DISCRETE MATHEMATICS

III SEMESTER
CSE

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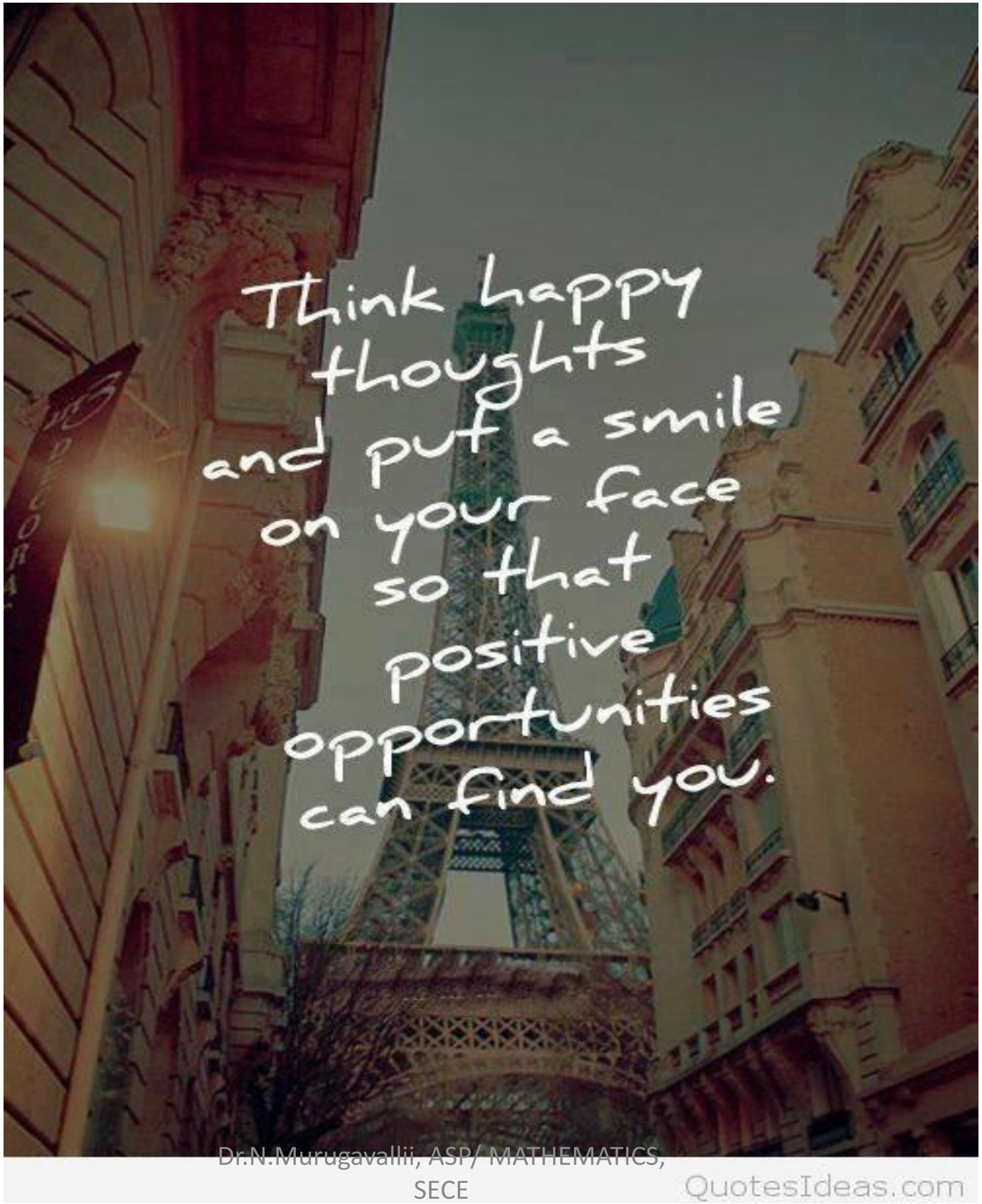
Module 1
Propositional calculus

COURSE OUTCOME:

CO 1: Apply principles and fundamental concepts of inference theory in proving and testing the logics


Think happy
thoughts
and put a smile
on your face
so that
positive
opportunities
can find you.
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CONNECTIVES

CONNECTORS	SYMBOL	NAME	OPERATOR NAME
And	^	Conjunction (And)	Binary
Or	˅	Disjunction (Or)	Binary
If and then	→	Conditionally imply (if ….. Then)	Binary
Iff (if and only if)	↔	Bi conditionally imply (iff)	Binary
Negation	¬	Not	Unary

TRUTH TABLE FOR CONNECTIVES

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

LAWS OF LOGIC

S. No	Name of the law	Primal form	Dual form
1	Idempotent law	$P \vee P \equiv P$	$P \wedge P \equiv P$
2	Identity law	$P \vee F \equiv P$	$P \wedge T \equiv P$
3	Dominant law	$P \vee T \equiv T$	$P \wedge F \equiv F$
4	Compliment law	$P \vee \neg P \equiv T$	$P \wedge \neg P \equiv F$
5	Commutative law	$P \vee Q \equiv Q \vee P$	$P \wedge Q \equiv Q \wedge P$
6	Associative law	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \equiv (P \wedge Q) \wedge R$
7	Distributive law	$(P \vee Q) \wedge R \equiv (P \wedge Q) \vee (P \wedge R)$	$(P \wedge Q) \vee R \equiv (P \vee Q) \wedge (P \vee R)$
8	Absorption law	$P \vee (P \wedge Q) \equiv P$	$P \wedge (P \vee Q) \equiv P$
9	De Morgan's law	$\neg (P \vee Q) \equiv \neg P \wedge \neg Q$	$\neg (P \wedge Q) \equiv \neg P \vee \neg Q$
10	Double nagation law	$\neg(\neg P) \equiv P$	

IMPLICATIONS:

I_1 : $P \wedge Q \Rightarrow P$ I_2 : $P \wedge Q \Rightarrow Q$

I_3 : $P \Rightarrow P \vee Q$ I_4 : $Q \Rightarrow P \vee Q$

I_5 : $\neg P \Rightarrow P \rightarrow Q$

I_6 : $Q \Rightarrow P \rightarrow Q$

I_7 : $\neg (P \rightarrow Q) \Rightarrow P$

I_8 : $\neg (P \rightarrow Q) \Rightarrow \neg Q$

I_9 : $P, Q \Rightarrow P \wedge Q$

I_{10} : $\neg P, P \vee Q \Rightarrow Q$ & $\neg Q, P \vee Q \Rightarrow P$ (DISJUNCTIVE SYLLOGISM)

I_{11} : $P, P \rightarrow Q \Rightarrow Q$ (MODUS PONENS)

I_{12} : $\neg Q, P \rightarrow Q \Rightarrow \neg P$ (MODUS TOLLENS)

I_{13} : $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (HYPOTHETICAL SYLLOGISM)

I_{14} : $P \vee R, P \rightarrow R, Q \rightarrow R \Rightarrow R$

EQUIVALENCES

E_1 : $P \rightarrow Q \equiv \neg P \vee Q$

E_2 : $\neg(P \rightarrow Q) \equiv P \vee \neg Q$ (CONDITIONAL AS DISJUNCTION)

E_3 : $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$ (CONTRAPOSITIVE LAW)

E_4 : $P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$

E_5 : $\neg (P \leftrightarrow Q) \equiv P \leftrightarrow (\neg Q)$

E_6 : $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$ (BICONDITIONAL AS DISJUNCTION)

E_7 : $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

E_8 : $(P \rightarrow Q) \wedge (R \rightarrow Q) \equiv (P \vee R) \rightarrow Q$

E_9 : $P \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$

E_{10} : $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \vee R)$

E_{11} : $P \vee Q \equiv \neg P \rightarrow Q$ E_{12} : $P \wedge Q \equiv \neg (P \rightarrow \neg Q)$

E_{13} : $(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$

E_{14} : $(P \rightarrow Q) \vee (P \rightarrow R) \equiv P \rightarrow (Q \vee R)$

E_{15} : $(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$

E_{16} : $P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$

Proposition

A proposition or statement is a declarative sentence (or assertion) which is either true or false, but not both. It is denoted by the letters P, Q, R, S…… / p, q, r, s except T and F …..

Example:

Mathematics is base for Science. - T

Covid 19 is an infectual disease - T

5 > 6 - F

Krishna wears blue shirt – T / F

Peacock is our national bird - T

Primary (or) Atomic Statement:

A statement can not be splitted further to simple statements is said to be simple or atomic or primitive statement.

Molecular / Compound Statement:

A statement which is the combination of one or more statements is called Compound or Molecular statement.

Converse, Inverse, Contra positive statements

Let $P \rightarrow Q$ be a conditional statement, then

$Q \rightarrow P$ is converse of $P \rightarrow Q$

$\neg P \rightarrow \neg Q$ is inverse of $P \rightarrow Q$

$\neg Q \rightarrow \neg P$ is contra positive of $P \rightarrow Q$

Symbolize the following statement:

1. The automated reply can not be sent when file system is full.

Let A : The automated reply can be sent

B : The file system is full

Then $\mathbf{B} \rightarrow \neg \mathbf{A}$

2. Let p, q, r represent the following propositions,

p : It is raining

q : The sun is shining

r : There are clouds in the sky

Symbolize the follwing statements

(i) It is raining, then there are clouds in the sky

Ans: $p \rightarrow r$

(ii) If it is not raining, then the sun is not shining and there are clouds in the sky.

Ans: $\neg p \rightarrow (\neg q \wedge r)$

(iii) The sun is shining if and only if it is not raining.

Ans: $q \leftrightarrow \neg p$

3. If you are CSE major or you are not a fresh man then you can access internet from campus.

C: you are CSE major

M: you are a fresh man

I: you can access internet from campus

Then $\mathbf{C} \vee \neg\mathbf{M} \rightarrow \mathbf{I}$

Note: If there are n variables in truth table in a given statement formula then there will be 2^n rows in truth table.

Construction of Truth tables:

$\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Tautology

A statement formula which is always true is called tautology.

Example: $P \vee \neg P$

Contradiction

A statement formula which is always false is called Contradiction.

Example: $P \wedge \neg P$

Contingency

A statement formula which is neither tautology nor contradiction is called contingency.

Fallacy

A statement formula which has only once false entry is called fallacy.

Prove that $P \vee Q \leftrightarrow Q \vee P$ is tautology.

P	Q	$P \vee Q$	$Q \vee P$	$P \vee Q \leftrightarrow Q \vee P$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

2. Construct the truth table for $\neg(P \vee (Q \wedge R)) \leftrightarrow (P \vee Q) \wedge (P \vee R)$

P	Q	R	Q ^ R	P ˅(Q ^ R)	¬(P ˅(Q ^ R))	P ˅ Q	P ˅ R	(P ˅ Q) ^ (P ˅ R)	Given formula
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	T	T	F
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	T	T	F
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	F	F	F
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	F	F	F

Therefore the above statement is contradiction.

