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MODULE IV

ALGEBRAIC STRUCTURES

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ALGEBRAIC STRUCTURES

BINARY OPERATION

In the set of natural numbers N, we can add any two numbers a and b get a unique number a+b. The operation addition combines two numbers and yield a third number and so it is a binary operation. Suppose such an operation is to be defined in a set S, we have to view addition in different way.

i.e., $+ \ : N \times N \rightarrow N$ is defined by +(a,b) = a+b.

Definition: Binary operation

Let S be a non empty set. A binary operation $*$ on S is a function $* \ : S \times S \rightarrow S$. The image of any ordered pair (a, b) of elements of S under $*$ is defined by $a * b$.

Note: $+$, $-$, $\times$, $\div$, $\cup$, $\cap$, $\circ$, $*$,…… are some binary operations.

Definition: Algebraic structure (or) Algebraic system:

A non-empty set A together with one or more n-ary operations $*$ defined on it is called algebraic system and it is denoted by (A, $*$).

Example:

(Z, +, $*$) is an algebraic system where + and $*$ are the operations of addition and multiplication on Z.

Example :

The usual addition + on natural number set is a binary operation.

The number set N = Natural number set

= the set of positive numbers

N= {1,2,3,4,5,….}.

(N, +) is an algebraic structure since the sum of any two numbers in N is also in N.

i.e., If 3, 56 $\in$ N then 3+56 = 59 $\in$ N

but (N, $-$) is not an algebraic structure since the difference of any two numbers in N is not in N.

i.e., If 5, 9 $\in$ N then 5-9 = -4 $\notin$ N.

Notations:

N = the set of positive numbers ={1,2,3,4,5,……}

Z = the set of all integers ={0,±1, ±2, ±3, ±4, ±5,……}

R = the set of real numbers

Q^+ = the set of positive real numbers

C = the set of complex numbers

Q = the set of rational numbers = $\left\{ \frac{p}{q}\ such\ that\ p,q\ \in Z\ and\ q \neq 0 \right\}$

Q^+ = the set of positive rational numbers

Note: (R, +), (Z, +), (Z,−) and (C, +) are an algebraic structures.

Properties of Binary operations:

Let the binary operation be $* \ : G \times G \to G$. It is denoted by (G, $*$).

CLOSURE PROPERTY:

For all $,b\ \in G$, $a * b\ \in G$

For example, addition on N is closed since $5, 9\ \in N$, $5 + 9 = 14\ \in N$. Therefore (N,+) is closed

COMMUTATIVE PROPERTY:

For all $,b\ \in G$, $a * b = b * a$

For example, multiplication on Z is commutative since $-6, 9\ \in Z$, $(-6) \times 9 = 9 \times (-6) = -54\ \in Z$. Therefore (Z,×) is commutative.

Note: (Z, ÷) is not commutative.

ASSOCIATIVE PROPERTY:

For all $,b,c\ \in G$, $a * (b * c) = (a * b) * c$

For example, multiplication on N is associative since 2,5,8 $\in N$, $2 \times (5 \times 8) = (2 \times 5) \times 8 = 80 \in N$. Therefore (N, ×) is associative.

EXISTENCE OF IDENTITY:

An algebraic structure (G, $*$) is said to have an identity element $e \in G$ if $a * e = e * a = a$ for all $a\ \in G$

For example, In the algebraic structure (Z,+), 0 is the identity element because $a + 0 = 0 + a = a\ for\ all\ a\ \in Z$

EXISTENCE OF INVERSE ELEMENT:

If $a * b = b * a = e$ for any $a,b \in G$ then 'b' is called the inverse of 'a' and it is denoted by $b = a^{-1}$. (here e is identity element and $e \in G$)

For example, In the algebraic structure (Z,+), inverse of any element a is -a because $a + (-a) = (-a) + a = 0 \; for\; all\; a \in Z$.

Note: The set of real numbers R with usual + and x as binary operations is an algebraic struture or algebraic system.

Semigroup: A non empty set S together with binary operation $*$ an algebraic structure (S, $*$) is called semigroup if $*$ satisfies the following properties

(i) Closure property: For all $a,b \in S$, $a * b \in S$

(ii) Associative property: For all $a,b,c \in S$, $a * (b * c) = (a * b) * c$

Example: The set of all rational numbers Q is a semi group for the operation $*$ defined by a $*$ b = $\frac{ab}{2}$ $\forall$ a, b $\in$ Q.

Monoid : A non empty set M together with binary operation $*$ (or) an algebraic structure (M, $*$) is called monoid if $*$ satisfies the following properties

(i) Closure property: For all $a,b \in M$, $a * b \in M$

(ii) Associative property: For all $a,b,c \in M$, $a * (b * c) = (a * b) * c$

(iii) Identity property: There exists an element $e \in M$ such that $a * e = e * a = a$ for all $a \in M$

Group : A non empty set G together with binary operation $*$ (or) an algebraic structure (G, $*$) is called a group if $*$ satisfies the following properties

(i) Closure property: For all $a,b \in G$, $a * b \in G$

(ii) Associative property: For all $a,b,c \in G$, $a * (b * c) = (a * b) * c$

(iii) Identity property: There exists an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$

(iv) Inverse: For each $a \in G$, there exists an element a' such that $a * a' = a' * a = e$

Abelian group : A non empty set G together with binary operation $*$ (or) an algebraic structure (G, $*$) is called an abelian group if $*$ satisfies the following properties

(i) Closure property: For all $a,b \in G$, $a * b \in G$
(ii) Associative property: For all $a,b,c \in G$, $a * (b * c) = (a * b) * c$
(iii) Identity property: There exists an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$
(iv) Inverse: For each $a \in G$, there exists an element a' such that $a * a' = a' * a = e$
(v) Commutative property: For all $a,b \in G$, $a * b = b * a$

In other words, a group (G, $*$) is called abelian group if it satisfies commutative property i.e., for all $a,b \in G$, $a * b = b * a$

Order of a Group:

Let G be a group under the operation $*$. Then the number of elements in G is called the order of the group G and is denoted by O(G) (or) $|G|$.

If G has n elements then O(G) = n.

For example, If A = {a,e,i,o,u} then O(A) = 5 (or) $|A| = 5$

Finite and Infinite Group:

If the O(G) is finite then G is called a finite group. Otherwise it is called infinite group.

Subgroup:

Let (G, $*$) be a group . A non empty set H of G is said to be a subgroup of G if H is itself group under the same operation $*$ of G.

Cyclic Group:

A group (G, $*$) is said to be a cyclic group if for every element $x \in G$ can be expressed as $x = a^m$ or $x = ma$ for some $a \in G$ and m $\in Z$.

Order of a group:

The number of elements in a group $(G, *)$ is called order of a Group and is denoted by $O(G)$.

Order of an element:

Let $(G, *)$ be a group and $a \in (G, *)$ **.** Then the least positive integer n such that $a^n = e$ is called the order of the element *a* .
(i.e., $a^n = e \Leftrightarrow O(a) = n$ **).**

Cosets:

Let (H, $*$) be a subgroup of a group (G, $*$).

Left coset of H: For any $a \in G$, the left coset of H is defined by

$a * H = \{a * h : h \in H\}$.

Right coset of H: For any $a \in G$, the right coset of H is defined by

$H * a = \{h * a : h \in H\}$.

NORMAL SUBGROUPS

A subgroup (H, $*$) of a group (G, $*$) is said to be a normal subgroup, for every $x \in G$ and for $h \in G$ if $x * h * x^{-1} \in H$ $i.e,$ $x * H * x^{-1} \subseteq H$.

Another form of definition: A subgroup (H, $*$) of a group (G, $*$) is called a normal subgroup if $x * h = h * x \; \forall \, x \in G$.

(or) A subgroup H of a group G is called a normal subgroup if
$xH = Hx \; \forall \, x \in G$.

Group Homomorphism:

Let (G, $*$) and (H, Δ) be any two groups. A mapping *f : G* *H* is called a group homomorphism if $f(a * b) = f(a) \, \Delta \, f(b)$ for all $a,b \in G$.

(or) Let (G, $*$) and (G', $*$) be two groups. A mapping *f : G* $G'$is called a group homomorphism if $f(a * b) = f(a) * f(b)$ for all $a,b \in G$.

Isomorphism:

Let (G, $*$) and (H, Δ) be any two groups. A mapping *f : G* *H* is called an isomorphism if

(i) *f* is homomorphism *i.e.,* $f(a * b) = f(a) \, \Delta \, f(b)$ for all $a,b \in G$.

(ii) f is ono to one (injective)

(iii) f is onto (surjective).

In otherwords, a bijective homomorphism is said to be an isomorphism.

Kernel of a homomorphism:

Let f : G $\rightarrow$ G' be a group homomorphism . The set of elements of G which are mapped into e' (i.e., e' is an identity element of G') is called the kernel of f and it is denoted by ***ker(f)***.

i.e., *ker(f)* ={ $x \in G$ / *f(x)* =*e′* }

Quotient group or Factor group:

Let (H, $*$) be a normal subgroup of a group (G, $*$) and G/H denotes the set of all left (or right) cosets of H in G. i.e., G/H = $\{a * H : \forall a \in G \}$. Then an algebraic structure (G/H, $\oplus$)is said to be a quotient group if $(a * H) \oplus (b * h) = (a * b) * H \;\; \forall a,b \in G$.

