Dynamics:

Newton's laws:

everything in inertial (or world) frame!

① $\sum {}^{w}\vec{F}_i = \dfrac{d\,{}^{w}\vec{p}}{dt}$

② $\sum {}^{w}\vec{\tau}_i = \dfrac{d\,{}^{w}\vec{\ell}}{dt}$

$$\sum {}^{w}\vec{F}_i = {}^{w}\vec{F} = \frac{d\,(m\,{}^{w}\vec{v})}{dt} = m\,\frac{d\,{}^{w}\vec{v}}{dt}$$

$$= m\,{}^{w}\dot{\vec{v}}$$

↳ ①

$$\left\{\begin{aligned} {}^{w}\vec{v} &= Q\,{}^{B}\vec{v} \\ {}^{w}\dot{\vec{v}} &= \dot{Q}\,{}^{B}\vec{v} + Q\,{}^{B}\dot{\vec{v}} \end{aligned}\right.$$

${}^{B}\dot{\vec{x}} = 0$ (by definition of rigid body)

but ${}^{B}\dot{\vec{v}} \neq 0$

also, $\dot{Q} = Q\,{}^{B}\hat{\omega}$ (see "Rotations.pdf") $\}$

$$\Rightarrow\ {}^{w}\vec{F} = m\left(\dot{Q}\,{}^{B}\vec{v} + Q\,{}^{B}\dot{\vec{v}}\right)$$

$$= m\left(Q\,{}^{B}\hat{\omega}\,{}^{B}\vec{v} + Q\,{}^{B}\dot{\vec{v}}\right)$$

$${}^{w}\vec{F} = m\,Q\left({}^{B}\vec{\omega} \times {}^{B}\vec{v} + {}^{B}\dot{\vec{v}}\right)$$

$$Q^{T}\,{}^{w}\vec{F} = Q^{T}Q\left({}^{B}\vec{\omega} \times {}^{B}\vec{v} + {}^{B}\dot{\vec{v}}\right)$$

$$\underbrace{Q^T\,{}^W\vec{F}}_{={}^B\vec{F}} = m\,\underbrace{Q^T Q}_{=I_{3\times 3}}(\vec{\omega}\times\vec{v}+\dot{\vec{v}})$$

$= I_{3\times3}$ by definition of Rotation Matrix

$$\Rightarrow \boxed{{}^B\vec{F} = m\left({}^B\vec{\omega}\times{}^B\vec{v} + {}^B\dot{\vec{v}}\right)}$$

↳ 2

$$\left\{\begin{aligned} &{}^W\vec{l} = Q\,{}^B\vec{l} \\ &\Rightarrow {}^W\dot{\vec{l}} = \dot{Q}\,{}^B\vec{l} + Q\,{}^B\dot{\vec{l}} \end{aligned}\right.$$

we know, $\dot{Q} = Q\,{}^B\hat{\omega}$

$$\Rightarrow {}^W\dot{\vec{l}} = Q\,{}^B\hat{\omega}\,{}^B\vec{l} + Q\,{}^B\dot{\vec{l}} \quad \}$$

$$\Rightarrow \sum {}^W\vec{\tau}_i = {}^W\vec{\tau} = \frac{d\,{}^W\vec{l}}{dt} = {}^W\dot{\vec{l}}$$

$${}^W\vec{\tau} = Q\left({}^B\hat{\omega}({}^B\vec{l}) + {}^B\dot{\vec{l}}\right)$$

$$\underbrace{Q^T\,{}^W\vec{\tau}}_{{}^B\vec{\tau}} = \underbrace{Q^T Q}_{=I_{3\times 3}}\left({}^B\hat{\omega}\,{}^B\vec{l} + {}^B\dot{\vec{l}}\right)$$

