
# U19MA203- DISCRETE MATHEMATICS

## UNIT – III
## GRAPH THEORY

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# UNIT- III – GRAPH THEORY

## Course Outcomes:

Apply integrated  approach to set theory and boolean algebra provide a firm basis.

# Definitions:

## Graph

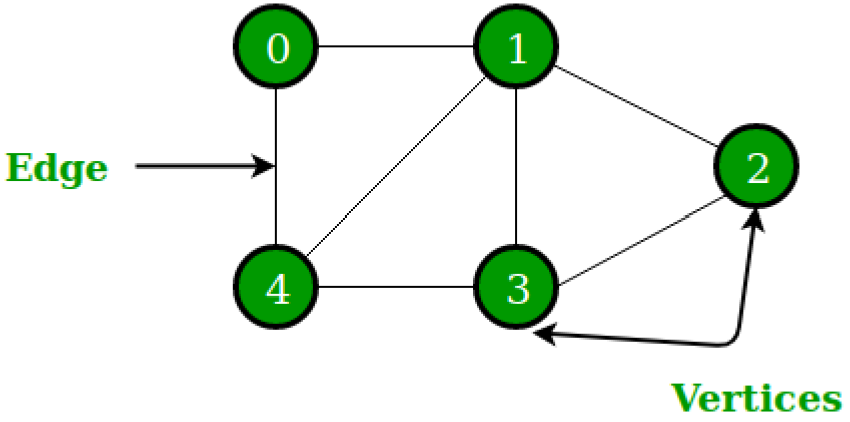
A graph is a pair of set (V, E) where V= Set of vertices and E = Set of Edges.

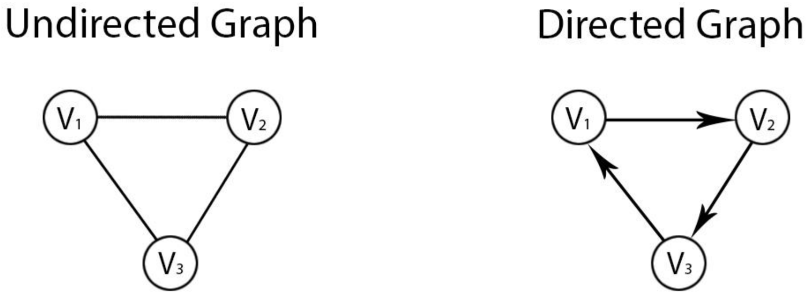
## Un directed graph

A graph in which every edges are undirected is called an undirected graph.

## Directed graph

A graph in which every edge is directed is called Digraph or directed graph.



## Self loop

Self loop is an edge from a vertex to itself.

## Parallel edges

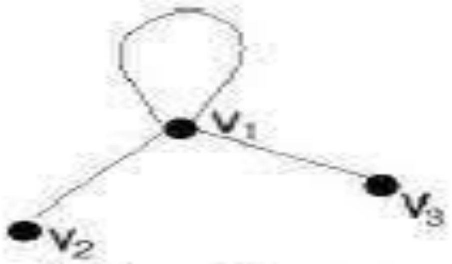
If two edges have same end points then the edges are called parallel edges.

## Incident

If the vertex $v_i$ is an end vertex of some edge $e_k$ then $e_k$ is said to be incident with $v_i$

## Adjacent edges and vertices

Two edges are said to be adjacent if they are incident on a common vertex.

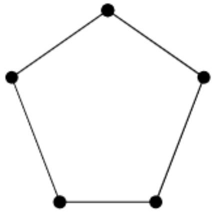



(a) self loop

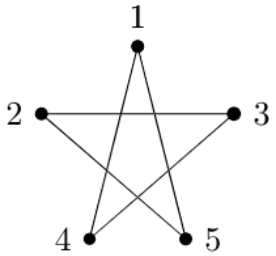

(b) parallel edges

## Self complementary graph

A graph G is said to be self complementary, if G is isomorphic to its complement.



$C_5$




Complement of $C_5$

Imagine 1 is below 4 and 5 and then flip 2 and 3. Imagine the structure, it is same as $C_5$

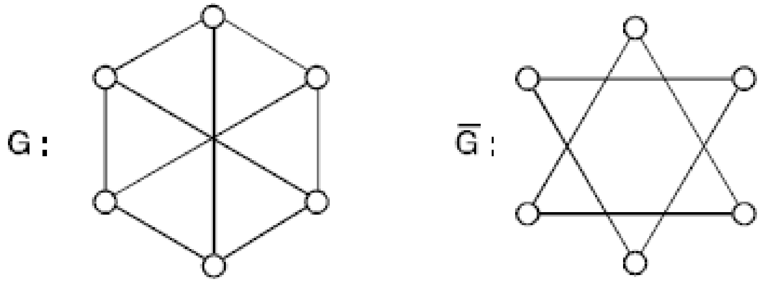
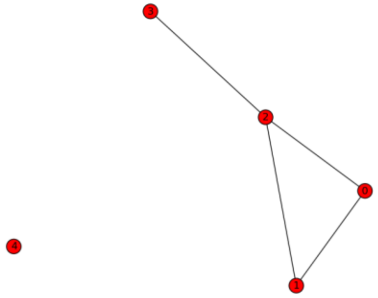



**Fig. 28. A graph and its complement**

## Isolated Vertex :

A vertex having no edge is incident on it is called an isolated vertex.

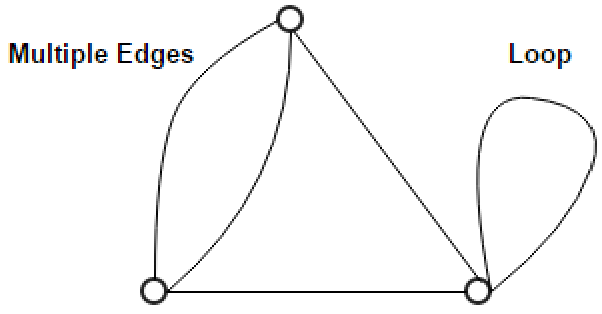


# Pendent vertex

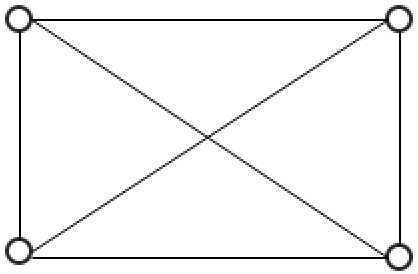
If the degeree of any vertex is one, then the vertex is called pendant vertex.

# Simple graph

A graph which has neither self loops nor parallel edges is called a simple graph.




Not a Simple Graph



Simple Graph

# Pendent vertex

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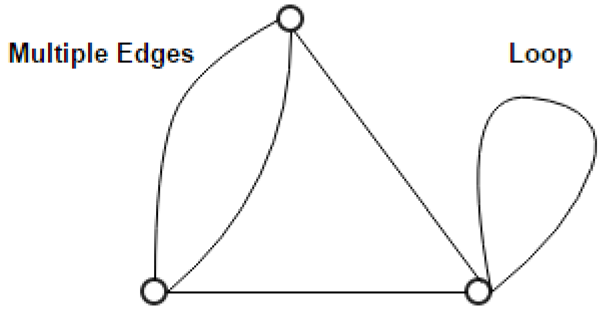
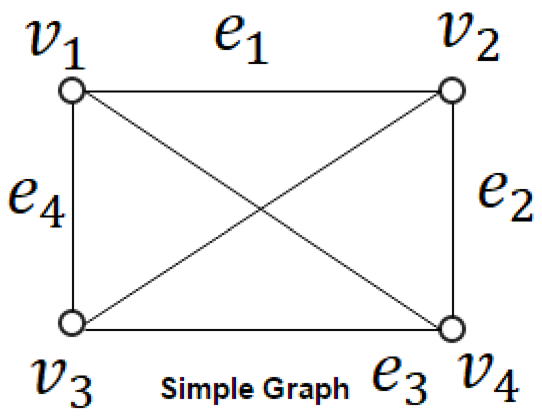
# Simple graph

A graph which has neither self loops nor parallel edges is called a simple graph.



Not a Simple Graph



Simple Graph

## Mixed graph

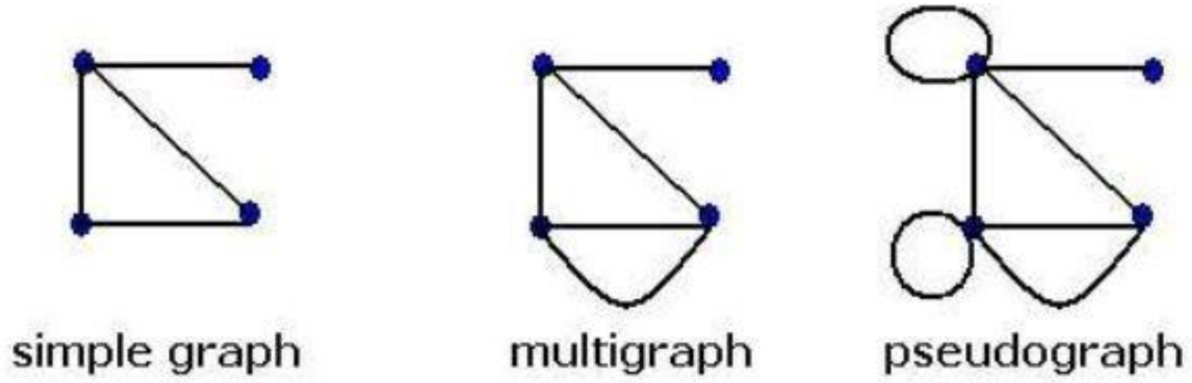
If some edges are directed and some edges are undirected, the graph is called mixed graph.

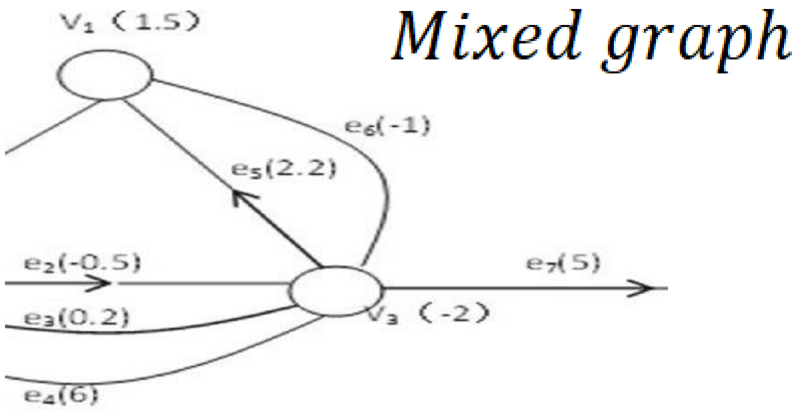
## Multi graph

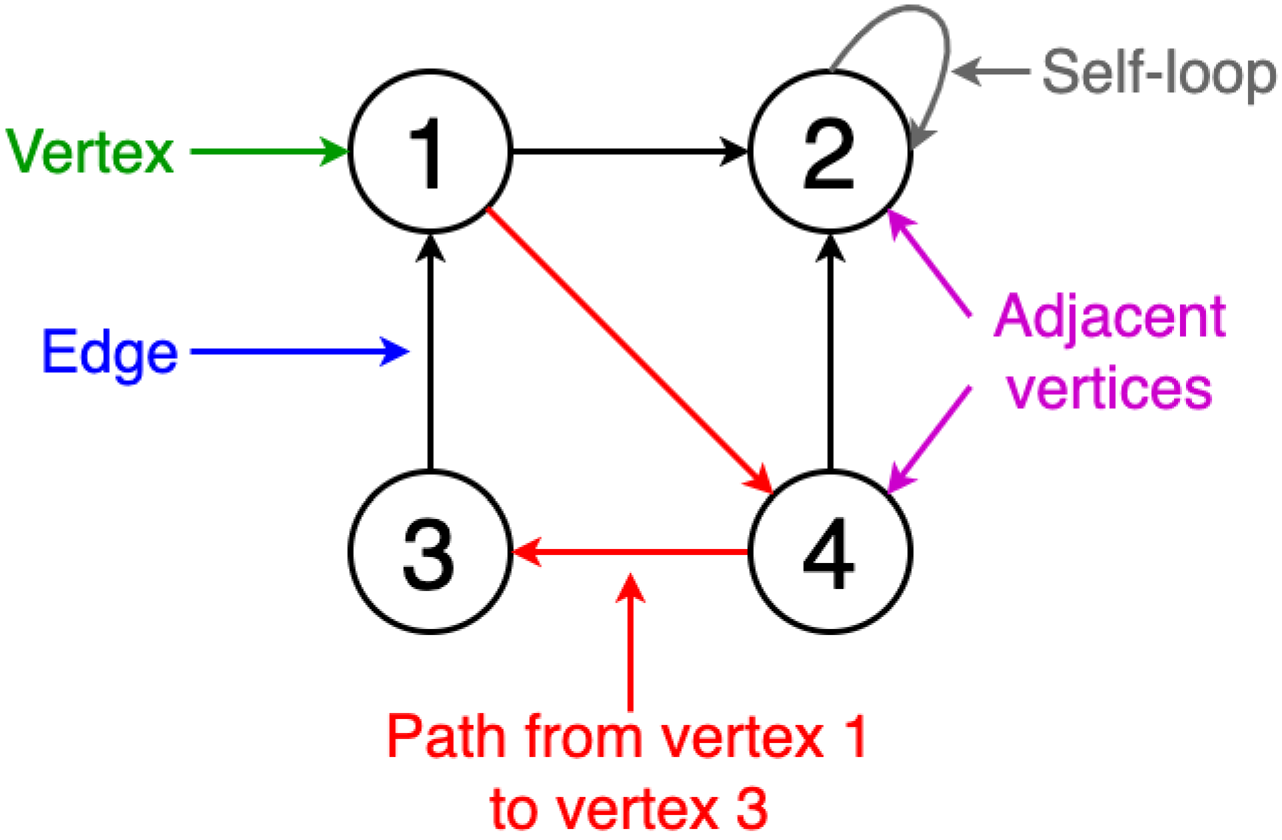
A graph which contains some parallel edges is called a multi graph.