Try this

a). Construct the truth table for $Q \wedge (P \rightarrow Q) \rightarrow P$.

b). Construct the truth table for $(P \leftrightarrow R) \wedge (\neg Q \rightarrow S)$.

Logical Equivalance

Two propositions P and Q are said to be logically equivalent if they have identical truth values (or) $P \leftrightarrow Q$ is a tautology.

i.e $P \equiv Q$ iff $P \leftrightarrow Q$ is a tautology.

Example: $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

Notation : $P \equiv Q$ **or** $P \Leftrightarrow Q$

Duality law

Two statement formulas P and P* are said to be dual of each other if one can be derived from the other by replacing ∧ as ∨ and ∨ as ∧, **T** as **F** and **F** as **T**.

Example:

1. The duality of $\neg (P \vee \neg Q) \rightarrow (P \leftrightarrow Q)$ is $\neg (P \wedge \neg Q) \rightarrow (P \leftrightarrow Q)$.
2. The dual of $(T \rightarrow \neg P) \vee (P \wedge F)$ is $(F \rightarrow \neg P) \wedge (P \vee T)$.

Tautological Implication

A statement formula A is said to be tautologically imply (i.e) $A \Rightarrow B$ if and only if $A \to B$ is a tautology.

Note: $A \Rightarrow B$ and $B \Rightarrow A$ then $A = B$.

Implication Rules

1. $P \wedge Q \Rightarrow P$ **(Simplification)**
   $P \wedge Q \Rightarrow Q$
2. $P \Rightarrow P \vee Q$ **(Addition)**
   $Q \Rightarrow P \vee Q$
3. $P, Q \Rightarrow P \wedge Q$ **(Conjunction)**
4. $P, (P \to Q) \Rightarrow Q$ **or** $(P \wedge (P \to Q) \Rightarrow Q)$ **(Modus ponens)**
5. $\neg Q, (P \to Q) \Rightarrow \neg P$ **or** $\neg Q \wedge (P \to Q) \Rightarrow \neg P$ **(Modus tollens)**
6. $\neg P, (P \to Q) \Rightarrow \neg P$ **or** $\neg P \wedge (P \to Q) \Rightarrow \neg P$
7. $(P \vee Q) \wedge \neg P \Rightarrow Q$ **(Disjunctive syllogism)**
8. $(P \vee Q) \wedge (\neg P \vee R) \Rightarrow (Q \vee R)$ **(Resolution)**
9. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$ **(DiLemma)**

PROBLEMS BASED ON LOGICAL EQUIVALENCES USING TRUTH TABLES

USING TRUTH TABLE PROVE THE FOLLOWING

1.Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Proof

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Hence, $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

2.Verify whether $(P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.

Verification

P	Q	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since all truth values are true, hence the given formula is a tautology.

Hence, $(P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.

3. Prove that $(\neg p \rightarrow q) \rightarrow (q \rightarrow p)$ is neither tautology nor a contradiction

Proof

p	q	$(\neg p \rightarrow q)$	$q \rightarrow p$	$(\neg p \rightarrow q) \rightarrow (q \rightarrow p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	T

Since the last column is neither all false nor true therefore the given formula is neither a tautology not a contradiction.

4. Prove that $(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Proof

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

From the truth table, we see that $(p \leftrightarrow q)$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ have same truth values. Hence, $(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

PROBLEMS BASED ON LOGICAL EQUIVALENCES WITHOUT USING TRUTH TABLES

Without using truth table, prove the following

1. Show that

$$[\neg P \wedge (\neg Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)] \equiv R$$

SOLUTION:

$[\neg P \wedge (\neg Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)]$

$\equiv [(\neg P \wedge \neg Q) \wedge R] \vee [(Q \vee P) \wedge R)]$ [BY (1) & (2)]

$\equiv [(\neg P \wedge \neg Q) \vee (P \vee Q)] \wedge R$ [BY (2)]

$\equiv [\neg(P \vee Q) \vee (P \vee Q)] \wedge R$ [BY (5)]

$\equiv T \wedge R$ [BY (3)]

$\equiv R$ [BY (4)]

Hence, $[\neg P \wedge (\neg Q \wedge R)] \vee [(Q \wedge R) \vee [(P \wedge R)] \equiv R$

NOTE: FORMULAE

ASSOCIATIVE LAW

$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R \quad \rightarrow (1)$

DISTRIBUTIVE LAW

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)] \rightarrow (2)$

COMPLEMENT LAW: $P \vee \neg P \equiv T \quad \rightarrow (3)$

IDENTITY LAW: $P \wedge T \equiv P \quad \rightarrow (4)$

DE MORGAN'S LAW : $\neg(P \vee Q) \equiv \neg P \wedge \neg Q \quad \rightarrow (5)$

2. Show that $P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$

Solution:

TO PROVE: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) (OR) (1)≡(2)≡(3)

CLAIM: (1)≡(2)

(1)$\equiv P \to (Q \to R) \equiv P \to (\neg Q \vee R)$ [Since $P \to Q \equiv \neg P \vee Q$]

(1)≡(2)

CLAIM: (2)≡(3)

(2) $\equiv P \to (\neg Q \vee R) \equiv \neg P \vee (\neg Q \vee R)$ [Since $P \to Q \equiv \neg P \vee Q$]

$\equiv (\neg P \vee \neg Q) \vee R$ [AL:$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$]

$\equiv \neg (P \wedge Q) \vee R$ [DL:$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$]

$\equiv (P \wedge Q) \to R$ [Since $P \to Q \equiv \neg P \vee Q$]

(2)≡(3)

Hence, (1)≡(2)≡(3)

3. Show that $\neg(P \wedge Q) \rightarrow [\neg P \vee (\neg P \vee Q)] \equiv (\neg P \vee Q)$

Solution:

$\neg(P \wedge Q) \rightarrow [\neg P \vee (\neg P \vee Q)]$

$\equiv \neg(P \wedge Q) \rightarrow [(\neg P \vee \neg P) \vee Q]$ [AL: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$]

$\equiv \neg(P \wedge Q) \rightarrow [\neg P \vee Q]$ **[Since** $P \vee P \equiv P$ **]**

$\equiv (P \wedge Q) \vee [\neg P \vee Q]$ **[Since** $P \rightarrow Q \equiv \neg P \vee Q$]

$\equiv [(P \wedge Q) \vee \neg P] \vee Q$ [AL: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$]

$\equiv [(P \vee \neg P) \wedge (Q \vee \neg P)] \vee Q$ [DL: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$]

$\equiv [T \wedge (Q \vee \neg P)] \vee Q$ [$P \vee \neg P \equiv T$]

$\equiv [(Q \vee \neg P)] \vee Q$ [$T \wedge P \equiv P$]

$\equiv (\neg P \vee Q) \vee Q$ [Associative law]

$\equiv \neg P \vee (Q \vee Q)$ [Associative law]

$\equiv (\neg P \vee Q)$. [$P \vee P \equiv P$]

4. Show that $((P \vee Q) \wedge [\neg P \wedge (\neg P \wedge Q)]) \equiv (\neg P \wedge Q)$

Solution:

$(P \vee Q) \wedge [\neg P \wedge (\neg P \wedge Q)]$

$\equiv (P \vee Q) \wedge [(\neg P \wedge \neg P) \wedge Q)]$ [Associative law]

$\equiv (P \vee Q) \wedge (\neg P \wedge Q)$ [since $P \wedge P \equiv P$]

$\equiv [(P \vee Q) \wedge \neg P] \wedge Q$ [Associative law]

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \wedge Q$

[DL: $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$]

$\equiv [F \vee (Q \wedge \neg P)] \wedge Q$ [since $P \wedge \neg P \equiv F$]

$\equiv [(F \vee Q) \wedge (F \vee \neg P)] \wedge Q$

[DL: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$]

$\equiv [Q \wedge \neg P] \wedge Q$ [since $F \vee P \equiv P$]

$\equiv \neg P \wedge (Q \wedge Q)$ [Associative law]

$\equiv \neg P \wedge Q$ [since $P \wedge P \equiv P$]

5. Show that

$\neg(P \leftrightarrow Q) \equiv (P \wedge \neg Q) \vee (Q \wedge \neg P) \equiv (P \vee Q) \wedge \neg (P \wedge Q)$

Solution:

$\neg(P \leftrightarrow Q) \equiv \neg [(P \to Q) \wedge (Q \to P)]$

$[P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)]$

$\equiv \neg [(\neg P \vee Q) \wedge (\neg Q \vee P)]$

$[P \to Q \equiv \neg P \vee Q]$

$\equiv [\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)]$

[De Morgan's law $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$]

$\equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$

[De Morgan's law $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$]

$\equiv [(P \wedge \neg Q) \vee Q] \wedge [(P \wedge \neg Q) \vee \neg P]$

[DL: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$]

$\equiv [(P \vee Q) \wedge (\neg Q \vee Q)] \wedge [(P \vee \neg P) \wedge (\neg Q \vee \neg P)]$ [DL]

$\equiv [(P \vee Q) \wedge T] \wedge [T \wedge (\neg Q \vee \neg P)]$ [since $\neg P \vee P \equiv T$]

$\equiv (P \vee Q) \wedge (\neg Q \vee \neg P)$ [since $P \wedge T \equiv P$]

$\equiv (P \vee Q) \wedge \neg (Q \wedge P)$

[De Morgan's law $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$]

$\equiv (P \vee Q) \wedge \neg (P \wedge Q)$.

6. Show that $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \Rightarrow r$.

Solution:

To prove: $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \Rightarrow r$ it is enough to prove $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \to r$ is a tautology.

Now consider

$((p \vee q) \wedge (p \to r) \wedge (q \to r)) \to r$

$\equiv ((p \vee q) \wedge ((p \vee q) \to r)) \to r$

[since $(P \to R) \wedge (Q \to R) \equiv (P \vee Q) \to R$]

$\equiv r \to r$ [$p \wedge (p \to q) \equiv q$ here $p = p \vee q$]

$\equiv T$ [$p \to p \equiv T$]

Hence proved.

To prove Equivalence, $(p \vee q) \wedge (p \to r) \wedge (q \to r)$ $\equiv (p \vee q) \wedge ((p \vee q) \to r) \equiv r$.

Try this:

1. Show that
   $[((p \vee \neg p) \to q) \to ((p \vee \neg p) \to r)] \Rightarrow (q \to r)$.
2. Show that $(P \to Q) \Rightarrow [P \to (P \wedge Q)]$
3. Show that $[P \to (Q \to R)] \Rightarrow [(P \to Q) \to (P \to R)]$
4. Show that
   $[p \to (q \vee r)] \equiv \neg r \to (p \to q) \equiv (p \wedge \neg q) \to r$.

PRINCIPAL NORMAL FORMS

MINTERMS:

Let P and Q be two statement variables. Let us construct all possible formulas which consist of conjunction of P or its negation and Q or its negation. None of the formulas should contain both a variable and its negation. Using commutative law if any two terms are equivalent, choose any one of the term. Collect remaining terms. They are called minterms.

MAXTERMS:

Let P and Q be two statement variables. Let us construct all possible formulas which consist of disjunction of P or its negation and Q or its negation. None of the formulas should contain both a variable and its negation. Using commutative law if any two terms are equivalent, choose any one of the term. Collect remaining terms. They are called minterms.

