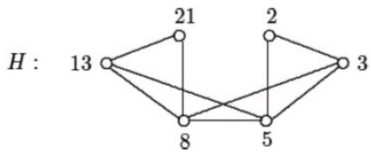
UNIT - V
Graph Theory

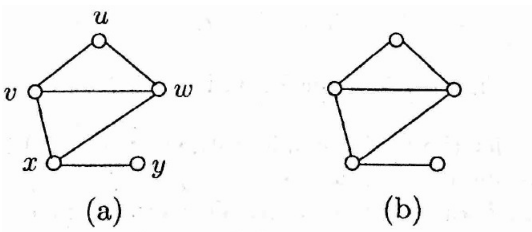
1 Introduction

Definition. *(Graph). A graph G consists of a fnite or countable vertex set* $V := V(G)$ *and an edge set* $E := E(G) \subset V \times V$*, So a graph* $\boldsymbol{G}$ *is a pair* $\boldsymbol{G = \{ V, E \}}$.

Example 1. *Consider the sequence of integers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ..., known as the **Fibonacci numbers**. Consider the set S = { 2, 3, 5, 8, 13, 21 } of six specific Fibonacci numbers. There are some pairs of distinct integers belonging to S whose sum or difference(in absolute value) also belongs to S, namely, { 2, 3 }, { 2, 5 }, { 3, 5 }, { 3, 8 }, { 5, 8}, { 5, 13 }, { 8, 13 }, { 8, 21 } and { 13, 21 }. We can visualize these pairs, by the graph H. In this case V(H) = { 2, 3, 5, 8, 13, 21 } and E(H) = { 2, 3 }, { 2, 5 }, { 3, 5 }, { 3, 8 }, { 5, 8}, { 5, 13 }, { 8, 13 }, { 8, 21 }, { 13, 21 }.*



The number of vertices in G is often called the **order (n)** of G, while the number of edges is its **size (m)**.
The above graph H has order-6 and size-9. We often use n and m for order and size, respectively, of a graph.
A graph G with $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, uw, vw, vx, wx, xy\}$ is shown below.



: A labeled graph and an unlabeled graph

In this case, graph (a) is drawn without labeling its vertices. For this reason, the graph (a) is a **labeled graph** and (b) is an **unlabeled graph**.

1.1 Terminology

- The two vertices u and v are **end vertices** of the edge (u, v).
- Edges that have the same end vertices are **parallel edges**.
- An edge of the form (v, v) is a **loop**.
- A graph is **simple graph** if it has no parallel edges or loops.
- A graph with no edges (i.e. E is empty) is **empty graph**.
- A graph with no vertices (i.e. V and E are empty) is a **null graph**.
- A graph with only one vertex is **trivial graph**.
- Edges are **adjacent edges** if they share a common end vertex.
- Two vertices u and v are **adjacent vertices** if they are connected by an edge, in other words, (u, v) is an edge.
- The **degree** of the vertex v, written as $d(v)$, is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.
- A **pendant vertex** is a vertex whose degree is one.
- An edge that has a pendant vertex as an end vertex is a **pendant edge**.
- An **isolated vertex** is a vertex whose degree is zero.

1.2 Degree

The **minimum degree** of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$; the **maximum degree** of G is denoted by $\Delta(G)$. So if G is a graph of order n and v is any vertex of G, then $0 \leq \delta(G) \leq deg\ v \leq \Delta(G) \leq n - 1$.

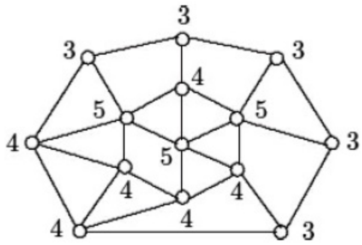
Theorem 2. *The First Theorem of Graph Theory: If G is a graph of size m, then* $\sum_{v \in V(G)} deg\ v = 2m$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices.

$\square$

Example 3. *A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5?*
Ans: Let x be the number of vertices of G having degree 3. Then,
$3 \cdot x + 4 \cdot 6 + 5 \cdot (8 - x) = 2 \cdot 27 \implies x = 5$. *Hence G has five vertices of degree 3 and three vertices of degree 5.*

Another way of soling the given problem is by drawing the graph as below.



OR

$x + y = 8$ and $3x + 4 \cdot 6 + 5y = 2 \cdot 27 = 54 \implies x = 5$ and $y = 3$ satisfies the equations.

Theorem 4. *Every graph has an even number of odd degree vertices.*

Proof. Let G be a graph of size m. Divide $V(G)$ into two subsets V_1 and V_2, where V_1 consists of the odd vertices of G and V_2 consists of the even vertices of G. By the First Theorem of Graph Theory,

$$\sum_{v \in V(G)} deg\ v = \sum_{v \in V_1} deg\ v + \sum_{v \in V_2} deg\ v = 2m$$

The number $\sum_{v \in V_2} deg\ v$ is even since it is a sum of even integers. Thus

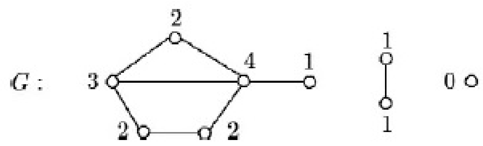
$$\sum_{v \in V_1} deg\ v = 2m - \sum_{v \in V_2} deg\ v$$

which implies $\sum_{v \in V_1} deg\ v$ is even. Since each of the numbers $deg\ v, v \in V_1$ is odd, the number of odd vertices of G is even.

$\square$

Degree Sequences

It is typical for the vertices of a graph to have a variety of degrees. If the degrees of the vertices of a graph G are listed in a sequence s, then s is called a degree sequence of G. For example, $s : 4, 3, 2, 2, 2, 1, 1, 1, 0$; $s^{'} : 0, 1, 1, 1, 2, 2, 2, 3, 4$; $s^{''} : 4, 3, 2, 1, 2, 2, 1, 1, 0$
all of the sequences are degree sequences of the graph G of the below figure, each of whose vertices is labeled by its degree.