## Psuedo graph:

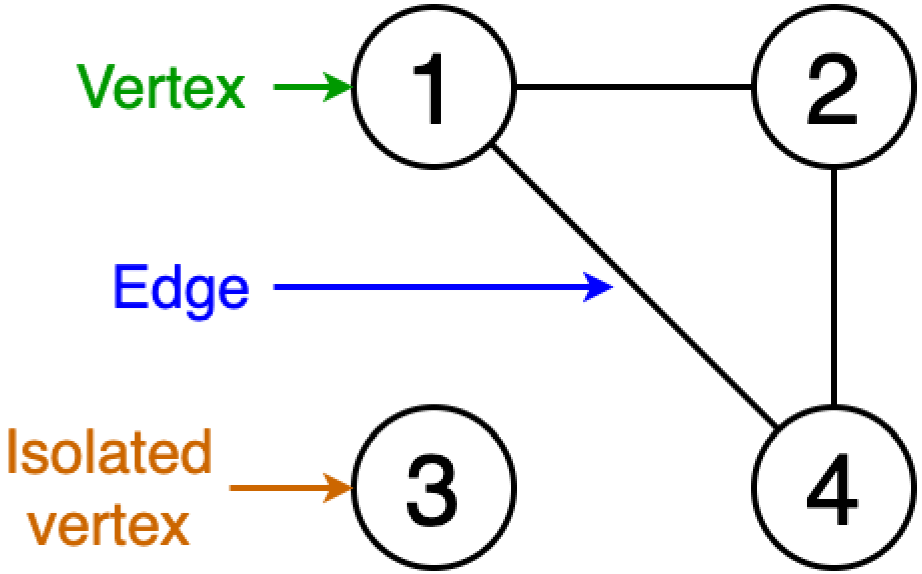
A graph in which loops and parallel edges are allowed is called psuedo graph.



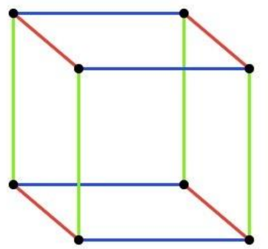






**Directed Graph**




**Undirected Graph**



# Graph Terminology:

## Degree of a vertex

The number of edges incident at the vertex $v_i$ is called degree of the vertex with self loop counted twice & it is denoted by $d(v_i)$.

## In-degree and Out-degree of a directed vertex

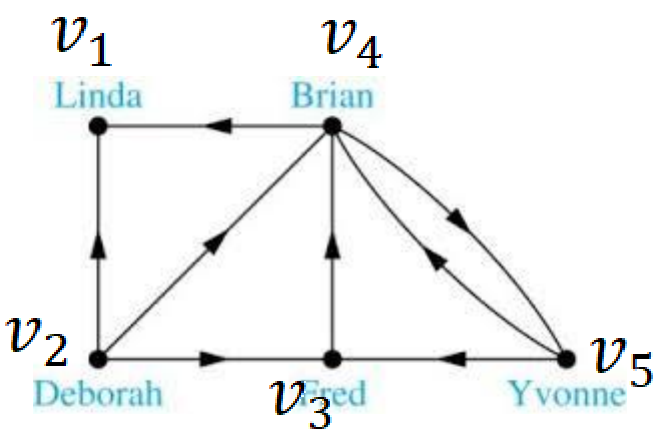
In a directed graph, the in-degree of a vertex v denoted by $deg^-(v)$ and defined by the number of edges with v as their terminal vertex.

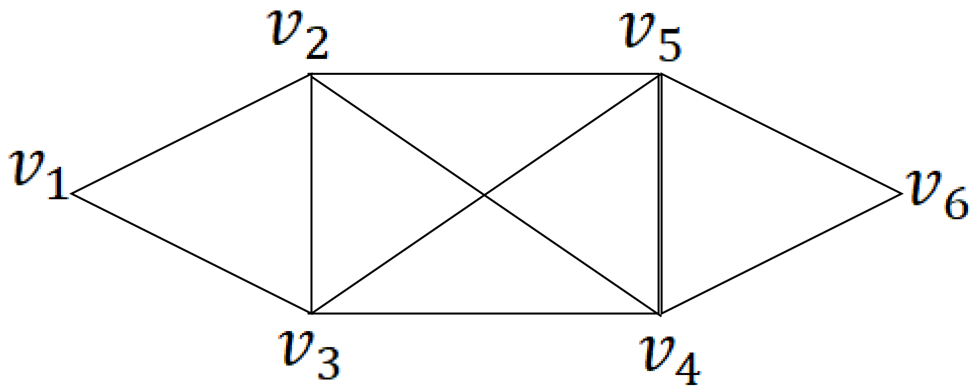
The out-degree of a vertex v denoted by $deg^+(v)$ and defined by the number of edges with v as their initial vertex.

## Note

A loop of a vertex contributes 1 to both in-degree & out-degree of this vertex.

# Find the degree of the following graphs




## For directed graph

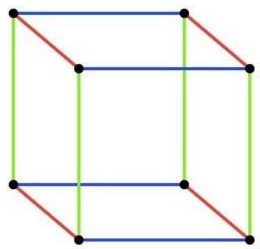
$\deg^-(v_1) = 2$
$\deg^-(v_2) = 0$
$\deg^-(v_3) = 2$
$\deg^-(v_4) = 3$
$\deg^-(v_5) = 1$

$\deg^+(v_1) = 0$
$\deg^+(v_2) = 3$
$\deg^+(v_3) = 1$
$\deg^+(v_4) = 2$
$\deg^+(v_5) = 2$

$v_1 = 2$
$v_2 = 3$
$v_3 = 3$
$v_4 = 5$
$v_5 = 3$

## For un directed graph

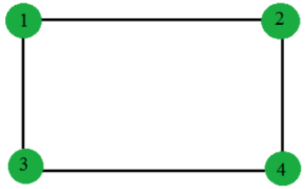
$\deg(v_1) = 2$
$\deg(v_2) = 4$
$\deg(v_3) = 4$
$\deg(v_4) = 4$
$\deg(v_5) = 4$
$\deg(v_6) = 2$



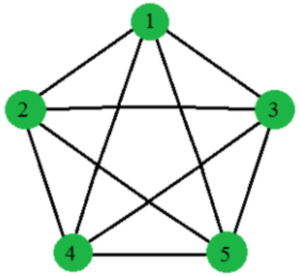
# Special types of Graphs:

## Regular graph

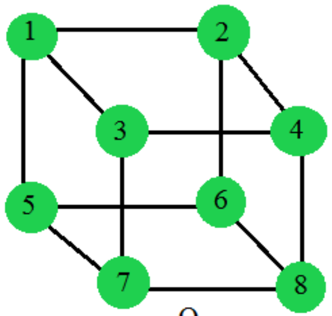
If every vertex of a simple graph has same degree, then the graph is known as regular graph.




2 Regular




4 Regular

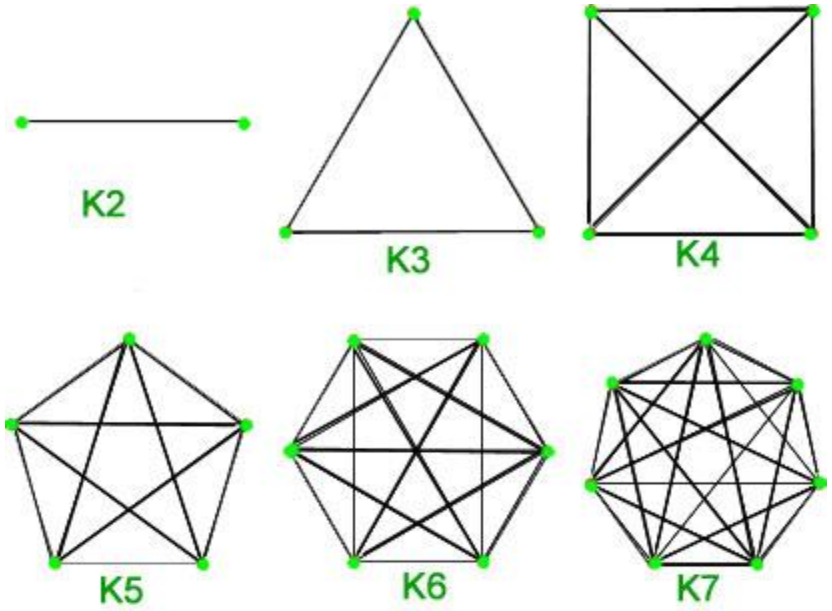



$Q_3$

## Complete graph

A simple graph in which there is exactly one edge between every pair of distinct vertex is called complete graph

**Kn – Complete graph with n vertices**

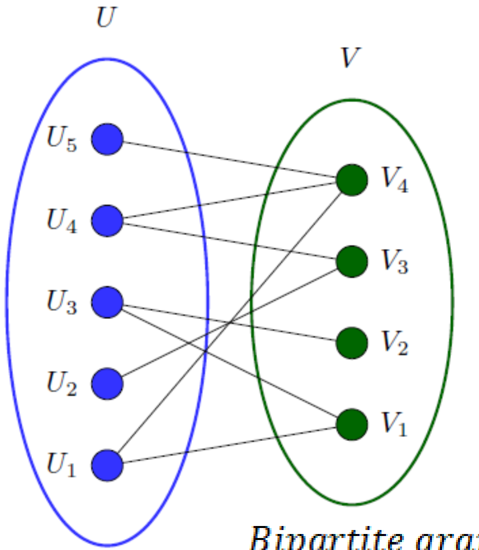


# Bipartite graph

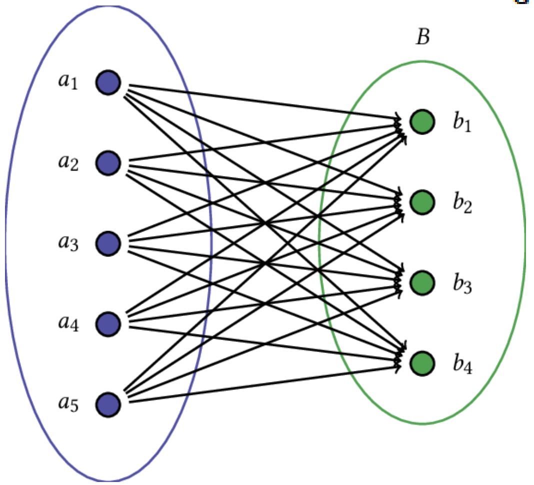
If the vertex set V of the simple graph can be partitioned into subset of $v_1$ and $v_2$ such that every edge of G connects a vertex in $v_1$ and a vertex in $v_2$. Then the graph is called as bipartite graph.

# Complete bipartite graph

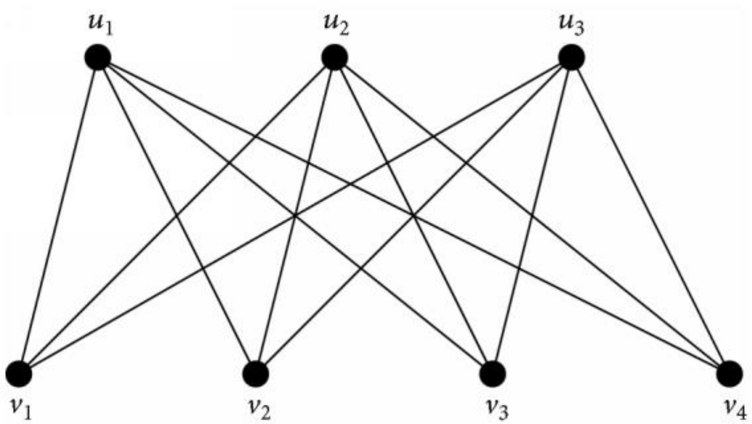
In a bipartite graph, if every vertex of $v_1$ is connected with every vertex of $v_2$ by an edge then the graph is called as completely bipartite graph.




