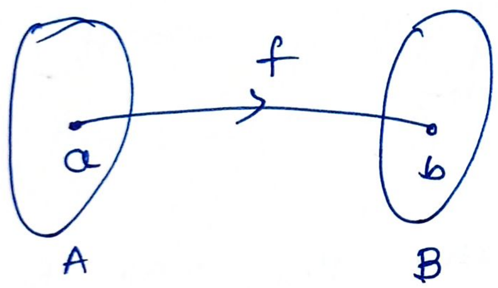
Functions

Let A and B are two non-empty sets. Then a function f from A to B is a relation from A to B such that for each $a \in A$ there is a unique b in B such that $(a, b) \in f$.



→ A is called domain of f and B is called Co-domain of f.

→ Subset of B consisting of all images of elements of A under f is called range of f.

ie The Range of $f: A \to B$ is

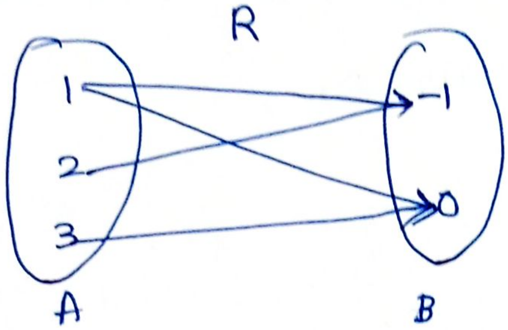
$$f(A) = \{ f(x) \mid x \in A \}.$$

→ For $f: A \to B$, if $b \in B$ and $f^{-1}(b)$ is defined as

$$f^{-1}(b) = \{ x \in A \mid f(x) = b \}.$$

problem Let $A = \{1,2,3\}$, $B = \{-1,0\}$ and R be a relation from A to B defined as $R = \{(1,-1)$ $(1,0)$ $(2,-1)$, $(3,0)\}$. Is R a function from A to B.

Sol:



$\therefore$ we observe that, under R, the element $1 \in A$ is related to two elements -1 and 0 of B

$\therefore$ R is not a function.

problem Let a function $f: R \to R$ defined as

$$f(x) = x^2+1.$$

Determine ranges of the following subsets of R.

(i) $A_1 = \{2,3\}$ (ii) $A_2 = \{-2,0,3\}$

(iii) $A_3 = (0,1)$ (iv) $A_4 = [-6,3]$.

Sol: (i) $f(2) = 5$, $f(3) = 10$

$\therefore$ $f(A_1) = \{5,10\}$.

(ii)

$f(-2) = 5, \quad f(0) = 1, \quad f(3) = 10.$

$\therefore \quad f(A_2) = \{5, 1, 10\}.$

(iii)

Since $A_3 = \{x \in \mathbb{R} \mid 0 < x < 1\}.$

$$f(A_3) = \{ f(x) \mid 0 < x < 1 \}$$

$$= \{ x^2 + 1 \mid 0 < x < 1 \}.$$

(iv) $f(A_4) = \{ f(x) \mid -6 \leq x \leq 3 \}$

$$= \{ (x^2 + 1) \mid -6 \leq x \leq 3 \}.$$

problem Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{6, 7, 8, 9, 10\}$

If a function $f : A \to B$ is defined as

$$f = \{ (1,7), (2,7), (3,8), (4,6), (5,9), (6,9) \}.$$

Determine $f^{-1}(6)$, $f^{-1}(9)$. If $B_1 = \{7, 8\}$

$B_2 = \{8, 9, 10\}$, find $f^{-1}(B_1)$, $f^{-1}(B_2)$.