Natural Homomorphism:

Let (H, $*$) be a normal subgroup of a group (G, $*$). A mapping *f : G* $\rightarrow$ *G/H such that* $f(x) = H * x$, $\forall x \in G$ is called a natural homomorphism of the group G onto the quotient group G/H.

PROPERTIES OF GROUPS

PROPERTY 1: In a group (G,$*$), the identity element is unique.

Proof: If possible, let e_1 *and* e_2 be two identity elements in the group (G,$*$).

Since e_2 is an identity and $e_1 \in$ G,

we have $e_2 * e_1 = e_1 * e_2 = e_1$ $\rightarrow$ (1)

Since e_1 is an identity and $e_2 \in$ G,

we have $e_1 * e_2 = e_2 * e_1 = e_2$ $\rightarrow$ (2)

From (1) and (2) we have $e_1 = e_2$

Hence the identity element of a group is unique.

PROPERTY 2: The inverse of every element in a group is unique.

Proof: Let (G,$*$) be a group.

Let b and c be inverses of the element "a" $\forall$ a,b, c $\in$ G

Then a $*$ b = e $\rightarrow$ (1)

and a $*$ c = e $\rightarrow$ (2)

To prove : b = c

b = b ∗ e [since e is the identity element in G]

= b ∗ (a ∗ c) [by (2)]

= (b ∗ a) ∗ c [associative]

= (a ∗ b) ∗ c [Commutative]

= e ∗ c [by (1)]

= c.

∴ b = c.

∴ The inverse is unique.

PROPERTY 3: [INVOLUTION LAW]

In a group $(G,*)$, $\left(a^{-1}\right)^{-1} = a\ ,\forall\, a \in G$

OR

In a group $(G,*)$, the inverse of a^{-1} is a .

Proof: Let $(G,*)$ be a group.

Let e be the identity element of $(G,*)$

Let $a \in (G,*)$.

We know that a has unique inverse say a^{-1} .

Therefore $a^{-1} * a = a * a^{-1} = e$ ---------(1)

To prove : $\left(a^{-1}\right)^{-1} = a\ ,\forall\, a \in G$

Consider

$\left(a^{-1}\right)^{-1} * (a^{-1} * a) = \left(a^{-1}\right)^{-1} * e = \left(a^{-1}\right)^{-1}$ ---------(2)

$\left(a^{-1}\right)^{-1} * (a^{-1} * a) = \left(\left(a^{-1}\right)^{-1} * a^{-1}\right) * a = e * a = a$ ---------(3)

From (2) & (3) , $\left(a^{-1}\right)^{-1} = a\ ,\forall\, a \in G$.

PROPERTY 4 : [CANCELLATION LAW]

In a group $(G,*)$, for $a,b,c \in (G,*)$

(i) $a * b = a * c \Rightarrow b = c$ **[Left cancellation law]**

(ii) $b * a = c * a \Rightarrow b = c$ **[Right cancellation law]**

Proof: Let $(G,*)$ be a group.

Let $a \in (G,*) \Rightarrow a^{-1} \in (G,*)$ since every element in a group has unique inverse.

To prove: LEFT CANCELLATION LAW

Let $a * b = a * c$

Pre-operating by a^{-1} on both sides, we get

$$a^{-1} * (a * b) = a^{-1} * (a * c)$$
$$(a^{-1} * a) * b = (a^{-1} * a) * c$$
$$e * b = e * c$$
$$b = c$$

To prove: RIGHT CANCELLATION LAW

Let $b * a = c * a$

Post-operating by a^{-1} on both sides, we get

$$(b * a) * a^{-1} = (c * a) * a^{-1}$$
$$b * (a * a^{-1}) = c * (a * a^{-1})$$
$$b * e = c * e$$
$$b = c$$

PROPERTY 5 :

In a group, identity element is the only idempotent element.

Note: An element $a \in (G, *)$ is called and **Idempotent element** if $a * a = a$

(i.e., **An element operated with itself gives same element)**

Proof:

Let $(G, *)$ be a group.

Let e be the identity element of $(G, *)$

Clearly e is idempotent since $e * e = e$.

To prove: e is the only idempotent element.

If possible let $a \in (G, *)$ be another idempotent element.

Therefore we have, $a * a = a$ **.**

Consider

$$a = a * e$$
$$= a * (a * a^{-1})$$
$$= (a * a) * a^{-1}$$
$$= a * a^{-1}$$
$$= e$$

Hence, e is the only idempotent element.

PROPERTY 6: In a group $(a * b)^2 = a^2 * b^2 \ \forall\ a, b \in G$ if G is abelian.

Proof: Assume that (G, ∗) is abelian.

$(a * b)^2 = (a * b) * (a * b)$

$= a * (b * (a * b))$

$= a * ((b * a) * b)$

$(a * b)^2 = a * ((a * b) * b)$

[Since G is abelian, $(a * b) = (b * a)$]

$\therefore\ (a * b)^2 = (a * a) * (b * b) = a^2 * b^2$

PROPERTY 7: In a group G if $(a * b)^2 = a^2 * b^2 \ \forall\ a, b \in G$ then G is abelian.

Proof: To prove that (G, ∗) is abelian.

Now $(a * b)^2 = a^2 * b^2$

$(a * b) * (a * b) = (a * a) * (b * b)$

$a * (b * (a * b)) = a * (a * (b * b))$

$\Rightarrow\ b * (a * b) = a * (b * b)$ (by left cancellation law)

$\Rightarrow\ (b * a) * b = (a * b) * b$

$\Rightarrow\ b * a = a * b$ (by right cancellation law)

$\therefore$ G is abelian.

PROPERTY 8 : A group (G, ∗) is an abelian if and only if $(a * b)^2 = a^2 * b^2 \ \forall\ a, b \in G$.

Solution: Let us assume that (G, ∗) is abelian.

$(a * b)^2 = (a * b) * (a * b)$

$= a * (b * (a * b))$

$= a * (b * a) * b$

Since G is abelian, $(a * b) = (b * a)$

$$(a * b)^2 = a * ((a * b) * b)$$

$$= (a * a) * (b * b)$$

$$(a * b)^2 = a^2 * b^2$$

Conversely, assume that $(a * b)^2 = a^2 * b^2$

To prove: G is abelian.

$$(a * b)^2 = a^2 * b^2$$

$$(a * b) * (a * b) = (a * a) * (b * b)$$

$$a * (b * (a * b)) = a * (a * (b * b))$$

$$b * (a * b) = a * (b * b) \quad \text{[by left cancellation law]}$$

$$(b * a) * b = (a * b) * b$$

$$b * a = a * b \quad \text{[by right cancellation law]}$$

$\therefore a * b = b * a \ \forall\, a, b \in G$. Hence G is an abelian.

PROPERTY 9: A group $(G, *)$ is an abelian if and only if $(a * b)^{-1} = a^{-1} * b^{-1} \ \forall\, a, b \in G$.

Proof: Assume that $(G, *)$ is an abelian

$$\therefore a * b = b * a \ \forall\, a, b \in G$$

Now $(a * b)^{-1} = (b * a)^{-1} = a^{-1} * b^{-1}$

Conversely, assume that $(a * b)^{-1} = a^{-1} * b^{-1}$

But $a^{-1} * b^{-1} = (b * a)^{-1}$. $\therefore (a * b)^{-1} = (b * a)^{-1}$

Taking inverse on both sides, $((a * b)^{-1})^{-1} = ((b * a)^{-1})^{-1}$

$\Rightarrow a * b = b * a \ \forall\, a, b \in G$. Hence G is an abelian

PROPERTY 10 : If $(G, *)$ is an abelian group, then $(a * b)^n = a^n * b^n \ \forall\, a, b \in G$ where n is a positive integer.

Proof: Proof follows by Mathematical induction

Let $P(n) = (a * b)^n = a^n * b^n$

To Prove: P(1) is true

Since $(G, *)$ is an abelian group, $a * b = b * a$

$\forall\, a, b \in G \quad \rightarrow$ (i)

For $a, b \in G$, we have $(a * b)^1 = a^1 * b^1$ by (i)

and $\quad (a * b)^2 = (a * b) * (a * b)$

$= a * (b * a) * b$ [by associative law]

$= a * (a * b) * b$ by (i)

$= (a * a) * (b * b)$ [by associative law]

$= a^2 * b^2$.

Thus the required result is true for n = 1, 2.

Assume that the result is true for P(m).

i.e. $(a * b)^m = a^m * b^m \rightarrow$ (ii)

To Prove: P(m+1) is true

Now, $(a * b)^{m+1} = (a * b)^m * (a * b)$

$= (a^m * b^m) * (a * b)$ by (ii)

$= a^m * (b^m * a) * b$ [by associative law]

$= a^m * (a * b^m) * b$ since G is abelian.

