_4 |$ = [100/(3*5*7)] = [100/105] =0

$| A_1 \cap A_2 \cap A_3 \cap A_4 |$ = [100/(2*3*5*7)] = [100/215] = 0

$| A_1 \cup A_2 \cup A_3 \cup A_4 |$

= 50 + 33 +20 +14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 + 0 - 0

= 78

$n(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}) = 99 - 78 = 21$

The number of primes not exceeding 100 = $4 + n(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4})$

= 4 + 21

= 25

Example : 5

Find the number of integers between 1 and 2000 inclusive that are not divisible by 2,3,5 or 7.

Solution:

Let

A_1 be the set of integers that are divisible by 2

A_2 be the set of integers that are divisible by 3

A_3 be the set of integers that are divisible by 5

A_4 be the set of integers that are divisible by 7

To find n($\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}$) by principle of inclusion and exclusion.

n($\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}$) =N –n(A_1) – n(A_2) – n(A_3) –n(A_4) +n($A_1 \cap A_2$) +n($A_1 \cap A_3$)+n($A_1 \cap A_4$)+n($A_2 \cap A_3$) + n($A_3 \cap A_4$) + n ($A_2 \cap A_4$) - n($A_1 \cap A_2 \cap A_3$) - n($A_1 \cap A_3 \cap A_4$)- n($A_1 \cap A_2 \cap A_4$) – n($A_2 \cap A_3 \cap A_4$)+ n($A_1 \cap A_2 \cap A_3 \cap A_4$)

$| A_1 | = \left[\frac{2000}{2}\right] = 1000$; $| A_2 | = \left[\frac{2000}{3}\right] = 666$

$| A_3 | = \left[\frac{2000}{5}\right] = 400$; $| A_4 | = \left[\frac{2000}{7}\right] = 285$

$| A_1 \cap A_2 | = \left[\frac{2000}{2\times 3}\right] = 333$; $| A_1 \cap A_3 | = \left[\frac{2000}{2\times 5}\right] = 200$

$| A_1 \cap A_4 | = \left[\frac{2000}{2\times 7}\right] = 143$

$| A_2 \cap A_3 | = \left[\frac{2000}{3\times 5}\right] = 133$

$| A_2 \cap A_4 | = \left[\frac{2000}{3\times7}\right] = 95$; $| A_3 \cap A_4 | = \left[\frac{2000}{5\times7}\right] = 57$

$| A_1 \cap A_2 \cap A_3 | = \left[\frac{2000}{2X3X5}\right] = 66$; $| A_1 \cap A_2 \cap A_4 | = \left[\frac{2000}{2X3X7}\right] = 47$

$| A_1 \cap A_3 \cap A_4 | = = \left[\frac{2000}{2X5X7}\right] = 28$; $| A_2 \cap A_3 \cap A_4 | = = \left[\frac{2000}{3X5X7}\right] = 19$

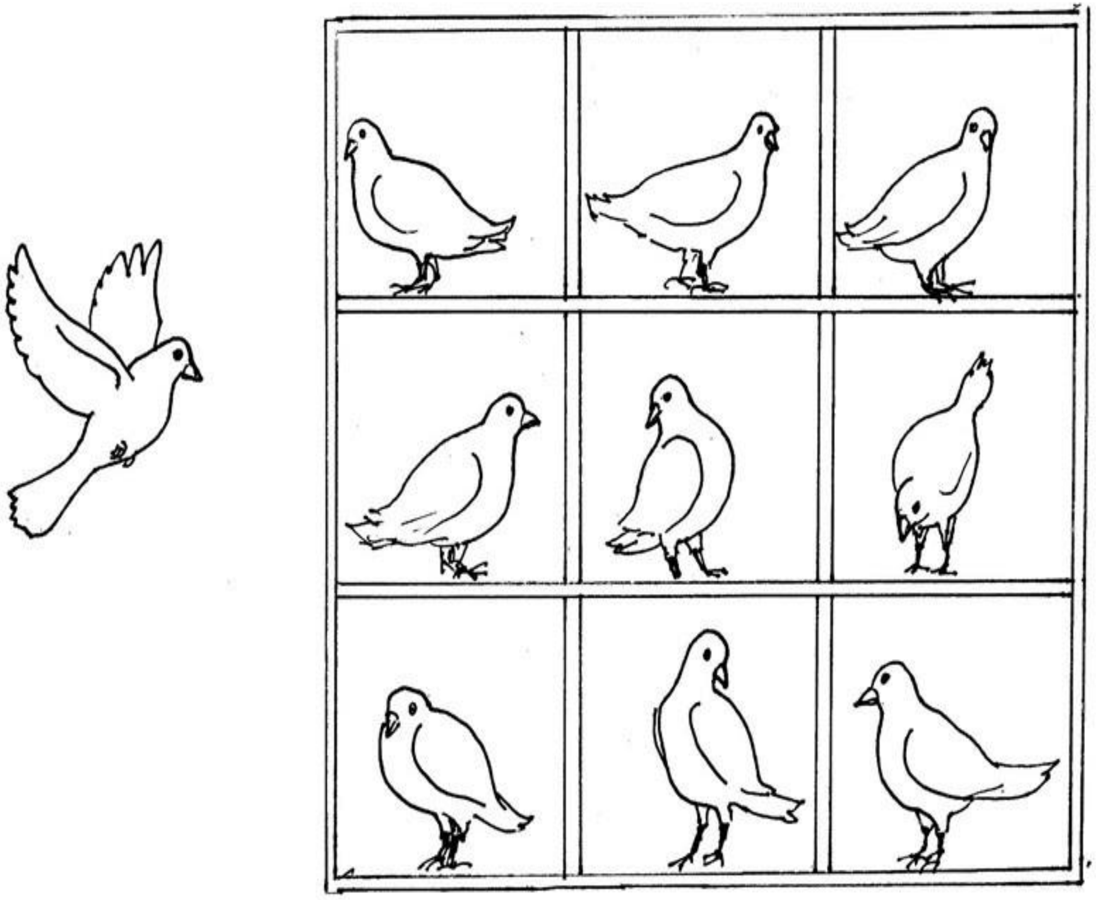
$| A_1 \cap A_2 \cap A_3 \cap A_4 | = = \left[\frac{2000}{2X3X5X7}\right] = 9$