Sol:

$$f^{-1}(6) = \{ x \in A \mid f(x) = 6 \}$$

$$= \{4\}.$$

$\cdot\ f^{-1}(9) = \{x \in A \mid f(x) = 9\}$

$= \{5, 6\}$

$\cdot\ f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$

$= \{x \in A \mid f(x) = 7,\ f(x) = 8\}$

$= \{1, 2, 3\}.$

$\cdot\ f^{-1}(B_2) = \{x \in A \mid f(x) \in B_2\}$

$= \{x \in A \mid f(x) = 8,\ f(x) = 9,\ f(x) = 10\}$

$= \{3, 5, 6\}.$

<u>problem</u>

Let $f: R \to R$ be defined as

$$f(x) = \begin{cases} 3x - 5 & \text{for } x > 0 \\ -3x + 1 & \text{for } x \leq 0. \end{cases}$$

(i) Determine $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$

(iii) Find $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$.

From definition of $f(x) = 3x-5$; $x > 0$

$-3x+1$; $x \leq 0.$

$\therefore$ For $x > 0$, Satisfying $3x-5 = -3.$

$x = \frac{2}{3}.$

~~For~~ NO $x \leq 0$, Satisfying $-3x+1 = -3$

$x = \frac{2}{3}.$

$\therefore \ f^{-1}(-3) = \{\frac{2}{3}\}.$

- $f^{-1}(6) = \emptyset$, because there is no $x \in R$ satisfying $f(x) = -6.$

(iii)

$f^{-1}([-5,5]) = \{x \in R \mid f(x) \in [-5,5]\}$

$= \{x \in R \mid -5 \leq f(x) \leq 5\}.$

when $x > 0$, $f(x) = 3x-5$

$\therefore$

$-5 \leq 3x-5 \leq 5$

$0 \leq x \leq \frac{10}{3}$

when $x \leq 0$, $f(x) = -3x+1$

$\therefore$ $-5 \leq -3x+1 \leq 5$

$-\frac{4}{3} \leq x \leq 2$

$\therefore \quad f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le 2 \text{ or } 0 \le x \le \frac{10}{3}\}$

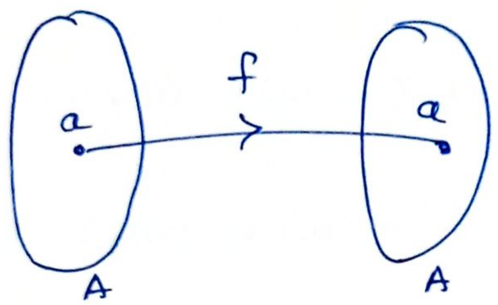
$= \{x \in \mathbb{R} \mid -\frac{4}{3} \le x \le \frac{10}{3}\}$

$= [-\frac{4}{3}, \frac{10}{3}].$

Similarly $\quad f^{-1}([-6,5]) = [-\frac{4}{3}, \frac{10}{3}].$

→ <u>Identity function :</u>

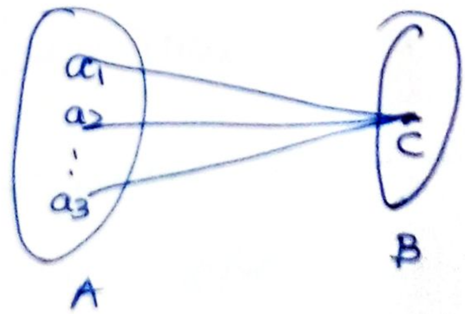
A function $f : A \to A$ Such that $f(a) = a$, $\forall a \in A$ is Called Identity function / Identity mapping on A.



Identity function on set A is usually denoted with I_A.

→ <u>Constant function :</u>

A function $f : A \to B$ Such that $f(a) = c$, $\forall a \in A$ where c is a fixed element of B is Called a Constant function.



→ <u>one-to-one function (Injective function)</u>

A function $f: A \to B$ is Said to be one-to-one function, if different elements of A have different images in B under B.

ie $a_1, a_2 \in A$ with $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

Equivalently, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

→ <u>Onto function (Surjective function)</u>

A function $f: A \to B$ is Said to be an onto function if for every element 'b' in B there is an element 'a' of A Such that $f(a) = b$.

→ <u>one-to-one Correspondance: (Bijective function)</u>

A function which is both one-to-one and onto is Called one-to-one Correspondance.