$= (a^m * a) * (b^m * b)$

$= a^{m+1} * b^{m+1}$.

Hence by Mathematical induction, the result is true for all positive integer n.

Hence $(a*b)^n = a^n * b^n, \forall\, a, b \in G$ is true for every *n.*

PROPERTY 11 : If for any element 'a' in a group (G, $*$), $a^2 = e$ then G is an abelian group.

Proof: Let a, b $\in$ G. Then $(a * b) \in G$ so that $(a * b)^2 = e$.

Since $a \in G$, $a^2 = e \Rightarrow a * a = e$

$b \in G$, $b^2 = e \Rightarrow b * b = e$

Now $(a * b)^2 = e$

$\Rightarrow (a * b) * (a * b) = e * e$

$= (a * a) * (b * b)$

$a * (b * (a * b)) = a * (a * (b * b))$

$\Rightarrow b * (a * b) = a * (b * b)$ [by left cancellation law]

$(b * a) * b = (a * b) * b$

$b * a = a * b$ [by right cancellation law]

Hence G is an abelian group.

PROPERTY 12 :If G is a finite group of order n and a $\in$ G then $a^n = e$.

Solution: $(G, *)$ is an finite group of order n.

$\therefore$ The element $a \in G$ is of finite order.

Let $O(a) = m$. Then m is the least positive integer such that

$a^m = e$.

$\therefore$ $O(a)$ divides $O(G)$, m divides n.

$\therefore$ $n = mq$ for some integer q.

$\therefore$ $a^n = a^{mq} = (a^m)^q = e^q = e. \quad \Rightarrow a^n = e.$

PROPERTY 13 :In a group $(G, *)$, the equations $x * a = b$ and $a * y = b$ have unique solutions. (OR)

If a, b $\in$ G, the equation $a * x = b$ has the unique solution $x = a^{-1} * b$. Similarly the equation $a * y = b$ has the unique solution $y = b * a^{-1}$.

Proof: Consider $x * a = b$. Post multiplying by a^{-1}, $x * a = b$

$$x * a * a^{-1} = b * a^{-1}$$

$$x * e = b * a^{-1}$$

$$x = b * a^{-1}$$

To prove uniqueness

Let x_1 and x_2 be two solutions of $x * a = b$. Then $x_1 * a = b$ and $x_2 * a = b$.

$\therefore$ $x_1 * a = x_2 * a$.

$\Rightarrow$ $x_1 = x_2$ by right cancellation law.

In a similar manner, the equation $a * y = b$ has a solution $y = a^{-1} * b$ and this solution is unique.

PROBLEMS ON GROUPS

PROBLEM 1: If $*$ is the binary operation defined on the set R of real numers defined by a $* b = a + b + 2ab$ for all $a,b \in R$.

(a) Verify (R, $*$) is monoid or not?

(b) Is it commutative?

(c) Which elements have inverse and what are they?

Solution:

To verify (a)

(i) Closure property:

For all $a,b \in R$, $a + b \in R$ $and\ 2ab \in R$

Therefore, $a + b + 2ab \in R \Rightarrow$ a $* b \in R$

‘ $*$ ‘ satisfies closure property

(R, $*$) is closure.

(ii) Associative property:

To prove : $a * (b * c) = (a * b) * c \ \ \forall\ a,b,c \in R$

Now ,

$$a * (b * c) = a * (b + c + 2bc)$$
$$= a + (b + c + 2bc) + 2a(b + c + 2bc)$$
$$= a + (b + c + 2bc) + 2ab + 2ac + 4abc$$
$$a * (b * c) = a + b + c + 2ab + 2bc + 2ac + 4abc \ \ ......(1)$$

Consider

$$(a * b) * c = (a + b + 2ab) * c$$
$$=(a + b + 2ab) + c + 2c\,(a + b + 2ab)$$
$$(a * b) * c = a + b + c + 2ab + 2bc + 2ac + 4abc \ \ ....(2)$$

From (1) and (2), $a * (b * c) = (a * b) * c\ \forall\ a,b,c \in R$

‘ $*$ ‘ satisfies associative property

(R, $*$) is associative.

(iii) To find the Identity element:

Let e be the identity element of R

Now $a * e = a \ \ \forall\ \ a \in R$

a+e+2ae = a $\Rightarrow$ (1+2a)e =0 $\Rightarrow$ e =0 $\in R$

Identity element exist.

Since ‵ $*$ ‵ satisfies Closure, Associative and identity properties.

(R, $*$) is a monoid.

(b) To verify commutative property

Consider $a * b$ =a+b+2ab = b+a+2ba = $b * a$

Therefore, $a * b = b * a \ \forall \ a,b \in R$

(R, $*$) is commutative.

(c) To find the inverse element:

Let a' be the inverse element of $a \in R$.

Then $a * a' = e \Rightarrow$ a+a′+2aa′ = e ⇒ a+(1+2a) a′ =0 (since e =0)

⇒ (1+2a) a′ = -a ⇒ a′ = $\frac{-a}{(1+2a)}$ if a $\neq -\frac{1}{2}$.

Hence the inverse element of $a \in R$ is a′ = $\frac{-a}{(1+2a)}$ except a = $-\frac{1}{2}$

PROBLEM 2: Show that (Q^+, $*$) is an abelian group where $*$ defined by $a * b = \frac{ab}{2}$ for all $a,b \in Q^+$.

Solution:

(i) Closure property:

For all $a,b \in Q^+ \Rightarrow ab \in Q^+ \Rightarrow \frac{ab}{2} \in Q^+$

Therefore, $a * b = \frac{ab}{2} \in Q^+ \Rightarrow a * b \in Q^+$

‵ $*$ ‵ satisfies closure property

(Q^+, $*$) is closure.

(ii) Associative property:

To prove : $a * (b * c) = (a * b) * c \ \forall \ a,b,c \in Q^+$

Now ,

$$a * (b * c) = a * \left(\frac{bc}{2}\right)$$

$$= \frac{\frac{abc}{2}}{2}$$

$$a * (b * c) = \frac{abc}{4} \quad \ldots..(1)$$

Consider

$$(a * b) * c = \left(\frac{ab}{2}\right) * c$$

$$= \frac{\frac{abc}{2}}{2}$$

$$(a * b) * c = \frac{abc}{4} \quad \ldots\ldots.(2)$$

From (1) and (2), $a * (b * c) = (a * b) * c \ \forall \ a,b,c \in Q^+$

' $*$ ' satisfies associative property.

$(Q^+, *)$ is associative.

(iii) To find the Identity element:

Let e be the identity element of R

Now, $a * e = a \ \forall \ a \in Q^+$

$\left(\frac{ae}{2}\right)$ = a $\Rightarrow \left(\frac{e}{2}\right)$ = 1 $\Rightarrow$ e =2 $\in Q^+$

Identity element exist.

(iv) To find the inverse element:

Let a' be the inverse element of $a \in Q^+$.

Then $a * a' = e \Rightarrow \left(\frac{a\,a'}{2}\right)$ = e $\Rightarrow \left(\frac{a\,a'}{2}\right)$ = 2 (since e =2)

$\Rightarrow a\,a' = 4 \quad \Rightarrow$ a$'$ = $\frac{4}{a} \in Q^+$

(v) To verify commutative property

Consider $a * b = \left(\frac{ab}{2}\right) = \left(\frac{ba}{2}\right) = b * a$

Therefore, $a * b = b * a \ \forall \ a,b \in Q^+$

$(Q^+, *)$ is commutative.

Hence $(Q^+, *)$ is an abelian group

PROBLEM 3: If S is the set of all ordered pairs (a,b) of real numbers with the binary operation $\oplus$ defined by $(\boldsymbol{a},\boldsymbol{b}) \oplus (\boldsymbol{c},\boldsymbol{d}) = (\boldsymbol{a}+\boldsymbol{c},\ \boldsymbol{b}+\boldsymbol{d})$ where a,b,c,d are real, prove that $(S,\ \oplus)$ is a commutative group.

Solution:

Given S = {(a,b): a, b ∈ R}

Let x,y,z ∈ S where x= (a,b), y=(c,d), z=(e,f) and a,b,c,d,e,f are real numbers.

(i) Closure property:

Let x, y ∈ S

$x \oplus y = (a,b) \oplus (c,d) = (a+c,\ b+d) \in$ S (since a+c, b+d ∈ R)

$\Rightarrow x \oplus y \in$ S

' $\oplus$' satisfies closure property

$(S,\ \oplus)$ is closure.

(ii) Associative property:

To prove : $x \oplus (y \oplus z) = (x \oplus y) \oplus z \quad \forall\ x,y,z \in S$

Now ,

$$x \oplus (y \oplus z) = (a,b) \oplus ((c,d) \oplus (e,f))$$

$$= (a,b) \oplus (c+e,\ d+f)$$

$$x \oplus (y \oplus z) = (a+c+e,\ b+d+f) \text{ ....(1)}$$

$$(x \oplus y) \oplus z = ((a,b) \oplus (c,d)) \oplus (e,f)$$

$$= (a+c, b+d) \oplus (e,f)$$

$$(x \oplus y) \oplus z = (a+c+e,\ b+d+f) \quad \text{.....(2)}$$

From (1) and (2), $a * (b * c) = (a * b) * c \ \forall\ a,b,c \in Q^+$

' $\oplus$ ' satisfies associative property

$(S, \oplus)$ is associative.