Note:

For n- variables there are 2^n minterms or maxterms

For example, let P and Q be two variables. Then minterms and maxterms of P and Q (i.e., $2^2 = 4$) are

Minterms	***Maxterms***
$P \wedge Q$	$P \vee Q$
$P \wedge \neg Q$	$P \vee \neg Q$
$\neg P \wedge Q$	$\neg P \vee Q$
$\neg P \wedge \neg Q$	$\neg P \vee \neg Q$

For example, let P, Q and R be three variables. Then possible minterms and maxterms of P, Q and R (i.e., $2^3 = 8$) are

Minterms	***Maxterms***
$P \wedge Q \wedge R$	$P \vee Q \vee R$
$\neg P \wedge Q \wedge R$	$\neg P \vee Q \vee R$
$P \wedge \neg Q \wedge R$	$P \vee \neg Q \vee R$
$P \wedge Q \wedge \neg R$	$P \vee Q \vee \neg R$
$\neg P \wedge \neg Q \wedge R$	$\neg P \vee \neg Q \vee R$
$P \wedge \neg Q \wedge \neg R$	$P \vee \neg Q \vee \neg R$
$\neg P \wedge Q \wedge \neg R$	$\neg P \vee Q \vee \neg R$
$\neg P \wedge \neg Q \wedge \neg R$	$\neg P \vee \neg Q \vee \neg R$

Principal Conjunctive Normal Forms (PCNF) and Principal Disjunctive Normal forms (PDNF):

Principal Disjunctive Normal Form (PDNF):

Sum of minterms is known as Principal Disjunctive Normal Form

Example: (minterm) ∨ (minterm) ∨ (minterm) ∨..... ∨(minterm)

Principal Conjunctive Normal Form (PCNF):

Product of maxterms is known as Principal Conjunctive Normal Form

Example:

(maxterm) ∧ (maxterm) ∧ (maxterm) ∧.... ∧ (maxterm)

Note:

(i) PDNF is also called as sum of products canonical form.

(ii) PCNF is also called as Product of sums canonical form

WORKING RULE TO OBTAIN PDNF:

Step 1: By applying various equivalence rules simplify the given statement formula,

if possible (i.e., Rewrite the given statement formula interms of $\vee, \wedge$ and $\neg$ alone)

Step 2: Apply the fact (Each term) $\wedge$ T

($P \wedge T \Leftrightarrow P$ this is possible)

Step 3: Instead of T, apply $P \vee \neg P$($P \vee \neg P \Leftrightarrow T$)

Step 4: Apply Distributive rules.

Step 5: Apply commutative rules.

***WORKING RULE TO OBTAIN PCNF*:**

Step 1: By applying various equivalence rules simplify the given statement formula,

if possible (i.e., Rewrite the given statement formula interms of $\vee$, $\wedge$ and $\neg$ alone)

Step 2: Apply the fact (Each term) $\vee$ F

($P \vee F \Leftrightarrow P$ this is possible)

Step 3: Instead of F, apply $P \wedge \neg P$($P \wedge \neg P \Leftrightarrow F$)

Step 4: Apply Distributive rules.

Step 5: Apply commutative rules.

TO OBTAIN PCNF FROM PDNF:

Step 1: Let S = PDNF of given statement.

Step 2: Write the sum of remaining minterms in S and it is denoted by $\neg$S.

Step 3: Take negation for $\neg$S. (ie)$\neg(\neg S)$=(PCNF of S)

TO OBTAIN PDNF FROM PCNF:

Step 1: Let S = PCNF of given statement.

Step 2: Write the product of remaining maxterms in S and it is denoted by $\neg$S.

Step 3: **T**ake negation for $\neg$S. (ie)$\neg(\neg S)$=(PDNF of S)

NOTE

(i) If a given formula is tautology, then its PDNF includes all the possible minterms of the variable. There is no PCNF.

(ii) If a given formula is contradiction, then its PCNF includes all the possible maxterms of the variables. There is no PDNF.

Obtain the principal disjunctive normal form of $(P\wedge Q) \vee (\neg P\wedge R) \vee (Q\wedge R)$. Also find PCNF

Solution:

$(P\wedge Q) \vee (\neg P\wedge R) \vee (Q\wedge R)$	Reasons
$\Leftrightarrow ((P\wedge Q) \wedge T) \vee ((\neg P\wedge R) \wedge T) \vee ((Q\wedge R) \wedge T)$	$\because$ $P\wedge T\Leftrightarrow P$
$\Leftrightarrow ((P\wedge Q) \wedge (R\vee\neg R)) \vee ((\neg P\wedge R) \wedge (Q\vee\neg Q)) \vee ((Q\wedge R) \wedge (P\vee\neg P))$	$\because$ $P\vee\neg P\Leftrightarrow T$
$\Leftrightarrow (P\wedge Q\wedge R) \vee (P\wedge Q\wedge\neg R) \vee (\neg P\wedge R\wedge Q) \vee (\neg P\wedge R\wedge\neg Q) \vee(Q\wedge R\wedge P) \vee (Q\wedge R\wedge\neg P)$	Distributive law
$\Leftrightarrow (P\wedge Q\wedge R) \vee (P\wedge Q\wedge\neg R) \vee (\neg P\wedge Q\wedge R) \vee(\neg P\wedge\neg Q\wedge R)$	Commutative law & Idempotent law

The required PDNF is

$(P\wedge Q\wedge R) \vee (P\wedge Q\wedge\neg R) \vee(\neg P\wedge Q\wedge R) \vee(\neg P\wedge\neg Q\wedge R)$

To find PCNF:

S = PDNF of given statement

$S = (P\wedge Q\wedge R) \vee (P\wedge Q\wedge \neg R) \vee (\neg P\wedge Q\wedge R) \vee (\neg P\wedge \neg Q\wedge R)$

$\neg S$ = **Sum of remaining minterms in S**

$\neg S = (P\wedge \neg Q \wedge R) \vee (P\wedge \neg Q \wedge \neg R) \vee (\neg P\wedge Q \wedge \neg R) \vee (\neg P\wedge \neg Q \wedge \neg R)$

PCNF of S $= \neg(\neg S)$

$= \neg((P\wedge \neg Q \wedge R) \vee (P\wedge \neg Q \wedge \neg R) \vee (\neg P\wedge Q \wedge \neg R) \vee (\neg P\wedge \neg Q \wedge \neg R))$

$= (\neg(P\wedge \neg Q \wedge R) \wedge \neg(P\wedge \neg Q \wedge \neg R) \wedge \neg(\neg P\wedge Q \wedge \neg R) \wedge \neg(\neg P\wedge \neg Q \wedge \neg R))$

PCNF of S $= (\neg P\vee Q\vee \neg R) \wedge (\neg P\vee Q\vee R) \wedge (P\vee \neg Q\vee R) \wedge (P\vee Q\vee R)$

<u>Minterms in three variables P, Q & R.</u>

$P\wedge Q \wedge R$, $\neg P\wedge Q \wedge R$, $P\wedge \neg Q \wedge R$,

$P\wedge Q \wedge \neg R$, $P\wedge \neg Q \wedge \neg R$, $\neg P\wedge \neg Q \wedge R$,

$\neg P\wedge Q \wedge \neg R$, $\neg P\wedge \neg Q \wedge \neg R$

Obtain the PCNF of the formula$(\neg P \to R) \wedge (Q \leftrightarrow P)$. Also find PDNF

SOLUTION:

$(\neg P \to R) \wedge (Q \leftrightarrow P)$	**Reasons**
$(\neg P \to R) \wedge ((Q \to P) \wedge (P \to Q))$	$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$
$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$	$\because \quad P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow ((P \vee R) \vee F) \wedge ((\neg Q \vee P) \vee F) \wedge ((\neg P \vee Q) \vee F)$	$\because \quad P \vee F \Leftrightarrow P$
$\Leftrightarrow ((P \vee R) \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P) \vee (R \wedge \neg R)) \wedge (\neg P \vee Q) \vee (R \wedge \neg R))$	$F \Leftrightarrow P \wedge \neg P$
$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$	Distributive law
$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$	Commutative law

The required PCNF is

$(P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

To find PDNF:

S = PCNF of given statement

$S = (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\neg S$ = **Product of remaining maxterms in S**

$\neg S = (\neg P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee \neg R)$

PDNF of S = $\neg(\neg S)$

$$= \neg((\neg P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee \neg R))$$

$$= (\neg (\neg P \vee \neg Q \vee R) \vee \neg (P \vee Q \vee \neg R) \vee \neg (\neg P \vee \neg Q \vee \neg R))$$

$$= (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R)$$

Maxterms in three variables P, Q & R.

$P \vee Q \vee R$, $\neg P \vee Q \vee R$, $P \vee \neg Q \vee R$,

$P \vee Q \vee \neg R$, $P \vee \neg Q \vee \neg R$, $\neg P \vee \neg Q \vee R$,

$\neg P \vee Q \vee \neg R$, $\neg P \vee \neg Q \vee \neg R$

Without using truth table find PCNF & PDNF of $P\to (Q\wedge R) \wedge (\neg P\to (\neg Q\wedge \neg R))$

SOLUTION:

$P\to (Q\wedge R) \wedge (\neg P\to (\neg Q\wedge \neg R))$	Reasons
$\Leftrightarrow (\neg P\vee(Q\wedge R)) \wedge (P\vee (\neg Q\wedge \neg R))$	$P\to Q \Leftrightarrow \neg P\vee Q$
$\Leftrightarrow (\neg P\vee Q)\wedge(\neg P\vee R) \wedge ((P\vee\neg Q)\wedge(P\vee\neg R))$	∵ ∵ Distributive Rule
$\Leftrightarrow ((\neg P\vee Q) \vee F) \wedge ((\neg P \vee R) \vee F) \wedge$ $((P \vee \neg Q) \vee F)\wedge((P \vee \neg R) \vee F)$	∵ $P\vee F \Leftrightarrow P$
$\Leftrightarrow ((\neg P\vee Q)\vee(R\wedge\neg R)) \wedge ((\neg P\vee R)\vee(Q\wedge\neg Q)) \wedge$ $((P\vee\neg Q)\vee(R\wedge\neg R)) \wedge ((P\vee\neg R)\vee(Q\wedge\neg Q))$	$(P\wedge\neg P) \Leftrightarrow F$
$\Leftrightarrow (\neg P\vee Q\vee R) \wedge (\neg P\vee Q \vee\neg R) \wedge (\neg P\vee R\vee Q)\wedge$ $(\neg P\vee R\vee\neg Q) \wedge (P\vee\neg Q\vee R) \wedge (P\vee\neg Q\vee\neg R)\wedge$ $(P\vee\neg R\vee Q) \wedge (P\vee\neg R\vee\neg Q)$	Distributive Rule
$\Leftrightarrow (\neg P\vee Q\vee R) \wedge (\neg P\vee Q \vee\neg R) \wedge (\neg P\vee\neg Q\vee R)\wedge$ $(P\vee\neg Q\vee R) \wedge (P\vee\neg Q\vee\neg R) \wedge (P\vee Q\vee\neg R)$	Commutative Rule

The required PCNF is

$(\neg P\vee Q\vee R)\wedge (\neg P\vee Q \vee\neg R)\wedge(\neg P\vee\neg Q\vee R)\wedge (P\vee\neg Q\vee R)\wedge(P\vee\neg Q\vee\neg R)\wedge(P\vee Q\vee\neg R)$

To find PDNF:

S = PCNF of given statement

S = $(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R)$

$\neg$**S = Product of remaining maxterms in S**

$\neg S = (P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

PDNF of S $= \neg(\neg S)$

$= \neg((P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R))$

$= (\neg(P \vee Q \vee R) \wedge \neg(\neg P \vee \neg Q \vee \neg R))$

$= (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

Maxterms in three variables P, Q & R.