If $f: A \to B$ is bijective, then every element of A has a unique image in B and ~~A~~ every element in B has a unique preimage in A.

Number of one-to-one and onto functions:

If A and B are finite sets with $|A| = m$ and $|B| = n$ where $m \geq n$. Then

→ The number of one-to-one functions possible from A to B $= \dfrac{n!}{(n-m)!}$

→ The number of onto functions from A to B

$$= \sum_{k=0}^{n} (-1)^k \cdot {}^{n}C_{n-k} \, (n-k)^m$$

problem Let A and B be finite sets. If there are 60 one-to-one functions from A to B and $|A| = 3$ what is $|B|$.

Sol: Here $m = 3$, $\dfrac{n!}{(n-m)!} = 60$

$$\therefore \frac{n!}{(n-3)!} = 60$$

$$\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 60$$

$$n(n-1)(n-2) = 60$$

Clearly for $n=5$, $5 \times 4 \times 3 = 60$

$\therefore \boxed{n=5}$ $\therefore |B| = 5$

problem Let $A = \{1,2,3,4\}$, $B = \{1,2,3,4,5,6\}$

(i) Find how many functions are there from A to B.

(ii) How many one-to-one functions from A to B.

(iii) How many functions are there from B to A?

How many of these are one-one? How many are onto?

Sol:

$|A| = m = 4$, $|B| = n = 6$.

(i) Number of functions $= n^m = 6^4 = 1296$

Number of one-one $= \dfrac{n!}{(n-m)!} = \dfrac{6!}{2!} = 360$

(ii) Number of functions from B to A $= 4^6 = 4096$

Number of one-one from B to A $= 0$.

Number of onto functions $= \sum_{K=0}^{4} (-1)^K \, 4C_{4-K} \, (4-K)^6$

$= 4^6 - 4 \times 3^6 + 6 \times 2^6 - 4$

$= 1560$.

problem The function $f : (Z \times Z) \to Z$ Such that

$f(x,y) = 2x+3y$. Verify that f is

onto but not one-to-one.

Sol:

Consider ~~some~~ $n \in Z$

Then $n = 4n - 3n$

$= 2(2n) + 3(-n)$

$= f(2n, -n)$

Hence, for every $n \in Z$ has a preimage $(2n, -n)$

in $Z \times Z$ under the mapping f.

$\therefore$ f is onto function.

Now, Consider $f(0,2) = 2 \times (0) + 3(2)$

$= 6$

$f(3,0) = 2(3) + 3(0)$

$= 6.$

ie. $f(0,2) = f(3,0)$

But $(0,2) \neq (3,0)$

$\therefore$ f is not one-to-one.

problem: Let $f: R \to R$ and $g: R \to R$ defined as

$f(x) = 3x+7, \forall x \in R$

$g(x) = x(x^3-1), \forall x \in R$. verify that 'f' is one-to-one but 'g' is not.

Sol:

For any $x_1, x_2 \in R$

$$f(x_1) = 3x_1+7$$
$$f(x_2) = 3x_2+7$$

Consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow 3x_1+7 = 3x_2+7$$

$$\Rightarrow 3x_1 = 3x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ f is one-to-one function.