(iii) To find the Identity element:

Let x $\in S$ and e=(e_1, e_2) be the identity element of S where $e_1, e_2 \in R$

Now $x \oplus e = x \;\; \forall \;\; x \in S$

$\Rightarrow (a, b) \oplus (e_1, e_2) = (a, b)$

$\Rightarrow (a + e_1,\ b + e_2) = (a, b) \Rightarrow a + e_1 = a$ and $b + e_2 = b$

$\Rightarrow e_1$ =a-a =0 and e_2 =b-b =0

$\Rightarrow (e_1, e_2)$ = (0,0)

Identity element of S is e = (e_1, e_2) = (0,0)

(iv) To find the inverse element:

Let $x' = (a', b') \in S$ where $a', b' \in R$ and $x' = (a', b')$ be the inverse element of $x = (a, b) \in S$.

Now $x \oplus x' = e$

$\Rightarrow (a, b) \oplus (a', b') = (e_1, e_2)$

$\Rightarrow (a + a',\ b + b') = (0, 0) \quad \Rightarrow a + a' = 0$ and $b + b' = 0$

$\Rightarrow a'$ =0-a =-a and b' =0-b =-b

$\Rightarrow x' = (a', b')$ = (-a.-b)

Therefore, the inverse of $x = (a, b)$ is x' = (-a.-b)

So, the inverse axiom is satisfied.

Hence $(S, \oplus)$ is a group

(v) To verify commutative property

Consider $x \oplus y = (a,b) \oplus (c,d) = (a + c,\ b + d)$

$= (c + a,\ d + b)$

$= y \oplus x$

Therefore, $x \oplus y = y \oplus x \;\; \forall\, x, y \in S$

$(Q^+, *)$ is commutative.

Hence $(S, \oplus)$ is an abelian group

PROBLEM 4: Examine G $= \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \neq 0 \in R \right\}$ is a commutative group under matrix multiplication where R is the set of real numbers.

Solution:

To verify (G·, ·) is commutative group

Given G $=\left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \neq 0 \in R\right\}$

Let A = $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$, B = $\begin{pmatrix} b & b \\ b & b \end{pmatrix}$ and C = $\begin{pmatrix} c & c \\ c & c \end{pmatrix}$ be any three matrices in G and $a \neq 0, b \neq 0, c \neq 0 \in R$.

(i) Closure property:

Let A,B ∈ G

$AB = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix}$ ∈ G (since 2ab ∈ R)

$\Rightarrow AB \in$ G

$(G, \cdot)$ is closure.

(ii) Associative property:

WKT, matrix multiplication is associative

Therefore A(BC)=(AB)C ∈ G

$(G, \cdot)$ is associative.

(iii) To find the Identity element:

Let I $=\begin{pmatrix} x & x \\ x & x \end{pmatrix} \in G$ be the identity element of G where $x \neq 0 \in R$

Now $AI = A \Rightarrow \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2ax & 2ax \\ 2ax & 2ax \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

$\Rightarrow 2ax = a \Rightarrow x = \frac{1}{2}$

Identity element of G is I = $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Therefore, the identity element exist for $(G, \cdot)$

(iv) To find the inverse element:

Let $A' = \begin{pmatrix} a' & a' \\ a' & a' \end{pmatrix} \in G$ where $a' \neq 0 \in R$ and $A' = \begin{pmatrix} a' & a' \\ a' & a' \end{pmatrix}$ be the inverse element of $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \in G$.

Now $A.A' = I$

$\Rightarrow \begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} a' & a' \\ a' & a' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2aa' & 2aa' \\ 2aa' & 2aa' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\Rightarrow 2aa' = \frac{1}{2} \Rightarrow a' = \frac{1}{4a}$

Therefore, the inverse of $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ is $A' = \begin{pmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{pmatrix}$

So, the inverse axiom is satisfied.

Hence $(G, \cdot)$ is a group

(v) To verify commutative property

Since ab = ba $\forall\ a, b \in R$, for any $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}, B = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \in G$,

we have AB = BA

$(G, \cdot)$ is commutative.

Hence $(G, \cdot)$ is an abelian group or commutative group

PROBLEM 5: Show that the set G ={0,1,2,3,4,5} is a group under addition modulo 6.

Solution:

To Prove: (G, $+_6$) is a group

Given G ={0,1,2,3,4,5}

We can form Cayley table and verify the group axioms

The Cayley table is

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(i) Closure property:

The body of the table contain only elements of G once is each row and column.

Therefore, $(G, +_6)$ is closure.

(ii) Associative property:

Since usual addition is associative, $+_6$ is associative

[For example, let $2, 3, 5 \in G$

Then $2 +_6 (3 +_6 5) = 2 +_6 2 = 4$

$(2 +_6 3) +_6 5 = 5 +_6 5 = 4$

$2 +_6 (3 +_6 5) = (2 +_6 3) +_6 5$]

$(G, +_6)$ is associative.

(iii) To find the Identity element:

0 is the identity element.

Therefore, the identity element exist for $(G, +_6)$,

(iv) To find the inverse element:

Inverse of 0 is 0.
Inverse of 1 is 5.
Inverse of 2 is 4.
Inverse of 3 is 3.

Hence $(G, +_6)$ is a group

To Prove: $(G\cdot, +_6)$ is an abelian group

(v) To verify commutative property

Since usual addition is commutative, $+_6$ is commutative.

{For example, let $3, 4 \in G$

Then $(3 +_6 4) = 1$

$(4 +_6 3) = 1$

$(3 +_6 4) = (4 +_6 3)$ }

$(G, +_6)$ is commutative.

Hence $(G, +_6)$ is an abelian group or commutative group

PROBLEM 6: Let S = Q x Q be the set of all ordered pairs of rational numbers and given by (a, b) ∗ (x, y) = (ax, ay + b).

(i) Check (S, ∗) is a semi group. Is it associative?
(ii) Also find the identity element of S.

Solution:

To prove: Closure property:

For all $(\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d}) \in \mathbf{Q} \times \mathbf{Q}$,

$(a, b) * (x, y) = (ax, ay + b) \in \boldsymbol{Q} \times \boldsymbol{Q}$

' ∗ ' satisfies closure property

(S, ∗) is closure.

To prove: Associative property

Consider [(a, b) ∗ (x, y)] ∗ (c, d) = [(ax, ay + b) ∗ (c, d)]

= [axc, axd + (ay + b)]

= [acx, adx + ay + b] → (i)

Now, (a, b) ∗ [(x, y) ∗ (c, d)] = (a, b) ∗ [xc, xd + y]

= [axc, a(xd + y) + b]

= [axc, axd + ay + b]

= [acx, adx + ay + b] → (ii)

From (i) and (ii), we have

[(a, b) ∗ (x, y)] ∗ (c, d) = (a, b) ∗ [(x, y) ∗ (c, d)]

∴ ∗ is associative.

∴ (S, ∗) is a semi group.

To prove: Commutative property.

(a, b) ∗ (x, y) = (ax, ay + b) → (iii)

(x, y) ∗ (a, b) = (xa, xb + y)

= (ax, bx + y) → (iv)

From (iii) and (iv) (a, b) ∗ (x, y) ≠ (x, y) ∗ (a, b)

∴ (S, ∗) is not commutative.

Existence of identity property.

Let (e_1, e_2) be the identity element of $(S, *)$.

Then for any $(a, b) \in S$, $(a, b) * (e_1, e_2) = (a, b)$

$(a e_1, ae_2 + b) = (a, b) \Rightarrow ae_1 = a$ and $ae_2 + b=b$

$\Rightarrow e_1 = 1$ and $e_2 = \frac{b-b}{a} = 0$ $(a \neq 0)$

$\therefore$ The identity element $= (e_1, e_2) = (1, 0)$.

PROBLEM 7: If Z_6 is the set of equivalence classes generated by the equivalence relation " congruence modulo 6 ", prove that $\{Z_6, x_6\}$ is a monoid where the operation x_6 on Z_6 is defined as $[i] \ x_6 \ [j] = [(i \times j) \ (\text{mod } 6)]$ for any $[i], [j] \in Z_6$. Which elements of the monoid are invertible?

Solution:

Congruence table:

x_6	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

The operation x_6 is associative.

For example, $\{[2] \ x_6 \ [4]\} \ x_6 \ [5] = [2] \ x_6 \ [5] = [4]$

Also, $[2] \ x_6 \ \{[4] \ x_6 \ [5]\} = [2] \ x_6 \ [2] = [4]$

We see that [1] is the identity element of $\{Z_6, x_6\}$ as $[1] \ x_6 \ [1] = [1]$ and $[5] \ x_6 \ [5] = [1]$

$\therefore$ The elements [1] and [5] alone are invertible and their inverses are [1] and [5] respectively.