$n(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4})$

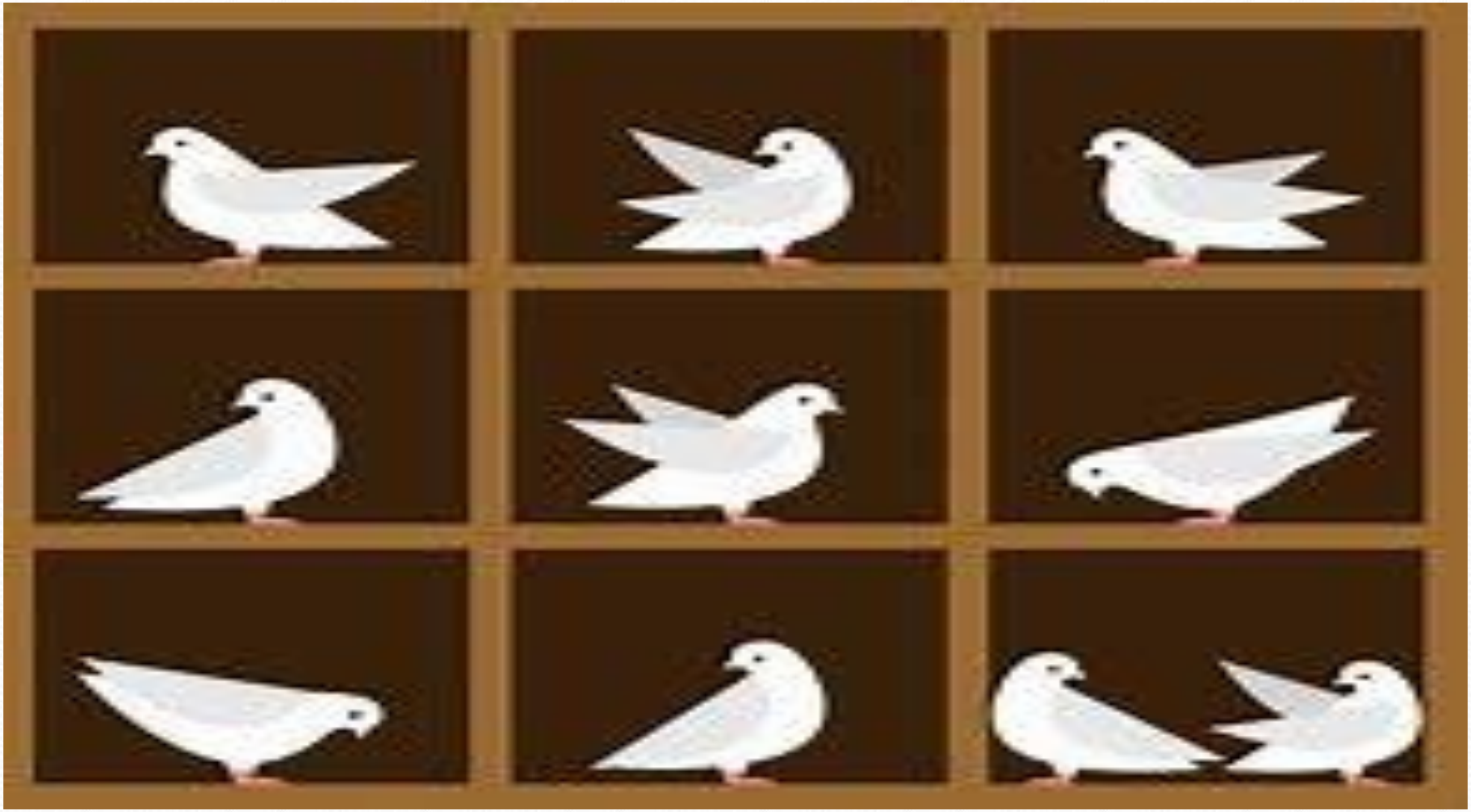
= 2000 - 1000 - 666 - 400 - 285 + 333 + 200 + 143 + 133 + 95 + 57 - 66 - 47 - 28 - 19 + 9

= 458.

Pigeonhole Principle



Pigeonhole Principle



The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects. ◀

Generalized Pigeonhole principle is also taken in the following form

If m pigeons are accommodated in n pigeonholes and m > n then one of the pigeonholes must contain atleast $\left[\frac{m-1}{n}\right] + 1$ pigeons.

Note: **[x] denotes the greatest integer less than or equal to x.**

Pigeonhole principle**:**

If (n+1) pigeon occupies ‘n’ holes then at least one hole has more than one pigeon.

Note:

If kn+1 or more pigeons occupy k pigeonholes,there will be more than n pigeons in atleast one pigeonholes where n is a positive integer.

The pigeonhole principle is also called as Dirichlet drawer principle.

What is the minimum number of students required in a Discrete mathematics class to be sure that at least six will receive the same grade if there are five possible grades A,B,C,D & E?

Solution:

Number of pigeonholes = Number of grades

i.e. $n = 5$

Since atleast six students (pigeons) to receive the same grade in Discrete mathematics class

$k+1 = 6$

$\Rightarrow k = 5$

$\therefore$ **The smallest integer $N = kn+1 = 5.5 + 1 = 26$.**

Hence, minimum no. of students = 26.

Show that 102 students must be in a class to guarantee that at least two students receive the same score on the final exam if the exam is graded on a scale from 0 to 100 points?

Solution:

Given exam is graded on a scale from 0 to 100.

Therefore there are 101 possible scores(Pigeonholes). i.e., n = 101

Since atleast two students (pigeons) to receive the same score on the final exam

$$k+1 = 2 \quad \Rightarrow k = 1$$

$\therefore$ **The smallest integer N = kn+1 = 1$\times$101 + 1 = 102.**

By Pigeonhole principle we have among 102 students there must be atleast two students with the same score.

The class should contain minimum 102 students.

If seven colours are used to paint 50 bicycles then show that at least 8 bicycles will be of the same colour.

Solution:

Here Number of pigeon = m = Number of bicycles = 50

Number of holes = n = Number of colours = 7

By Pigeon hole principle, we have

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{50-1}{7} \right\rfloor + 1 = 8$$

Therefore, atleast 8 bicylces will be of the same colour.

If we select any group of 1000 students on campus, show that at least three of them must have same birthday?

Solution:

The maximum number of days in a year = 366.

Here Number of students = Number of pigeons = m = 1000

Number of days in a year = Number of holes = n = 366

By Generalized Pigeon hole principle,

Minimum number of students having same birthday is

$$\left[\frac{m-1}{n}\right]+1=\left[\frac{1000-1}{366}\right]+1=2+1=3.$$

Hence at least 3 must have same birthday.

Prove that an equilateral triangle; whose sides are of length 1 unit, if any five points are chosen, then at least two of them lies in a triangle whose sides apart is less than ½.

Solution:

Let D,E and F are mid points of the side AB,BC and AC respectively. So that the triangle ABC divided into four equilateral triangles of each side ½.

Now Number of pigeon = Number of interior points = 5

Number of pigeon holes = Number of triangles = 4

By Pigeonhole principle, at least one triangle has more than 1 point.

Since each triangle side is ½ the distance between two interior points of any sub triangle is less than ½.

Prove that there exists a positive integer n such that m divides 2^n-1 where m being a positive odd integer.