*Bipartite graph*

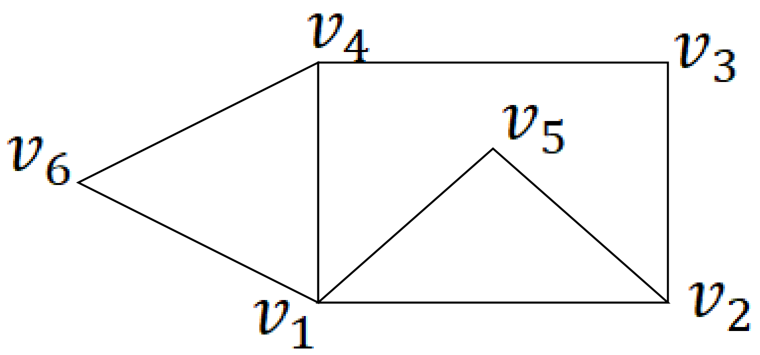



*Complete bipartite graph*

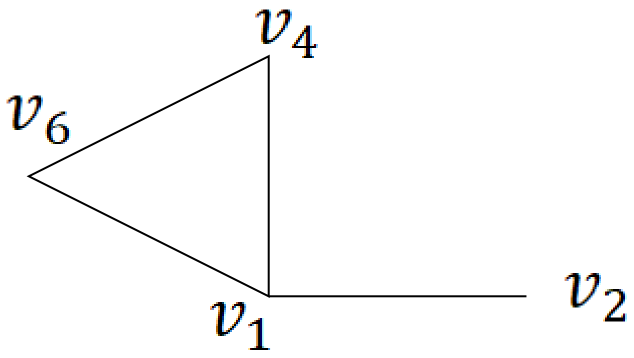


# Sub graph

A graph H =(V',E') is said to be sub graph of G =(V,E) if V' $\subseteq$ V and E' $\subseteq$ E.



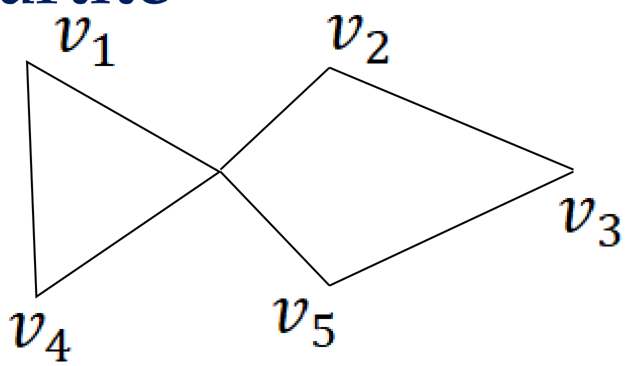

The sub graphs are



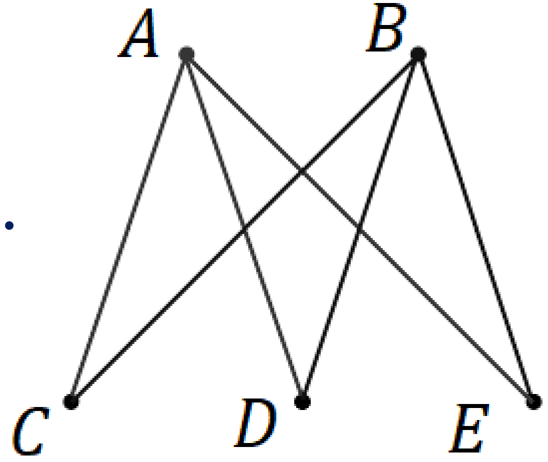

*Subgraph*

# Check whether the following graphs are bipartite or completely Bipartite

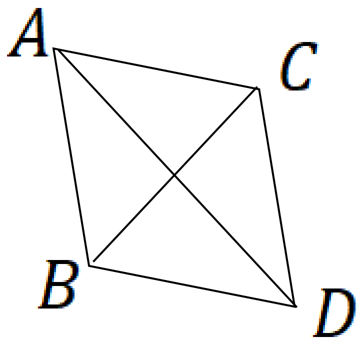
1.



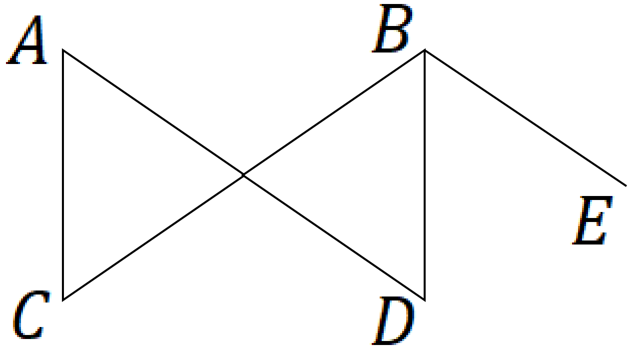

2.

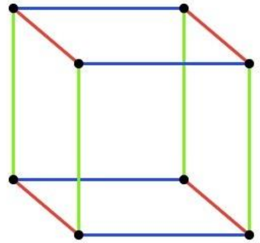



3.




4.





## Solution:

1. $A = \{v_1, v_2, v_3, v_5\}$ and $B = \{v_4, v_5, v_3\}$

   $A \cap B = \{v_3 \; v_5\} \neq \varphi$. Therefore G is not bipartite.

2. $v_1 = \{A, B\}$ and $v_2 = \{C, D, E\}$

   $v_1 \cap v_2 = \varphi$. Therefore G is bipartite. Also every vertex in $v_1$ is connected with ever y vertex in $v_2$ . Hence Completely bipartite.

3. $v_1 = \{A, B, D\}$ and $v_2 = \{B, C, D\}$

   $v_1 \cap v_2 = \{B, D\} \neq \varphi$. Therefore G is not bipartite.

4. $v_1 = \{A, B\}$ and $v_2 = \{C, D, E\}$

   $v_1 \cap v_2 = \varphi$. Therefore G is bipartite. Also every vertex in $v_1$ is connected with every vertex in $v_2$ . Hence it is not completely bipartite

# Self Complementary graph

A graph G and its complement G′ are said to be self complementary If there exists an isomorphism between them.

## Matrix representation of Graph

## Adjacency Matrix:

Let G be a simple graph with n vertices then the adjacency matrix is given by

$$A_{nXn} = [a_{ij}] = \begin{cases} 1, & \textit{if there is an edge between } v_1, v_2 \\ 0, & ow \end{cases}$$

Note:

For simple graph : Entries are 0 and 1

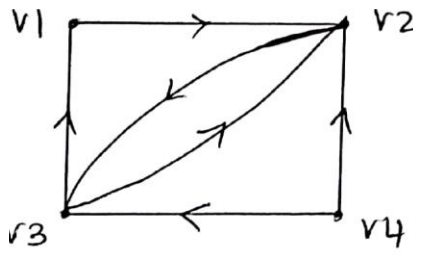
For Pseudo graph : Entries are >1

Undirected graph : Symmetric matrix

Directed graph : Non symmetric matrix.

# Find the adjacency matrix of G and write the degree.

(i)




Soln:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

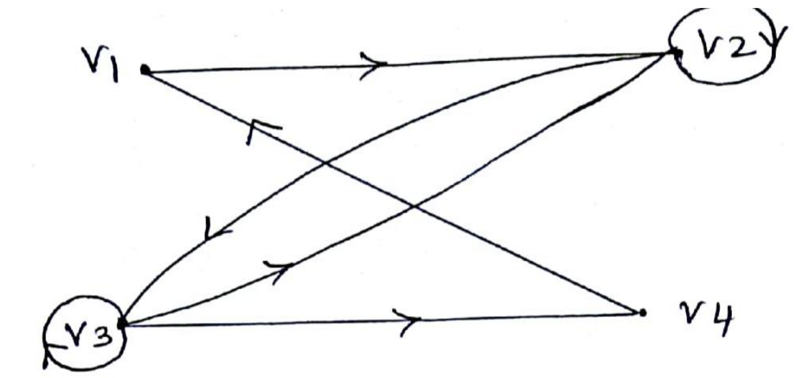
$d(v_1) = 1$

$d(v_2) = 1$

$d(v_3) = 2$

$d(v_4) = 2$

(ii)




Soln:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

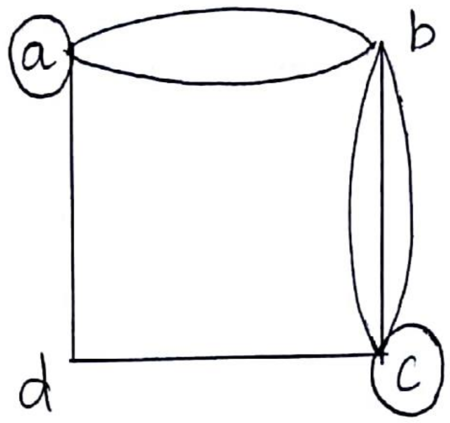
$d(v_1) = 1$

$d(v_2) = 2$

$d(v_3) = 3$

$d(v_4) = 1.$

(iii)




Soln:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$d(v_1) = 4$
$d(v_2) = 5$
$d(v_3) = 5$
$d(v_4) = 2$

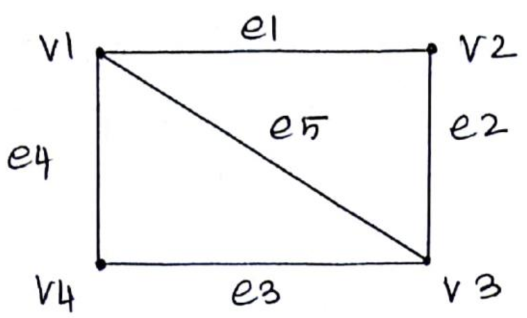
## Incidence Matrix of an undirected graph

If G is a undirected graph with n vertices and m edges then the incidence matrix A is nxm matrix defined by

$$A = \begin{cases} 1, & \text{When edge } e_j \text{ is incident on } v_i \\ 0, & \text{otherwise} \end{cases}$$

## Problems:

## (i) Write the incident matrix of the following graph

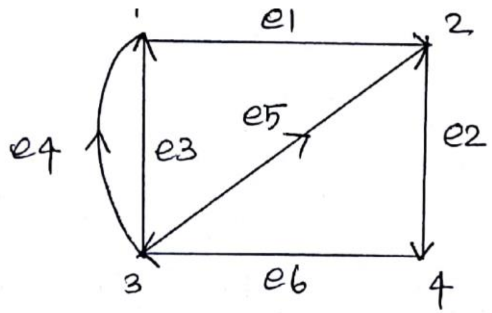



Soln:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

### Incident matrix of a directed graph

$$A = \begin{cases} 1, & \text{If edge } e_j \text{ is directed from } v_i \\ -1, & \text{If edge } e_j \text{ is directed to } v_i \\ 0, & \text{Otherwise} \end{cases}$$



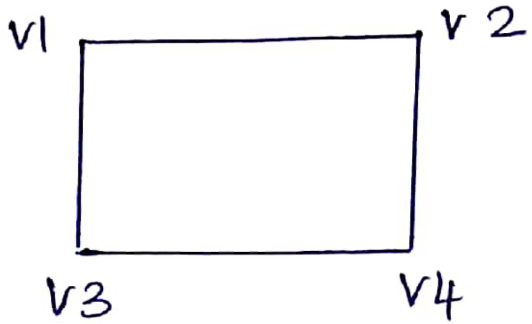

Soln:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} e1 & e2 & e3 & e4 & e5 & e6 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cancel{0}\,-1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

1. Draw the graph with the adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

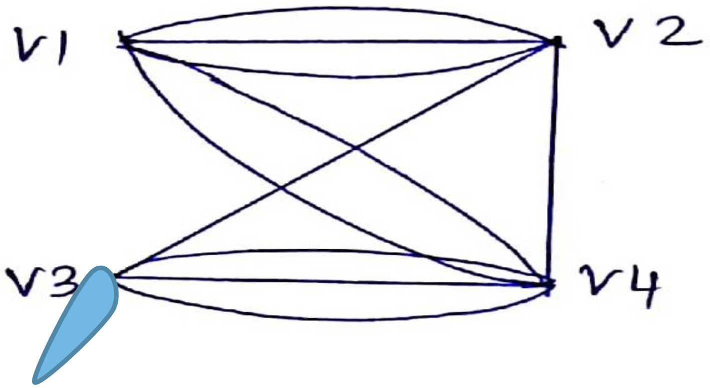
Soln:



2. Draw a pseudo graph with adjacency matrix

$$\begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

Soln :

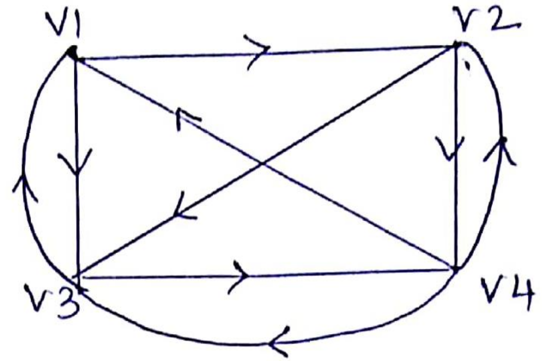



3. Write the adjacency matrix of the digraph G={($v_1$,$v_3$), ($v_1$,$v_2$), ($v_2$,$v_4$) , ($v_3$,$v_1$), ($v_2$,$v_3$) ,($v_3$, $v_4$), ($v_4$ ,$v_1$), ($v_4$,$v_2$),($v_4$,$v_3$)} and draw the graph

Soln :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

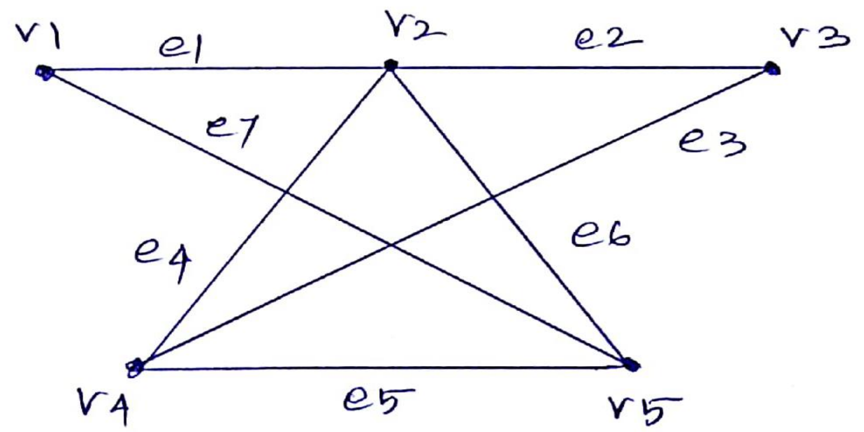
Graph :