The sequence s is non-increasing, s' is non-decreasing and s'' is neither.
Suppose that we are given a finite sequence s of nonnegative integers. This finite sequence of nonnegative integers is called **graphical** if it is a degree sequence of some graph.

Example 5. *Which of the following sequences are graphical?*

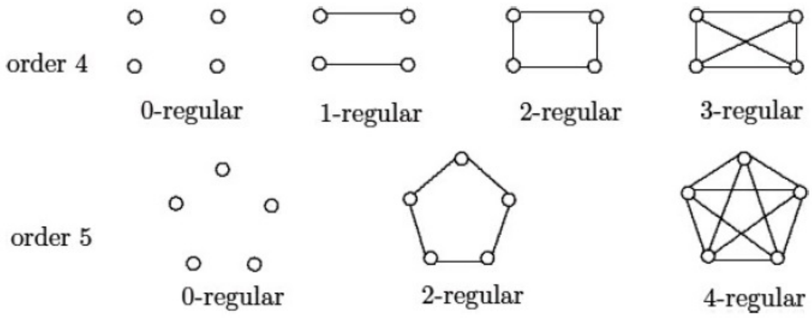
1. $s_1 : 3, 3, 2, 2, 1, 1$
2. $s_2 : 6, 5, 5, 4, 3, 3, 3, 2, 2$
3. $s_3 : 7, 6, 4, 4, 3, 3, 3$
4. $s_4 : 3, 3, 3, 1$

Ans: (i) Yes (ii) No (iii) No (iv) No.

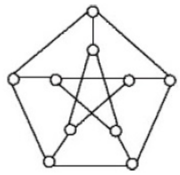
Definition. ***Regular Graphs***

*We have already mentioned that $0 \leq \delta(G) \leq \Delta(G) \leq n - 1$ for every graph G of order n. If $\delta(G) = \Delta(G)$ then the vertices of G have the same degree and G is called **regular**.*

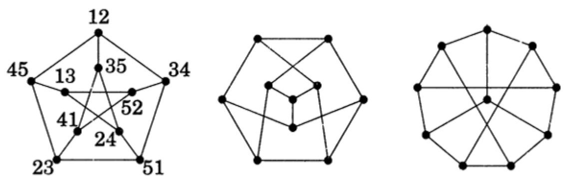
If $deg\ v = r$ for every vertex v of G, where $0 \leq r \leq n - 1$, then G is **r-regular** or **regular of degree r**. The only regular graphs of order 4 or 5 are shown in the below figure. There is no 1-regular or 3-regular graph of order 5, as no graph contains an odd number of odd vertices by a theorem.



A 3-regular graph is also referred to as a **cubic graph**. The best known cubic graph may very well be the **Petersen graph**, shown in the below figure.



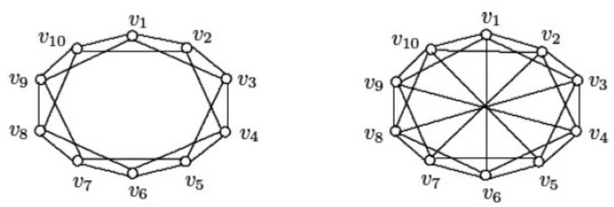
Petersen graph is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.



By a theorem, there are no r-regular graphs of order n if r and n are both odd.

Theorem 6. *Let r and n be integers with $0 \leq r \leq n-1$. There exists an r-regular graph of order n if and only if at least one of r and n is even.*

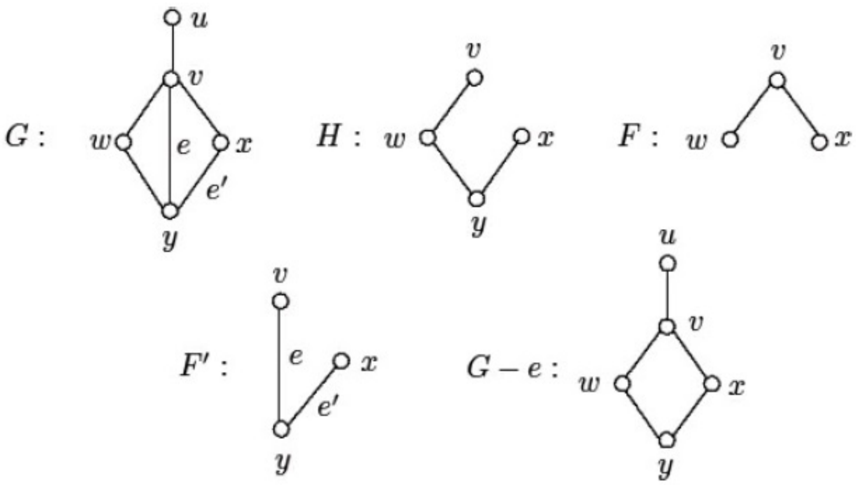
A 4-regular graph and a 5-regular graph, both of order 10 are shown below.



(a): $H_{4,10}$ (b): $H_{5,10}$

Definition. *A graph H is called a* ***subgraph*** *of a graph G, written $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.*

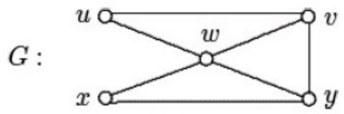
We also say that G contains H as a subgraph. If $H \subset G$ and either $V(H)$ is proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G. If a subgraph of a graph G has the same vertex set as G, then it is a **spanning subgraph** of G.



A subgraph F of a graph G is called an **induced subgraph** of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. If S is a nonempty set of vertices of a graph G, then the **subgraph of G induced by S** is the induced subgraph with vertex set S. The induced subgraph is denoted by $\langle S \rangle$. To emphasize that this is an induced subgraph of G, we sometimes denote this subgraph by $\langle S \rangle_G$. For a nonempty set X of edges, the subgraph $\langle X \rangle$ induced by X has edge set X and consists of all vertices that are incident with at least one edge in X. This subgraph is called an **edge-induced subgraph** of G. Sometimes $G[S]$ and $G[X]$ are used for $\langle S \rangle$ and $\langle X \rangle$ respectively.
Any proper subgraph of a graph G can be obtained by removing vertices and edges from G. For an edge e of G, we write $G - e$ for the spanning subgraph of G whose edge set consists of all edges of G except e. More generally, if X is a set of edges of G, then $G - X$ is the spanning subgraph of G with $E(G - X) = E(G) - X$.