Now

$x_1=0, x_2=1$, we have $g(0)=0$

$g(1)=0$

Since $g(x_1) = g(x_2)$ but $x_1 \neq x_2$

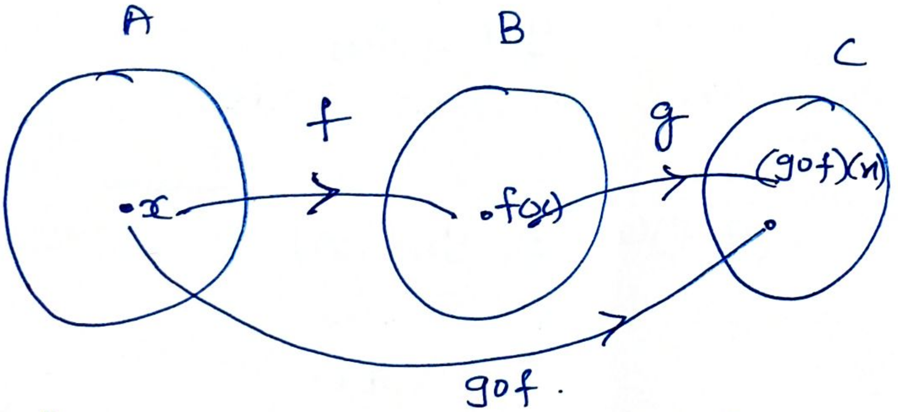
$\therefore$ $g(x)$ is not one-to-one function.

Composition of functions:

Let $f: A \to B$ and $g: B \to C$ be functions with the property that Range of f is a subset of the domain of g. Then $gof: A \to C$ defined as

$$(gof)(x) = g(f(x)) \quad , \forall x \in A$$

is called composition of f and g.



For function $f: A \to A$, fof is denoted as f^2.

problem For the following functions compute gof, fog. Also verify $gof = fog$.

(i) $f(x) = x^5$, $g(x) = x^{1/5}$ for all real x.

(ii) $f(x) = 2x$, $g(x) = \lfloor \frac{x}{2} \rfloor$ for all integers x.

Sol:

(i) $(g \circ f)(x) = g(f(x))$

$= g(x^5)$

$= (x^5)^{1/5}$

$= x$

|||ly

$(f \circ g)(x) = f(g(x))$

$= f(x^{1/5})$

$= (x^{1/5})^5$

$= x.$

$\therefore \quad g \circ f = f \circ g.$

(ii) $(g \circ f)(x) = g(f(x))$

$= g(2x)$

$= \left\lfloor \frac{2x}{2} \right\rfloor$

$= \lfloor x \rfloor$

$= x$ for all integers x.

$\lfloor\ \rfloor$ - Denotes floor function.

$(f \circ g)(x) = f(g(x))$

$= f\left(\left\lfloor \frac{x}{2} \right\rfloor\right)$

$= 2\left\lfloor \frac{x}{2} \right\rfloor$ for all integers x.

$\therefore \quad f \circ g \neq g \circ f.$

problem Consider the functions $f(x) = x^3$, $g(x) = x^2+1 \ \forall x \in R$

Find gof, fog, f^2 and g^2.

Sol:

- $(gof)(x) = g(f(x)) = g(x^3) = (x^3)^2+1 = x^6+1.$

- $(fog)(x) = f(g(x)) = f(x^2+1) = (x^2+1)^3$

- $f^2(x) = (fof)(x) = f(f(x)) = f(x^3) = (x^3)^3 = x^9$

- $g^2(x) = (gog)(x) = g(g(x)) = g(x^2+1) = (x^2+1)^2+1$

problem Let f, g, h be the functions from Z to Z defined as

$$f(x) = x-1, \quad g(x) = 3x \text{ and } h(x) = \begin{cases} 0 & : \text{if } x \text{ is Even} \\ 1 & : \text{if } x \text{ is ODD.} \end{cases}$$

Verify that $fo(goh) = (fog)oh$. Determine h^{500}.

Sol:

Consider

$$fo(goh)(x) = f\{(goh)(x)\}$$

$$= \begin{cases} 0-1 & : x \text{ is Even} \\ 3-1 & : x \text{ is ODD} \end{cases}$$

$$= \begin{cases} -1 & : x \text{ is Even} \\ 2 & : x \text{ is ODD.} \end{cases}$$

Since

$$(goh)(x) = g(h(x))$$

$$= \begin{cases} 0 & : x \text{ is Even} \\ 3 & : x \text{ is ODD} \end{cases}$$

$[(fog)oh](x) = (fog)[h(x)]$

$$= \begin{cases} (fog)(0) & : \text{if } x \text{ is Even} \\ (fog)(1) & : \text{if } x \text{ is ODD.} \end{cases}$$