$P \vee Q \vee R$, $\neg P \vee Q \vee R$, $P \vee \neg Q \vee R$,
$P \vee Q \vee \neg R$, $P \vee \neg Q \vee \neg R$, $\neg P \vee \neg Q \vee R$,
$\neg P \vee Q \vee \neg R$, $\neg P \vee \neg Q \vee \neg R$

Obtain PDNF of $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

Solution :

$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$	
$\Rightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$	($\because P \to Q \Rightarrow \neg P \vee Q$ & Demorgan's law)
$\Rightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P))$	(Distributive law)
$\Rightarrow \neg P \vee ((F \wedge Q) \vee (Q \wedge P))$	($P \wedge \neg P \Leftrightarrow F$ & Idempotent)
$\Rightarrow \neg P \vee (F \vee (Q \wedge P))$	($P \wedge F \Leftrightarrow F$)
$\Rightarrow \neg P \vee (Q \wedge P)$	($P \vee F \Leftrightarrow P$)
$\Rightarrow (\neg P \wedge T) \vee (Q \wedge P)$	($P \wedge T \Leftrightarrow P$)
$\Rightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge P)$	($T \Leftrightarrow P \vee \neg P$)
$\Rightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P)$	(Distributive law)

$(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

which is the required PDNF.

Obtain PDNF and PCNF for the statement formula $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

SOLUTION: Let $S = \neg(P \vee Q) \leftrightarrow (P \wedge Q)$

STEP No	$S = \neg(P \vee Q) \leftrightarrow (P \wedge Q)$	REASONS
1	$\equiv [\neg(P \vee Q) \rightarrow (P \wedge Q)] \wedge [(P \wedge Q) \rightarrow \neg(P \vee Q)]$	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
2	$\equiv [(P \vee Q) \vee (P \wedge Q)] \wedge [\neg(P \wedge Q) \vee \neg(P \vee Q)]$	$P \rightarrow Q \equiv \neg P \vee Q$
3	$\equiv [((P \vee Q) \vee P) \wedge ((P \vee Q) \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$	(1) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ (2) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
4	$\equiv [(P \vee Q) \wedge (P \vee Q)] \wedge [((\neg P \vee \neg Q) \vee \neg P) \wedge ((\neg P \vee \neg Q) \vee \neg Q)]$	$(P \vee Q) \vee P \equiv (P \vee P) \vee Q \equiv (P \vee Q)$
5	$\equiv [P \vee Q] \wedge [(\neg P \vee \neg Q) \wedge (\neg P \vee \neg Q)]$	$P \wedge P \equiv P$
6	$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$	$P \wedge P \equiv P$

PCNF of $S = (P \vee Q) \wedge (\neg P \vee \neg Q)$

PCNF of $\neg S$ = Product of remaining max-terms

$= (\neg P \vee Q) \wedge (P \vee \neg Q)$

PDNF $S = \neg$ (**PCNF of** $\neg S$) $= (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

Find the PCNF and PDNF of $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

Solution :

We first note that

$$P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

The PCNF is obtained as follows :

$$\neg(P \vee Q) \leftrightarrow (P \wedge Q)$$

$$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee (\neg\neg(P \vee Q) \wedge \neg(P \wedge Q))$$

$$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q)$$

$$\Leftrightarrow (P \wedge \neg P \wedge Q \wedge \neg Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q)$$

$$\Leftrightarrow (F \wedge F) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$

$$\Leftrightarrow F \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$

$$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$$

The PDNF is obtained as follows :

$$\neg(P \vee Q) \leftrightarrow (P \wedge Q)$$

$$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$$

$$\Leftrightarrow ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$$

$$\Leftrightarrow (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$$

$$\Leftrightarrow F \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee F$$

$$\Leftrightarrow (Q \wedge \neg P) \vee (P \wedge \neg Q)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$$

Alternately, the PDNF can be obtained as follows : Let S denote $\neg (P \vee Q) \leftrightarrow (P \wedge Q)$. The PCNF of $\neg S$ is

$$\neg S \Leftrightarrow (P \vee \neg Q) \wedge (\neg P \vee Q)$$

and hence

$$
\begin{aligned}
S &\Leftrightarrow \neg\neg S \\
&\Leftrightarrow \neg ((P \vee \neg Q) \wedge (\neg P \vee Q)) \\
&\Leftrightarrow \neg (P \vee \neg Q) \vee \neg (\neg P \vee Q) \\
&\Leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q).
\end{aligned}
$$

Obtain the PCNF of the following $P \rightarrow (\neg P \wedge (Q \rightarrow P))$

Solution :

$P \rightarrow (\neg P \wedge (Q \rightarrow P))$	
$\Leftrightarrow \neg P \vee (\neg P \wedge (\neg Q \vee P))$	$(\because P \rightarrow Q \Rightarrow \neg P \vee Q)$
$\Leftrightarrow (\neg P \vee \neg P) \wedge (\neg P \vee \neg Q \vee P)$	(Distributive law)
$\Leftrightarrow \neg P \wedge (T \vee \neg Q)$	$(\because$ Idempotent & $P \vee \neg P \Leftrightarrow T)$
$\Leftrightarrow \neg P \wedge T$	$(P \wedge T \Leftrightarrow T)$
$\Leftrightarrow \neg P$	$(P \wedge T \Leftrightarrow P)$
$\Leftrightarrow (\neg P \wedge F)$	$(P \vee F \Leftrightarrow P)$
$\Leftrightarrow \neg P \vee (Q \wedge \neg Q)$	$(F = P \wedge \neg P)$
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$	(Distributive law)

which is the required PCNF.

Obtain the PCNF & PDNF of $(Q \to P) \wedge (\neg P \wedge Q)$

SOLUTION:

Let S: $(Q \to P) \wedge (\neg P \wedge Q)$

STEP NO.	S: $(Q \to P) \wedge (\neg P \wedge Q)$	REASON
1	$S \equiv (\neg Q \vee P) \wedge (Q \wedge \neg P)$	**Conditional as disjunction & Commutative law**
2	$\equiv (\neg Q \vee P) \wedge (\neg(\neg Q \vee P))$	**Demorgan's law** $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
3	$\equiv F$	$P \wedge \neg P \equiv F$

Therefore, the given formula is contradiction.
Since, the given formula is contradiction, the PCNF includes all possible maxterms of the variable. There is no PDNF.
Therefore PCNF OF $S \equiv (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$

PDNF AND PCNF USING TRUTH TABLE

Using truth table we can easily find PDNF and PCNF of given statement formulas.

WORKING RULE TO OBTAIN PDNF:

Step 1: Construct truth table for the given statement formula

Step 2: Choose each and every row in which the final column value is "True"

Step 3: In the selected row, if the truth value of each individual variable value is True select the variable and False then select the negation of that variable. In such a way collect all possible minterms.

Step 4: Sum all the minterms gives the required PDNF.

WORKING RULE TO OBTAIN PCNF:

Step 1: Construct truth table for the given statement formula

Step 2: Choose each and every row in which the final column value is "False"

Step 3: In the selected row, if the truth value of each individual variable value is "False" select the variable and "True" then select the negation of that variable. In such a way collect all possible maxterms.

Step 4: Product all the maxterms gives the required PDNF

Obtain the PDNF of $P \to Q$ using truth table.

Solution:

P	Q	$P \to Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	-
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

PDNF of given statement is $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

Obtain the Principal disjunctive normal forms of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ using truth table.

Solution: Let S: $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

P	Q	R	$\neg P$	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	S	MINTERM
T	T	T	F	T	F	T	T	$(P \wedge Q \wedge R)$
T	T	F	F	T	F	F	T	$(P \wedge Q \wedge \neg R)$
T	F	T	F	F	F	F	F	-
T	F	F	F	F	F	F	F	-
F	T	T	T	F	T	T	T	$(\neg P \wedge Q \wedge R)$
F	T	F	T	F	F	F	F	-
F	F	T	T	F	T	F	T	$(\neg P \wedge \neg Q \wedge R)$
F	F	F	T	F	F	F	F	-

The PDNF of S is $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$.

Find the PCNF and the PDNF of the proposition

$$P \wedge (Q \rightarrow R)$$

Solution :

The truth table for the proposition $P \wedge (Q \rightarrow R)$ is given below:

P	Q	R	$Q \rightarrow R$	$P \wedge (Q \rightarrow R)$	Minterm	Maxterm
F	F	F	T	F	—	$P \vee Q \vee R$
F	F	T	T	F	—	$P \vee Q \vee \neg R$
F	T	F	F	F	—	$P \vee \neg Q \vee R$
F	T	T	T	F	—	$P \vee \neg Q \vee \neg R$
T	F	F	T	T	$P \wedge \neg Q \wedge \neg R$	—
T	F	T	T	T	$P \wedge \neg Q \wedge R$	—
T	T	F	F	F	—	$\neg P \vee \neg Q \vee R$
T	T	T	T	T	$P \wedge Q \wedge R$	—

The PDNF is

$$(P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R)$$

Also the PCNF is

$$(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$$
$$\wedge (P \vee \neg Q \vee \neg R) \vee (\neg P \vee \neg Q \vee R)$$

Obtain PDNF of $P \to ((P \to Q) \vee (\neg Q \vee \neg P))$\`

Solution :

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \vee \neg P$	$(P \to Q) \vee (\neg Q \vee \neg P)$	$P \to ((P \to Q) \vee (\neg Q \vee \neg P)$	Minterms
F	F	T	T	T	T	T	T	$\neg P \wedge \neg Q$
F	T	T	F	T	T	T	T	$\neg P \wedge Q$
T	F	F	T	F	T	T	T	$P \wedge \neg Q$
T	T	F	F	T	F	T	T	$P \wedge Q$

The PDNF is $(\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (P \wedge Q) \vee (P \wedge \neg Q)$

INFERENCE THEORY OF STATEMENT CALCULUS

Premises or **Hypothesis**

A statement which is assumed to be true is known as premises **or** hypothesis.

Conclusion

A statement which is obtained finally from the given set of premises is known as conclusion

Inference Theory

It deals with the rules that are applied to reach the conclusion from the premises.