Let's start at some vertex u of a graph G. If we proceed from u to a neighbour of u and then to a neighbour of that vertex, and so on, until we finally come to a stop at a vertex, v, then we have just described a walk from u to v in G. More formally, a $u - v$ **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as $W : u = v_0, v_1, ..., v_k = v$, where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, ..., k - 1$. Each vertex v_i, $(0 \leq i \leq k)$ and each edge

$v_iv_{i+1}(0 \leq i \leq k-1)$ is said to lie on or belong to W. If $u = v$, then the walk W is **closed**; while if $u \neq v$, then W is **open**. As we move from one vertex of W to the next, we are actually encountering or traversing edges of G, possibly traversing some edges of G more than once. The number of edges encountered in a walk(including multiple occurrences of an edge) is called the **length** of the walk.



In the above graph $W : x, y, w, y, v, w$ is therefore a walk, indeed an $x - w$ walk of length 5. A walk of length 0 is a **trivial walk**. So $W : v$ is a trivial walk.
A $u - v$ **trail** in a graph G is a $u - v$ walk in a graph in which no edge is traversed more than once. $T : u, w, y, x, w, v$ is a $u - v$ trail in the graph G.
A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ **path**. $P : u, w, y, v$ is a $u - v$ path. If no vertex in a walk is repeated(thereby producing a path), then no edge is repeated either. Hence every path is a trail.

Theorem: If a graph G contains a $u - v$ walk of length l, then G contains a $u - v$ path of length at most l.

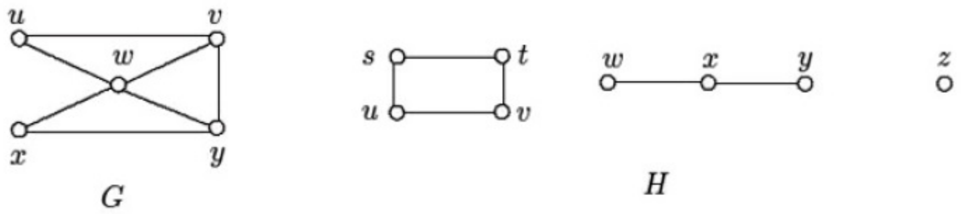
Definition. *A **circuit** in a graph G is a closed trail of length 3 or more. Hence a circuit begins and ends at the same vertex but repeats no edges. A circuit can be described by choosing any of its vertices as the beginning(and ending) vertex provided the vertices are listed in the same cyclic order. In a circuit, vertices can be repeated, in addition to the first and last.*
In the above graph: $C : y, w, u, v, w, x, y$ or $C : x, y, w, u, v, w, x$ or $C : w, x, y, w, u, v, w$ is a circuit.
*A circuit that repeats no vertex, except for the first and last, is a **cycle**. A **k-cycle** is a cycle of length k. A 3-cycle is also referred to as a **triangle**. A cycle of odd length is called an **odd cycle**; while, a cycle of even length is called an **even cycle**.*
In the above graph: $C' : x, y, v, w, x$ is a cycle, namely a 4-cycle.
If a vertex of a cycle is deleted, then a path is obtained. This is not true for circuits, however.

Let G be a graph of order n and size m, where $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = \{e_1, e_2, ..., e_m\}$.

If G contains a $u - v$ path, then u and v are said to be **connected** and u is connected to v(and v is connected to u). So, saying that u and v are connected only means that there is some $u - v$ path in G. A graph G is **connected** if every two vertices of G are connected, that is, if

G contains a $u-v$ path for every pair u, v of distinct vertices of G.
Note: A graph G is connected if and only if G contains a $u-v$ walk for every pair u, v of vertices of G. Since every vertex is connected to itself, the trivial graph is connected.

A graph G that is not connected is called **disconnected**. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G. A graph G is then connected if and only if it has exactly one component.



The above graph G is connected, where as the graph H is disconnected.
The graph H has three components, namely H_1, H_2 and H_3.

In general for subgraphs $G_1, G_2, ..., G_k, k \geq 2$, of a graph G, with mutually disjoint vertex sets, we write $G = G_1 \cup G_2 \cup ...G_k$ if every vertex and every edge of G belong to exactly one of these subgraphs. In particular, we write $G = G_1 \cup G_2 \cup ...G_k$ if $G_1, G_2, ..., G_k$ are components of G. Therefore, we can write $H = H_1 \cup H_2 \cup H_3$ for the graph in the above figure.

Theorem 7. *Let $n_1, n_2, \ldots, n_k \in \mathbb{N}$. Then*

$$\sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right).$$

Theorem 8. *A simple graph with n vertices and k components can have at most have $(n-k)(n-k+1)/2$ edges.*

Proof. Let X be a graph with k components. Let n_i be the number of vertices in the i^{th} component, where $1 \leq i \leq k$. Then, the number of edges in the graph is equal to sum of the edges in each of its components. Thus, X has maximum number of edges if each component is a complete graph. Hence, the maximum possible number of edges in the graph X is $\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$.