$\because$

$\boxed{\begin{array}{l} (fog)(x) \\ \Rightarrow f(g(x)) \\ \Rightarrow f(3x) \\ \Rightarrow 3x-1 \end{array}}$

$$= \begin{cases} 3(0)-1 & : \text{if } x \text{ is Even} \\ 3(1)-1 & : \text{if } x \text{ is ODD} \end{cases}$$

$$= \begin{cases} -1 & : \text{if } x \text{ is Even} \\ 2 & : \text{if } x \text{ is ODD} \end{cases}$$

$\therefore \quad (fog)oh = fo(goh).$

Since $\quad h^2(x) = (hoh)(x) = h[h(x)]$

$$= \begin{cases} 0 & : x \text{ is Even} \\ 1 & : x \text{ is ODD} \end{cases}$$

$$h^{500}(x) = h[h^{499}(x)] = \begin{cases} 0 & : x \text{ is Even} \\ 1 & : x \text{ is ODD.} \end{cases}$$

<u>problem</u> Let $f, g : R \to R$ where $f(x) = ax+b$, $g(x) = 1-x+x^2$

If $(gof)(x) = 9x^2-9x+3$, determine a, b.

Sol:

$$(gof)x = g(f(x))$$

$$= g(ax+b)$$

$$= 1-(ax+b)+(ax+b)^2$$

$$= a^2x^2 + (2ab-a)x + b^2-b+1$$

hence $\quad a^2 = 9 \longrightarrow (1)$

$2ab - a = -9 \longrightarrow (2)$

$b^2 - b + 1 = 3. \longrightarrow (3)$

from (1) $\quad a = \pm 3.$

when $a = +3, \quad 2(3)b - 3 = -9$

$\Rightarrow \quad b = -1$

$\therefore \quad \boxed{a = +3, \ b = -1.}$

when $a = -3, \quad 2(-3)b - (-3) = -9$

$b = 2$

~~But b=2, does not satisfies equ (3)~~

$\therefore$

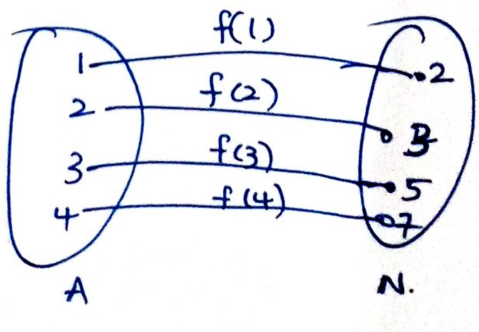
$\left.\begin{array}{l} a = +3, \ b = -1 \\ a = -3, \ b = 2 \end{array}\right\}$ are possible values.

problem Let $g: N \to N$ be defined as $g(n) = 2n$.

If $A = \{1, 2, 3, 4\}$ and $f: A \to N$ is given by

$f = \{(1,2), (2,3), (3,5), (4,7)\}$. Find gof.

Sol:



Now

$$(g \circ f)(x) = g(f(x))$$

$$\therefore \; g(f(1)) = g(2) = 2 \cdot 2 = 4$$

$$g(f(2)) = g(3) = 2 \cdot 3 = 6$$

$$g(f(3)) = g(5) = 2 \cdot 5 = 10$$

$$g(f(4)) = g(7) = 2 \cdot 7 = 14$$

problem

Let S be the set of all strings in a's and b's.

and $L : S \to Z$ be the length function for all strings in S. ie $L(s)$ = number of characters in s.

$T : Z \to \{0, 1, 2\}$ is defined as $T(n) = n \bmod 3$.

what $(T \circ L)(abaa)$?