Methods

- Direct method- $(P_1 \wedge P_2 \wedge .... \wedge P_n) \rightarrow C$
- Indirect method- $[(P_1 \wedge P_2 \wedge .... \wedge P_n) \wedge \neg C] \rightarrow F$
- In consistent premises. – $(P_1 \wedge P_2 \wedge .... \wedge P_n) \rightarrow F$

Rules of Inference:

Rule P: **A premise may be introduced at any point of derivation**

Rule T: **A formula S may be introduced at any point in a derivation if S is tautologically implied by any one or more of the preceding formulas** (Tautological rules)

Rule CP: **If S can be derived from R and set of premises then R S can be derived from the set premises alone** [Only for if …. then form. Introduce LHS as additional premises].

Note: Rule CP means Rule of Conditional Proof

TYPE 1: Problems on Direct method

When a conclusion is derived from a set of premises is called direct method

1. Show that R is a valid inference from the premises $P \rightarrow Q$, $Q \rightarrow R$ and P.

Solution:

S.No	Premises	RULE	Reason
1	$P \rightarrow Q$	P	Premises
2	$Q \rightarrow R$	P	Premises
3	$P \rightarrow R$	T , (1, 2)	$(P \rightarrow Q) , (Q \rightarrow R) \Rightarrow P \rightarrow R$
4	P	P	Premises
5	R	T , (3, 4)	$P , (P \rightarrow Q) \Rightarrow Q$

2. Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$, $(A \wedge \neg B) \rightarrow (R \vee S)$.

Solution:

S.No	Premises	RULE	Reason
1	$(C \vee D) \rightarrow \neg H$	P	Premises
2	$C \vee D$	P	Premises
3	$\neg H$	T, (1, 2)	$P, (P \rightarrow Q) \Rightarrow Q$
4	$\neg H \rightarrow (A \wedge \neg B)$	P	Premises
4	$(A \wedge \neg B)$	T, (3, 4)	$P, (P \rightarrow Q) \Rightarrow Q$
5	$(A \wedge \neg B) \rightarrow (R \vee S)$	P	Premises
6	$(R \vee S)$	T, (4, 5)	$P, (P \rightarrow Q) \Rightarrow Q$

3. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$, and $\neg M$.

Solution

S.No	Premises	RULE	Reason
1	$P \rightarrow M$	P	Premises
2	$\neg M$	P	Premises
3	$\neg P$	T, (1, 2)	$P, (P \rightarrow Q) \Rightarrow Q$
4	$P \vee Q$	P	Premises
5	Q	T, (3, 4)	$(P \vee Q) \wedge \neg P \Rightarrow Q$
6	$Q \rightarrow R$	P	Premises
7	R	T, (6, 5)	$P, (P \rightarrow Q) \Rightarrow Q$
8	$R \wedge (P \vee Q)$	T, (4, 7)	$P, Q \Rightarrow P \wedge Q$

4. Show that $S \vee R$ is tautologically implied $P \vee Q$, $P \rightarrow R$, $Q \rightarrow S$.

Solution

P	Premises	RULE	Reason
1	$P \vee Q$	P	Premises
2	$\neg P \rightarrow Q$	T, (1)	$\neg P \vee Q \Rightarrow P \rightarrow Q$
3	$Q \rightarrow S$	P	Premises
4	$\neg P \rightarrow S$	T, (2, 3)	$(P \rightarrow Q), (Q \rightarrow R) \Rightarrow P \rightarrow R$
5	$\neg S \rightarrow P$	T, (4)	$P \rightarrow Q \Rightarrow \neg Q \rightarrow \neg P$
6	$P \rightarrow R$	P	Premises
7	$\neg S \rightarrow R$	T, (5, 6)	$(P \rightarrow Q), (Q \rightarrow R) \Rightarrow P \rightarrow R$
8	$S \vee R$	T , (7)	$\neg P \vee Q \Rightarrow P \rightarrow Q$

5. Conclude D from the premises $(A \to B) \wedge (A \to C)$, $\neg (B \wedge C)$, $D \vee A$

Solution

S.No	Premises	RULE	Reason
1	$(A \to B) \wedge (A \to C)$	P	Premises
2	$A \to (B \wedge C)$	T, (1)	$(A \to B) \wedge (A \to C) \Rightarrow A \to (B \wedge C)$
3	$\neg (B \wedge C)$	P	Premises
4	$\neg A$	T, (2, 3)	$\neg Q$, $(P \to Q) \Rightarrow \neg P$
5	$D \vee A$	P	Premises
6	D	T , (4, 5)	$(P \vee Q) \wedge \neg P \Rightarrow Q$

6. Conclude from the premises $R \to \neg Q$, $R \vee S$, $S \to \neg Q$, $P \to Q \Rightarrow \neg P$.

Solution

S.No	Premises	RULE	Reason
1	$R \vee S$	P	Premises
2	$\neg R \to S$	T, (1)	$P \vee Q \Rightarrow \neg P \to Q$
3	$S \to \neg Q$	P	Premises
4	$\neg R \to \neg Q$	T, (2, 3)	$(P \to Q) , (Q \to R) \Rightarrow P \to R$
5	$Q \to R$	T	$P \to Q \Rightarrow \neg Q \to \neg P$
6	$R \to \neg Q$	P	Premises
7	$Q \to \neg Q$	T , (5, 6)	$(P \to Q) , (Q \to R) \Rightarrow P \to R$
8	$\neg Q \vee \neg Q$	T, (7)	$\neg P \vee Q \Rightarrow P \to Q$
9	$\neg Q$	T, (8)	$\neg P \vee \neg P \Rightarrow \neg P$
10	$P \to Q$	P	Premises
11	$\neg P$	T, (9, 10)	$\neg Q , (P \to Q) \Rightarrow \neg P$

INFERENCE THEORY PROBLEMS ON PROPOSITION

TYPE 2: BASED ON RULE CP

Rule CP: If S can be derived from R and set of premises then $R \rightarrow S$ can be derived from the set premises alone

[Only for if …. then form. Introduce LHS as additional premises].

7. Show that $R \rightarrow S$ can be derived from , $\neg R \vee P$, $P \rightarrow (Q \rightarrow S)$, Q using CP rule.

Solution

S.No	Premises	RULE	Reason
1	R	AP	Additional Premises
2	$\neg R \vee P$	P	Premises
3	$R \rightarrow P$	T (2)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
4	P	T (1, 3)	$P , (P \rightarrow Q) \Rightarrow Q$
5	$P \rightarrow (Q \rightarrow S)$	P	Premises
6	$Q \rightarrow S$	T(4, 5)	$P , (P \rightarrow Q) \Rightarrow Q$
7	Q	P	Premises
8	S	T, (6, 7)	$P , (P \rightarrow Q) \Rightarrow Q$
9	$R \rightarrow S$	CP rule	

Without CP rule

S.No	Premises	RULE	Reason
1	$P \rightarrow (Q \rightarrow S)$	P	Premises
2	$\neg R \vee P$	P	Premises
3	$R \rightarrow P$	T (2)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
4	$R \rightarrow (Q \rightarrow S)$	T (1, 3)	$(P \rightarrow Q), (Q \rightarrow R) \Rightarrow P \rightarrow R$
5	$\neg R \vee (\neg Q \vee S)$	T, (4)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
6	$\neg Q \vee (\neg R \vee S)$	T, (5)	$P \vee (Q \vee R) \Rightarrow (P \vee Q) \vee R$
7	$Q \rightarrow (R \rightarrow S)$	T, (6)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
8	Q	P	Premises
9	$R \rightarrow S$	T, (7, 8)	$P, (P \rightarrow Q) \Rightarrow Q$

8. Show that $P \rightarrow (Q \rightarrow S)$ can be derived from $P \rightarrow (Q \rightarrow R)$, $Q \rightarrow (R \rightarrow S)$.

Solution

S.No	Premises	RULE	Reason
1	P	AP	Additional Premises
2	$P \rightarrow (Q \rightarrow R)$	P	Premises
3	$Q \rightarrow R$	T, (1, 2)	$P, (P \rightarrow Q) \Rightarrow Q$
4	$Q \rightarrow (R \rightarrow S)$	P	Premises
5	$\neg Q \vee (\neg R \vee S)$	T, (4)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
6	$\neg R \vee (\neg Q \vee S)$	T, (5)	$P \vee (Q \vee R) \Rightarrow (P \vee Q) \vee R$
7	$R \rightarrow (\neg Q \vee S)$	T, (6)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
8	$Q \rightarrow (\neg Q \vee S)$	T, (3, 7)	$(P \rightarrow Q), (Q \rightarrow R) \Rightarrow P \rightarrow R$
9	$\neg Q \vee (\neg Q \vee S)$	T, (8)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
10	$\neg Q \vee S$	T, (9)	$P \vee P \Rightarrow P$
11	$Q \rightarrow S$	T, (10)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
12	$P \rightarrow (Q \rightarrow S)$	CP rule	

9. Show that $(P \vee Q) \rightarrow R \Rightarrow (P \wedge Q) \rightarrow R$

Solution

S.No	Premises	RULE	Reason
1	$(P \wedge Q)$	AP	Additional Premises
2	P	T, (1)	$P \wedge Q \Rightarrow P, Q$
3	$P \vee Q$	T (2)	$P \Rightarrow P \vee Q$
4	$(P \vee Q) \rightarrow R$	P	Premises
5	R	T, (3, 4)	$P , (P \rightarrow Q) \Rightarrow Q$

INFERENCE THEORY PROBLEMS ON PROPOSITION

TYPE 3: INDIRECT METHOD (OR) METHOD OF CONTRADICTION

Consider the negation of the conclusion as the additional premises and arrive at the conclusion as contradiction F.

i.e., $[(P_1 \wedge P_2 \wedge \ldots . \wedge P_n) \wedge \neg C] \rightarrow F$

10. Use indirect method to prove $\neg Q, P \rightarrow Q \Rightarrow \neg P$

Solution

S.No	Premises	RULE	Reason
1	$\neg(\neg P)$	$\neg$ Conclusion	
2	P	T (1)	$\neg(\neg P) = P$
3	$P \rightarrow Q$	P	Premises
4	Q	T (2, 3)	$P, (P \rightarrow Q) \Rightarrow Q$
5	$\neg Q$	P	Premises
6	$Q \wedge \neg Q$	T (4, 5)	$P, Q \Rightarrow P \wedge Q$
7	F	T (6)	$Q \wedge \neg Q \Rightarrow \mathbf{F}$

11. Use indirect method to prove $\neg Q, P \to Q \Rightarrow \neg P$

Solution

S.No	Premises	RULE	Reason
1	$\neg(\neg P)$	$\neg$ Conclusion	
2	P	T (1)	$\neg(\neg P) = P$
3	$P \to Q$	P	Premises
4	Q	T (2, 3)	$P, (P \to Q) \Rightarrow Q$
5	$\neg Q$	P	Premises
6	$Q \wedge \neg Q$	T (4, 5)	$P, Q \Rightarrow P \wedge Q$
7	F	T (6)	$Q \wedge \neg Q \Rightarrow \mathbf{F}$

12. Show that $R \rightarrow \neg Q$, $R \vee S$, $S \rightarrow \neg Q$, $P \rightarrow Q \Rightarrow \neg P$ using indirect method.