Solution

Consider $(m+1)$ positive integers $2^1-1, 2^2-1, \quad 2^3-1.....2^m-1,$ $2^{m+1}-1$.

By the pigeonhole priniciple, when these are divided by m two of the numbers will give the same remainder. [$(m+1)$ numbers are the $(m+1)$ pigeons and the m remainders, namely $0,1,2.....(m-1)$ are the m pigeons]

Let the two numbers be 2^s-1 and 2^t-1 which give the same remainder r, upon division by m, wher $1 \le s < t \le m+1$.

$$\therefore \quad 2^s-1 = q_1 m + r \text{ and } 2^t-1 = q_2 m + r$$

$$\therefore \quad 2^s-2^t = (q_1-q_2)m$$

$$\text{But} \quad 2^s-2^t = 2^t\left[2^{s-t}-1\right]$$

$$\therefore \quad 2^t\left[2^{s-t}-1\right] = (q_1-q_2)m$$

But m is odd and hence cannot be a factor of 2^t.

$\therefore$ m divides $2^{s-t}-1$

Taking $n=s-t$, m divides 2^n-1 Hence we get the result.

If n pigeon holes are occupied by (kn+1) pegeons, k is a positivie integer, prove that at least one pigeon hole is occupied by (k+1) or more pigeons. Hence find the minimum number of m integers to be selected from S= $\{1,2,....9\}$ so that the sum of two of the m integers are even.

Solution:

Assume n holes are occupied by (kn+1) pigeons.
Claim: At least one hole is occupied by (k+1) or more pigeon.
If not, (ie) At least one hole is not occupied by (k+1) or more pigeon.
$\Rightarrow$ Each hole contains at most k pigeons.
Since we have n holes the total number of pigeon is at most kn.
which is a contradiction
$\therefore$ at least one hole is occupied by (k+1) or more pigeon.
Let us divide set S = $\{1, 2,...9\}$ in to two subsets $\{1,3,5,7,9\}$ and $\{2,4,6,8\}$ which may be treated as pigeon holes.

$\therefore n = 2.$

At least two numbers must be chosen either from the first subset or from the second.
(ie) at least one pigeon hole contain two pigeons.
(ie) $k+1 = 2 \Rightarrow k = 1$
The minimum number of the pigeon required or minimum number of integers to be selected is equal to kn+1 = 3.

Mathematical Induction

Induction

- To understand the basic principle of mathematical induction,s suppose a set of thin rectangular tiles are placed in order as shown in figure.
- When the first tile is pushed in the indicated direction
- (a) The first tile falls, and
- (b) In the event that any tile falls its successor will necessarily falls.
- This is underlying principle of mathematical induction.

Problem 1:

Prove by mathematical induction for n, a positive integer,

$$1^2 + 2^2 + \ldots. + n^2 = \frac{n\,(n+1)(2n+1)}{6}$$

Solution :

Let p(n) : $1^2 + 2^2 + \ldots.+ n^2 = \frac{n\,(n+1)(2n+1)}{6}$

To Prove: P(1) is true.

$$P(1) : 1^2 = \frac{1\,(1+1)(2.1+1)}{6} = 1 \text{ is true.}$$

Assume that P(k) is true

i.e., P(k) : $1^2 + 2^2 + \ldots.+ k^2 = \frac{k\,(k+1)(2k+1)}{6}$ -------(1) is true,

where k is any integer.

Claim : P(k+1) is true.

i.e., to prove P (k+1) : $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)\,(k+1)+1)(2(k+1)+1)}{6}$

Consider $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k\,(k+1)(2k+1)}{6} + (k+1)^2$ using (1)

$$= \frac{(k+1)[k(2k+1)+6\,(k+1)]}{6}$$

$$= \frac{(k+1)\,(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\,(k+1)+1)(2(k+1)+1)}{6}$$

$\Rightarrow$ P(k+1) is true.

Hence by mathematical induction, for any n>0,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n\,(n+1)(2n+1)}{6}$$

Problem 2:

By mathematical induction show that $1 + 2 + 2^2 + \text{----} + 2^n = 2^{n+1} - 1$ for $n \geq 1$.

Solution:

Let P(n) : $\mathbf{1 + 2 + 2^2 + \text{----} + 2^n = 2^{n+1} - 1}$

To prove: P(1) is true

P (1) : $2^1 = 2^1 - 1 = 1$ is true.

Assume that P(k) : $1 + 2 + 2^2 + \text{----} + 2^k = \mathbf{2^{k+1} - 1}$ **is true**

Claim: P(k+1) is true.

$P(k+1) = 1+2+2^2+\ldots\ldots+2^k + 2^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1} = 2.2^{k+1} - 1 = 2^{k+2} - 1 = 2^{(k+1)+1} - 1$

$\therefore$ P(k+1) is also true.

Hence the proof.

Problem 3:

By using mathematical induction, show that $n! \geq 2^{n+1}$, $n = 5,6,7,\ldots.$

Solution :

Let $p(n) : n! \geq 2^{n+1}$

To prove: p(5) is true: $p(5) : 5! \geq 2^{5+1}$ is true.

Assume that $p(k) : k! \geq 2^{k+1}$ is true ----I

Claim: p(k+1) is true.

Using (1), $k! \geq 2^{k+1}$ is true

Multiply on both sides by 2 we get

$2(k!) \geq 2.\, 2^{k+1} \Rightarrow 2(k!) \geq 2^{k+2}$

$\Rightarrow (k+1)\, k! \geq 2(k!) \geq 2^{k+2}$ [By (*)]

$\Rightarrow (k+1)\, k! \geq 2^{k+2} \quad \Rightarrow (k+1)! \geq.\, 2^{k+2}$

[since $(k+1) \geq 2 \Rightarrow (k+1)\, k! \geq 2\, k!$ -------(*) {because $k > 5$}]

Therefore p(k+1) is true. Hence $n! \geq 2^{n+1}$

Problem 4: Prove by mathematical induction that for all $n \geq 1$, $n^3 + 2n$ is a multiple of 3.

Solution:

Let P(n) : $n^3 + 2n$ is a multiple of 3.

(i) To Prove: P(1) is true

P(1) : $1^3 + 2.1 = 3$ which is a multiple of 3.

(ii) Assume that P(k) is true. i.e., P(k) : $k^3 + 2k$ is a multiple of 3 --(1)

Claim : To Prove: P(k+1) is true.

Consider $(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 2k + 3$

$= (k^3 + 2k) + 3(k^2 + k + 1)$ ------- (2)

Since $k^3 + 2k$ is a multiple of 3, (2) is also a multiple of 3.

By the Principle of Mathematical induction, the proof follows

Problem 5: Using Mathematical induction show that $n^3 - n$ is divisible by 3 for $n \in Z^+$.