$$ {}^B\vec{\tau} = {}^B\hat{\omega}\,{}^B\vec{\ell} + {}^B\dot{\vec{\ell}} $$

$$ \left\{ {}^B\vec{\ell} = \underbrace{J\,{}^B\vec{\omega}}_{[\;]_{3\times3}[\;]_{3\times1}} \Rightarrow {}^B\dot{\vec{\ell}} = \underbrace{\dot{J}}_{=0}\,{}^B\vec{\omega} + J\,{}^B\dot{\vec{\omega}} \right. $$

becuz, Inertia tensor doesn't change in Body frame

$$ \left. \therefore\ {}^B\dot{\vec{\ell}} = J\,{}^B\dot{\vec{\omega}} \right\} $$

$$ \Rightarrow {}^B\vec{\tau} = {}^B\hat{\omega}\,{}^B\vec{\ell} + {}^B\dot{\vec{\ell}} $$

$$ = {}^B\hat{\omega}\,J\,{}^B\vec{\omega} + J\,{}^B\dot{\vec{\omega}} $$

$$ \left\{ \vec{a}\times\vec{b} = \underbrace{\hat{a}\,b}_{[\;]_{3\times3}[\;]_{3\times1}} \right\} $$

$$ \Rightarrow \boxed{ {}^B\vec{\tau} = {}^B\vec{\omega} \times J\,{}^B\vec{\omega} + J\,{}^B\dot{\vec{\omega}} } $$

$$ \underline{ {}^B\vec{F} = m\left({}^B\vec{\omega}\times{}^B\vec{v} + {}^B\dot{\vec{v}}\right) } $$

$$ \underline{ {}^B\vec{\tau} = {}^B\vec{\omega}\times J\,{}^B\vec{\omega} + J\,{}^B\dot{\omega} } $$

$$ {}^B\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = {}^B\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} J_{xx} & & 0 \\ & J_{yy} & \\ 0 & & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} $$

$$ + J\,{}^B\begin{pmatrix} \dot{\omega}_x \end{pmatrix} $$

$$+ \; J \begin{bmatrix} \vdots \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}$$

$$= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ J_{xx}\omega_x & J_{yy}\omega_y & J_{zz}\omega_z \end{bmatrix} + \begin{bmatrix} J_{xx}\,{}^{B}\dot{\omega}_x \\ J_{yy}\,{}^{B}\dot{\omega}_y \\ J_{zz}\,{}^{B}\dot{\omega}_y \end{bmatrix}$$

$$= \begin{bmatrix} (J_{zz}-J_{yy})\,\omega_y\omega_z \\ (J_{xx}-J_{zz})\,\omega_x\omega_z \\ (J_{yy}-J_{xx})\,\omega_x\omega_y \end{bmatrix} + \begin{bmatrix} J_{xx}\,\dot{\omega}_x \\ J_{yy}\,\dot{\omega}_y \\ J_{zz}\,\dot{\omega}_z \end{bmatrix}$$

$$\boxed{\begin{bmatrix} {}^{B}\tau_x \\ {}^{B}\tau_y \\ {}^{B}\tau_z \end{bmatrix} = \begin{bmatrix} (J_{zz}-J_{yy})\,\omega_y\omega_z + \dot{\omega}_x J_{xx} \\ (J_{xx}-J_{zz})\,\omega_x\omega_z + \dot{\omega}_y J_{yy} \\ (J_{yy}-J_{xx})\,\omega_x\omega_y + \dot{\omega}_z J_{zz} \end{bmatrix}}$$

choice-1 choice-2

Drone State Space:

Choice 1

$$X = \begin{bmatrix} {}^{W}\vec{r} \\ q \\ {}^{B}\vec{v} \\ {}^{B}\vec{\omega} \end{bmatrix} \begin{matrix} (3\times1) \\ (4\times1) \\ (3\times1) \\ (3\times1) \end{matrix} \rightarrow 13\times1$$

Choice 2

$$X = \begin{bmatrix} {}^{W}\vec{r} \\ q \\ {}^{W}\vec{v} \\ {}^{B}\vec{\omega} \end{bmatrix} \rightarrow 13\times1$$