Solution

S.No	Premises	RULE	Reason
1	$\neg(\neg P)$	$\neg$ Conclusion	
2	P	T (1)	$\neg(\neg P) = P$
3	$P \rightarrow Q$	P	Premises
4	Q	T (2, 3)	$P, (P \rightarrow Q) \Rightarrow Q$
5	$R \rightarrow \neg Q$	P	Premises
6	$\neg R$	T, (4, 5)	$P, (P \rightarrow Q) \Rightarrow Q$
7	$R \vee S$	P	Premises
8	$\neg R \rightarrow S$	T, (7)	$P \rightarrow Q \Rightarrow \neg P \vee Q$
9	S	T, (6, 8)	$P, (P \rightarrow Q) \Rightarrow Q$
10	$S \rightarrow \neg Q$	P	Premises
11	$\neg Q$	T, (9, 10)	$P, (P \rightarrow Q) \Rightarrow Q$
12	$Q \wedge \neg Q$	T (4, 11)	$P, Q \Rightarrow P \wedge Q$
13	F	T (12)	$Q \wedge \neg Q \Rightarrow F$

INFERENCE THEORY PROBLEMS ON PROPOSITION

TYPE 4 : INCONSISTENT PREMISES

Consider the premises and arrive at the conclusion as contradiction F.

i.e., $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow F$

13. Show that the following premises are inconsistent, $p\rightarrow q$, $q\rightarrow r$, $s\rightarrow \neg r$, $q \wedge s$.

Solution

S.No	Premises	RULE	Reason
1	$p\rightarrow q$	P	Premises
2	$q\rightarrow r$	P	Premises
3	$s\rightarrow \neg r$	P	Premises
4	$p\rightarrow r$	T, (1, 2)	$P, (P \rightarrow Q) \Rightarrow Q$
5	$\neg r \rightarrow \neg p$	T, (4)	$P \rightarrow Q \Rightarrow \neg Q \rightarrow \neg P$
6	$s\rightarrow \neg p$	T, (3, 5)	$P, (P \rightarrow Q) \Rightarrow Q$
7	$q \wedge s$	P	Premises
8	q	T, (7)	$P \wedge Q \Rightarrow P, Q$
9	s	T, (7)	$P \wedge Q \Rightarrow P, Q$
10	r	T, (2, 8)	$P, (P \rightarrow Q) \Rightarrow Q$
11	$\neg r$	T, (3, 10)	$P, (P \rightarrow Q) \Rightarrow Q$
12	$r \wedge \neg r$	T (10, 11)	$P, Q \Rightarrow P \wedge Q$
13	F	T (12)	$Q \wedge \neg Q \Rightarrow F$

Aliter :

S.No	Premises	RULE	Reason
1	$q \rightarrow r$	P	Premises
2	$s \rightarrow \neg r$	P	Premises
3	$q \wedge s$	P	Premises
4	q	T, (3)	$P \wedge Q \Rightarrow P, Q$
5	s	T, (3)	$P \wedge Q \Rightarrow P, Q$
6	r	T, (1,4)	$P, (P \rightarrow Q) \Rightarrow Q$
7	$\neg r$	T, (2, 5)	$P, (P \rightarrow Q) \Rightarrow Q$
8	$r \wedge \neg r$	T (6, 7)	$P, Q \Rightarrow P \wedge Q$
9	F	T (8)	$Q \wedge \neg Q \Rightarrow \mathbf{F}$

1. Show that the following premises are inconsistent.

If Jack misses many classes through illness then he fails high school. If Jack fails high school then he is uneducated. If Jack reads a lot of books then he is not uneducated. Jack misses many classes through illness and reads a lot of books.

Solution:

Define J : Jack misses many classes through illness

S : Jack fails high school

U : Jack is uneducated

B : Jack reads a lot of books

Premises: $J \rightarrow S$, $S \rightarrow U$, $B \rightarrow \neg U$, $J \wedge B$

Premises: $J \rightarrow S$, $S \rightarrow U$, $B \rightarrow \neg U$, $J \wedge B$

S.No	Premises	Rule	Reason
1	$J \rightarrow S$	P	Premises
2	$S \rightarrow U$	P	Premises
3	$J \rightarrow U$	T, (1,2)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \equiv P \rightarrow R$
4	$B \rightarrow \neg U$	P	Premises
5	$U \rightarrow \neg B$	T, (4)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
6	$J \rightarrow \neg B$	T, (3,5)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \equiv P \rightarrow R$
7	$\neg J \vee \neg B$	T, (6)	$P \rightarrow Q \equiv \neg P \vee Q$
8	$\neg (J \wedge B)$	T, (7)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
9	$J \wedge B$	P	Premises
10	$(J \wedge B) \wedge \neg (J \wedge B) = F$	T, (8,9)	$P, Q \Rightarrow P \wedge Q$

Premises: $(\neg R \vee \neg D) \rightarrow (N \wedge A)$, $N \rightarrow M$, $\neg M$

Conclusion : R

S.No	Premises	Rule	Reason
1	$N \rightarrow M$	P	Premises
2	$\neg M$	P	Premises
3	$\neg N$	T, (1,2)	$\neg Q$, $(P \rightarrow Q) \Rightarrow \neg P$
4	$(\neg R \vee \neg D) \rightarrow (N \wedge A)$	P	Premises
5	$\neg(N \wedge A) \rightarrow \neg(\neg R \vee \neg D)$	T, (4)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
6	$(\neg N \vee \neg A) \rightarrow (R \wedge D)$	T, (5)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
7	$\neg N \vee \neg A$	T, (3)	$P \equiv P \vee Q$
8	$R \wedge D$	T (7,6)	$P, (P \rightarrow Q) \Rightarrow Q$
9	R	T, (8)	$P \wedge Q \equiv P$

3. “ If A works hard, then B or C will enjoy themshelves.
If B enjoys himself, then A will not work hard.
If D enjoys himself, then C will not enjoy.
There fore , if A works hard D will not enjoy himself.”

Show that these statements constitute a valid argument.

Solution:

Define: P : A works hard
Q : B enjoys himself
R : C enjoys himself
S : D enjoys himself

Premises: $P \rightarrow (Q \vee R)$, $Q \rightarrow \neg P$, $S \rightarrow \neg R$

Conclusion : $P \rightarrow \neg S$

Premises: $P \rightarrow (Q \vee R)$, $Q \rightarrow \neg P$, $S \rightarrow \neg R$

Conclusion : $P \rightarrow \neg S$

S.No	Premises	Rule	Reason
1	P	AP	Additional Premises
2	$P \rightarrow (Q \vee R)$	P	Premises
3	$Q \vee R$	T, (1,2)	$P, (P \rightarrow Q) \Rightarrow Q$
4	$Q \rightarrow \neg P$	P	Premises
5	$P \rightarrow \neg Q$	T, (4)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
6	$\neg Q$	T, (1,5)	$P, (P \rightarrow Q) \Rightarrow Q$
7	$\neg Q \rightarrow R$	T, (3)	$P \rightarrow Q \equiv \neg P \vee Q$
8	R	T, (6, 7)	$P, (P \rightarrow Q) \Rightarrow Q$
9	$S \rightarrow \neg R$	P	Premises
10	$R \rightarrow \neg S$	T, (8)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
11	$\neg S$	T, (8, 10)	$P, (P \rightarrow Q) \Rightarrow Q$
12	$P \rightarrow \neg S$	CP rule	

4. Show that the hypothesis

" If you send me an e-mail message, then I will finish writing the program". " If you do not send me an e-mail message, then I will go to sleep early "and " If I go to sleep early then I will wake up feeling refreshed lead to the conclusion ' If do not finish writing the program, then I will wake up feeling refreshed."

Solution:

Define p : If you send me an e-mail message

q : I will finish writing the program

r : I will go to sleep early

s : I will wake up feeling refreshed

Premises: $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$

Conclusion : $\neg q \rightarrow s$.

Premises: $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$

Conclusion : $\neg q \rightarrow s.$

S.No	Premises	Rule	Reason
1	$\neg q$	AP	Additional Premises
2	$p \rightarrow q$	P	Premises
3	$\neg q \rightarrow \neg p$	T, (2)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
4	$\neg p$	T, (1,3)	$P, (P \rightarrow Q) \Rightarrow Q$
5	$\neg p \rightarrow r$	T, (4)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
6	r	T, (5)	$P, (P \rightarrow Q) \Rightarrow Q$
7	$r \rightarrow s$	T, (3)	$P \rightarrow Q \equiv \neg P \vee Q$
8	s	T, (6, 7)	$P, (P \rightarrow Q) \Rightarrow Q$
9	$\neg q \rightarrow s.$	CP rule	

5. Determine the validity of the following argument.

" My father praises me only if I can proud myself. Either I do well in sports or I cannot be proud of myself. If I study well, then I can't well in sports. Therefore if father praises me then I do not study well."

Solution:

Define: A : My father praises me

B : I can proud myself

C : I do well in sports

D : I study well

Premises: $A \rightarrow B$, $C \vee \neg B$, $D \rightarrow \neg C$

Conclusion : $A \rightarrow \neg D$.

Premises: $A \rightarrow B$, $C \vee \neg B$, $D \rightarrow \neg C$

Conclusion : $A \rightarrow \neg D$.

S.No	Premises	Rule	Reason
1	A	AP	Additional Premises
2	$A \rightarrow B$	P	Premises
3	B	T, (1, 2)	$P, (P \rightarrow Q) \Rightarrow Q$
4	$C \vee \neg B$	P	Premises
5	$\neg C \rightarrow \neg B$	T, (4)	$P \rightarrow Q \equiv \neg P \vee Q$
6	$B \rightarrow C$	T, (5)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
7	C	T, (3)	$P, (P \rightarrow Q) \Rightarrow Q$
8	$D \rightarrow \neg C$	P	Premises
9	$C \rightarrow \neg D$	T, (8)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
10	$\neg D$	T, (7,9)	$P, (P \rightarrow Q) \Rightarrow Q$
11	$A \rightarrow \neg D$	CP rule	

6. Show that the hypothesis

"It is not sunny this afternoon and it is colder than yesterday". "We will go swimming only if it is sunny". "If we do not go swimming, then we will take a canoe trip" and "if we take a canoe trip then we will be home by sunset" lead to the conclusion " we will be home by sunset"

Solution:

Define: A : It is sunny this afternoon

B : It is colder than yesterday

C : We will go swimming

D : We will take a canoe trip

E : We will be home by sunset

Premises: $\neg A \wedge B$, $C \rightarrow A$, $\neg C \rightarrow D$, $D \rightarrow E$

Conclusion : E.

Premises: $\neg A \wedge B$, $C \rightarrow A$, $\neg C \rightarrow D$, $D \rightarrow E$

Conclusion : E.