2. CYCLIC GROUP

A group $(G, *)$ is called a cyclic group if for every $x \in G$ can be expressed as $x = a^m$ or $x = ma$ for some $a \in G$ and $m \in Z$.

Here '*a*' is called the generator of the cyclic group G.

Note: A cyclic group can have more than one generator.

Example

Consider the group $(G, \times) = (\{1, -1, i, -i\}, \times)$ under usual multiplication.

We have , $i^1 = i,\ i^2 = -1, i^3 = -1,\ i^4 = 1$

So i is a generator of the group.

Similarly, we have $-i$ is another generator of the group.

Order of a group:

The number of elements in a group $(G, *)$ is called order of a Group and is denoted by $O(G)$ **.**

Order of an element:

Let $(G, *)$ be a group and $a \in (G, *)$ **.** Then the least positive integer n such that $a^n = e$ is called the order of the element *a* .
(i.e., $a^n = e \Leftrightarrow O(a) = n$ **).**

Example

Consider the group $(\{1, -1, i, -i\}, \times)$

$i^1 = i\ ; i^2 = -1\ ; i^3 = -i\ ; i^4 = 1\ ; i^8 = 1\ ; i^{12} = 1, \ldots$
Therefore $O(i) = 4$

THEOREM 2.1: Every cyclic group is an abelian.

PROOF:
Let $(G, *)$ be a cyclic group with generator 'a'.

To prove: $(G,*)$ is abelian .i.e $(G,*)$ is commutative.

Let $b, c \in (G,*)$

$\Rightarrow b = a^m, c = a^n$ since a is the generator of G

Consider

$b * c = a^m * a^n$

$= a^{m+n}$

$= a^{n+m}$

$= a^n * a^m$

$= c * b$

Therefore, $(G,*)$ is commutative.

Hence, $(G,*)$ is abelian.

Hence, every cyclic group is an abelian.

Note: The converse of the above theorem need not be true.
i.e., Every abelian group need not be cyclic.

THEOREM 2.2:

If a is a generator of the cyclic group $(G,*)$, then a^{-1} is also a generator of $(G,*)$.

PROOF:

Let $(G,*)$ be a cyclic group with generator 'a'.

Then every element $x \in G$ can be written as $x = a^m$, where m is an integer.

$x = a^m$

$x = (a^{-1})^{-m}$, where $-m$ is an integer

$\Rightarrow a^{-1}$ is also a generator of $(G,*)$

THEOREM 2.3:

Let $(G,*)$ be a finite cyclic group generated by $a \in G$.

If $O(G)=n$ then $a^n = e$ so that $G = \{a, a^2, ..., a^n = e\}$ where e is the identity element for $*$ in G.

Furthermore, n is the least positive integer for which $a^n = e$.

PROOF:

Let $(G,*)$ be a cyclic group of order n. i.e., $O(G)=n$

Let $a \in G$ be the generator of G.

To prove this theorem, we should prove two result:

Claim 1: n is the least positive integer for which $a^n = e$.

Claim 2: Every element of $G = \{a, a^2, \ldots, a^n = e\}$ are distinct.

To prove: claim1: n is the least positive integer for which $a^n = e$.

Contradictorily, assume that there exist $m < n$ such that $a^m = e$.
Let $x \in G$
Then $x = a^k$ since a is the generator of G.

Divide *k* by *m*.

$$m)\ k\ (q$$
$$\overline{\quad r \quad}$$

By division algorithm, $k = mq + r,\ 0 \le r < m$

Consider,

$$x = a^k = a^{mq+r} = a^{mq} * a^r = \left(a^m\right)^q * a^r = e^q * a^r = a^r$$

For $x \in G$ We get, $x = a^r,\ 0 \le r < m$.

Therefore every element of G is of the form $x = a^r,\ 0 \le r < m$.

$$\Rightarrow G = \{a^0, a^1, a^2, \ldots, a^{m-1}\}$$

$\Rightarrow O(G) = m < n$ which is a contradiction to the fact that $O(G) = n$.

Therefore our assumption is wrong.

Hence n is the least positive integer for which $a^n = e$.

To prove: claim 2: Every element of $G = \{a, a^2, \ldots, a^n = e\}$ are distinct.

Contradictorily, assume that

$$a^i = a^j \text{ for some } i < j$$

Post operating by a^{-i} on both sides

$$a^i * a^{-i} = a^j * a^{-i}$$

$$e = a^{j-i}$$

Hence we get $a^{j-i} = e$ for $j - i < n$ which is a contradiction to **claim-1.**

Hence our assumption is wrong.

Therefore All the elements of G are distinct.

Hence the Theorem.

3. SUBGROUP

Definition: Subgroup: Let $(G, *)$ be a group and $H \subseteq G$.

$(H, *)$ is called a Sub-group of $(G, *)$ if $(H, *)$ is itself a group.

i.e., $(H, *)$ is (i) closed (ii) Associative (iii) Existence of identity
(iv) Every element in H has Inverse with respect to $*$.

Example:

(i) $(Q, +)$ is a subgroup of $(R, +)$.
(ii) $(R, +)$ is a subgroup of $(C, +)$.

THEOREM 3.1

The identity element of a subgroup is the same as the identity element of the group.

PROOF:

Let $(G, *)$ be group.
Let $(H, *)$ be a sub-group.
Let e be the identity element of the group $(G, *)$.

To Prove: e is the identity element of the subgroup $(H, *)$.

If possible assume that e' be the identity element of $(H, *)$.
Let $a \in H \Rightarrow a \in G$
Since e is the identity element of $(G, *)$, we have

$a * e = a$ --------------------------(1)

Since e' is the identity element of $(H, *)$, we have

$a * e' = a$ --------------------------(2)

From (1) & (2) we have

$$a * e = a * e'$$

$$e = e' \quad [\ left\ cancellation\ law\ ]$$

Hence identity element of a subgroup is the same as the identity element of the group.

THEOREM 3.2:

NECESSARY AND SUFFICIENT CONDITION FOR A SUBGROUP

The necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is $a, b \in H \Rightarrow a * b^{-1} \in H, \forall a, b \in H$.

Necessary Part:

Assume that H is a subgroup of G.
$\Rightarrow$ H is a group itself. i.e., H is closed, associative, has identity and inverse exist.

To Prove: $a, b \in H \Rightarrow a * b^{-1} \in H, \forall a, b \in H$

Let $a, b \in H$

We have $b \in H \Rightarrow b^{-1} \in H$ (Existence of inverse)

Hence $a, b^{-1} \in H \Rightarrow a * b^{-1} \in H$ (Closure Property)

Therefore $a, b \in H \Rightarrow a * b^{-1} \in H, \forall a, b \in H$.

Sufficient Part:

Assume that $a, b \in H \Rightarrow a * b^{-1} \in H, \forall a, b \in H$

To Prove: H is a subgroup of G.

i.e., To prove $(H, *)$ is (i) close (ii) Associative(iii) Existence of identity (iv) Every element in H has Inverse with respect to $*$.

Existence of identity:

Choose $b = a$

$So,\ a, a \in H \Rightarrow a * a^{-1} \in H \Rightarrow e \in H$

Hence Existence of identity.

Every element in H has Inverse with respect to $*$:

Let $e, a \in H \Rightarrow e * a^{-1} \in H \Rightarrow a^{-1} \in H$

Therefore $a \in H \Rightarrow a^{-1} \in H$.

Hence every element has inverse in H.

Closure:

Let $a, b \in H$

We have $b \in H \Rightarrow b^{-1} \in H$ (Existence of inverse)

Hence $a, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H$ (Closure Property)

$\Rightarrow a * b \in H$

Therefore $a, b \in H \Rightarrow a * b \in H$

Hence H is closed.

Associative:

Since $(G, *)$ is a group, $(G, *)$ is Associative.

Hence $(H, *) \subset (G, *)$ is also associative.

Hence $(H, *)$ is a subgroup of $(G, *)$.

Another form of proof:

Prove that the necessary and sufficient conditions for a non-empty subset H of a group G to be a subgroup is a, b $\in$ H $\Rightarrow$ a $*$ b^{-1} $\in$ H.

(OR)

A non-empty subset H of a group G is a subgroup of G iff a $*$ b^{-1} $\in$ H $\forall$ a, b $\in$ H

Proof:

Necessary condition: Let us assume that H is a subgroup of G.

Then H itself is a group under $*$.

$\therefore$ a, b $\in$ H $\Rightarrow$ a $*$ b $\in$ H (closure property)

Since b $\in$ H, b^{-1} $\in$ H

$\therefore$ for a, b $\in$ H, a, b^{-1} $\in$ H $\Rightarrow$ a $*$ b^{-1} $\in$ H

Sufficient condition: Let a $*$ b^{-1} $\in$ H for a, b $\in$ H

Now we prove that H is a subgroup of G.