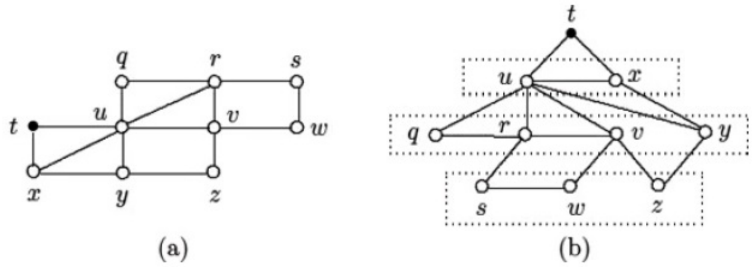
But from Theorem 7 we have

$$\frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \leq \frac{1}{2}\left[\left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right)\right] - \frac{1}{2}\sum_{i=1}^{k} n_i$$
$$= \frac{1}{2}(n-k)(n-k+1) \quad \square$$

$\square$

Distance in graphs:

Let G be a connected graph of order n and let u and v be two vertices of G . The **distance** between u and v is the smallest length of any $u-v$ path in G and is denoted by $d_G(u,v)$ or simply $d(u,v)$ if the graph G under consideration is clear. Hence if $d(u,v) = k$, then there exists a $u-v$ path $P : u = v_0, v_1, ..., v_k = v$ of length k in G, but no $u-v$ path of smaller length exists in G. A $u-v$ path of length $d(u,v)$ is called a $u-v$ **geodesic**. In fact, since the path P is a $u-v$ geodesic, not only is $d(u,v) = d(u,v_k) = k$ but $d(u,v_i) = i$ for every i with $0 \leq i \leq k$.

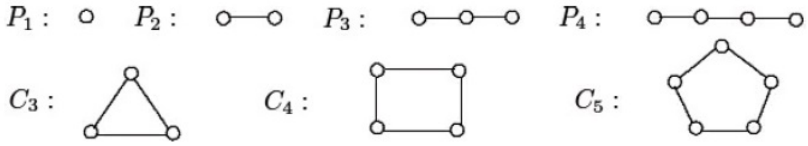


In the above figure graph (a) is redrawn as graph (b).

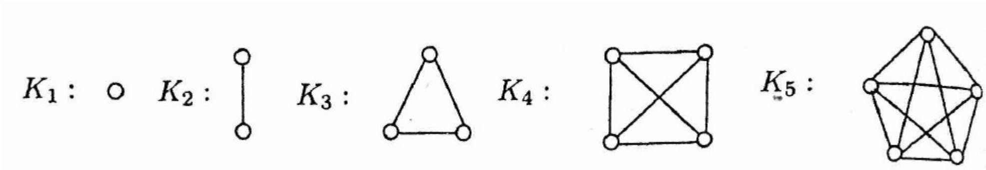
The greatest distance between any two vertices of a connected graph G are called the **diameter** of G and is denoted by $diam(G)$. The diameter of the above graph is 3. The path $P' : y, u, r, s$ is a $y-s$ geodesic whose length is $diam(H)$.
If G is a connected graph such that $d(u,v) = diam(G)$ and $w \neq u, v$, then no $u-w$ geodesic can contain v, for otherwise $d(u,w) > d(u,v) = diam(G)$, which is impossible.

Common Classes of Graphs

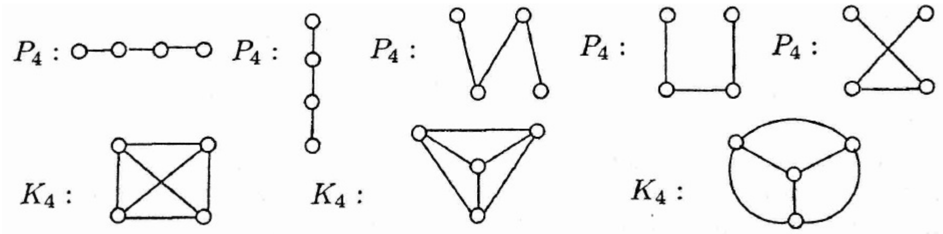
If a graph G of order n can be labeled(or relabeled) $v_1, v_2, ..., v_n$ so that its edges are $v_1v_2, v_2v_3, ..., v_{n-1}v_n$, the G is called a **path**; while if the vertices of a graph G of order $n \geq 3$ can be labeled(or relabeled) $v_1, v_2, ..., v_n$, so that its edges are $v_1v_2, v_2v_3, ..., v_{n-1}v_n$ and v_1v_n, then G is called a **cycle**. A graph that is a path of order n is denoted by P_n, while a graph that is a cycle of order $n \geq 3$ is denoted by C_n. Several paths and cycles are shown in the below figure.



A graph G is **complete** if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. Therefore, K_n has the maximum possible size for a graph with n vertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices in K_n is $\binom{n}{2}$ and so the size of K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.



The above graphs are complete graphs.



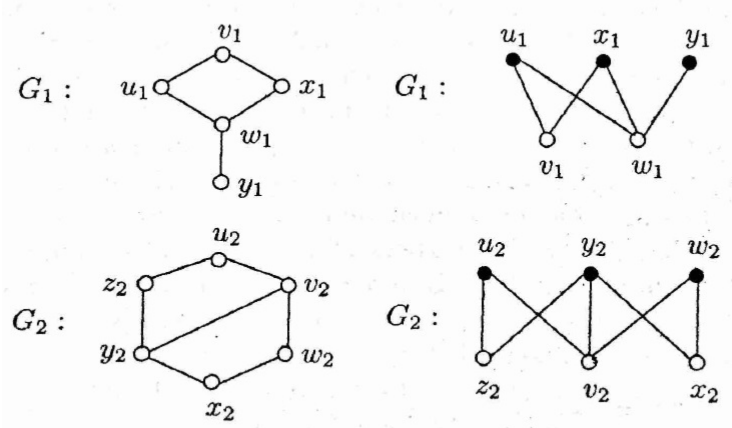
Given any graph it can be drawn in different ways.

The radius and diameter are easily computed for simple graphs:

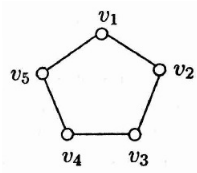
1. Complete graphs: $\text{diam}(K_n) = \text{rad}(K_n) = 1$ (for $n \geq 2$).
2. Path on n vertices: $\text{diam}(P_n) = n - 1$; $\text{rad}(P_n) = \lceil (n-1)/2 \rceil$.
3. Cycle on n vertices: $\text{diam}(C_n) = \text{rad}(C_n) = \lfloor n/2 \rfloor$.