S.No	Premises	Rule	Reason
1	$\neg A \wedge B$	P	Premises
2	$\neg A$	T, (1)	$P \wedge Q \equiv P$
3	$C \rightarrow A$	P	Premises
4	$\neg A \rightarrow \neg C$	T(3)	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
5	$\neg C$	T(2, 4)	$P, (P \rightarrow Q) \Rightarrow Q$
6	$\neg C \rightarrow D$	P	Premises
7	D	T, (5, 6)	$P, (P \rightarrow Q) \Rightarrow Q$
8	$D \rightarrow E$	P	Premises
9	E	T, (7,8)	$P, (P \rightarrow Q) \Rightarrow Q$

7. Show that the following set of premises are inconsistent.
If Rama gets his degree, he will go for a job.
If he goes for a job, he will get married soon.
If he goes for higher study, he will not get married.
Rama gets his degree and goes for higher study.

8. Prove the following argument. " If I study or I am a genius, then I will pass the course. I will not allow to take next course. If I pass the course then I will be allowed to take next course. Therefore I will not study."

9. Test the validity of the following argument. If Ram studies, then he will pass Discrete Mathematics. If Ram does not play tennis, then he will study. Ram failed in Discrete mathematics. Therefore Ram play tennis.

PREDICATE CALCULUS

The predicate calculus study with predicates

Consider the following statement **" Ram is a good boy"**

In the above sentence **" is a good boy "** is the predicate and the name **" Ram "** is the subject.

If we denote the predicate " is a good boy" by B and subject "Ram" by r then the statement " Ram is a good boy" can be represented by **B(r)**.

Remark:

- Always we denote predicate by capital letters and the subjects by small letter.
- In general, any statement of the type " S is P " where P is predicate and S is subject can be denoted by P(s).

SOME MORE EXAMPLES:

(i) " x is an integer "

Here predicate is **" is an integer "** and it is denoted by I. **Subject is " x"** and it is denoted by x.

Therefore, **the given statement is can be denoted by I(x).**

(ii) **Pavithra is good in Mathematics and Geetha is good in Physics.**

- The statement " Pavithra is good in Mathematics " can be represented by M(p) where M represents predicate " is good in Mathematics " and p represents subject " Pavithra "
- The statement "Geetha is good in Physics " can be represented by P(g) where P represents predicate " is good in Physics " and ' g ' represents subject "Geetha "

The given statement "Pavithra is good in Mathematics and Geetha is good in Physics " can be symbolized as $\mathbf{M(p) \wedge P(g)}$.

1- Place predicate, 2 – Place predicate, 3 - Place predicate …….

Consider the following statements

(a) Nandhini is an intelligent girl

(b) Muruga is shorter than Vijaya

(c) Vimal sits between Ramesh and Ajay.

Definition: n – place predicates:

An n- place predicate is a predicate requiring n names of objects where n > 0. It is denoted by P(x_1, x_2, x_3, -------x_n) where x_1, x_2, x_3, -------x_n are the names of ‘ n ‘ objects associated with predicate P.

- In the statement (a), the predicate “is an intelligent girl“ is a ‘1 – Place ‘predicate because there is only one name of object “Nandhini “ associated with a predicate.
  - ✓ **It can be symbolized as G(n)** where G represents predicate “is an intelligent girl“ and n represents subject “Nandhini “.
- In the statement (b), the predicate “is shorter than “ is a ‘2 – Place’ predicate because there are two names of objects “Muruga and Vijaya “ associated with a predicate.
  - ✓ **It can be symbolized as S(m,n)** where S represents predicate “is shorter than “ and m and n represents subjects “Muruga and Vijaya “ .
- In the statement (c), the predicate “sits between “ is a ‘3– Place' predicate because there are three names of objects “Vimal, Ramesh and Ajay “ associated with a predicate.
  - ✓ **It can be symbolized as S(v, r, a)** where S represents predicate “sits between “ and v, r and a represents subjects “Vimal, Ramesh and Ajay “

STATEMENT FUNCTIONS

Definition: Statement functions:

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable.

Remark: A statement function becomes a statement when the variable replaced by the name of any object.

Example: F(x) : x is a fruit.--------(*)

Which represents statement function.

Replacing x by name say Apple in (*) , we get the statement Apple is a fruit.

i.e., F(Apple) : Apple is a fruit

Replacing x by name say Orange in (*) , we get the statement Orange is a fruit.

i.e., F(Orange) : Orange is a fruit.

Quantifiers

Quantifier is one which is used to quantify the nature of variables.

Quantifiers are words that refer to quantities such as **”some”** or **”all”** and tell for how many elements a given predicate is true.

Types of Quantifiers

- Universal quantifier (**∀**-for all, for every) (**(∀x) or (x)**)
- Existential quantifier (**∃**- there exists, for some) (**(∃x)**)

Universal quantifier:

The quantifier " for all x" is called the Universal quantifier . It is denoted by the symbol " **(∀x) or (x)** " . The universal quantifier is equivalent to each of the following phrases

(i) For all x
(ii) For every x
(iii) For each x
(iv) Everything x is such that
(v) Each thing x is such that

Example: " Every apple is red "

The above statement can be restated as follows.

For all x, if x is an apple then x is red.

Now, we shall translate it into symbolic form using universal quantifier.

Define: **A(x) : x is an apple**

R(x) : x is red.

Therefore, we can write given statement as (x) (A(x) → R(x)).

Try this! " Everything is yellow " in symbolic form.

Existential quantifier:

The quantifier " some x" is called the existential quantifier . It is denoted by the symbol " **($\exists$x)** " . The existential quantifier is equivalent to each of the following phrases.

(i) For some x
(ii) some x such that
(iii) There exists an x such that
(iv) There is an x is such that
(v) There is atleast one x is such that

Example: " Some men are clever "

The above statement can be restated as follows.

" There is a x such that x is a man and x is clever.

Now, we shall translate it into symbolic form using existential quantifier.

Define: **M(x) : x is a man and C(x) : x is clever**

Therefore, we can write given statement as ($\exists$x) (M(x) $\wedge$ C(x)).

Try this! " Some people are honest " in symbolic form

UNIVERSE OF DISCOURSE

It is defined as a set which contains all the values taken by a variable.

Example: Let $P(x) : x = x^2$ be the statement.

If the universe of discourse is the set of integers, what are the truth values of **(a) P(-1)** **(b) (x) P(x).**

Solution:

Given $P(x) : x = x^2$.

Universe of discourse = {0, ± 1, ± 2, ± 3, ...}.

(a) False. Since x = -1 does not satisfy $x = x^2$

Therefore, truth value of P(-1) is false.

(b) False. Since x = 2 does not satisfy $x = x^2$

Therefore, truth value of **(x)** P**(x)** is false.

Predicate Formulas

Consider a Predicate P with n variables as $P(x_1, x_2, x_3, ..., x_n)$. Here P is n-place predicate and $x_1, x_2, x_3, ..., x_n$ are n individuals variables. This n-place predicate is known as atomic formula of predicate calculus.

For Example: P(x), Q(x, y), R(x,y,z)

Well Formed Formula

Well Formed Formula (WFF) is a predicate holding any of the following −

- All propositional constants and propositional variables are WFFs
- If x is a variable and Y is a WFF, ∀ x Y and ∀ x Y are also WFF
- Truth value and false values are WFFs
- Each atomic formula is a WFF
- All connectives connecting WFFs are WFFs

Free, Bound and Scope – variable

Bound Variable: The variable is said to be bound if it is concerned with either universal ($\forall x$) or existential ($\exists x$) quantifiers.

Free Variable: The variable which is not concerned with any quantifier is called free variable.

Scope : The part of the logical expression to which a quantifier is applied is called the scope of a quantifier .

Example:

1. **Consider the statement (x) P(x, y).**
   In this formula, x is said to be bound , y is said to be free variable and Scope of the quantifier is P(x, y).

2. **Consider the statement ((∃x) M(x)) ∧ C(x).**
   In this formula, x is said to be bound in M(x) , x is said to be free variable in C(x) and Scope of the quantifier (∃x) is M(x).

Try this! (i) (∀x) P(x) → P(y) (ii) (∃y) (P(y)→ (∀x) P(x))

Symbolic Representation of Expressions using Quantifiers

1. Write in symbolic form the statement, "All lions are dangerous"

Solution

P(x): x is dangerous

Q(x): x is a lion

Universe of discourse : Set of all living things

Symbolic form: $(x)\ (Q(x) \rightarrow P(x))$.

2. "Some animals are dangerous". Write in Symbolic form.

Solution

A(x): x is an animal

B(x): x is dangerous

Symbolic form $(\exists x)\ (A(x) \wedge B(x))$

3. Symbolize the statement

(i) All dogs bark

(ii) All men should be educated.

(iii) All complex numbers composed of real and imaginary parts.

Solution :

Symbolic form:

(i) **P(x): x** is a dog

Q(x): x barks

$\Rightarrow (\forall x)(P(x) \to Q(x))$

Universe of discourse: Set of animals.

(ii) For all x, if x is a man, then x should be educated

M(x): x is a man

E(x): x is educated

$\Rightarrow (\forall x)(M(x) \to E(x))$

(iii) **For all x, if x is a complex number, then x is composed of real & imaginary parts.**

C(x): x is a Complex number

R(x): x is composed of real and imaginary parts.

$\Rightarrow (\forall x)\ (C(x) \rightarrow R(x))$

Try this!

(i) Write the symbolic form of the expression

" Some men are giant "

(ii) Write the symbolic form of the expression

" Some real numbers are rational numbers "

INFERENCE THEORY

PREMISES OR **HYPOTHESIS**

A statement which is basically assumed as true is known as premises **or** hypothesis.

CONCLUSION

A statement which is obtained finally from the given set of premises is known as conclusion

INFERENCE THEORY

It deals with the rules that are applied to reach the conclusion from the premises.

METHODS

- Direct method - $(P_1 \wedge P_2 \wedge \ldots. \wedge P_n) \rightarrow C$
- Indirect method - $[P_1 \wedge P_2 \wedge \ldots. \wedge P_n \wedge \neg C] \rightarrow F$
- In consistent premises. – $(P_1 \wedge P_2 \wedge \ldots. \wedge P_n) \rightarrow F$

THE THEORY OF INFERENCE FOR PREDICATE CALCULUS

The rules of equivalent formulae, tautological implications, Rule P, Rule T, Rule CP, indirect method of proof for statement calculus can also be used here. Apart from this, we have some more rules to be followed, involving quantifiers. The **rules of specification called US and ES** are used **to eliminate quantifiers**, whereas **the rules of generalization called UG and EG** are used **to insert quantifiers**.