Solution:

Let $P(n) = n^3 - n$ is divisible by 3.

To Prove: P(1) is true:

$P(1) : 1^3 - 1$ is divisible by 3 is true.

Assume that $P(k) : k^3 - k$ is divisible by 3 ---(1)

Claim: P(k+1) is true.

$$P(k+1) = (k+1)^3 - (k+1)$$

$$= (k^3 - k) + 3(k^2 + k) \quad \text{-------}(2)$$

$\Rightarrow$ $k^3 - k$ is divisible by 3 and $3(k^2 + k)$ is divisible by 3, we have equation (2) is divisible by 3.

$\therefore$ P(k+1) is true.

By the principle of Mathematical induction $n^3 - n$ is divisible by 3

Problem 6:

Prove by induction that $2n^3 + 3n^2 + n$ is divisible by 6 for all integers $n \geq 0$.

Solution:

Let $P(n)$ be the statement $2n^3 + 3n^2 + n$ is divisible by 6.

To prove: $P(n)$ is true $\forall\ n \geq 0$.

Basis step: To prove P(0) is true [Here $n_0 = 0$]

$\therefore\ P(0)$: 0 is divisible by 6, which is true

So, $P(0)$ is true

Inductive step: Assume P(k) is true, $k > 0$.

$\Rightarrow 2k^3 + 3k^2 + k$ is divisible by 6 is true

$\Rightarrow 2k^3 + 3k^2 + k = 6x$ --------- (1)

where x is an integer.

To prove: $P(k+1)$ is true.

That is to prove $2(k+1)^3 + 3(k+1)^2 + (k+1)$ is divisible by 6 is true. Now

$$2(k+1)^3 + 3(k+1)^2 + (k+1) = 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + k + 1$$

$$= (2k^3 + 3k^2 + k) + 6k^2 + 6k + 2 + 6k + 3 + 1$$

$$= (2k^3 + 3k^2 + k) + 6k^2 + 12k + 6$$

$$= 6x + 6(k^2 + 2k + 1) \qquad \text{[using(1)]}$$

$$= 6[x + k^2 + 2k + 1],$$

where $x + k^2 + 2k + 7$ is an integer.

$\therefore\ P(k+1)$ is true.

Thus $P(k)$ is true $\Rightarrow P(k+1)$ is true.

Hence by first principle of induction $P(n)$ is true $\forall\ n \geq 0$.

Problem 7:

If the sum of the cubes of three consecutive integers is a cube k^3 prove that $3 \mid k$.

Solution:

Let n, $n + 1$, $n + 2$ be the three consecutive integers.

Given $n^3 + (n + 1)^3 + (n + 2)^3$ is a cube k^3

$$n^3 + n^3 + 3n^2 + 3n + 1 + n^3 + 3n^2 \cdot 2 + 3n \cdot 2^2 + 2^3 = k^3$$

$$\Rightarrow 3n^3 + 9n^2 + 15n + 9 = k^3$$

$$\Rightarrow 3(n^3 + 3n^2 + 5n + 3) = k^3$$

$$\Rightarrow 3 \mid k^3 \rightarrow 3 \mid k \cdot k \cdot k$$

Since 3 is a prime, $3 \mid k$.

Problem 8:

Prove by induction $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{(2n+1)}$

Solution:

$$\text{Let } P(n) = \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{(2n+1)}$$

Basis

$$P(1) \; : \; \frac{1}{3} = \frac{1}{(2(1)+1)} = \frac{1}{2+1} = \frac{1}{3}, \text{ which is true.}$$

$\therefore$ P(1) is true

Induction

Assume that P(K) is true.

i.e., $$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{(2k+1)}$$

Consider P(K+1)

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots\ldots+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$=\left(\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots\ldots+\frac{1}{(2k-1)\,(2k+1)}\right)+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$=\frac{k}{(2k+1)}+\left[\frac{1}{[2(k+1)-1][2(k+1)+1]}\right]$$

$$= \frac{k}{(2k+1)} + \left[\frac{1}{(2k+1)\ (2k+3)}\right] \quad = \frac{1}{(2k+1)} + \left[k + \frac{1}{(2k+3)}\right]$$

$$= \frac{1}{(2k+1)} + \left[\frac{2k^2+3k+1}{(2k+3)}\right] \quad = \frac{(2k+1)\ (k+1)}{(2k+1)\ (2k+3)}$$

$$= \frac{(k+1)}{(2k+3)} \quad = \frac{(k+1)}{[2(k+1)+1]}$$

$\therefore$ P(K+1) is true

i.e., $P(K) \Rightarrow P(K+1)$

$\therefore$ By mathematical induction,

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots + \frac{1}{(2n-1)\ (2n+1)} = \frac{n}{(2n+1)}$$

Problem 9:

Prove that for all n ≥ 6, $4n < n^2$-7

Solution:

Let P(n) = $4n < n^2 - 7$ for $n \geq 6$

Basis

P(6) : $4(6) < 36 - 7$

$24 < 29$, which is true.

∴ P(6) is true.

Induction

Assume P(K) is true.

i.e., $4k < k^2 - 7$ is true.

Consider P(K+1)

$$4(k+1) < (k+1)^2 - 7$$

consider $4(k+1) = 4k + 4$

Since $4k < k^2 - 7$

$4(k+1) < k^2 - 7 + 4$

But for $k \geq 6$, $2k + 1 > 4$

$\therefore \quad 4(k+1) < k^2 - 7 + (2k+1)$

$$= (k^2 + 2k + 1) - 7$$

$$= (k+1)^2 - 7$$

$\therefore \quad 4(k+1) < (k+1)^2 - 7$

$\therefore$ P(K+1) is true.

$P(K) \Rightarrow P(K+1)$

Hence by induction, $4n < n^2 - 7$ for $n \geq 6$

Problem 10:

Prove that for all n $\geq 4, 2^n < n!$ *by induction*

Solution:

Let P(n) : $2^n < n!$

Basis

P(4) : 4 = $2^4 = 16 < 4! = 24$

$\therefore$ $16 < 24$ is true.

i.e., P(4) is true.

Induction

Assume P(K) : $2^k < k!$ is true

To show P(K+1) : $2^{k+1} < (k+1)!$

Consider $2^{k+1} = 2^k \cdot 2$

Since $2^k < k!$

$2^{k+1} < 2(k!)$

i.e., $2^{k+1} < k\,!\,(k)$ since $k > 2$

$= 2^{k+1} < k\,!\,(k)$

$= (k+1)!$

$\therefore$ $2^{k+1} < (k+1)!$

$\therefore$ P(K+1) is true

Hence $P(K) \Rightarrow P(K+1)$

$\therefore$ By mathematical induction $2^n < n\,!$ for $n \geq 4$

Problem 11:

Prove *by induction* $13^n - 6^n$ *is divisible by* 7.