## 4. Draw the graph G whose incidence matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

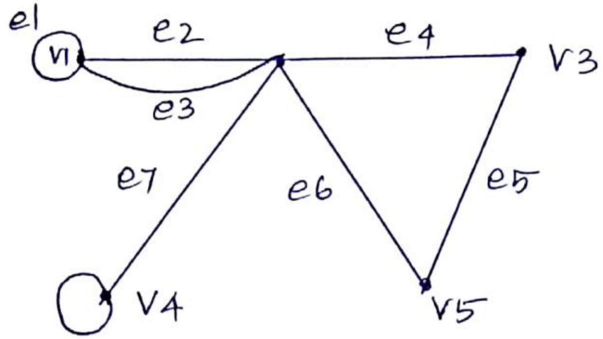
Soln:




## 5. Draw the Pseudo graph with the incidence matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Soln:

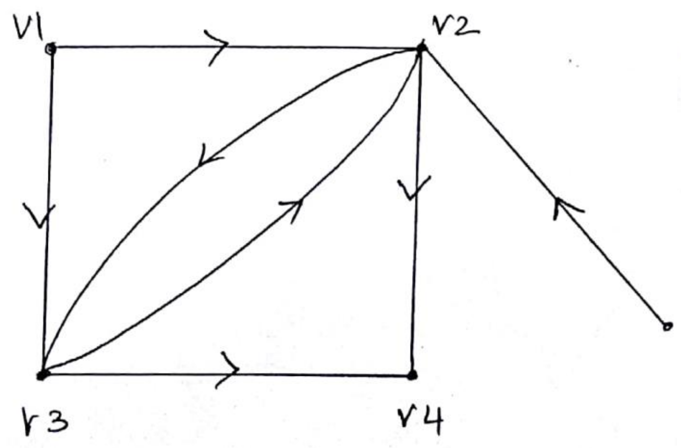


# Path matrix

If G is a graph with ‘n’ vertices then the path matrix p is defined as

$$P_{ij} = \begin{cases} 1 & \text{If there exists a path from } v_i \text{ to } v_j \\ 0 & \text{Other wise} \end{cases}$$

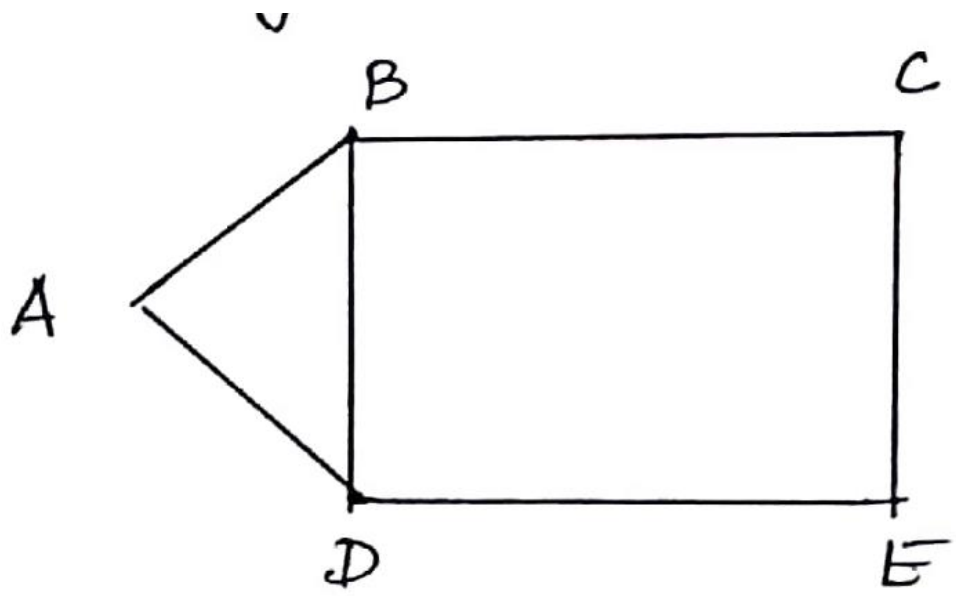
1. Find the path matrix of




Soln:

$$P = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Find all the simple paths from A to E and all the cycles with respect to vertex A of the given graph.




Soln: Simple paths : (i) A → B → C → E

(ii) A → B → D → E

(iii) A → D → E

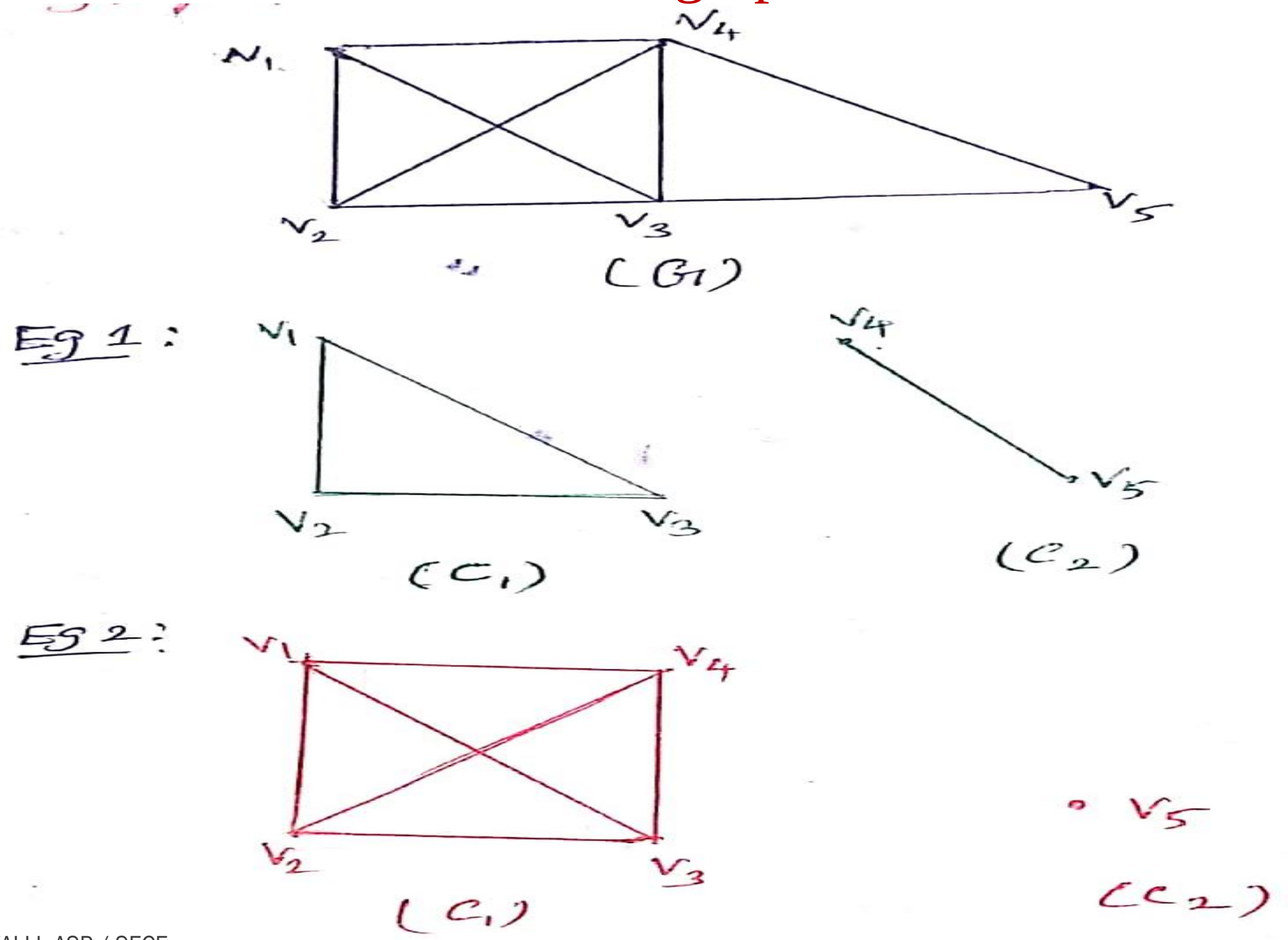
(IV) A → D → B → C → E

The cycles are (i) A → B → C → E → D → A

(ii) A → D → E → C → B → A

**CONNECTED GRAPH:** A graph G is said to be **connected** if every pair of vertices are joined by path.

Otherwise, it is called disconnected graph



## Isomorphism

Two graphs $G_1$ and $G_2$ are said to be isomorphic if there exists one to one corresponded between the vertex sets which preserves adjacency of vertices.

## Method 1:

Two graphs are isomorphic iff their vertices can be labled in such a way that the corresponding adjacency matrices are equal.

## Isomorphism

Two graphs $G_1$ and $G_2$ are said to be isomorphic if there exists one to one corresponded between the vertex sets which preserves adjacency of vertices.

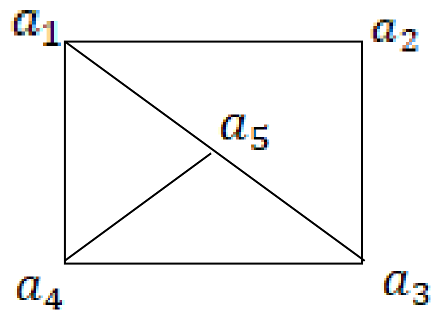
## Method 1:

Two graphs are isomorphic iff their vertices can be labled in such a way that the corresponding adjacency matrices are equal.

1. Check whether the following graphs are isomorphic.

Step 1:

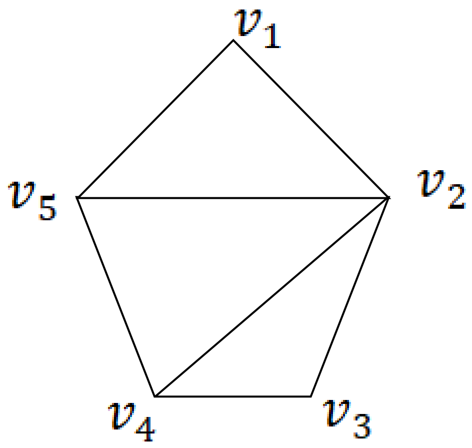
| In Graph 1 | In Graph 2 |
|---|---|
| No. of Vertices : 5 | No. of Vertices : 5 |
| No. of Edges : 7 | No. of Edges : 7 |




$G_1$

Step 2:

| In Graph 1 | In Graph 2 |
|---|---|
| deg $a_1$ = 3 | deg $v_1$ = 2 |
| deg $a_2$ = 2 | deg $v_2$ = 4 |
| deg $a_3$ = 3 | deg $v_3$ = 2 |
| deg $a_4$ = 3 | deg $v_4$ = 3 |
| deg $a_5$ = 3 | deg $v_5$ = 3 |




$G_2$

Equal number of vertices does not have equal degree. Therefore $G_1$ and $G_2$ are not isomorphic to each other.

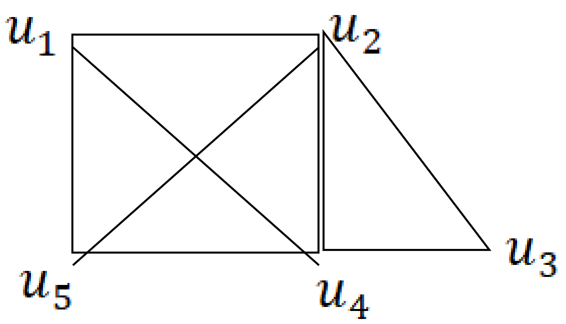
2. Check whether the following graphs are isomorphic.