Sol:

$$(T \circ L)(abaa) = T(L(abaa))$$

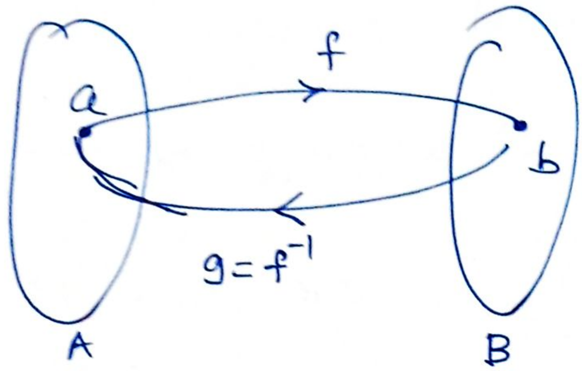
$$= T(4)$$

$$= 4 \bmod 3$$

$$= 1$$

Invertible function:

If $f: A \to B$ Then f is Said to be invertible, if there is a function $g: B \to A$ Such that $g \circ f = I_A$ and $f \circ g = I_B$. where I_A, I_B are identity functions on A and B respectively.



Theorem: A function $f: A \to B$ is invertible, if and only if it is one-to-one and onto.

proof:

Assume that $f: A \to B$ is invertible.

Then, we have a Unique function $g: B \to A$ with
$$g \circ f = I_A$$
$$f \circ g = I_B.$$

For $a_1, a_2 \in A$ with

$$f(a_1) = f(a_2)$$
$$\Rightarrow g(f(a_1)) = g(f(a_2))$$
$$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$$
$$\Rightarrow I_A(a_1) = I_A(a_2)$$
$$\Rightarrow a_1 = a_2$$

Now Take any element $b \in B$.

Then $g(b) \in A$ and

$$b = I_B(b)$$
$$= (f \circ g)(b)$$
$$= f(g(b))$$

Thus b is the ~~element~~ Image of $g(b)$ under f

$\therefore$ f is one-to-one and onto.

Conversely, Suppose that f is 1-1 and onto.

Then for each $b \in B$, there is a unique $a \in A$ Such that $b = f(a)$.

Now Consider the function $g : B \to A$ defined by $g(b) = a$.

Now $(g \circ f)(a) = g(f(a))$

$$= g(b)$$
$$= a$$
$$I_A(a)$$

$(f \circ g)(b) = f(g(b))$

$$= f(a)$$
$$= b$$
$$= I_B(b)$$

$\therefore$ f is invertible ~~and~~ with g as the inverse.

problem Find the Inverse of $f(x) = e^x$ defined from R to R^+.

Sol: For any $x_1, x_2 \in R^+$

$$f(x_1) = f(x_2)$$

$$\Rightarrow e^{x_1} = e^{x_2}$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ f is one-to-one.

For every $y \in R^+$, $x = \log_e y$ as its preimage in R under f

$\therefore$ f is onto also

$\therefore$ f is invertible.

$$f^{-1}(y) = \log_e y \text{ for all } y \in R^+.$$

problem Let $A = B = R$, the set of all real numbers and the functions $f: A \to B$, $g: B \to A$ defined as

$$f(x) = 2x^3 - 1, \quad \forall x \in A$$

$$g(y) = \left(\frac{y+1}{2}\right)^{1/3}, \quad \forall y \in B.$$

Show that each of f and g is inverse of the other.

Sol:

For any $x \in A$,

$$(gof)(x) = g(f(x))$$

$$= \cancel{g\left(\frac{y+1}{2}\right)} = g(2x^3 - 1)$$

$$= \cancel{g(f} = \left\{\frac{1}{2}(2x^3 - 1 + 1)\right\}^{1/3}$$

$$= x.$$

$\therefore \quad (gof)(x) = I_A$

For any $y \in B$,

$$(fog)(y) = f(g(y))$$

$$= f\left\{\left[\frac{1}{2}(y+1)\right]^{1/3}\right\}$$

$$= 2\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]^3 - 1$$

$$= 2\left[\frac{1}{2}(y+1)\right] - 1$$

$$= y$$

$\therefore \quad fog = I_B.$

$\therefore$ f and g are inverse of each other.

problem Let $A=B=C=R$ and $f: A \to B$, $g: B \to C$ are defined as

$$f(a) = 2a+1$$
$$g(b) = \frac{b}{3} \quad \forall a \in A, \ \forall b \in B.$$

Compute $g \circ f$ and show that $g \circ f$ is invertible. What is $(g \circ f)^{-1}$?