Definition. *A graph G is a **bipartite graph** if $V(G)$ can be partioned into two subsets U and W, called **partite sets**, such that every edge of G joins a vertex of U and a vertex of W.*

The connected graphs G_1 and G_2 of the below figure are bipartite, as every edge of G_1 joins a vertex of $U_1 = \{u_1, x_1, y_1\}$and a vertex of $W_1 = \{v_1, w_1\}$, while every edge of G_2 joins a vertex of $U_2 = \{u_2, w_2, y_2\}$ and a vertex of $W_2 = \{v_2, x_2, z_2\}$. By letting $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, we see that every edge of $G = G_1 \cup G_2$ joins a vertex of U and a vertex of W. This illustrates the observation that a graph is bipartite if and only if each of its components is bipartite.



The below graph C_5 is not bipartite.



If For every $u \in U$ and every $v \in W$, the edge (u, v) is in E. then G is called **complete bipartite graph** and it is denoted by $K_{p,q}$, the above graph $G_2 = K_{3,3}$ is a complete bipartite graph.

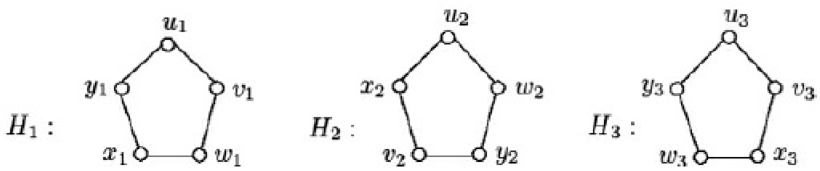
- $\text{diam}(K_{p,q}) = \text{rad}(K_{p,q}) = 2$ (if p or q is at least 2).
- Number of vertices in $K_{p,q}$ is $n = p + q$
- Number of edges in $K_{p,q}$ is $m = pq$

Theorem 9. *If a nontrivial, connected graph G is bipartite, then G contains no odd cycles.*

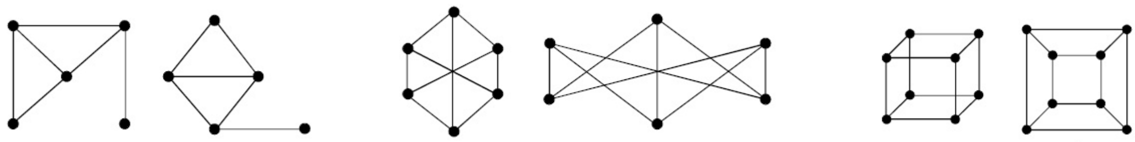
Proof. Suppose G is a nontrivial, connected, bipartite graph containing an odd cycle $v_1, v_2, \ldots,$ v_{2k+1}, v_1, for some integer k. Let V_o and V_e be the partite sets. Wlog, say $v_1 \in V_o$. Since v_1v_2 is an edge, $v_2 \in V_e$. Since v_2v_3 is an edge, $v_3 \in V_o$. Continuing this reasoning $2k$ times we conclude that $v_{2k+1} \in V_o$. Since $v_{2k+1}v_1$ is an edge, $v_1 \in V_e$. Now we have $v_1 \in V_o$ and $v_1 \in V_e$, a contradiction. □ □

2 Isomorphic Graphs

Two(labeled) graphs G and H are **isomorphic**(have the same structure) if there exists a one-to-one correspondence from $V(G)$ to $V(H)$ such that if $uv \in E(G)$ then, $f(u)f(v) \in E(H)$ and $uv \notin E(G)$ then, $f(u)f(v) \notin E(H)$. In this case, f is called an **isomorphism** from G to H. Thus, if G and H are isomorphic graphs, then we say that G is isomorphic to H and we write $G \cong H$. If G and H are unlabeled, then they are isomorphic if, under any labeling of their vertices, they are isomorphic as labeled graphs. If two graphs G and H are not isomorphic, then they are called **non-isomorphic** graphs and we write $G \ncong H$. The below graphs are isomorphic



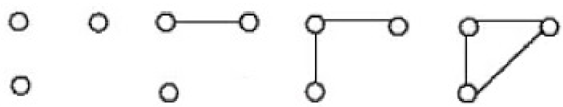
The necessary conditions for two graphs to be isomorphic are
1. Both must have the same number of vertices.
2. Both must have the same number of edges.
3. Both must have equal number of vertices with the same degree.
4. They must have the same degree sequence and same cycle vector $(c_1, c_2, ..., c_n)$, where c_i is the number of cycles of length i. Below are some pairs of non-isomorphic graphs.



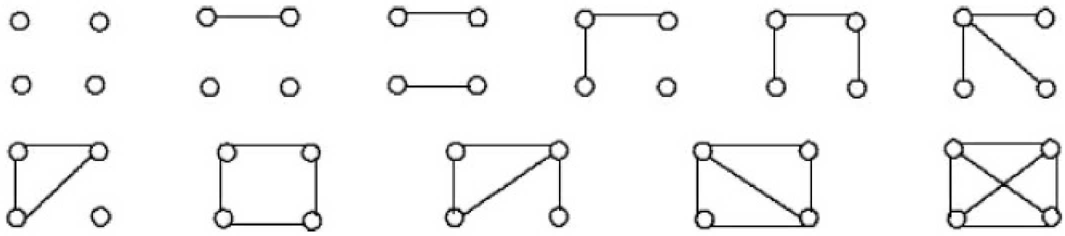
There is only one non-isomorphic graph of order 1,

two non-isomorphic graphs of order 2

and four non-isomorphic graphs of order 3.