Solution:

Let $P(n)$: $13^n - 6^n$ is divisible by 7.

Basis

$P(1) = 13^1 - 6^1 = 13 - 6 = 7$ which is divisible by 7.

$\therefore$ $P(1)$ is true

Induction

Assume $P(K)$: $13^k - 6^k$ is divisible by 7.

To show $P(K+1)$: $13^{k+1} - 6^{k+1}$ is divisible by 7.

Consider $13^{k+1} - 6^{k+1}$

$= 13^{k+1} - 13^k . 6 + 13^k . 6 - 6^{k+1}$

$= 13^k (13 - 6) + 6(13^k - 6^k)$

$= 13^k (13 - 6) + 6m(13 - 6)$ $[\because 13^k - 6^k = m(13-6)]$

$= (13^k + 6m)(13 - 6)$

$= 7(13^k + 6m)$

$\therefore$ $13^{k+1} - 6^{k+1}$ is divisible by 7

$\therefore$ $P(K+1)$ is true

$\therefore$ $13^{k+1} - 6^{k+1}$ is divisible by 7.

$P(K) \Rightarrow P(K+1)$

Strong Mathematical Induction

To conclude that we can reach every rung by strong induction:

BASIS STEP: $P(1)$ holds.

INDUCTIVE STEP: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for an arbitrary integer k, and show that $P(k+1)$ must also hold.

We will have then shown by strong induction that for every positive integer n, $P(n)$ holds, i.e., we can reach the nth rung of the ladder.

Strong Mathematical Induction

INDUCTIVE STEP: The inductive hypothesis is $P(i)$ is true for all integers i with $2 \leq i \leq k$. To show that $P(k+1)$ must be true under this assumption, two cases need to be considered:

1. If $k+1$ is prime, then $P(k+1)$ is true.
2. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k+1$ can also be written as the product of those primes.

Strong Mathematical Induction

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as a product of primes.

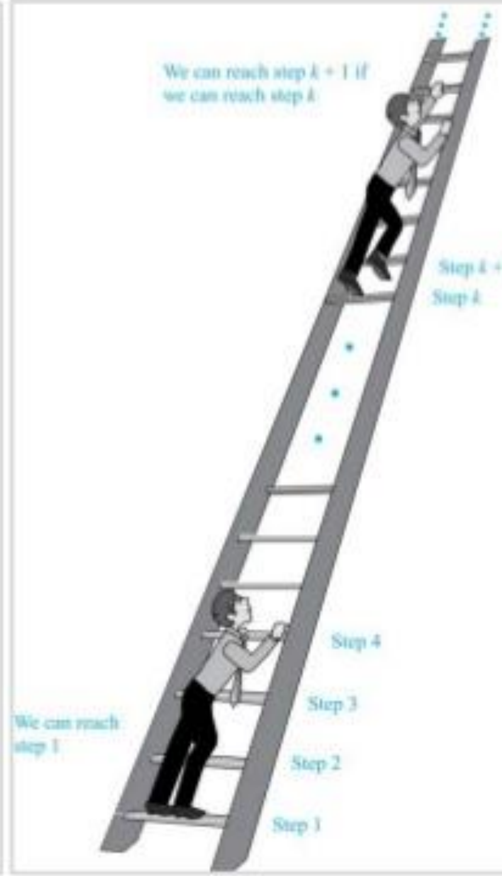
BASIS STEP: $P(2)$ is true since 2 itself is prime.

INDUCTIVE STEP: The inductive hypothesis is $P(i)$ is true for all integers i with $2 \leq i \leq k$. To show that $P(k+1)$ must be true under this assumption, two cases need to be considered:

Strong Mathematical Induction

Strong induction tells us that we can reach all rungs if:

1. We can reach the first rung of the ladder.
2. For every integer k, if we can reach the first k rungs, then we can reach the $(k+1)$st rung.



Example for strong induction

Show that if n is an integer than 1, then n can be written as the product of primes.

Solution

Let $p(n)$ be the proposition that n can be written as the product of primes

Basis step

$p(2)$ is true since 2 can be written as the product of one prime.

Induction step

Assume that $p(j)$ is true for all integers j with $j \leq k$. Consider $p(k+1)$.

The integer $k+1$ is prime or composite.

If $k+1$ is prime, we immediately see that $p(k+1)$ is true. If $k+1$ is composite, then $k+1=ab$ with $2 \le a \le b \le k$. By induction hypothesis, both a and b can be written as the product of primes. Thus if $k+1$ is composite, it can be written as the product of primes namely, those primes in the factorization of a and those in the factorization of b.

Recurrence Relations

- Sequences are ordered lists of elements.
  - $1, 2, 3, 5, 8$
  - $1, 3, 9, 27, 81, \ldots$
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to <u>represent sequences</u>.

Recurrence Relations

Definition: A sequence is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \ldots\}$ or $\{1, 2, 3, 4, \ldots\}$) to a set S.

- The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of $f(n)$ where f is a function from $\{0,1,2, \ldots\}$ to S. We call a_n a term of the sequence.

Example: Consider the sequence $\{a_n\}$ where $a_n = 1/n$. Then

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Recurrence Relations

Definition: A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
- The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Recurrence Relations

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, 4, \ldots$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here the initial condition is $a_0 = 2$.]

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Fibonacci Numbers

Definition: the Fibonacci sequence, $f_0, f_1, f_2, \dots$, is defined by:

- Initial Conditions: $f_0 = 0,\ \ f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_4, f_5, f_6, f_7 and f_8.

Recurrence Relations

- Finding a formula for the nth term of the sequence generated by a recurrence relation is called solving the recurrence relation.
- Such a formula is called a explicit (closed) formula.
- One technique for finding an explicit formula for the sequence defined by a recurrence relation is backtracking.

Recurrence Relations

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$. Find an explicit formula for the sequence.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3 \\
&= (a_{n-3} + 3) + 2 \cdot 3 = a_{n-3} + 3 \cdot 3 \\
&\cdots \\
&= a_2 + (n-2) \cdot 3 = (a_1 + 3) + (n-2) \cdot 3 \\
&= a_1 + (n-1) \cdot 3 = 2 + (n-1) \cdot 3 \\
&= 3 \cdot n - 1
\end{aligned}
$$

Fibonacci Numbers

Definition: the Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by:

- Initial Conditions: $f_0 = 0, \;\; f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_4, f_5, f_6, f_7 and f_8.

Linear Homogeneous Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

where $r_1, r_2, \ldots, r_k$ are real numbers, and $r_k \neq 0$.

- it is linear because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- it is homogeneous because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
- the degree is k because a_n is expressed in terms of the previous k terms of the sequence.