Let a $\in$ H. $\therefore$ a^{-1} $\in$ H $\Rightarrow$ a $*$ a^{-1} $\in$ H $\Rightarrow$ e $\in$ H

Hence the identity element 'e' exists in H

If a $\in$ H is any element then a, e $\in$ H $\Rightarrow$ e $*$ a^{-1} $\in$ H $\Rightarrow$ a^{-1} $\in$ H

Every element 'a' of H has its inverse a^{-1} in H.

Theorem 3.3:

The intersection of two subgroups of a group G is also a subgroup of G.

Proof: Let H and K be two subgroups of a group (G, $*$). As e $\in$ H and e $\in$ K, e $\in$ H $\cap$K where e is the identity element of G.

So H $\cap$K is non-empty $\rightarrow$ (i).

Let a, b $\in$ H $\cap$K. Then a, b $\in$ H and a, b $\in$ K. Since H and K are subgroups of G,

a $*$ b^{-1} $\in$ H and a $*$ b^{-1} $\in$ k. $\therefore$ a $*$ b^{-1} $\in$ H $\cap$K $\rightarrow$ (ii)

From (i) and (ii) , H $\cap$K is a subgroup of G.

RESULT:

The union of two subgroups need not be a subgroup.

PROOF:

The proof is given by the following example:

Consider the group $(Z,+)$, where Z is the set of integers.

Consider the following two subgroups $(H_1,+)$ and $(H_2,+)$ of $(Z,+)$ where

$H_1=\{.........., -4, -2, 0, 2, 4 ,..........\}$ and

$H_2=\{.........., -6, -3, 0, 3, 6 ,..........\}$

$\Rightarrow H_1 \cup H_2=\{.........., -6, -4, -3, -2, 0, 2, 3, 4, 6 ,..........\}$

Clearly $2, 3 \in H_1 \cup H_2$.

But $2 + 3 = 5 \notin H_1 \cup H_2$

$\Rightarrow H_1 \cup H_2$ is not closed.

Therefore $H_1 \cup H_2$ is not a subgroup.

Hence, The union of two subgroups need not be a subgroup.

4. COSETS

DEFINTION: Let (H, $*$) be a subgroup of a group (G, $*$).

Left coset of H: For any $a \in G$, the left coset of H is defined by

$a * H = \{a * h : h \in H\}$.

Right coset of H: For any $a \in G$, the right coset of H is defined by

$H * a = \{h * a : h \in H\}$.

Note: The right and left cosets are also denoted by aH and Ha respectively.

Example: Consider the group G = {1, -1 ,i, -i} and subgroup H = {1, -1} under the usual multiplication. (i.e $*$ $is\ multiplication$)

The left cosets are

1* H =1H = {1, -1}

-1*H =-1H= {-1 ,1} = {1, -1} = 1H

i * H = iH = {i, -i}

-i * H = -iH = {-i, i} = {i, -i} =iH.

Therefore ,the distinct left cosets are 1H = {1, -1} and iH = {i, -i}.

Note:

(i) Both left and right cosets of H in G is non empty
(ii) Since $e \in H$, $e * H = H * e = H$
(iii) $H * a$ and $a * H$ are also subsets of G
(iv) If G is abelian, then $a * H = H * a$
(v) The union of all left or right cosets of H in G is equal to G
(vi) Cosets are either disjoint or identical

Example: consider the group $Z_4 = \{[0],[1],[2],[3]\}$ of integers modulo 4. Let H = $\{[0], [2]\}$ be a subgroup of Z_4 under $+_4$ (addition modulo 4).

Solution: Given $(H, +_4)$ is a subgroup of a group $(Z_4, +_4)$

The left cosets of H are

[0] + H = $\{[0], [2]\}$ = H

[1] + H = $\{[1], [3]\}$

[2] + H = $\{[2], [4]\}$ =$\{[2], [0]\}$ =$\{[0], [2]\}$ = H

[3] + H = $\{[3], [5]\}$ = $\{[3], [1]\}$ = $\{[1], [3]\}$ = [1] + H

Therefore , the two distinct left cosets are [0] + H and [1] + H of H in Z_4 .

Note: The union of all distinct left or right cosets form a group

In previous example, union of [0] + H and [1] + H is Z_4

since [0] + H U [1] + H =$\{[0], [2]\}$ U $\{[1], [3]\}$ = $\{[0],[1],[2],[3]\}$ = Z_4

Theorem 4.1: Any two right (or left) cosets of H are either identical or disjoint.

Proof: Let $H * a \; and \; H * b$ be two right cosets of a subgroup h of G.

Let $a,b \in G$.

To prove : $(H * a) \cap (H * b) = \emptyset$ (or) $(H * a) = (H * b)$

Suppose $(H * a) \cap (H * b) \neq \emptyset$, then there exists an element

$x \in (H * a) \cap (H * b)$

$\Rightarrow x \in (H * a) \; and \; x \in (H * b)$

If $x \in (H * a) \;\; then \; H * x = H * a$ -----(1)

[by theorem, $if \; a \in (H * b) \;\; then \;\; H * a = H * b$]

If $x \in (H * b) \;\; then \; H * x = H * b$………(2)

From (1) and (2)

$(H * a) = (H * b)$.

Hence the proof.

Theorem : Let (H, $*$) be a subgroup of a group (G, $*$).The set of left (or right) cosets of H in G forms a partition of G.

Proof: Given (G, $*$) be a group and (H, $*$) be a subgroup.

Let us first prove that every element of G appears in atleast one left coset.

Let $a * H = \{a * h : h \in H\}$ be a left coset of H for all $a \in G$.

For $e \in H \Rightarrow a * e \in a * H \Rightarrow a \in a * H$.

Therefore, every element of G appears in atleast one left coset.

We know that , any two left (or right) cosets of H are either identical or disjoint.

Hence, each element of G appears in exactly one and only one left coset of H in G.

Since the union of all distinct left (or right) cosets of H in G is equal to G, the set of left (or right) cosets forms a partition of G.

Lagrange's theorem: If H is a subgroup of a finite group then order of H divides order of G [i.e., O(H)/O(G)]

(or)

The order of each subgroup of a finite group is a divisor of the order the group.

Proof: Let (G, $*$) be a finite group of order n. i.e., O(G) = n

and (H, $*$) be a subgroup of order m. i.e., O(H) = m.

To prove : $O(H)$ *divides* $O(G)$ *i.e.,* $O(H)/O(G)$ *i.e.,* m/n

i.e., $\frac{n}{m} = k$, k *a constant* i.e., $\frac{O(G)}{O(H)} = k$

Since G is a finite group of order n, the number of left cosets of H in G is finite.

Let k be the number of **distinct** left cosets of H in G.

Let the k cosets be $a_1 * H, a_2 * H, a_3 * H, \ldots\ldots a_k * H$.

We know that the left cosets of H form a partition of G.

Therefore , G = $(a_1 * H) \cup (a_2 * H) \cup (a_3 * H) \ldots\ldots \cup (a_k * H)$

$\Rightarrow O(G) = O[(a_1 * H) \cup (a_2 * H) \cup (a_3 * H) \ldots\ldots \cup (a_k * H)]$

$\Rightarrow O(G) = O(a_1 * H) + O(a_2 * H) + O(a_3 * H) \ldots\ldots + O(a_k * H)$

$\Rightarrow O(G) = O(H) + O(H) + O(H) \ldots\ldots + O(H)$ [since $(a * H) = O(H)$]

$\Rightarrow O(G) = k\, O(H)$

$\Rightarrow$ n = km

$\Rightarrow \frac{n}{m} = k$

$\Rightarrow \frac{O(G)}{O(H)} = k$

$\Rightarrow \; O(H)/O(G)$

$\Rightarrow \; O(H) \; divides \; O(G)$

Hence the proof.

5. NORMAL SUBGROUPS

A subgroup (H, $*$) of a group (G, $*$) is said to be a normal subgroup, for every $x \in G$ and for $h \in G$ if $x * h * x^{-1} \in H$ $i.e,\; x * H * x^{-1} \subseteq H$.

Another form of definition: A subgroup (H, $*$) of a group (G, $*$) is called a normal subgroup if $x * h = h * x \; \forall \; x \in G$.

(or) A subgroup H of a group G is called a normal subgroup if $xH = Hx \; \forall \; x \in G$.

Theorem 5.1: A subgroup (H, $*$) of a group (G, $*$) is normal subgroup if and only if $\mathrm{x * h * x^{-1} = H} \;\; \forall \, \mathrm{x \in G}$ and $\mathrm{h \in H}$

Proof:

Necessary Part:

$Let\; x * h * x^{-1} = H \;\; \Rightarrow \;\; x * H * x^{-1} \subseteq H \;\; \forall \, x \in G$

$\Rightarrow$ H is a normal subgroup of G

Sufficient Part:

Conversely, assume that H is a normal subgroup of G.