Choice-1: Dynamics:

$$\dot{X} = f(X, u)$$

$$\dot{X} = \begin{bmatrix} {}^{W}\dot{\vec{r}} \\ \dot{q} \\ {}^{B}\dot{\vec{v}} \\ {}^{B}\dot{\vec{\omega}} \end{bmatrix} = \begin{bmatrix} Q\,{}^{B}\vec{v} \\ \frac{1}{2}\left(q \otimes \begin{bmatrix} 0 \\ {}^{B}\vec{\omega} \end{bmatrix}\right) \\ \left(\frac{{}^{B}\vec{F}}{m} - {}^{B}\vec{\omega} \times {}^{B}\vec{v}\right) \\ \text{extract from } \; {}^{B}\vec{\tau} = {}^{B}\vec{\omega} \times J\,{}^{B}\vec{\omega} + J\,{}^{B}\dot{\vec{\omega}} \end{bmatrix}$$

Choice-2:

$$\dot{X} = \begin{bmatrix} {}^{W}\dot{\vec{r}} \\ \dot{q} \\ {}^{W}\dot{\vec{v}} \\ {}^{B}\dot{\vec{\omega}} \end{bmatrix} = \begin{bmatrix} \boxed{{}^{W}\vec{v}} \\ \frac{1}{2}\left(q \otimes \begin{bmatrix} 0 \\ {}^{B}\vec{\omega} \end{bmatrix}\right) \\ \boxed{Q\,{}^{B}\dot{\vec{v}} = Q\left(\frac{{}^{B}\vec{F}}{m} - {}^{B}\vec{\omega} \times {}^{B}\vec{v}\right)} \\ \text{extract from } \; {}^{B}\vec{\tau} = {}^{B}\vec{\omega} \times J\,{}^{B}\vec{\omega} + J\,{}^{B}\dot{\vec{\omega}} \end{bmatrix}$$

${}^{W}\vec{r} \Rightarrow$ Motion Capture

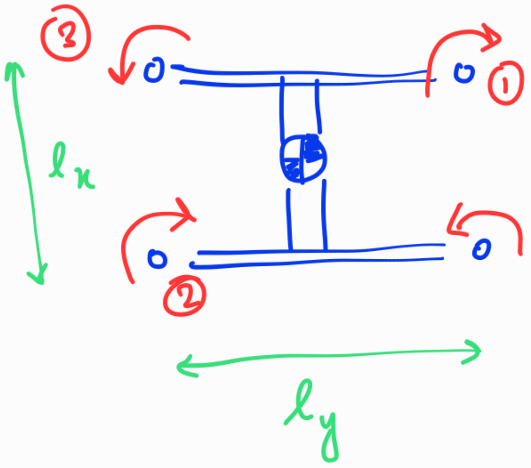
$q \rightarrow$ integrating Rotation Matrix "Q" from known initial orientatn "Q_0"

will give us orientation (Q or quaternion "q") at any future time given that we have IMU (gyroscope) readings $({}^{B}\vec{\omega}_i)$ at every time step.

${}^{B}\vec{v} \rightarrow$ Motion capture? Integration?

${}^{B}\vec{\omega} \rightarrow$ gyroscope readings of IMU sensor.

Finding ${}^{B}\vec{\mathbb{F}}$ & ${}^{B}\vec{\tau}$ for a drone...



$${}^{B}\vec{\mathbb{F}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_{wind}$$

$${}^{B}\tau_x = {}_{w}\vec{M}_x + \{F_2 + F_3 - (F_1 + F_4)\}$$

$${}^{B}\tau_y = {}_{w}\vec{M}_y + \{F_1 + F_3 - (F_2 + F_4)\}$$

$${}^{B}\tau_z \quad {}_{w}\vec{M}_z + \{\underbrace{M_1 + M_2 - (M_3 + M_4)}_{\text{Yaw Torque of propellers working against air}}\}$$

If wind induces moments too,
then

then

↳ $_w\vec{M}_x$ $_w\vec{M}_y$ $_w\vec{M}_z$ are moments generated due to wind)