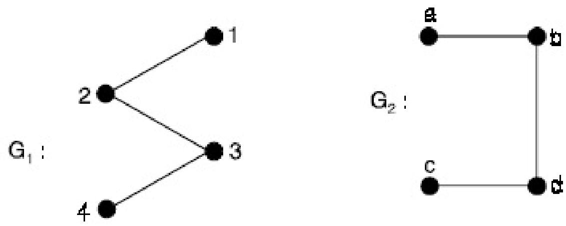


There are eleven non-isomorphic graphs of order 4 and these are shown in the below figure.

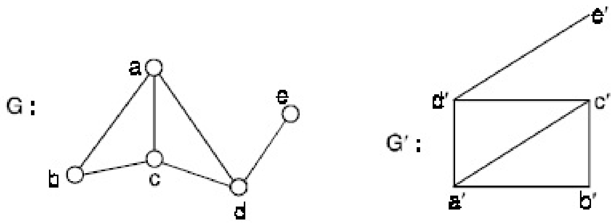


Problem:

Show that the below two graphs are isomorphic. (i)

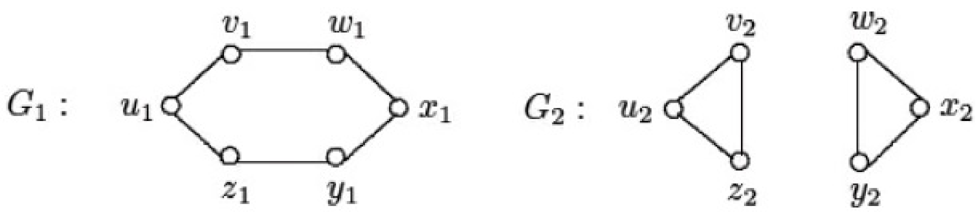


(ii)

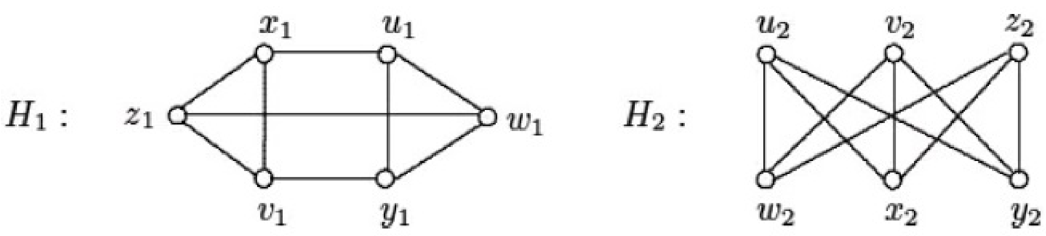


Problem:

Are the two graphs given below isomorphic? (i)

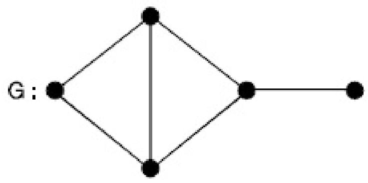


(ii)

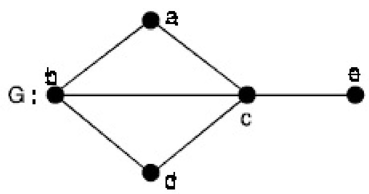


In the above figures G_1 and G_2 are non-isomorphic.
H_1 and H_2 are complements of G_1 and G_2 respectively.
H_1 and H_2 are non-isomorphic as well.
Note: Two graphs G and H are isomorphic if and only if their complements and are isomorphic.

Problem:
Construct three non-isomorphic spanning subgraphs of the graph G shown below.



Problem: Find all possible non-isomorphic induced subgraphs of the following graph G corresponding to the three element subsets of the vertex set of G.



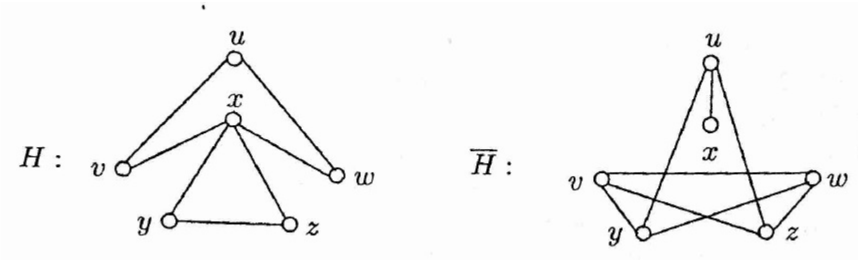
3 Operations on Graph

Definition. *The **complement** $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ such that for each pair u, v of vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.*

Observe that if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. The graph $\overline{K_n}$ then has n vertices and no edges; it is called the **empty graph** of order n. Therefore, empty graph have empty edge sets. In fact, if G is any graph of order

n, then $G-E(G)$ is the empty graph $\overline{K_n}$. By definition, no graph can have an empty vertex set.

A graph H and its complement are shown in the below figure.

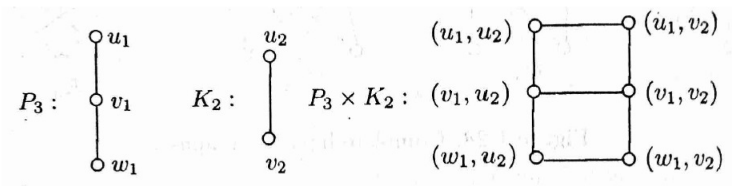


Definition. *If G is isomorphic to $\overline{G}$ then , G is called self complementary graph.*

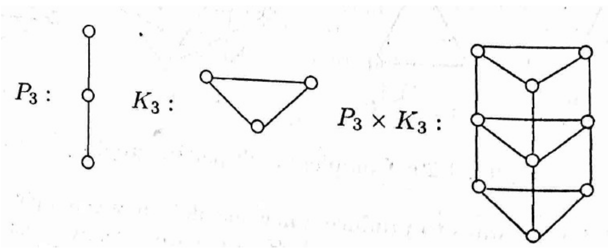
Example 10. *P_4 and C_5 are self complementary graphs.*