Step 1:

In Graph 1 — In Graph 2

No. of Vertices : 5 — No. of Vertices : 5

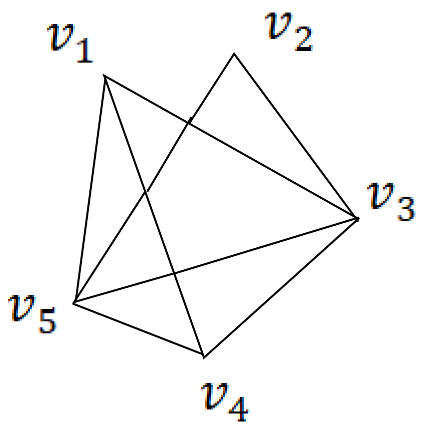
No. of Edges : 8 — No. of Edges : 8




$G_1$

Step 2:

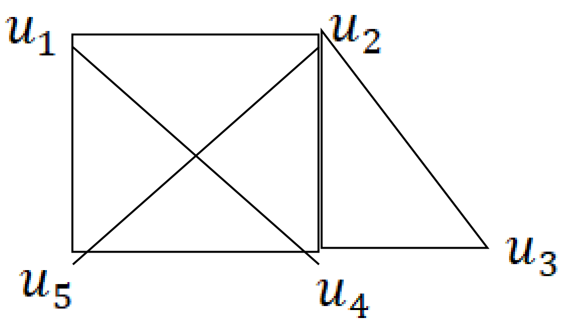
| In Graph 1 | In Graph 2 |
|---|---|
| deg $u_1 = 3$ | deg $v_1 = 3$ |
| deg $u_2 = 4$ | deg $v_2 = 2$ |
| deg $u_3 = 2$ | deg $v_3 = 4$ |
| deg $u_4 = 4$ | deg $v_4 = 3$ |
| deg $u_5 = 3$ | deg $v_5 = 4$ |




$G_2$

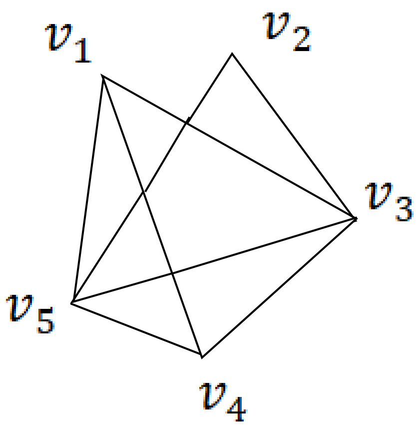
Equal number of vertices have equal degree.

$$
A(G_1) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$




$G_1$

$$
A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$




$G_2$

$A(G_1) \neq A(G_2)$

So we can alter $A(G_2)$

**$G_1$**

$$u_1 = \left\{\begin{matrix} u_2 \\ u_4 \\ u_5 \end{matrix}\right. \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} \qquad u_5 = \left\{\begin{matrix} u_1 \\ u_2 \\ u_4 \end{matrix}\right. \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

$$u_2 = \left\{\begin{matrix} u_1 \\ u_3 \\ u_4 \\ u_5 \end{matrix}\right. \begin{pmatrix} 3 \\ 2 \\ 4 \\ 3 \end{pmatrix} \qquad u_3 = \left\{\begin{matrix} u_2 \\ u_4 \end{matrix}\right. \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

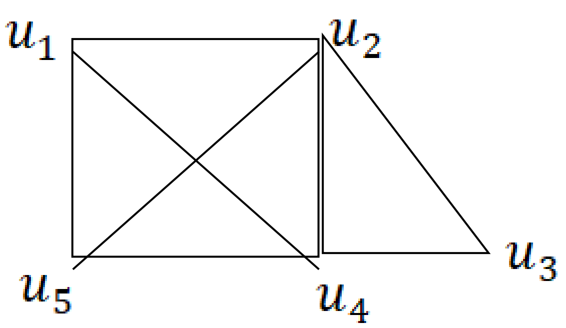
$$u_4 = \left\{\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_5 \end{matrix}\right. \begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \end{pmatrix}$$

**$G_2$**

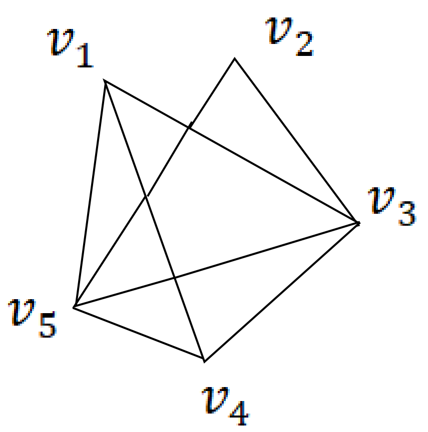
$$v_1 = \left\{\begin{matrix} v_3 \\ v_4 \\ v_5 \end{matrix}\right. \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} \qquad v_4 = \left\{\begin{matrix} v_1 \\ v_3 \\ v_5 \end{matrix}\right. \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

$$v_2 = \left\{\begin{matrix} v_3 \\ v_5 \end{matrix}\right. \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad v_5 = \left\{\begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix}\right. \begin{pmatrix} 3 \\ 2 \\ 4 \\ 3 \end{pmatrix}$$

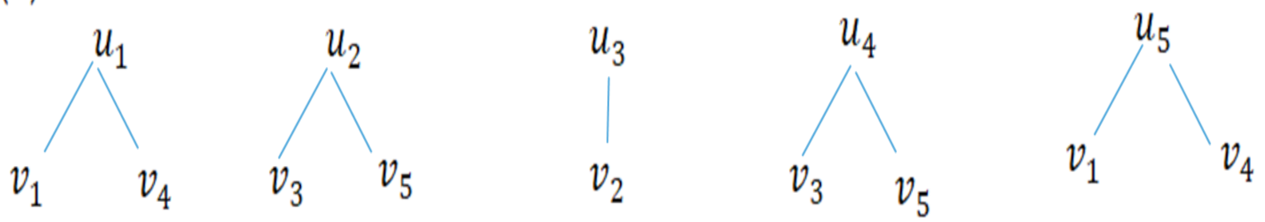
$$v_3 = \left\{\begin{matrix} v_1 \\ v_2 \\ v_4 \\ v_5 \end{matrix}\right. \begin{pmatrix} 3 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

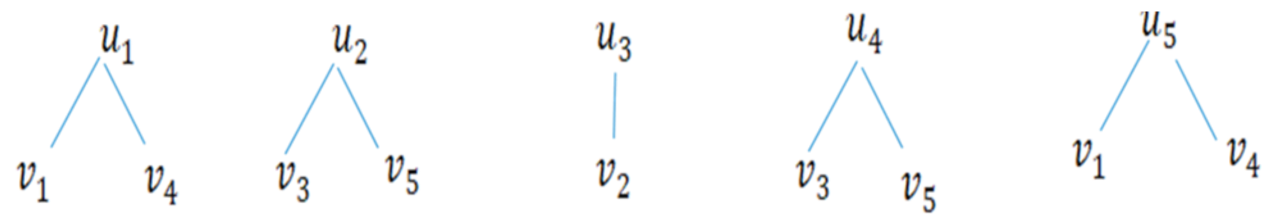



$G_1$




$G_2$






$u_1 \longrightarrow v_1$

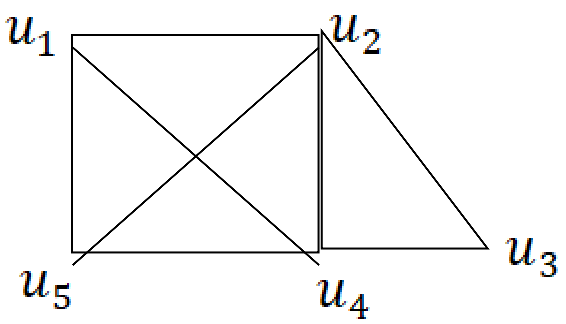
$u_2 \longrightarrow v_3$

$u_3 \longrightarrow v_2$

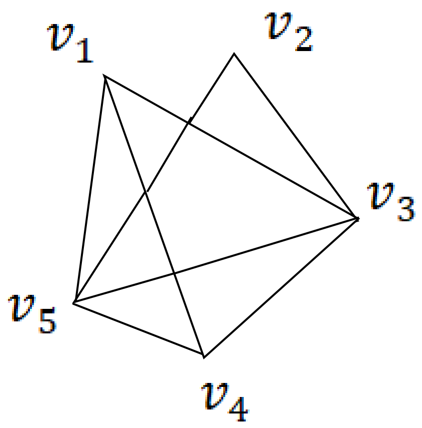
$u_4 \longrightarrow v_5$

$u_5 \longrightarrow v_4$

$$A(G_2) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$




$G_1$




$G_2$

$$A(G_1) = A(G_2) \qquad \therefore\ G_1 \cong G_2$$

$$u_1 \longrightarrow v_1$$
$$u_2 \longrightarrow v_3$$
$$u_3 \longrightarrow v_2$$
$$u_4 \longrightarrow v_5$$
$$u_5 \longrightarrow v_4$$

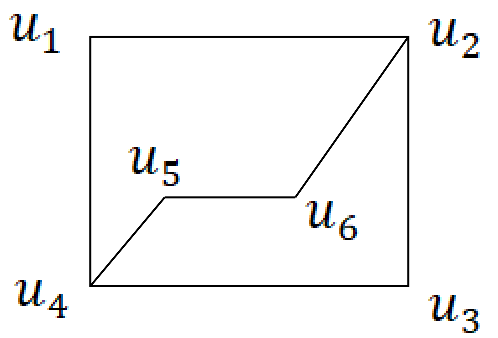
$$A(G_2) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A(G_1) = A(G_2) \qquad \therefore\ G_1 \cong G_2$$

3. Check whether the following graphs are isomorphic.

Step 1:

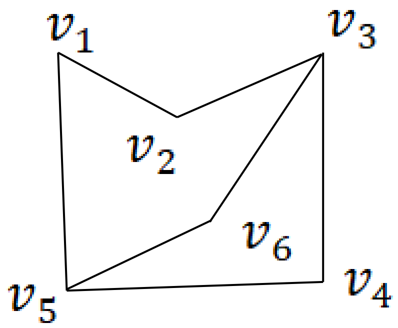
| In Graph 1 | In Graph 2 |
|---|---|
| No. of Vertices : 6 | No. of Vertices : 6 |
| No. of Edges : 7 | No. of Edges : 7 |




$G_1$

Step 2:

| In Graph 1 | In Graph 2 |
|---|---|
| deg $u_1 = 2$ | deg $v_1 = 2$ |
| deg $u_2 = 3$ | deg $v_2 = 2$ |
| deg $u_3 = 2$ | deg $v_3 = 3$ |
| deg $u_4 = 3$ | deg $v_4 = 2$ |
| deg $u_5 = 2$ | deg $v_5 = 3$ |
| deg $u_6 = 2$ | deg $v_6 = 2$ |




$G_2$

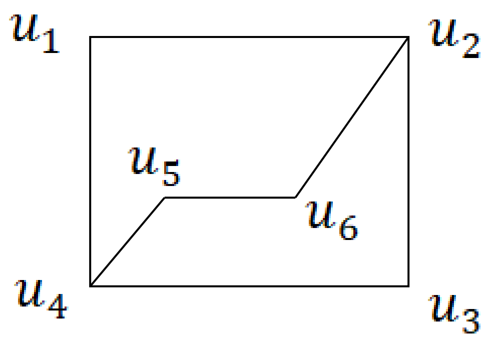
Equal number of vertices have equal degree.

STEP - 3 :

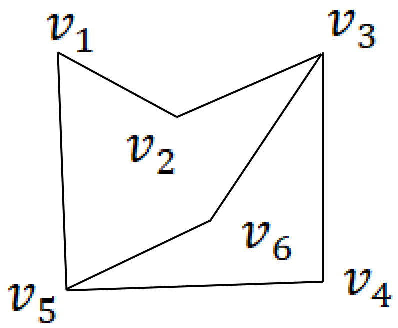
$$A(G_1) = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

$$A(G_1) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$




$G_1$



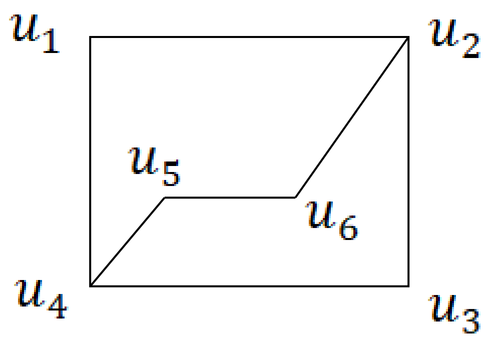

$G_2$

## $G_1$          $G_2$

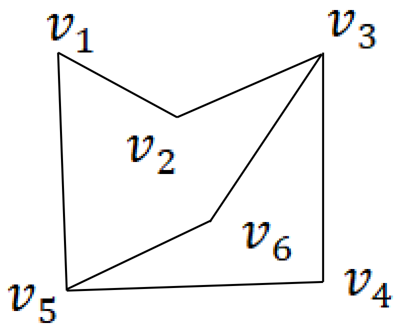
$$u_1 = \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad u_4 = \begin{Bmatrix} u_1 \\ u_3 \\ u_5 \end{Bmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad v_1 = \begin{Bmatrix} v_2 \\ v_5 \end{Bmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad v_4 = \begin{Bmatrix} v_3 \\ v_5 \end{Bmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$u_2 = \begin{Bmatrix} u_1 \\ u_3 \\ u_6 \end{Bmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad u_5 = \begin{Bmatrix} u_1 \\ u_2 \\ u_4 \end{Bmatrix} \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} \qquad v_2 = \begin{Bmatrix} v_1 \\ v_3 \end{Bmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad v_5 = \begin{Bmatrix} v_1 \\ v_4 \\ v_6 \end{Bmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