Linear Homogeneous Recurrence Relations

- $p_n = (1.11)p_{n-1}$ linear homogeneous recurrence relation of degree 1
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree 2
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $h_n = 2h_{n-1} + 1$ not homogeneous
- $b_n = nb_{n-1}$ coefficients are not constants

RECURRENCE RELATION

Formation of Recurrence Relation

Problems :

1.Form the recurrence relation from $S(k) = 5 * 2^k, k > 0$.

Solution :

If $k > 1, S(k) = 5 * 2^k$

$$= 2 * 5 * 2^{k-1}$$

$$= 2 * S(k-1)$$

The recurrence relation is $S(k) - 2S(k-1) = 0$ **and the initial condition** $S(0) = 5$.

2. Find the recurrence relation for the Fibonacci sequence of numbers.

Solution :

The sequence of numbers

0,1,1,2,3,5,8,13...... is the Fibonacci sequence of numbers.

If $\boldsymbol{F_n}$ is the n-th term, then $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$

The recurrence relation is $F_n - F_{n-1} - F_{n-2} = 0$, $n \geq 2$

with initial conditions $F_0 = 0, F_1 = 1$.

3. Form the recurrence relation from $\mathbf{S(k) = 2k + 9}$.

Solution :

$$\mathrm{S(k)} \;\; = 2\mathrm{k} + 9$$

$$\mathrm{S(k-1)} = 2(\mathrm{k}-1) + 9$$

$$\mathrm{S(k-1)} = 2(\mathrm{k}) + 9 - 2$$

$$\Longrightarrow S(k-1) = S(k) - 2$$

$\therefore$ **The recurrence relation *is*** $S(k) - S(k-1) = 2.$

4. Find the recurrence relation satisfying the equation $Y_n = A(3)^n + B(-4)^n$

Solution :

Given $Y_n = A(3)^n + B(-4)^n$ ---------(1)

$Y_{n+1} = A3^{n+1} + B(-4)^{n+1}$

$= 3A3^n - 4 B(-4)^n$ ---------(2)

$Y_{n+2} = 9A3^n + 16 B(-4)^n$ ------------(3)

$(3) + (2) - (1) \times 12 \Rightarrow$

$Y_{n+2} + Y_{n+1} - 12 Y_n = 9A3^n + 16 B(-4)^n + 3A3^n - 4 B(-4)^n - 12[A(3)^n + B(-4)^n] = 0$

$Y_{n+2} + Y_{n+1} - \mathbf{12}Y_n \mathbf{= 0}$ which is the required solution.

Find the recurrence relation from $y_n = A.2^n + B.3^n$.

Solution

$$y_n = A.2^n + B.3^n \quad (1)$$

$$y_{n+1} = 2A.2^n + 3B.3^n \quad (2)$$

$$y_{n+2} = 4A\,2^n + 9B.3^n \quad (3)$$

From (1) and (2), $y_{n+1} - 2y_n = B.3^n$

From (2) and (3) $\quad y_{n+2} - 2y_{n+1} = 3B.3^n$

$$\therefore \quad y_{n+2} - 2y_{n+1} = 3\left(y_{n+1} - 2y_n\right)$$

$\therefore \quad y_{n+2} - 5y_{n+1} + 6y_n = 0$ is the required recurrence relation.

Find the recurrence relation obtained from $y_n = A2^n + B(-3)^n$.

Solution

$$y_n = A.2^n + B(-3)^n$$

$$y_{n-1} = A.2^{n-1} + B(-3)^{n-1}$$

$$y_{n-2} = A.2^{n-2} + B(-3)^{n-2}$$

Eliminating A and B in the above expressions

$$\begin{vmatrix} 4 & 9 & -y_n \\ 2 & -3 & -y_{n-1} \\ 1 & 1 & -y_{n-2} \end{vmatrix} = 0$$

$$4[3y_{n-2} + y_{n-1}] - 9(-2y_{n-2} + y_{n-1}) - y_n(2+3) = 0$$

$$30y_{n-2} - 5y_{n-1} - 5y_n = 0$$

$$\Rightarrow \quad y_n + y_{n-1} - 6y_{n-2} = 0$$

This is the required recurrence relation.

Find the recurrence relation from $A(k)=k^2-k$.

Solution

$$A(k) = k^2 - k$$

$$A(k-1) = (k-1)^2 - (k-1)$$

$$= k^2 - 2k + 1 - k + 1$$

$$= (k^2 - k) - 2k + 1$$

$$\therefore \quad A(k-1) = A(k) - 2k + 1$$

$$\text{Also } A(k-2) = A(k-1) - 2(k-1) + 1$$

$$= A(k-1) - 2k + 3$$

$$\therefore \quad A(k-2) - A(k-1) = (-2k+1) + 2$$

$$\Rightarrow \quad A(k-2) - A(k-1) = A(k-1) - A(k) + 2$$

$\therefore \quad A(k) - 2A(k-1) + A(k-2) = 2$ is the required relation.

Recurrence relation

If $\{a_n\}$, n ≥ 0 represents a sequence of numbers then an expression that relates a term of the sequence to one or more of its preceeding terms is called a recurrence relation.

For example , $a_{n+2} = a_{n+1} + 2a_n$ is a recurrence relation.

$F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ is called the Fibonacci recurrence relation for the sequence of numbers {1,1,2,3,5,8,13,…..}.

Order of recurrence relation

Order of a recurrence relation = Highest subscript – lowest subscript.

(i) $F_n - F_{n-1} - F_{n-2} = 0$ is a recurrence relation of order $n-(n-2)=2$

(ii) $2a_n - a_{n-1} = 5 * 2^n$ is a recurrence relation of order $n-(n-1)=1$.

General form of k-th order recurrence relation

Consider $\mathbf{c}_0 y_{n+k} + c_1 y_{n+k-1} + c_2 y_{n+k-2} + \cdots . + c_n y_n = f(n)$

is the general form of the k-th order recurrence relation ,where

$c_1, c_2, \ldots\ldots\ldots c_n$ are constants and $f(n)$ is a function of 'n' only.

This recurrence relation is non-homogeneous type. When

$f(n) = 0$, the recurrence relation is said to be homogeneous.

Solution of Recurrence Relation Using Generating Function

Solution of Recurrence Relation Using Generating Function

We first introduce the concept generating functions and then we will consider how generating function is used to solve a recurrence relation.

Generating Function

Let a_0, a_1, a_2,.......... be a sequence of real numbers.

The function $f(x) = a_0 + a_1x + a_2x^2$ +......... = $\sum_{i=0} a_i x^i$ is called the generating function for the given sequence .

For example, $(1-x)^n$ is the generating function of the sequence $\binom{m}{r}$, 0 ≤ r ≤ n.