RULES OF PREDICATES

UNIVERSAL SPECIFICATION (US RULE)

1) $(\forall x)(P(x)) \Rightarrow P(y)$
2) $(\forall x)(P(x)) \Rightarrow P(x)$, $P(a)$, $P(b)$

UNIVERSAL GENERALIZATION (UG RULE)

$$P(y) \Rightarrow (\forall x)(P(x))$$

EXISTENTIAL SPECIFICATION (ES RULE)

$$(\exists x)(P(x)) \Rightarrow P(y)$$

EXISTENTIAL GENERALIZATION (EG RULE)

$$P(y) \Rightarrow (\exists x)(P(x))$$

LOGICAL EQUIVALENCE AND IMPLICATION

EQUIVALENCE

1) $(\exists x)\,(A(x) \vee B(x)) \Leftrightarrow (\exists x)\,A(x) \vee (\exists x)B(x)$
2) $(\forall x)(\,A(x) \wedge B(x)) \Leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$
3) $\neg(\exists x)\,A(x) \Leftrightarrow (\forall x)\,\neg A(x)$
4) $\neg\,(\forall x)\,A(x) \Leftrightarrow (\exists x)\,\neg A(x)$

IMPLICATION

1) $(x)\,(A(x) \vee (x)\,B(x) \Rightarrow (x)(A(x) \vee B(x))$
2) $(\exists x)\,(A(x) \wedge B(x)) \Rightarrow (\exists x)\,A(x) \wedge (\exists x)B(x)$

PROBLEMS ON PREDICATE CALCULUS

TYPE 1 : DIRECT METHOD

TYPE 1 : DIRECT METHOD

Problem 1

Prove that $\neg [(\forall x)\, P(x) \to Q(x)] \Leftrightarrow (\exists x)[P(x) \wedge \neg Q(x)]$

Solution:

$$\neg [(\forall x)\, P(x) \to Q(x)] \Leftrightarrow (\exists x)\, \neg [\, P(x) \to Q(x)]$$

$$\Leftrightarrow (\exists x)\, \neg [\neg P(x) \vee Q(x)]$$

$$\Leftrightarrow (\exists x)[P(x) \wedge \neg Q(x)]$$

Hence, $\neg [(\forall x)\, P(x) \to Q(x)] \Leftrightarrow (\exists x)[P(x) \wedge \neg Q(x)]$

Problem 2:

Prove that the following implication

$(x)\ (P(x) \rightarrow Q(x)) \wedge (x)\ (Q(x) \rightarrow R(x)) \Rightarrow (x)\ (P(x) \rightarrow R(x))$

Solution:

Steps	Premises	Rule	Reason
1	$(x)\ (P(x) \rightarrow Q(x))$	P	Premise
2	$P(y) \rightarrow Q(y)$	US	(1)
3	$(x)\ (Q(x) \rightarrow R(x))$	P	Premise
4	$Q(y) \rightarrow R(y)$	US	(3)
5	$P(y) \rightarrow R(y)$	T	(2),(4) $(P\rightarrow Q) \wedge (Q\rightarrow R) \Rightarrow (P\rightarrow R)$
6	$(x)\ (P(x) \rightarrow R(x))$	UG	(5)

Problem 3 :

Show that $(\forall x)[H(x)\rightarrow M(x)]\wedge H(s)\Rightarrow M(s)$.

Solution:

Steps	Premises	Rule	Reason
1	$(\forall x)[H(x)\rightarrow M(s)]$	P	Premise
2	$H(s)\rightarrow M(s)$	T	US
3	$H(s)$	P	Premise
4	$M(s)$	P	(2) & (3) Modus Ponens $P(x)$, $P(x)\rightarrow Q(x) \Rightarrow Q(x)$

Hence, $(\forall x)[H(x)\rightarrow M(x)]\wedge H(s)\Rightarrow M(s)$.

Problem 4:

Prove that $(\exists x)\,(P(x) \wedge S(x))\,,\ (\forall x)\,(P(x) \rightarrow R(x)) \Rightarrow (\exists x)\,(R(x) \wedge S(x))$

Solution:

Steps	Premises	Rule	Reason
1	$(\forall x)\,(P(x) \rightarrow R(x))$	P	Premise
2	$P(y) \rightarrow R(y)$	US	(1)
3	$(\exists x)\,(P(x) \wedge S(x))$	P	Premise
4	$P(y) \wedge S(y)$	ES	(3)
5	$P(y)$	T	(4) , $P \wedge Q \Rightarrow P$
6	$R(y)$	T	(2),(5), $P \rightarrow Q\,,\ P \Rightarrow Q$
7	$S(y)$	T	(4) , $P \wedge Q \Rightarrow Q$
8	$R(y) \wedge S(y)$	T	(6),(7), $P,\ Q \Rightarrow P \wedge Q$
9	$(\exists x)\,(R(x) \wedge S(x))$	EG	(8)

Show that from (a) $(\exists x)\,(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$

(b) $(\exists y)\,(M(y) \wedge \neg W(y))$ the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

Solution:

S. N	Premises	Rule	Reason
1	$(\exists y)\,(M(y) \wedge \neg W(y))$	P	Premise
2	$M(x) \wedge \neg W(x)$	ES	(1)
3	$\neg(M(x) \rightarrow W(x))$	T	(2), $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
4	$(\exists y)\,\neg(M(y) \rightarrow W(y))$	EG	(3)
5	$\neg(y)\, M(y) \rightarrow W(y)$	T	(4) $(\exists x) \neg A(x) \Leftrightarrow \neg (x) A(x)$
6	$(\exists x)\,(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$	P	Premise
7	$\neg (\exists x)\,(F(x) \wedge S(x))$	T	(5),(6) $P(x) \rightarrow Q(x)$, $\neg Q(x) \Rightarrow \neg P(x)$
8	$(x) \neg (F(x) \wedge S(x))$	T	(7) $\neg(\exists x) A(x) \Leftrightarrow (x) \neg A(x)$
9	$\neg (F(y) \wedge S(y))$	US	(8)
10	$F(x) \rightarrow \neg S(x)$	T	(9) $\neg(P \wedge Q) \equiv P \rightarrow \neg Q$
11	$(x)\,(F(x) \rightarrow \neg S(x))$	UG	(10)

PROBLEMS ON PREDICATE CALCULUS

TYPE 2 : INDIRECT METHOD

INDIRECT METHOD (OR) METHOD OF CONTRADICTION

Consider the negation of the conclusion as the additional premises and arrive at the conclusion as contradiction F.

Problem 1: **Show that $(x)(P(x) \lor Q(x)) \Rightarrow (x)P(x) \lor (\exists x)Q(x)$ by indirect method**

Solution:

Steps	Premises	Rule	Reason
1	$\neg ((x)P(x) \lor (\exists x)Q(x))$	P	Additional Premise
2	$\neg (x)P(x) \land \neg (\exists x)Q(x))$	T	(1)
3	$\neg (x)P(x)$	T	(2)
4	$(\exists x)\neg P(x)$	T	(3)
5	$\neg (\exists x)Q(x)$	T	(2)
6	$(x)\neg Q(x)$	T	(5)
7	$\neg P(y)$	ES	(4)
8	$\neg Q(y)$	US	(6)
9	$\neg P(y) \land \neg Q(y)$	T	(7),(8)
10	$\neg (P(y) \lor Q(y))$	T	(9)
11	$(x) (P(x) \lor Q(x))$	P	Premise
12	$P(y) \lor Q(y)$	US	(11)
13	$\neg (P(y) \lor Q(y)) \land (P(y) \lor Q(y))$	T	(10),(12), $(P,Q \Rightarrow P \land Q)$
14	F	T	$(A \land \neg A)=F$

Problem 2: Prove that follows logically from $(\exists x)\, M(x)$ the premises $(\forall x)(H(x) \rightarrow M(x))$ and $(\exists x)\, H(x)$ by Method of contradiction

Solution:

Steps	Premises	Rule	Reason
1	$\neg (\exists x)\, M(x)$	P	Additional Premises
2	$(\forall x) \neg M(x)$	T	(1)
3	$\neg M(a)$	US	(2)
4	$(\exists x)\, H(x)$	P	Premises
5	$H(a)$	ES	(4)
6	$H(a) \wedge \neg M(a)$	T	(5),(3), $P, Q \Rightarrow P \wedge Q$
7	$\neg (H(a) \rightarrow M(a))$	T	(6), $\neg(P \rightarrow Q) \Rightarrow P \wedge \neg Q$
8	$(\forall x)(H(x) \rightarrow M(x))$	P	Premises
9	$H(a) \rightarrow M(a)$	US	(8)
10	$(H(a) \rightarrow M(a)) \wedge \neg(H(a) \rightarrow M(a))$	T	(9), (7)
11	F	T	$P \wedge \neg P = F$

PROBLEMS ON PREDICATE CALCULUS

TYPE 3 : CP RULE

TYPE 3 : CP RULE (OR) CONDITIONAL PROOF (OR) CONDITIONAL CONCLUSION

If the conclusion is a conditional statement $R(x)\rightarrow S(x)$ then we consider $R(x)$ as additional premises & we prove $S(x)$. We write the conclusion as $R(x)\rightarrow S(x)$.

Prove that the implication $(x)(P(x) \to Q(x)) \Rightarrow (x)P(x) \to (x)Q(x)$

Solution:

Steps	Premises	Rule	Reason
1	$(x)(P(x) \to Q(x))$	P	Premise
2	$P(y) \to Q(y)$	US	(1)
3	$(x)\ P(x)$	P	Additional Premises
4	$P(y)$	US	(3)
5	$Q(y)$	T	(3),(4) , P , $P \to Q \Rightarrow Q$
6	$(x)\ Q(x)$	UG	(5)
7	$(x)\ P(x) \to (x)Q(x)$	CP	(3), (6)

PROBLEMS ON PREDICATE CALCULUS

VALIDITY OF THE ARGUMENT

Problem 1 : Establish the validity of the following argument.
(This argument popularly known as "Socrates argument")

- All men are mortal
- Socrates is a man
- Therefore Socrates is mortal

Solution: Let's use the Predicate formulae the above statements.

- **H(x) : x is man**
- **M(x) : x is mortal.**
- **s: Socrates.**

Now the above statements can be represented as −

- All men are mortal - $(x)(H(x) \rightarrow M(x))$
- Socrates is a man - $H(s)$
- Socrates is mortal - $M(s)$

As a statement, we need to conclude −

$$(x)(H(x) \rightarrow M(x)) \wedge H(s) \Longrightarrow M(s)$$

Verify the validity of the following argument. Every living thing is a plant or animal. John's dog is alive and it is not a plant. All animals have hearts. Hence John's dog has a heart.

Solution:

Universe : Set of all living things

Predicate are $P(x)$: x is a plant

$A(x)$: x is an animal

$H(x)$: x is a heart

a : John's dog

Premises are : $\mathbf{(x)(P(x) \vee A(x)), \neg P(a), (x)(A(x) \rightarrow H(x))}$

Conclusion : $\mathbf{H(a)}$

Steps	Premises	Rule	Reason
1	$(x)(P(x) \vee A(x))$	P	Premises
2	$P(a) \vee A(a)$	US	(1)
3	$\neg P(a)$	P	Premises
4	$A(a)$	T	(2),(3), $(P \vee Q) \wedge \neg P \Rightarrow Q$
5	$(x)(A(x) \rightarrow H(x))$	P	Premises
6	$A(a) \rightarrow H(a)$	US	(5)
7	$H(a)$	T	(4),(6), $(P \rightarrow Q) \wedge P \Rightarrow Q$

Try this !

Establish the validity of the argument
“ All integers are rational numbers” ,
“Some integers are powers of 3”.
Therefore, “Some rational numbers are powers of 3”.

“The essence of Mathematics is not to make simple things complicated but to make complicaed things simple”

- Gudder


Thank You