$\Rightarrow x * H * x^{-1} \subseteq H \;\; \forall \, x \in G$ ....(1)

Now $x \in G \; \Rightarrow \; x^{-1} \in G$

i.e., $x^{-1} * H * (x^{-1})^{-1} \subseteq H \;\; \Rightarrow \; x^{-1} * H * x \subseteq H$

$\Rightarrow \; x * x^{-1} * H * x * x^{-1} \subseteq x * H * x^{-1}$

$\Rightarrow e * H * e \subseteq x * H * x^{-1}$

$\Rightarrow H \subseteq x * H * x^{-1}$ ....(2)

From (1) and (2), we get $x * h * x^{-1} = H \;\; \forall \, x \in G$ and $h \in H$

Theorem : *The intersection of any two normal subgroups of a group is a normal subgroup of a group.*

(or)

If H and K are normal subgroups of a group G then H ∩ K is also a normal subgroup of a group G.

Proof: Let (H, $*$) and (K, $*$) be two normal subgroups of a group (G, $*$).

Given H and K are normal subgroups

$\Rightarrow$ H and K are subgroups of G.

By theorem, the intersection of any two subgroups is also a subgroup.

Therefore, H ∩ K is a subgroup of G.

To Prove: H ∩ K is a normal subgroup of G.

Let $x \in G$ and $h \in \mathrm{H} \cap \mathrm{K}$

i.e., $x \in G$ and [$h \in \mathrm{H}$ & $h \in \mathrm{K}$]

$\Rightarrow$ $x \in G$, $h \in H$ and $\in G$, $h \in K$

$\Rightarrow x * h * x^{-1} \in H$ and $x * h * x^{-1} \in K$ (since H and K are normal subgroup of G)

$\Rightarrow x * h * x^{-1} \in \mathrm{H} \cap \mathrm{K}$

$\Rightarrow$ H ∩ K is a normal subgroup of G.

Theorem : *Every subgroup of an abelian group is a normal subgroup.*

Proof:

Let G be an abelian group and H be a subgroup of G.

To Prove: H is a Normal subgroup of G

Consider $x * H * x^{-1} = x * (H * x^{-1})$

$$= x * (x^{-1} * H) \ \ (\text{G is an abelian})$$

$$= (x * x^{-1}) * H$$

$$= e * H$$

$$x * H * x^{-1} = H$$

$\Rightarrow$ $x * H * x^{-1} = H$ for all $x \in G, h \in H$

Hence, H is a normal subgroup of G

6.HOMOMORPHISM

Group Homomorphism:

Let (G, $*$) and (H, Δ) be any two groups. A mapping $f : G \longrightarrow H$ is called group homomorphism if $f(a * b) = f(a)\, \Delta\, f(b)$ for all $a,b \in G$.

(or) Let (G, $*$) and (G′, $*$) be two groups. A mapping $f : G \longrightarrow G'$ is called group homomorphism if $f(a * b) = f(a) * f(b)$ for all $a,b \in G$.

Isomorphism:

Let (G, $*$) and (H, Δ) be any two groups. A mapping $f : G \longrightarrow H$ is called an isomorphism if

(i) f is homomorphism *i.e.,* $f(a * b) = f(a)\, \Delta\, f(b)$ for all $a,b \in G$.
(ii) *f is ono to one (injective)*
(iii) *f is onto (surjective).*

In otherwords, a bijective homomorphism is said to be an isomorphism.

Another definition of Isomorphism:

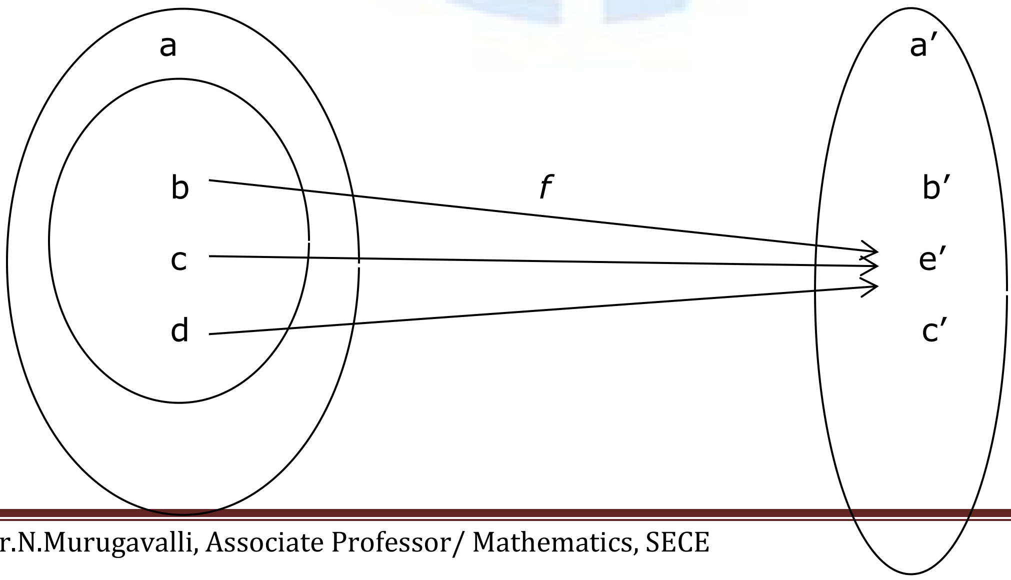
Let (G, $*$) and (G′, $*$) be any two groups. A homomorphism $f : G \longrightarrow G'$ is called an isomorphism if f is one to one and onto.

Then we say that G and G′ are isomorphic and it can be written as G≅G′.

Kernel of a homomorphism: Definition:

Let f : G $\longrightarrow$ G′ be a group homomorphism. The set of elements of G which are mapped into e′ (i.e., e′ is an identity element of G′) is called the kernel of f and it is denoted by ***ker(f)*** .i.e., *ker(f)* $= \{ x \in G \,/\, f(x) = e' \}$

for example,



e d'

G G'

Here *ker(f)* = {b,c,d}.

Quotient group or Factor group:

Let (H, $*$) be a normal subgroup of a group (G, $*$) and G/H denotes the set of all left (or right) cosets of H in G. i.e., G/H = $\{a * H : \forall\, a \in G\}$. Then an algebraic structure (G/H, $\oplus$) is said to be a quotient group if $(a * H) \oplus (b * h) = (a * b) * H \;\forall\, a,b \in G$.

Natural Homomorphism:

Let (H, $*$) be a normal subgroup of a group (G, $*$). A mapping *f : G* $\longrightarrow$ *G/H such that* $f(x) = H * x$, $\forall\, x \in G$ is called a natural homomorphism of the group G onto the quotient group G/H.

Theorem 6.1: Homomorphism preserves identities,inverses and subgroup.

Proof: Let f : (G, $*$) $\longrightarrow$ (G′, $*$) be a homomorphism

To prove: Homomorphism preserves identities

Let $a \in G$. Clearly $f(a) \in G'$.

Now $f(a) * e' = f(a)$ (since e′ is the identity element of G′)

$= f(a * e)$ (since e is the identity element of G)

$f(a) * e' = f(a) * f(e)$ (since f is homomorphism)

$\Rightarrow f(e) = e'$.

Therefore, f preserves identities

To prove: Homomorphism preserves inverses

Let $a \in G$. Since G is group, $a^{-1} \in G$

Now $f(e) = e'$

$f(a * a^{-1}) = e'$ [$a * a^{-1} = a^{-1} * \mathrm{a} = \mathrm{e}$]

$f(a) * f(a^{-1}) = e'$

Therefore $f(a^{-1})$ is the inverse of $f(a)$

$i.e., [f(a)]^{-1} = f(a^{-1}),$

Therefore, f preserves inverses.

To prove: Homomorphism preserves subgroup

Let H be a subgroup of G. Then $f(H) = \{f(h) / h \in H\}$

Since H is non empty, e $\in H$.

If $h,k \in H$ then $h * k^{-1} \in H$ [since H is a subgroup]

Let $h * k^{-1} = m$

Now we have to prove that f(H) is a subgroup of G'.

i.e., $h',k' \in f(H) \Rightarrow h' * (k')^{-1} \in f(H)$

Let $h',k' \in f(H)$. [$since\ f(h) = h'\ and\ f(k) = k'$]

Consider $h' * (k')^{-1} = f(h) * (f(k))^{-1}$

$= f(h) * (f(k^{-1})$

$= f(h * k^{-1})$

$= f(m) \in f(H)$

$h',k' \in f(H) \Rightarrow h' * (k')^{-1} \in f(H)$

Therefore , f(H) is a subgroup of G'.

Hence, f preserves subgroup

Thereom 6.2: If f : (G, $*$) $\longrightarrow$ (G', $*$) is a homomorphism then the Ker(f) is a normal subgroup of G

Proof: WKT, *ker(f)* = { $x \in G$ / *f(x)* = e' } , $e' \in G'$

To Prove: **Ker(f) is a normal subgroup of G**

i.e., we have to prove the following

(i) ker*(f)* is non-empty

(ii) *ker(f)* is a subgroup of G

(iii) *ker(f)* is a normal subgroup of G

Proof of (i): *ker(f)* is non-empty

Since f(e) = e' is always true.