Sol:

we have

$$(g \circ f)(a) = g(f(a))$$
$$= g(2a+1)$$
$$= \frac{1}{3}(2a+1)$$

Thus $g \circ f : A \to C$ defined as $(g \circ f)(a) = \frac{2a+1}{3}$

we know that $f^{-1}(b) = \frac{b-1}{2}$

$$g^{-1}(c) = 3c.$$

Then $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$

$$= f^{-1}(g^{-1}(c))$$
$$= f^{-1}(3c)$$
$$= \frac{3c-1}{2}$$

The Growth of functions:

Big 'oh' notation:

Big - O notation is a standard way to describe how much time and how much memory required to run an algorithm.

Let f and g be functions from ~~set~~ the set of integers or set of real numbers to set of real numbers. we say that $f(x)$ is $O(g(x))$, if there are constants C and k such that

$$|f(x)| \leq C\, g(x) \quad \text{whenever } x \geq k$$

~~This~~ This is read as " f(x) is big-oh of g(x).

Example: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

Sol:

$$x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2 \text{ for } x \geq 1.$$

$\therefore$ $f(x) < 4x^2$. ie $C = 4$

$k = 1$.

~~Big Theta Notation (Θ)~~

~~f(x) is said to be of Θ(g(x))~~

Big-Ω Notation:

- $f(x)$ is $\Omega(g(x))$, if there are positive constants c and k such that when $x > k$

$$|f(x)| \geq c\, g(x).$$

Big-Θ Notation (Big theta)

- $f(x)$ is $\Theta(g(x))$, if $f(x)$ is both $O(g(x))$ and $\Omega(g(x))$.

- For any function that is $\Theta(g(x))$, we will say that function is of order $g(x)$.

Ex: for function $2^x + x^2$

$$|2^x + x^2| = 2^x + x^2$$

$$\leq 2 \cdot 2^x. \qquad \because x^2 \leq 2^x, \; x \geq 4$$

$\therefore$ $2^x + x^2$ is $O(2^x)$. $\quad c = 2$, $k = 4$.

and also $|2^x + x^2| = 2^x + x^2$

$\geq 2^x$ for $x \geq 1$.

$\therefore$ $2^x + x^2$ is $\Omega(2^x)$ with $C = 1$

$K = 1$.

$\therefore$ $2^x + x^2$ is $\Theta(2^x)$.

Stirling Numbers of the Second Kind:

Stirling numbers of the second kind, written as $\left\{ {n \atop k} \right\}$ or $S(n,k)$, are the number of ways to partition n objects into k-disjoint non-empty sets.

Consider $\{a,b,c,d\}$

We want to find the number of ways to distribute these elements into 3 non-empty subsets. Such as:

① $\{a,b\}, \{c\}, \{d\}$

② $\{a,c\}, \{b\}, \{d\}$

③ $\{a,d\}, \{b\}, \{c\}$

④ $\{a\}, \{b,c\}, \{d\}$ $\therefore$

⑤ $\{a\}, \{b,d\}, \{c\}$

⑥ $\{a\}, \{b,c\}, \{d\}$.

Stirling Numbers and onto functions:

Let $|A| = m$, $|B| = m$ where $m \geq n$.

Then number of onto functions are

$$P(m,n) = \sum_{k=0}^{n} (-1)^k \, {}^{n}C_{n-k} \cdot (n-k)^m.$$

Then $\frac{P(m,n)}{n!}$ is Considered as Stirling number

Of Second Kind denoted as $S(m,n)$

$$\therefore S(m,n) = \frac{P(m,n)}{n!}$$

$$= \frac{\sum_{K=0}^{n} (-1)^K \, {}^nC_{n-K} \cdot (n-K)^m}{n!} \quad \text{for } m \geq n$$

→. There are $n! \cdot S(m,n)$ onto functions from A to B.