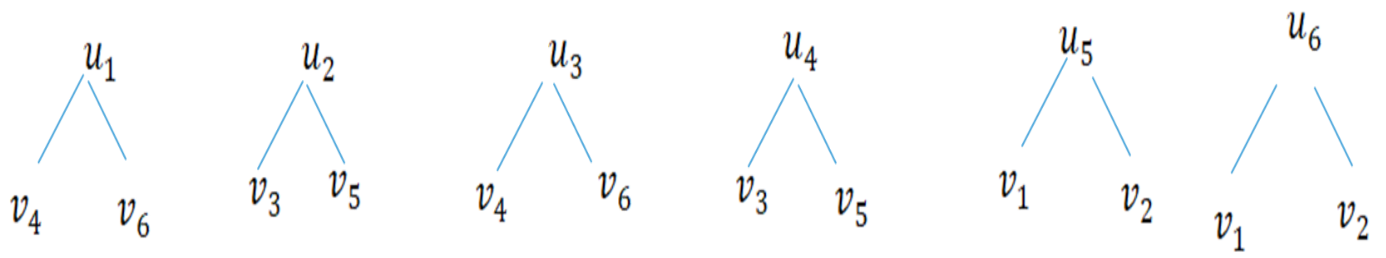
$$u_3 = \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad u_6 = \begin{Bmatrix} u_2 \\ u_5 \end{Bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad v_3 = \begin{Bmatrix} v_2 \\ v_4 \\ v_6 \end{Bmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad v_6 = \begin{Bmatrix} v_3 \\ v_5 \end{Bmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

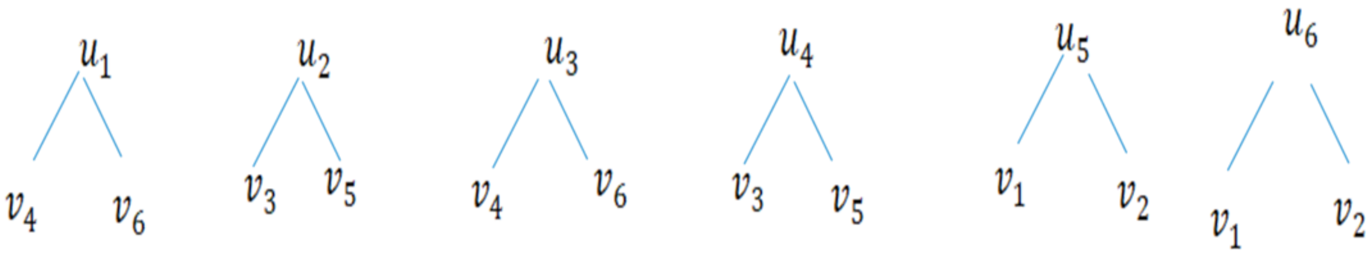



$G_1$




$G_2$






$u_1 \longrightarrow v_4$

$u_2 \longrightarrow v_3$

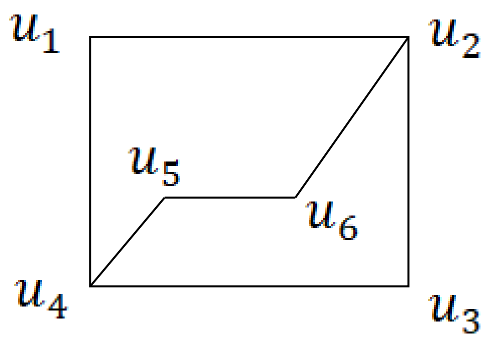
$u_3 \longrightarrow v_6$

$u_4 \longrightarrow v_5$

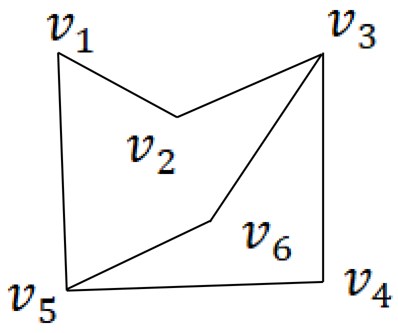
$u_5 \longrightarrow v_1$

$u_6 \longrightarrow v_2$

$$A(G_2) = \begin{array}{c} \\ v_4 \\ v_3 \\ v_6 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_4 & v_3 & v_6 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$




$G_1$




$G_2$

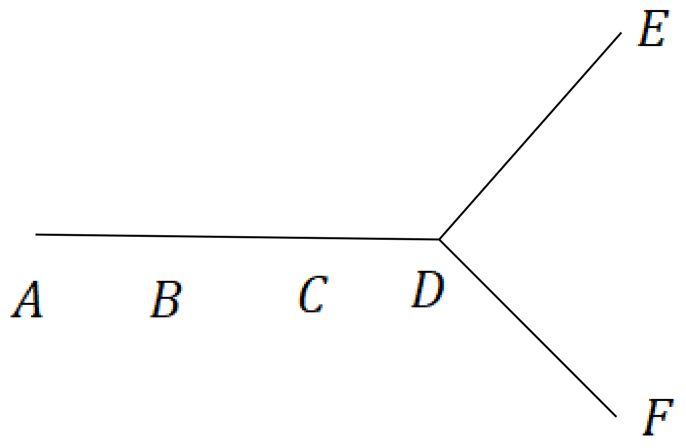
$$A(G_1) = A(G_2)$$

$$\therefore \; G_1 \cong G_2$$

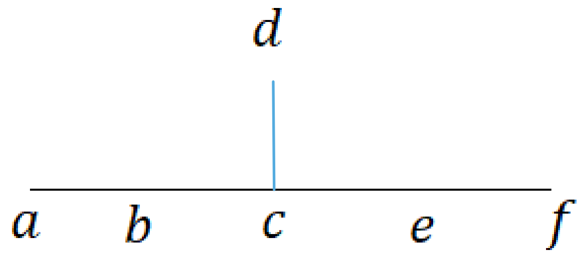
4. Check whether the following graphs are isomorphic.

Step 1:

| In Graph 1 | In Graph 2 |
|---|---|
| No. of Vertices : 6 | No. of Vertices : 6 |
| No. of Edges : 5 | No. of Edges : 5 |



$G_1$



$G_2$

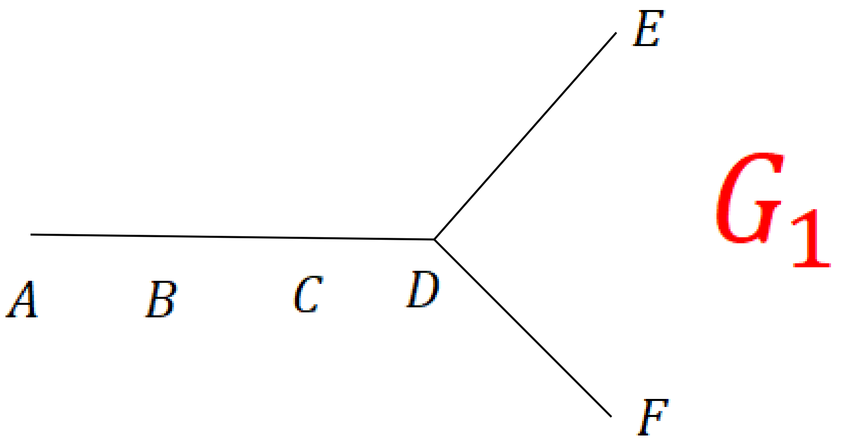
Step 2:

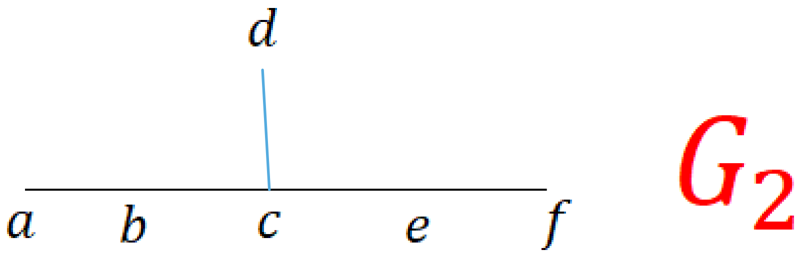
| In Graph 1 | In Graph 2 |
|---|---|
| deg A = 1 | deg a = 1 |
| deg B = 2 | deg b = 2 |
| deg C = 2 | deg c = 3 |
| deg D = 3 | deg d = 1 |
| deg E = 1 | deg e = 1 |
| deg F = 1 | deg f = 1 |

Equal number of vertices have equal degree.

STEP-3 :

$$A(G_1) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \quad \& \quad A(G_2) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



A – B (2)

B < A, C $\begin{pmatrix}1\\2\end{pmatrix}$

C < B, D $\begin{pmatrix}2\\3\end{pmatrix}$

D < C, E, F $\begin{pmatrix}2\\1\\1\end{pmatrix}$

E – D (3)

F – D (3)

a – b (2)

b < a, c $\begin{pmatrix}1\\3\end{pmatrix}$

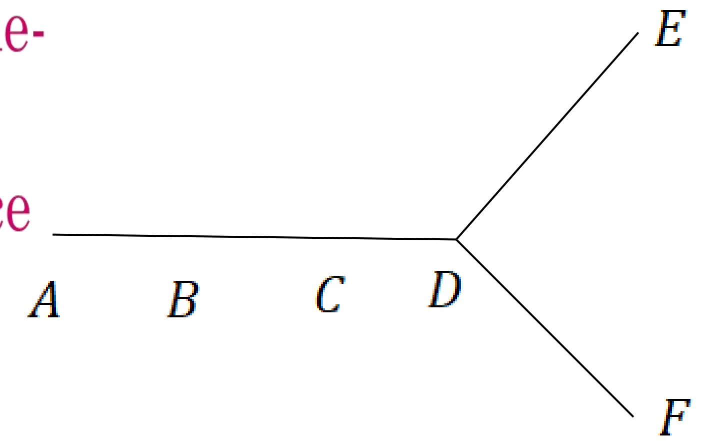
c < b, d, e $\begin{pmatrix}2\\1\\2\end{pmatrix}$

d – c (3)

e < c, f $\begin{pmatrix}3\\1\end{pmatrix}$

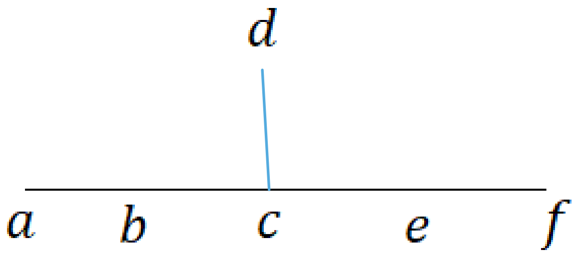
f – e (2)

There is no one-one correspondence between the vertices.




$G_1$

$$G_1 \ncong G_2$$




$G_2$

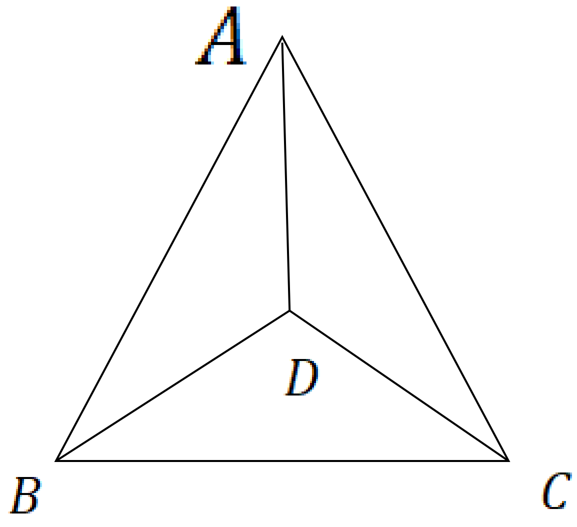
## Method 2:

## Isomorphism by circuits:

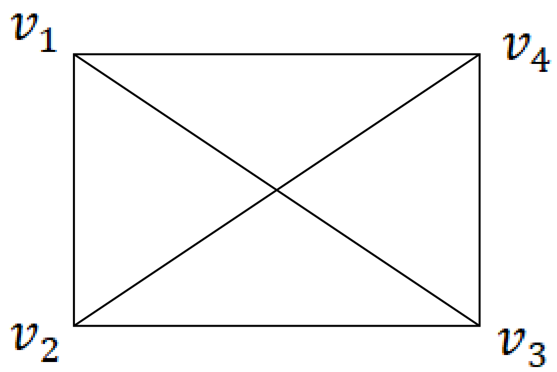
Two graphs are isomorphic if they contain same number of circuits of length k > 2 .