Definition. *For two(not necessarily vertex-disjoint) graphs G and H, the* ***Cartesian product*** *$G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$, that is , every vertex of $G \times H$ is an ordered pair (u, v), where $u \in V(G)$ and $v \in V(H)$.Two distinct vertices (u, v) and (x, y) are adjacent in $G \times H$ if either (i) $u = x$ and $vy \in E(H)$ or (ii) $v = y$ and $ux \in E(G)$.*

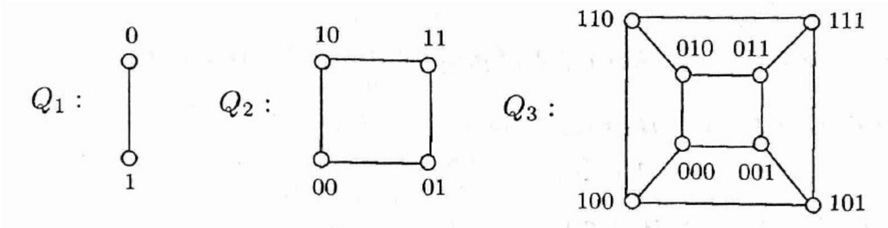
The below figure shows the Cartesian product of P_3 and K_2.



Note: The order in which the graphs G and H are written is structurally irrelevant, that is $G \times H$ and $H \times G$ are the same graph, that is, they are isomorphic graphs.
There is an informal way of drawing the graph $G \times H$(or $H \times G$) that doesn't require us to label the vertices. Replace each vertex x of G by a copy $H_x$of the graph H. Let u and v be two vertices of G. If u and v are adjacent in G, then we join corresponding vertices of H_u and H_v by an edge. If u and v are not adjacent in G, then we add no edges between H_u and H_v. This is illustrated in the below figure.



Notice that $K_2 \times K_2$ is the 4-cycle. The graph $C_4 \times K_2$ is often denoted by Q_3 and is called the **3-cube**. More generally, we define Q_1 to be K_2 and for $n \geq 2$, define Q_n to be $Q_{n-1} \times K_2$. The graphs Q_n are then called **n-cubes** or **hypercubes**. The n-cube can also be defined as that graph whose vertex set is the set of order n-tuples of 0s and 1s(commonly called the **n-bit strings**) and where two vertices are adjacent if their ordered n-tuples differ in exactly one position(coordinate). The n-cubes for $n = 1, 2, 3$ are shown in the below figure.



4 Matrix Representations of Graphs

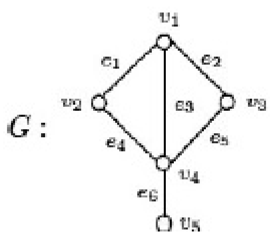
Definition. *The **adjacency matrix** of G is the $n \times n$ matrix $A = [a_{ij}]$, where*

$$a_{ij} = \begin{cases} 1 & \text{if} \quad v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Definition. *The **incidence matrix** of G is the $n \times m$ matrix $B = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} 1 & \text{if} \quad v_i \quad \text{is incident with} \quad e_j \\ 0 & \text{otherwise} \end{cases}$$

Example 11. *For the graph*

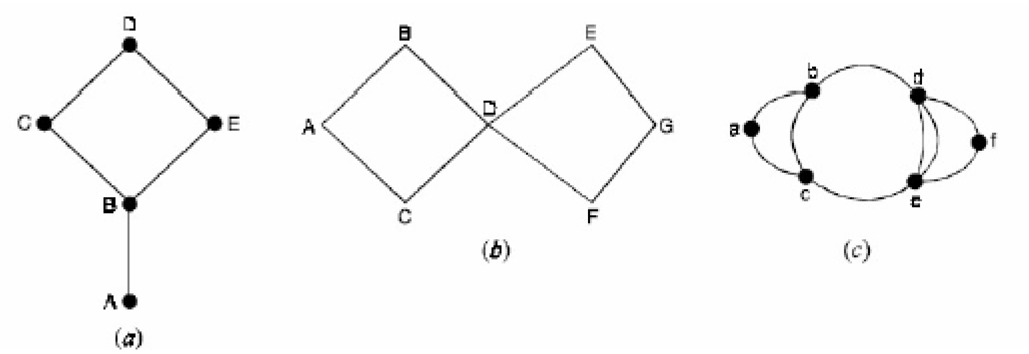


the adjacency matrix and the incidence matrix are given as below.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5 Eulerian and Hamiltonian Graphs

Definition. *A circuit C in a graph G is called an* ***Eulerian circuit*** *if C contains every edge of G. Since no edge is repeated in a circuit, every edge appears exactly once in an Eulerian circuit. A connected graph that contains an Eulerian circuit is called an* ***Eulerian graph****. In a connected graph G, an open trail that contains every edge of G as an* ***Eulerian trail****.*



(a) has an Euler trail but no Euler circuit

(b) has both Euler circuit and Euler trail

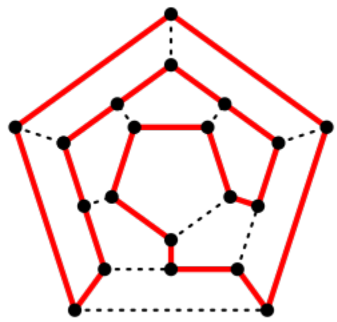
(c) has an Euler trail but no Euler circuit.

Theorem 12. *If the graph G Eulerian then, every vertex has even degree.*

Proof. If G is Eulerian then there is an Euler circuit, P, in G. Every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the one after it in the list. This circuit uses every edge exactly once. So every edge is accounted for and there are no repeats. Thus every degree must be even. □

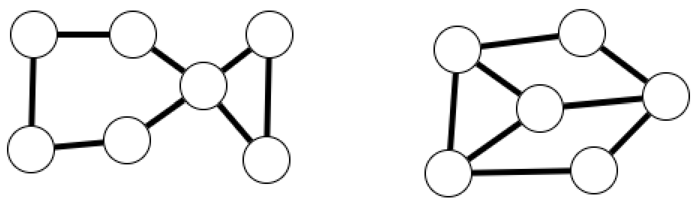
Definition. *A cycle in a graph G that contains every vertex of G is called a* ***Hamiltonian cycle*** *of G. Thus a Hamiltonian cycle of G is a spanning cycle of G. A* ***Hamiltonian graph*** *is a graph that contains a Hamiltonian cycle.*

The graph $C_n (n \geq 3)$ is Hamiltonian. Also, for $n \geq 3$, the complete graph K_n is a Hamiltonian graph. A path in a graph G that contains every vertex of G is called a **Hamiltonian path** in G. If a graph contains a Hamiltonian cycle, then it contains a Hamiltonian path. In fact, removing any edge from a Hamiltonian cycle produces a Hamiltonian path. If a graph contains a Hamiltonian path, however, it need not contain a Hamiltonian cycle.