Sequence	General terms (a_n)	General functions
$1,1,1,1,....$	$a_n = 1^n, \; n \geq 0$	$\frac{1}{1-x}$
$1,-1,1,-1,....$	$a_n = (-1)^n, \; n \geq 0$	$\frac{1}{1+x}$
$1,a,a^2,a^3,....$	$a^n, \; n \geq 0$	$\frac{1}{1-ax}$
$1,-a,a^2,-a^3,....$	$(-a)^n, \; n \geq 0$	$\frac{1}{1+ax}$
$0,1,2,3....$	$n, \; n \geq 0$	$\frac{x}{(1-x)^2}$
$1,2,3,4....$	$(n+1), \; n \geq 0$	$\frac{1}{(1-x)^2}$
$1^2,2^2,3^2....$	$(n+1)^2, \; n \geq 0$	$\frac{x+1}{(1-x)^3}$
$0^2,1^2,2^2,3^2....$	$n^2, \; n \geq 0$	$\frac{x(x+1)}{(1-x)^3}$
$1,1,\frac{1}{2!},\frac{1}{3!}....$	$\frac{1}{n!}, \; n \geq 0$	e^x
-	$n(n+1), \; n \geq 1$	$\frac{2x}{(1-x)^3}$

SOME USEFUL EXPANSION:

1. $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$

2. $\sum_{n=0}^{\infty} (-1)^n x^n = (1+x)^{-1}$

3. $\sum_{n=0}^{\infty} a^n x^n = (1-ax)^{-1}$

4. $\sum_{n=0}^{\infty} (-1)^n a^n x^n = (1+ax)^{-1}$

5. $\sum_{n=0}^{\infty} (n+1) x^n = (1-x)^{-2}$

6. $\sum_{n=0}^{\infty} \frac{(n+1)(n+1)}{2} x^n = (1-x)^{-3}$

Note

If $f(x)=\sum_{n=0}^{\infty} a_n x^n$, then the folowing results are very useful to

solve recurrence relations.

(1) $\sum_{n=0}^{\infty} a_{n+2} x^n = \frac{f(x)-a_0-a_1 x}{x^2}$

(2) $\sum_{n=0}^{\infty} a_{n+1} x^n = \frac{f(x)-a_0}{x}$

(3) $\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 f(x)$

(4) $\sum_{n=2}^{\infty} a_{n-1} x^n = x\left[f(x)-a_0\right]$

(5) $\sum_{n=2}^{\infty} a_n x^n = f(x)-a_0-a_1 x$


EXAMPLES

Example 1:

Solve the recurrence relation a_n - 7 a_{n-1} + 10 a_{n-2} = 0 by the method of generating functions

with the initial condition $a_0 = 3$ and $a_1 = 3$.

Solution :

Let a_n-7 a_{n-1}+10 a_{n-2}=0 , $n \geq 2$ and $a_0 = 3, a_1 = 3$.

Multiplying (1) by x^n and summing from n=2 to ∞,

$\sum_{n=2}^{\infty} a_n x^n$ -7 $\sum_{n=2}^{\infty} a_{n-1} x^n$ +10 $\sum_{n=2}^{\infty} a_{n-2} x^n$ = 0

Let G(x)= $\sum_{n=0}^{\infty} a_n x^n$

Therefore,

$[G(x) - a_0 - a_1 x]$ - 7 x [$G(x) - a_0]$ + 10 x^2 $G(x)$ = 0

$[G(x) - 3 - 3x]$ - 7 x[$G(x) - 3]$ + 10 x^2 $G(x)$ = 0

$$G(x)\,[\,1 - 7x + 10x^2] = 3 + 3x\text{-}21\,x$$

$(1-2x)(1-5x)\, G(x) = 3-18\,x$

$$G(x) = \frac{3-18\,x}{(1-2\,x)(1-5x)}$$

Using Partial Fraction Method,

$$\frac{3-18\,x}{(1-2x)(1-5x)} = \frac{A}{1-2x} + \frac{B}{1-5x} \text{-----------------------}*$$

$3-18\,x = A\,(1-5x) + B(1-2x)$

Put $x = \frac{1}{2}$, Then $A = 4$

Put $x = \frac{1}{5}$, Then $B = -1$

Put A,B values in *,we get

$$\frac{3-18\,x}{(1-2x)(1-5x)} = \frac{4}{1-2x} - \frac{1}{1-5x}$$

$G(x) = 4\,(1-2x)^{-1} - (1-5x)^{-1}$

$\sum_{n=0}^{\infty} a_n x^n = 4\sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} 5^n x^n$

$= \sum_{n=0}^{\infty}[4(2^n) - 5^n]x^n$

$a_n = 4(2^n) - 5^n$ is the required solution.

Solve the recurrence relation a_{n+2}-5 a_{n+1}+ $6a_n$=2, $n \geq 0$ by the method of generating functions with the initial condition $a_0 = 1$ and $a_1 = 2$.

Solution:

Given a_{n+2}-5 a_{n+1}+ $6a_n$=2, $n \geq 0$--------------(1)

Multiplying (1) by x^n and summing from n=0 to ∞,

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 5\sum_{n=0}^{\infty} a_{n+1} x^n + 6\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} x^n$$

Let G(x)= $\sum_{n=0}^{\infty} a_n x^n$

Therefore, $\frac{G(x)-a_0-a_1x}{x^2} - 5\frac{G(x)-a_0}{x} + 6G(x) = \frac{2}{1-x}$

Therefore $G(x) - a_0 - a_1x - 5x[G(x) - a_0] + 6x^2\, G(x) = \frac{2x^2}{1-x}$

Using a_0=1, a_1=2,

Therefore $G(x) - 1 - 2x - 5x[G(x) - 1] + 6x^2\, G(x) = \frac{2x^2}{1-x}$

$$G(x)[1-5x+6x^2]-1-2x+5x = \frac{2x^2}{1-x}$$

$$G(x)[1-5x+6x^2] = \frac{2x^2}{1-x} + 1 - 3x$$

$$= \frac{2x^2 + (1-4x+3x^2)}{1-x}$$

$$G(x)[(1-2x)(1-3x)] = \frac{1-4x+5x^2}{1-x}$$

$$G(x) = \frac{1-4x+5x^2}{(1-2x)(1-3x)(1-x)}$$

Let $\frac{5x^2-4x+1}{(1-2x)(1-3x)(1-x)} = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x}$ [Using Partial Fraction method]

$5x^2 - 4x + 1 = A(1-3x)(1-2x) + B(1-x)(1-3x) + + C(1-x)(1-2x)$------------*

Put $x=1$. Then $A=1$. Put $x=1/2$.Then $B=-1$. Put

$x=1/3$.Then $C=1$

Substitute A,B,C values in *,

we get

$$\frac{5x^2-4x+1}{(1-2x)(1-3x)(1-x)}=\frac{1}{1-x}-\frac{1}{1-2x}+\frac{1}{1-3x}$$

$$G(x)=(1-x)^{-1}-(1-2x)^{-1}+(1-3x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} 1x^n-\sum_{n=0}^{\infty} 2^n x^n+\sum_{n=0}^{\infty} 3^n x^n$$

$$=\sum_{n=0}^{\infty}(1-2^n+3^n)x^n$$

- $a_n=1-2^n+3^n$ **is the required solution.**

Solve the recurrence relation a_{n+2}-2 a_{n+1}+ a_n=2^n, $n \geq 0$ by the method of generating functions with the initial condition $a_0 = 2$ and $a_1 = 1$.