1. Determine whether the following graphs are isomorphic using circuit method.

$G_1$




$G_2$



Step 1:

In Graph 1

No. of Vertices : 4

No. of Edges : 6

In Graph 2

No. of Vertices : 4

No. of Edges : 6

Step 2:

In Graph 1

deg A = 3

deg B = 3

deg C = 3

deg D = 3

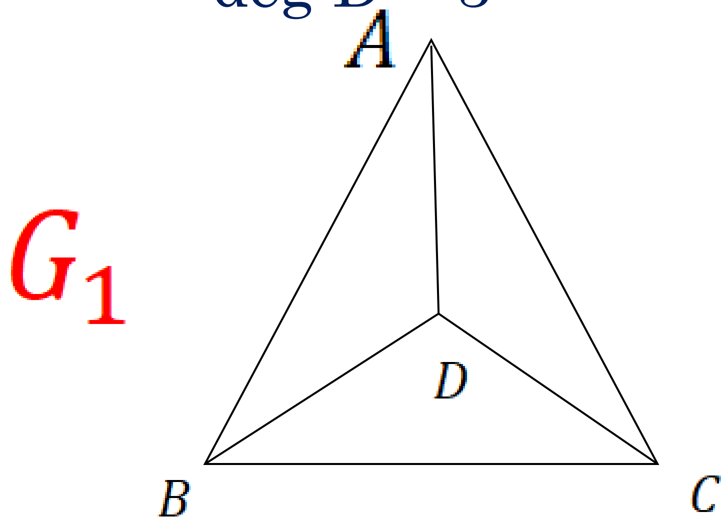
In Graph 2

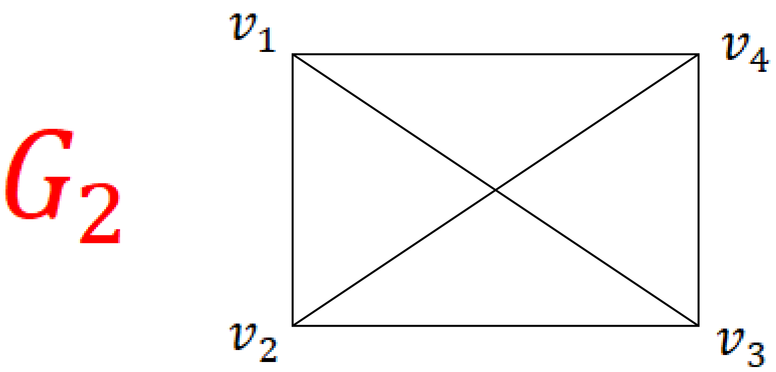
deg $v_1$ = 3

deg $v_2$ = 3

deg $v_3$ = 3

deg $v_4$ = 3. Equal number of vertices have equal degree. Hence, given two graphs are isomorphic

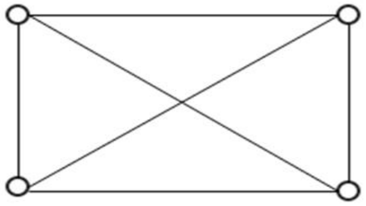


# Module-III
# Graph Theory

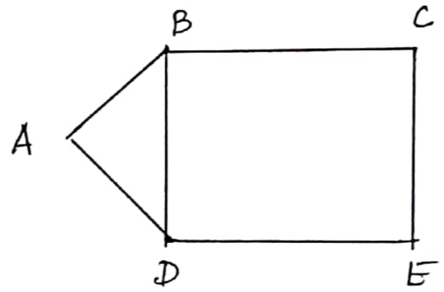
## *GRAPH THEOREMS*

## THEOREM 1:-HAND SHAKING THEOREM:



### Statement:

The sum of the degree of the vertices of an undirected graph G is twice the number of edge in G. i.e,$\sum_{i=1}^{n} \deg(v_i) = 2|E|$.




### Proof:

Each edge of G is incident with two vertices and hence contributes two to the sum of degree of all the vertices of the undirected graph G.

∴ the sum of degree of all the vertices in G is twice the number of edge in G

i.e,$\sum_{i=1}^{n} \deg(v_i) = 2|E|$ .

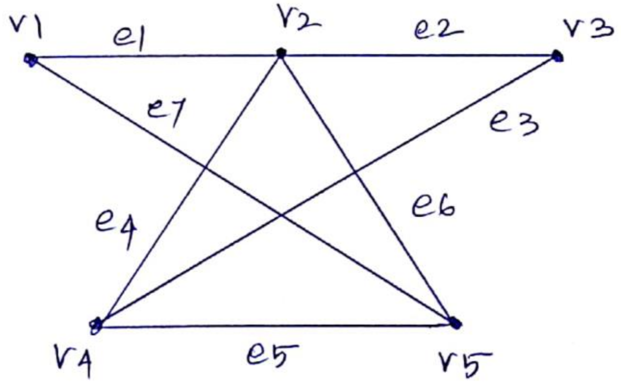
**Hence the proof**

## Theorem 2:

In an undirected graph G, the number of vertices of odd degree is even.

**Proof:**

Let G = (V, E) the vertex set V is divided into 2 sets O and E where O is set of odd degree vertices and E is set of even degree vertices.




Then $\sum_{i \in O} \deg(v_i) + \sum_{j \in E} \deg(v_j) = \sum_{i=1}^{n} \deg(v_i)$

By Handshaking Theorem,

$$\sum_{i \in O} \deg(v_i) + \sum_{j \in E} \deg(v_j) = 2|E|$$

$$\sum_{i \in O} \deg(v_i) = 2|E| - \sum_{j \in E} \deg(v_j)$$

$$= 2|E| - \text{Even number}$$

(since sum of even number is even)

Since each degree of $v_i$ is odd, the number of terms contained in

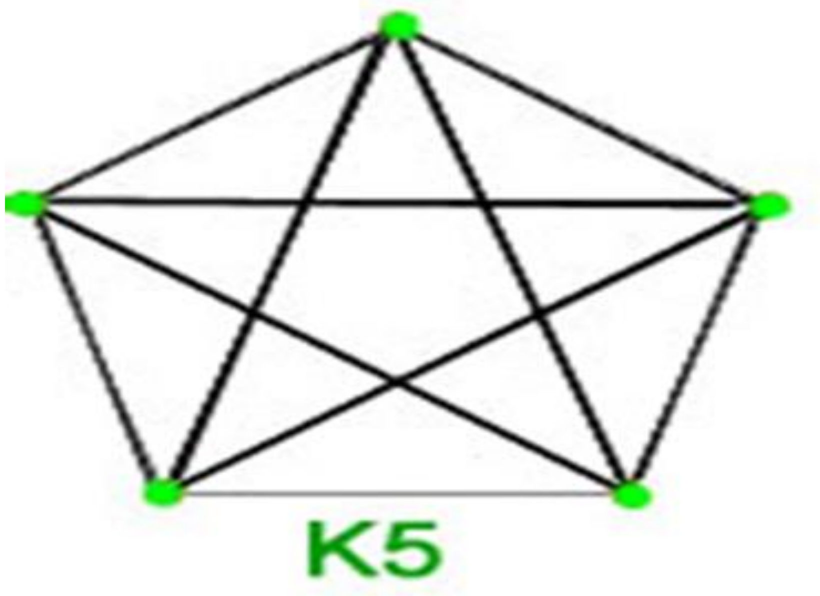
$\sum_{i \in O} \deg(v_i)$ is even.

i.e, number of vertices of odd degree is even.

**Hence the Proof**

**Theorem 3:**

The maximum number of edge in a complete graph with 'n' vertices is $\frac{n(n-1)}{2}$.



## Theorem 4:

Show that a graph G is self complementary if it has 4n or 4n+1 vertices (n is non-negative integers)

**Proof:**

Let $G = (V, E)$ be a self complementary graph with 'm' vertices .

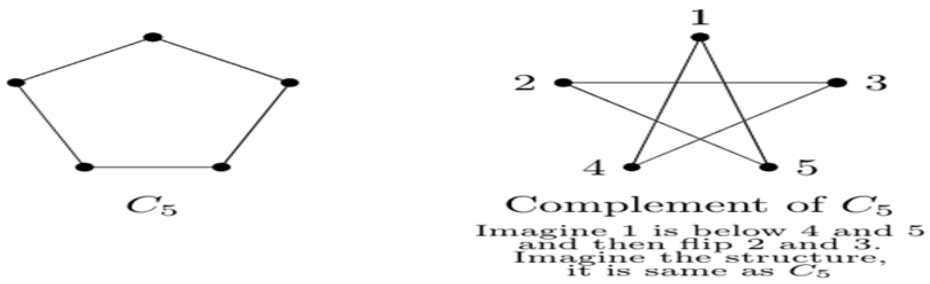
Since G is self complementary , G is isomorphic to $G'$

i.e, $|E(G)| = |E(G')|$

Since G has 'm' vertices, maximum number of edges in G is $\frac{m(m-1)}{2}$

$$|E(G)| + |E(G')| = \frac{m(m-1)}{2}$$

$$\Rightarrow 2|E(G)| = \frac{m(m-1)}{2}$$




[since G and $G'$ are self complementary graphs]

$$\Rightarrow |E(G)| = \frac{m(m-1)}{4}$$


$|E(G)| = \frac{m(m-1)}{2}$ is an odd integer and one of m or m-1 are odd

∴ m or m-1 is a multiple of 4

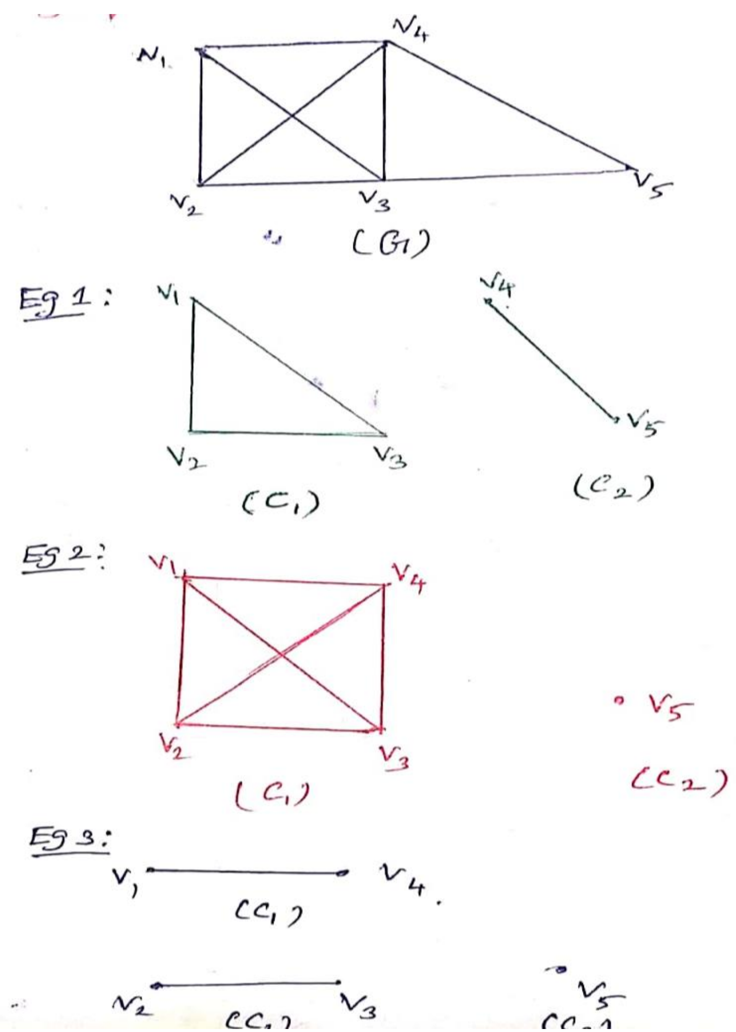
Hence m is the form of 4n (or) 4n+1

∴ G has 4n (or) 4n+1 vertices where n is non negative integer.

**Hence the proof.**

## Theorem 5 :

The maximum number of edges in simple connected graph with 'n' vertices and 'k' component is $\frac{(n-k)(n-k+1)}{2}$




**Proof:**

Let the number of vertices of 'i'-th component of G be $n_i$ $(n_i \geq 1)$.

Then $n_1 + n_2 + \ldots\ldots + n_k = n$

(OR) $\sum_{i=1}^{k} n_i = n$----------------(1)

$(n_1-1) + (n_2-1) + \ldots\ldots + (n_k-1) = n-k$

$$\sum_{i=1}^{k} n_i - 1 = n - k$$

Squaring on both sides,we get

$(\sum_{i=1}^{k}(n_i-1))^2 = (n-k)^2$

$\sum_{i=1}^{k}(n_i-1)^2 \leq [\sum_{i=1}^{k}(n_i-1)]^2 = (n-k)^2$  $\{Since \sum_{i=1}^{k} {n_i}^2 \leq [\sum_{i=1}^{k} n_i]^2 \}$

$\sum_{i=1}^{k}(n_i-1))^2 \leq n^2 + k^2$-2nk (*since* $n_i \geq 1$).

$\sum_{i=1}^{k}(n_i^2+1-2\,n_i) \leq \mathrm{n}^2 + \mathrm{k}^2$-2nk

$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq \mathrm{n}^2 + \mathrm{k}^2 - 2\mathrm{nk}$$

$\sum_{i=1}^{k} n_i^2 + k - 2n \leq \mathrm{n}^2 + \mathrm{k}^2 - 2\mathrm{nk}$ (since $\sum_{i=1}^{k} 1 = \mathrm{k}$)

$\sum_{i=1}^{k} n_i^2 \leq [n^2 + k^2 - 2nk - k + 2n]$----------------(2)

Maximum number of edges in the i-th component of G is $\frac{n_i(n_i-1)}{2}$

Maximum number of edges in G $= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \sum_{i=1}^{k} \frac{(n_i^2-n_i)}{2}$

$$= \sum_{i=1}^{k} \frac{n_i^2}{2} - \sum_{i=1}^{k} \frac{n_i}{2} = \frac{1}{2}\{\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\}$$

$$\leq \frac{n^2 + k^2 - 2nk - k + 2n - n}{2} \quad \text{from}[(1),(2)]$$

$$= \frac{n^2 + k^2 - 2nk - k + n}{2}$$

$$= \frac{(n-k)^2 - k + n}{2} = \frac{(n-k)^2 + (n-k)}{2} = \frac{(n-k)(n-k+1)}{2}$$

Maximum number of edges in G $\leq \frac{(n-k)(n-k+1)}{2}$

**Hence the proof**

## Theorem 6:

The number of edges in a bipartite graph with ‘n’ vertices is atmost $\left[\frac{n}{2}\right]^2$ .

**Proof:**

Let the vertex set can be partitioned into the subsets $V_1$ and $V_2$ . Let $V_1$ contains ‘$x$’ vertices. Then $V_2$ contains ‘$n$ - $x$’ vertices.