Therefore, atleast $e \in ker(f)$

Hence, ker(f) is non empty.

Proof of (ii): *ker(f)* is a subgroup of G i.e., it is enough to prove that $a,b \in ker(f) \Rightarrow a * b^{-1} \in \ker(f)$

Let $a,b \in ker(f)$. Then $f(a) = e'$ and $f(b) = e'$

$f(a * b^{-1}) = f(a) * f(b^{-1})$

[since *f* is a group homomorphism, $f(a * b) = f(a) * f(b)$]

$\Rightarrow f(a * b^{-1}) = f(a) * (f(b))^{-1}$

$\Rightarrow f(a * b^{-1}) = e' * (e')^{-1}$

$\Rightarrow f(a * b^{-1}) = e' * e'$

$\Rightarrow f(a * b^{-1}) = e'$

$\Rightarrow a * b^{-1} \in ker(f)$

Therefore, $a,b \in ker(f) \Rightarrow a * b^{-1} \in \ker(f)$

Hence, *ker(f)* is a subgroup of G

Proof of (iii): *ker(f)* is a normal subgroup of G.
i.e., $x * h * x^{-1} \in ker(f)$ for all $x \in G$, $h \in Ker(f)$

i.e., it is enough to prove that $f(x * h * x^{-1}) = e'\ \forall x \in G \text{ and } h \in ker(f)$

Now $f(x * h * x^{-1}) = f(x) * f(h) * f(x^{-1})$

[since *f* is a group homomorphism, $f(a * b) = f(a) * f(b)$]

$\Rightarrow f(x * h * x^{-1}) = f(x) * e' * f(x^{-1})$ [$h \in Ker(f),\ f(h) = e'$]

$\Rightarrow f(x * h * x^{-1}) = f(x) * f(x^{-1})$

$\Rightarrow f(x * h * x^{-1}) = f(x * x^{-1})$ [$since\ x * x^{-1} = e$]

$\Rightarrow f(x * h * x^{-1}) = f(e)$

$\Rightarrow f(x * h * x^{-1}) = e'$

$\Rightarrow x * h * x^{-1} \in ker(f)$ for all $x \in G$, $h \in Ker(f)$

Hence, *ker(f)* is a normal subgroup of G

Theorem 6.3: Fundamental theorem on homomorphism of groups:

Every homomorphic image of a group G is isomorphic to some quotient group of G

(or)

Let f : (G, $*$) $\longrightarrow$ (G′,$*$) be a onto homomorphism of groups with Kernel K. Then $G/K \cong$ G′.

Proof: Let f : (G, $*$) $\longrightarrow$ (G′,$*$) be a homomorphism of groups .

Let G′ be the homomorphic image of group G.

Then f is a homomorphism of G onto G′.

Let K be the Kernel of this homomorphism

i.e., K = *ker(f)* = { $x \in G$ / *f(x)* = *e′* } , $e' \in G'$.

Clearly K is a normal subgroup of G . (by theorem 1.2)

Define $\emptyset$: G/K $\longrightarrow$ G′ by $\emptyset(K * a) = f(a)$ for all $a \in G$.

To Prove: $G/K \cong$ G′ i.e., G/K is isomorphic to G′.

i.e., $f(a) \in G'$ $for\ all\ a \in G\ and\ K * a \in G/K$

Define $\emptyset$: G/K $\longrightarrow$ G′ by $\emptyset(K * a) = a$ for all $a \in G$.

To prove this, it is enough to prove the following

(i) $\emptyset$ is well defined
(ii) $\emptyset$ is one to one
(iii) $\emptyset$ is onto
(iv) $\emptyset$ is homomorphism

Claim (i): $\emptyset$ is well defined

i.e., We have to prove that $K * a = K * b \Rightarrow \emptyset(K * a) = \emptyset(K * b)$

Now $K * a = K * b$

$\Rightarrow a * b^{-1} \in K$ (since K is a normal subgroup)

$\Rightarrow f(a * b^{-1})$ = e′

$\Rightarrow f(a) * f(b^{-1}) = e'$

$\Rightarrow [f(a) * (f(b))^{-1}] * f(b) = e' * f(b)$

$\Rightarrow f(a) * [(f(b))^{-1} * f(b)] = e' * f(b)$

$\Rightarrow f(a) * e' = e' * f(b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \emptyset(K * a) = \emptyset(K * b)$

$K * a = K * b \Rightarrow \emptyset(K * a) = \emptyset(K * b)$

$\emptyset$ is well defined

Claim (ii): $\emptyset$ is one to one

To prove: $\emptyset(K * a) = \emptyset(K * b) \Rightarrow K * a = K * b$

Now $\emptyset(K * a) = \emptyset(K * b) \Rightarrow f(a) = f(b)$

$\Rightarrow f(a) * f(b^{-1}) = f(b) * f(b^{-1})$

$\Rightarrow f(a) * f(b^{-1}) = e'$

$\Rightarrow f(a * b^{-1}) = e'$

$\Rightarrow a * b^{-1} \in K$

$\Rightarrow K * a = K * b$

Therefore, $\emptyset(K * a) = \emptyset(K * b) \Rightarrow K * a = K * b$

$\emptyset$ is one to one.

Claim (iii) : $\emptyset$ is onto

Let $y \in G'$ be any element.

Since f is onto, y = f(a) ,$\forall\ a \in G$

$\emptyset(K * a) = f(a) = y.$

$\emptyset\ is\ onto.$

Claim (iv) : $\emptyset$ is homomorphism

Now $\emptyset((K * a) * (K * b)) = \emptyset(K * (a * b))$

$= f(a * b)$

$= f(a) * f(b)$

$\emptyset((K * a) * (K * b)) = \emptyset(K * a) * \emptyset(K * b)$

Therefore, $\emptyset$ is a bijective homomorphism

Hence, $\emptyset$ is an isomomorphism between G/K and G'.

i.e., $G/K \cong$ G'

Permutation Group:

Let S be a non empty set. A bijective function f: $S \to S$ is called permutation . If S has n elements, then the permutation is said to be of degree n.

Note:

1. The set of all permutations on a set of n symbols is denoted by S_n.

2. S_n is a group under composition of functions as operation. The group S_n is called the permutation group on n symbols. It is also known as symmetric group of degree n anf $O(S_n) = n!$.

Theorem 6.4: Cayley's Theorem:

Statement: Every finite group of order n is isomorphic to a permutation group of degree n.

Proof: Let G be a finite group of order n. i.e., O(G) = n.

Step 1: To form a permutation set

Let $a \in G$ be any element. Define a function $f_a : G \longrightarrow G$ by $f_a(x) = a * x$

Claim(i) : f_a is one to one

$f_a(x) = f_a(y) \Rightarrow a * x = a * y \Rightarrow x = y$.

f_a is one to one.

Claim(ii) : f_a is onto

Let $y \in G$. Then $f_a(a^{-1} * y) = a * (a^{-1} * y) = (a * a^{-1}) * y = e * y = y$.

f_a is onto.

Thus $f_a : G \to G$ is one to one and onto function and so, it is permutation set of degree n.

Therefore, permutation set G' = $\{f_a / \in G \}$.

Step 2: G' is a group under composition function operation.

Let G' = $\{f_a / \in G \}$.

Closure property:

Let $f_a , f_b \in G'$.

Now $(f_a \circ f_b)(x) = f_a[f_b(x)]$

$= f_a(b * x]$

$= a * (b * x)$

$= (a * b) * x$

$(f_a \circ f_b)(x) = f_{a*b}(x)$

$\Rightarrow (f_a \circ f_b) = f_{a*b} \in G'$ [since $a,b \in G \Rightarrow a * b \in G$]

$\Rightarrow (f_a \circ f_b) \in G'$.

Hence , G' is closed under composition function operation.

Associative property:

Composition function always satisfies associative property.

Identity and Inverse property:

It is obvious that $f_e \in G'$ is the identity element and $f_{a^{-1}} \in G'$ is the inverse of $f_a \in G'$.

Hence G' is a group under composition function operation.

Step 3: G and G' are isomorphic (or) $\emptyset$ is an isomorphism

Define $\emptyset : G \to G'$ by $\emptyset(a) = f_a \; \forall \; a \in G$.

Claim (a) : $\emptyset$ is one to one

To prove: $\emptyset(a) = \emptyset(b) \Rightarrow f_a = f_b \Rightarrow f_a(x) = f_b(x) \Rightarrow a * x = b * x \Rightarrow a = b$

Therefore, $\emptyset(a) = \emptyset(b) \Rightarrow a = b$.

Hence, $\emptyset$ is one to one

Claim (b) : $\emptyset$ is onto

Since f_a is onto, $\emptyset(a)$ is also onto.

Claim (c) : $\emptyset$ is a homomorphism

Consider for any $\forall \; a , b \in G$,

$\emptyset(a * b) = f_{a*b} = f_a \circ f_b = \emptyset(a) \circ \emptyset(b) \; \forall \; a,b \in G$.

Therefore, $\emptyset$ is a homomorphism.

Hence, G and G' are isomorphic.