Solution:

Given a_{n+2}**-2** a_{n+1}**+** a_n**=**2^n **---------------- (1)**

Multiply (1) by x^n **and summing from** $n = 0$ to ∞,

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 2\sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

Let G(x)= $\sum_{n=0}^{\infty} a_n x^n$

$$\frac{G(x)-a_0-a_1x}{x^2}-2\frac{G(x)-a_0}{x}+G(x)=\frac{1}{1-2x}$$

$$G(x)-a_0-a_1x\text{-}2\,x\,[G(x)-a_0]+x^2\,G(x)=\frac{x^2}{1-2x}$$

$$G(x)[1\text{-}2\,x+x^2]\text{-}2\text{-}x+4\,x=\frac{x^2}{1-2x}$$

$$G(x)(1-x)^2=\frac{x^2}{1-2x}+2\text{-}3x$$

$$=\frac{x^2+(1-2x)(2-3x)}{1-2x}$$

$$G(x)=\frac{7x^2-7x+2}{(1-x)^2(1-2x)}$$

Using Partial Fraction method,

$$\frac{7x^2-7x+2}{(1-x)^2(1-2x)}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{(1-2x)}$$

$7x^2-7x+2$ = A$(1-x)(1-2x)+B\,(1-2x)$+C $(1-x)^2$----------------*

Put x=1.Then B=-2

Put x=1/2,Then C=1

Now Equating the co-eff of constants,

2=A+B+C, 2=A-2+1, Therefore A=3

Substitute A,B,C values in $*$, we get

$$\frac{7x^2-7x+2}{(1-x)^2(1-2x)} = \frac{3}{1-x} - \frac{2}{(1-x)^2} + \frac{1}{(1-2x)}$$

$$G(x)=3\,(1-x)^{-1}-2\,(1-x)^{-2}+(1-2x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3x^n - 2\sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=0}^{\infty} 2^n\, x^n$$

$$= \sum_{n=0}^{\infty}(3-2(n+1)+2^n)x^n$$

$$= \sum_{n=0}^{\infty}(1-2n+2^n)x^n$$

- $a_n = 1-2n+2^n$ *is the required solution.*

Example 4:

Solve the recurrence relation $a_n = a_{n-1} = 2n$, $n \geq 1$ by the method of generating functions with the initial condition $a_0 = 0$.

Solution:

Given $a_n - a_{n-1} = 2n, \quad n \geq 1$ (1)

Multiplying (1) by x^n and summing from $n = 1$ to ∞

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 2\sum_{n=1}^{\infty} n x^n$$

Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$G(x) - a_0 - xG(x) = 2.\frac{x}{(1-x)^2}$$

Using $a_0 = 0$, $(1-x)G(x) = \frac{2x}{(1-x)^2}$

$$\therefore \quad G(x) = \frac{2x}{(1-x)^3}$$

$$= 2.\frac{(x-1)+1}{(1-x)^3}$$

$$= \frac{2}{(1-x)^3} - \frac{2}{(1-x)^2}$$

$$\therefore \quad G(x) = 2(1-x)^{-3} - 2(1-x)^{-2}$$

$$\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n - 2\sum_{n=0}^{\infty} (n+1) x^n$$

$$a_n = (n+1)(n+2) - 2(n+1)$$

$$= (n+1)[n+2-2] = (n+1)n$$

Hence $an = n^2 + n$ is the required solution

Example : 5

Solve the recurrence relation $a_n = a_{n-1} + (n-1)$, $n \geq 2$ by the method of generating functions with the initial condition $a_1 = 0$.

Soultion

The recurrence relation is

$$a_n - a_{n-1} = (n-1), \quad n \geq 2 \qquad (1)$$

Given

$a_1 = 0$ Taking $n = 1$, $a_1 = a_0 + 0$

$\therefore$ $a_0 = a_1 = 0$

Multiplying (1) by x^n and summing from $n = 2$ to ∞

$$\sum_{n=2}^{\infty} a_n . x^n - \sum_{n=2}^{\infty} a_{n-1} x^n = \sum_{n=2}^{\infty} (n-1) x^n$$

Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$G(x) - a_0 - a_1 x - x\left[G(x) - a_0\right] = x^2 + 2x^3 + 3x^4 + \ldots\ldots..$$

$$G(x)\left[1-x\right] = x^2\left[1 + 2x + 3x^2 + \ldots\ldots\right]$$

$$= x^2 (1-x)^{-2}$$

$$G(x) = \frac{x^2}{(1-x)^3} = \frac{x^2 - 1 + 1}{(1-x)^3}$$

$$= \frac{1}{(1-x)^3} - \frac{1+x}{(1-x)^2}$$

$$G(x) = \frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{x}{(1-x)^2}$$

We know that $\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n = (1-x)^{-3}$

$\sum_{n=0}^{\infty} (n+1)x^n = (1-x)^{-2}$ and $\sum_{n=0}^{\infty} n.x^n = \frac{x}{(1-x)^2}$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[\frac{(n+1)(n+2)}{2} - (n+1) - n\right] x^n$$

$$a_n = \frac{(n+1)(n+2)}{2} - 2n - 1$$

$$= \frac{n^2+3n+2}{2} - 2n - 1$$

$$= \frac{n^2+3n+2-4n-2}{2}$$

$a_n = \frac{n^2-n}{2}$ is the required solution.

Exercies

Using generating function, solve the follwing recurrence relations.

1. $a_k - 7a_{k-1} = 0$ with $a_0 = 5$

2. $a_k = 3a_{k-1} + 2$ with $a_0 = 1$

3. $a_k = 3a_{k-1} + 4^{k-1}$ with $a_0 = 1$

4. $a_n - 5a_{n-1} + 6a_{n-2} = 0$ with $a_0 = 6$ and $a_1 = 30$

5. Using generating functions, find an explicit formula for the Fibonacci numbers.

Answers

1. $a_k = 5(7^k)$

2. $a_k = 2(3^k) - 1$

3. $a_k = 4^k$

4. $a_n = 18(3^n) - 12(2^n)$

5. $F_K = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^K - \left(\frac{1-\sqrt{5}}{2}\right)^K\right]$


Thank you