The maximum number of edges of the graph can be obtained ,when each of the ‘$x$’ vertices in $V_1$ is connected to the each of the ‘$n$ - $x$’ vertices in $V_2$ .

∴ The maximum number of edges $f(x) = x(n - x)$ is a function of $x$.

Now ,we have to find the value of x for which $f(x)$ is maximum.

To Find Maxima:

$$f(x) = x(n-x)$$

$$\Rightarrow f(x) = nx - x^2$$

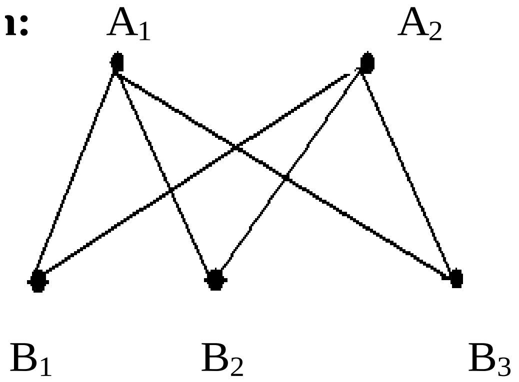
$$\Rightarrow f'(x) = n\text{-}2x$$

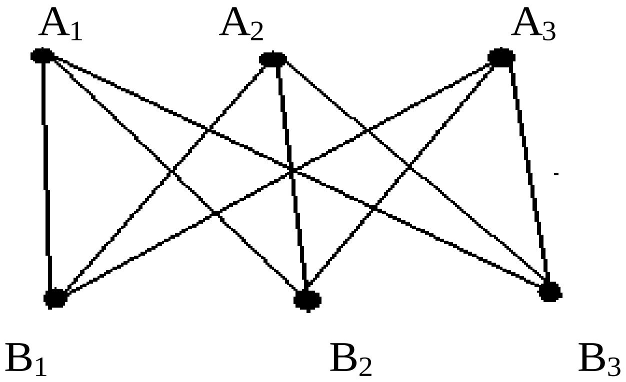
$$\Rightarrow f''(x) = \text{-}2 \quad (\textit{which is negative})$$

$$f'(x) = 0 \Rightarrow n\text{-}2x\text{=}0$$

$$\Rightarrow n\text{=}2x$$

$$\Rightarrow x\text{=}\frac{n}{2}$$




$\therefore f(x)$ is maximum at $x=\frac{n}{2}$ and

The maximum number of edges is $f(x) = x(n-x)$

$$= \frac{n}{2}\left[n - \frac{n}{2}\right] = \left[\frac{n}{2}\right]^2$$

**Hence the proof.**

## Theorem 7:

A graph G is connected if and only if for any partition of V into 2 subsets $V_1$ and $V_2$ there is an edge joining a vertex of $V_1$ to a vertex $V_2$.

**Proof:**

Let G be a connected graph and V = $V_1 \cup V_2$ be a partition of v into subsets.

Let u $\in V_1$ and v $\in V_2$ .Since the graph G is connected ,there is a path in G such that

u = $v_0, v_1, v_2, \ldots\ldots\ldots\ldots, v_n$ = v

Let i be the least positive integer such that $v_i \in V_2$. Then $v_{i-1} \in V_1$ and the vertices $v_{i-1}$ and $v_i$ are adjacent. Thus, there is an edge joining $v_{i-1} \in V_1$ and $v_i \in V_2$

Necessary Part:

Let us assume for any partition of V into subsets $V_1$ and $V_2$ ,there is an edge joining a vertex of $V_1$ to a vertex of $V_2$ and let us prove G is connected .

Conversely, let us assume that G is disconnected .

Then G contains atleast 2 components.

Let $V_1$ be the set of all vertices of one component and $V_2$ be the set of all remaining vertices of G. Clearly $V_1 \cup V_2$ = V and $V_1 \cap V_2 = \Phi$(null set).

Thus the collection $\{V_1, V_2\}$ is a partition of V and there is no edge joining any vertex of $V_1$ to any vertex of $V_2$ which is contradiction to our assumption there is an edge joining a vertex of $V_1$ to a vertex of $V_2$ . Hence G is connected.

**Hence the proof.**

## Theorem :8

A Graph G is disconnected if and only if the vertex $V$ is partitioned into two non-empty disjoint subsets $V_1$ and $V_2$ such that there is no edge in G where one end vertex is in subset of $V_1$ and the other is in subset of $V_2$

**Proof:**

Let us assume such partition exists.

Consider 2 arbitrary vertices $V_1$ and $V_2$ of graph $G = (V, E)$ such that $u \in V_1$ and $w \in V_2$.

As per our assumption , no path can exist between vertices $u$ and $w$.

Hence, if a partition exist, the graph G is disconnected

Conversely, assume that G is disconnected.

i.e., G has more than one components.

Let $G_1$ and $G_2$ be the two components. i.e., $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Clearly $V_1 \cup V_2$ = V and $V_1 \cap V_2 = \Phi$(null set).

Since graph G is disconnected,

***there is no path exists between the vertex set $V_1$ and $V_2$.***

Now, it is clear that no vertex in $V_1$ is joined to any vertex in $V_2$ by an edge.

Hence the partition exists.

**Hence the proof**

## Theorem :9

If $G$ is disconnected then $\overline{G}$ is also connected.

**Proof:**

Let $G$ be disconnected graph.

Then $G$ has more than one component

Let $u$, $v$ be any two vertices of $G$.

To Prove: There is a $u$-$v$ path in $\overline{G}$.

If $u$ and $v$ are in different components of $G$ , then $u,v$ are non adjacent in $\overline{G}$.

If $u$ and $v$ are in same components of $G$, then choose a vertex w in a in different component of $G$. Then $u$-$w$-$v$ is a $u$-$v$ path in $\overline{G}$.

**Hence the proof**

## Theorem :10

A connected graph with no loop is Eulerian iff the degree of each vertex is even.

**Proof:**

Assume G is disconnected graph with no loop and it is Euler.Then G has a Eulerian circuit.Any Eulerian circuit in G leaves each vertex as many time as it enters.So each vertex of G is even.

**Necessary part:**

Suppose G is disconnected graph in which each vertex is of even degree, we prove G is Eulerian by constructing an Euler circuit in it.

An Eulerian circuit is obtained in G by the procedure in which vertices are joined until to get the circuit.

Start from any vertex v.

Traverse distinct edges of G until return to v. Since each vertices are of even degree this is possible certainly.

Let $c_1$ be the circuit thus obtained .If $c_1$ contains all the edges of the graph G, then $c_1$ is an Eulerian circuit.

If $c_1$ is not an Eulerian circuit,consider a vertex u.

Now start from u and obtain a circuit $c_2$ by traversing distinct edges in $c_1$.

Note that the two circuits is no common edges ,even though they may have common vertices.

Case 1:If u=v

The two circuits can be joined together to from an enlarged circuit $c_3$

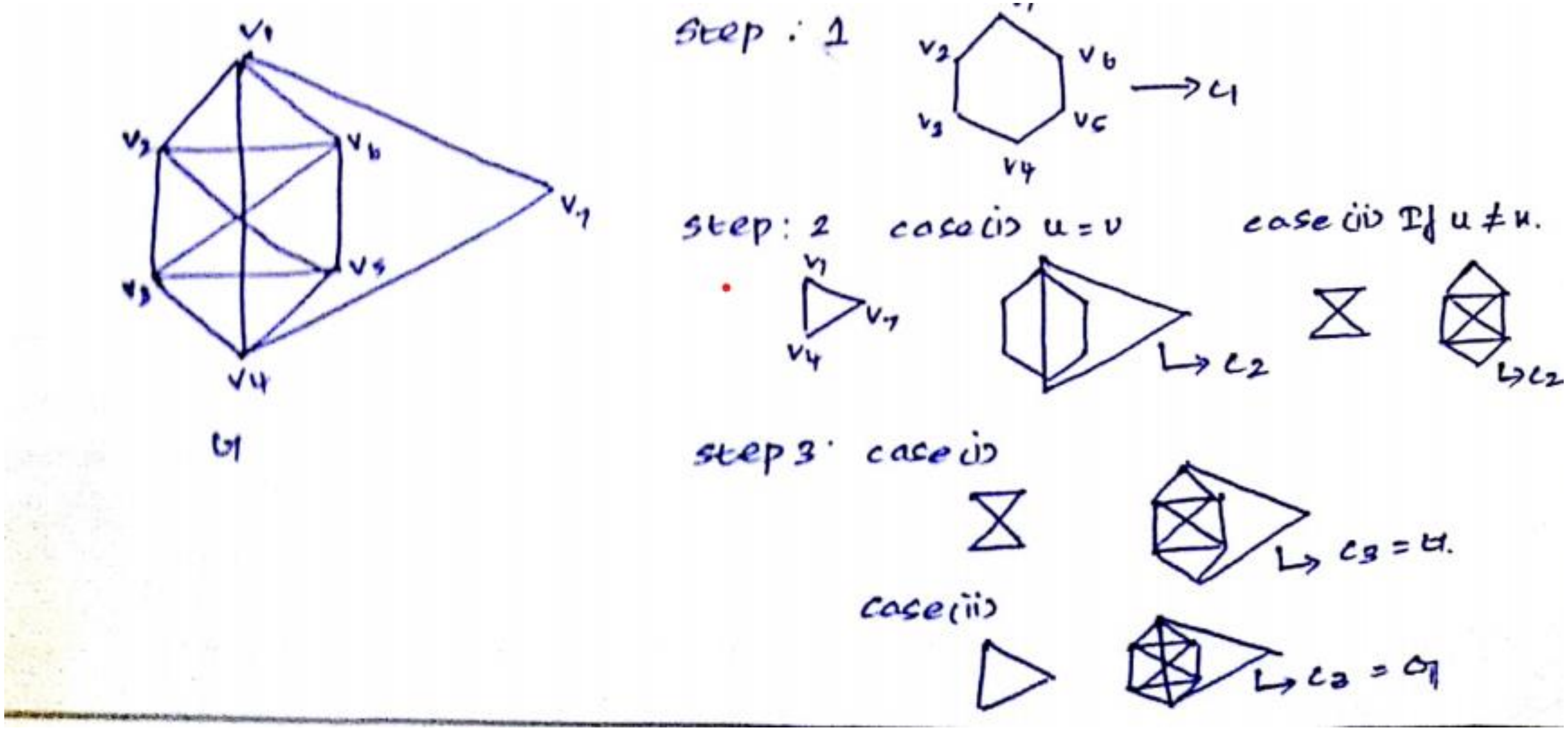
Case 1:If u≠v

Let P,Q be two distinct simple paths between u and v consisting of edges from $c_1$

The P,Q and $c_2$ are joined together to from a new circuit $c_3$ .

If $c_3$ contains all the edges of the graph G ,then $c_3$ is an Eulerian circuit.

Otherwise we will continue the step 2 procedure until to obtain a circuit that has all the edges of G.



## Theorem :11

Let $G$ be a simple indirected graph with $n$ vertices . Let $u,v$ be two non adjacent vertices in $G$ such that $deg(u)+deg(v) \geq n$ in $G$.

S.T G is Hamilton iff G+uv is Hamiltonian

**Proof:**

If $G$ is Hamiltonian,then obviously $G+uv$ is Hamiltonian

**Sufficient part:**

Suppose $G+uv$ is Hamiltonian, but $G$ is not, then by Dirac Theorem , If $G$ is a simple Graph with $n$ vertices with $n \geq 3$ such that degree of every verex is $G$ is atleast $n/2$,then $G$ has Hamiltonian circuit .

Hence $deg(u)+deg(v) < n$ which is contradiction for our assumption.

∴ If $G+uv$ is Hamiltonian then $G$ is Hamiltonian.

## Theorem :12

P.T a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

**Proof:**

Let G be a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not connected, then G must have atleast two component.Let it be $G_1$ and $G_2$.

Let $v_1$ be the vertex set of $G_1$ with $|v_1|$ =m.

Let $v_2$ be the vertex set of $G_2$ with $|v_2|$ =n-m.

Then (i) $1 \leq m \leq n-1$

(ii) There is no edge joining a vertex $v_1$ and vertex of $v_2$

(iii) $|v_2| = n-m \geq 1$

Now $|E(G)| = |E(G_1 \cup G_2)| = |E(G_1)| + |E(G_2)|$

$$= \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$ [By Theorem 3]

$$=\frac{(m^2-\mathrm{m})+\mathrm{n}(n-\mathrm{m}-1)-m(n-\mathrm{m}-1)}{2}$$

$$=\frac{(m^2-\mathrm{m})+\mathrm{n}(n-1)-mn-mn+m^2+\mathrm{m})}{2}$$

$$=\frac{\mathrm{n}(n-1)-(2n-2)+(2n-2)-2mn+2m^2)}{2}$$

$$=\frac{\mathrm{n}(n-1)-2(n-1)+2n-2-2mn+2m^2)}{2}$$

$$=\frac{(n-1)(n-2)-2n(m-1)+2(m^2-1)}{2}$$

$$=\frac{(n-1)(n-2)-2n(m-1)+2(m+1)(m-1)}{2}$$

$$=\frac{(n-1)(n-2)-2(m-1)(n-m-1)}{2}$$ since $(m-1)(n-m-1)\geq 0$ for $1\leq m\leq n-1$

which is a contradiction as G has more than $\frac{(n-1)(n-2)}{2}$ edges.

Hence G is connected graph.


Thank You!