Theorem 13. *Let G be a graph of order $n \geq 3$. If $\deg(u) + \deg(v) \geq n$ for each pair u, v of non-adjacent vertices of G, then G is Hamiltonian. (Converse need not be true)*

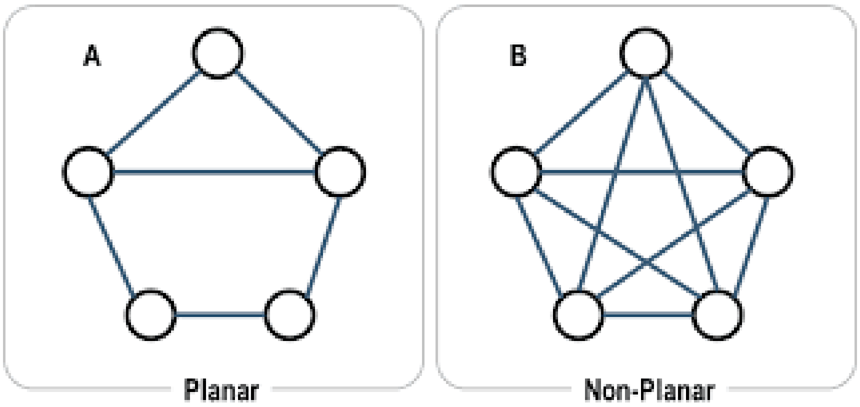
Theorem 14. *Let G be a graph of order $n \geq 3$. If $\deg(v) \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian. (Converse need not be true)*



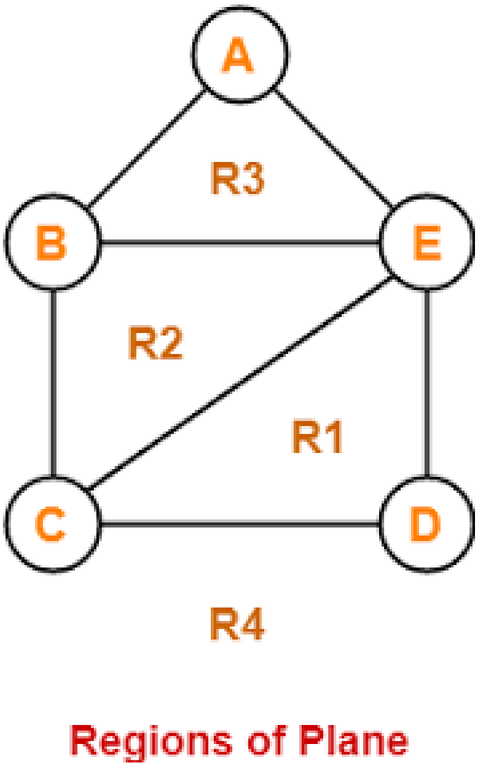
(i) Eulerian but not Hamiltonian graph , (ii) Hamiltonian but not Eulerian graph

6 Planar

Definition. *Planar Graph A Graph G is called **Planar graph** if G can be drawn in the plane so that no two of its edges cross each other, expect at an vertex. A graph that is not planer is called **Non-planar graph**.*



A planar graph divides the plane into connected pieces called **Regions**.



Theorem 15. *[The Euler Identity]*
If G is a connected planar graph of order n and size m and having r regions then $n - m + r = 2$.

Proof. [Induction; on r]

Step 1: If $r = 1$ then G cannot contain any cycle, since G is connected G is a tree by definition of tree $m = n - 1$

$$\begin{aligned} \therefore n - m + r &= 2 \\ n - (n-1) + 1 &= 2 \\ 2 &= 2 \end{aligned}$$

Step 2: Assume that the result holds for all the graph with $r \geq 2$.
Step 3: If $r \geq 2$, then G is not a tree and hence it has a cycle, let e be an edge on the cycle, e is on the boundary of two distinct regions S_1 and S_2, by removing the edge e the two regions S_1 and S_2 merge and form a new region S', Since $G - e$ now have $m' = m - 1$ edges and $r' = r - 1$ regions. Applying inductions hypothesis to $G' = G - e$ we get

$$\begin{aligned} 2 &= n - m' + r' \\ &= n - (m-1) + (r-1) \\ &= n - m + r \end{aligned}$$

Which completes the proof of the theorem. □

Theorem 16. *If G is a planar graph of order $n \geq 3$ and size m then*

1. $m \leq 3n - 6$
2. $m \leq 2n - 4$ *if G has no 3-cycles*

Remark:

1. If G is a graph with $n \geq 3$ and size $m > 3n - 6$ then G is a non planar graph graph with no triangles (cycle of length three).

2. If G is a graph with no triangles and $m > 2n - 4$ then G is a non planar graph.

Theorem 17. *Every planar graph contains a vertex of degree five or less.*

Proof. Suppose that G is a graph every vertex of which has degree or more, this implies that $n \geq 7$.

$$2m = \sum_{u \in V(G)}^{max} deg(u) \geq 6n$$

thus

$$m \geq 3n$$
$$\implies m > 3n - 6$$

Hence G is non planar graph.
therefore if G is a planar graph then contains a vertex of degree five or less □

Theorem 18. *The complete graph K_5 is non planar.*

Proof. In K_5, $n = 5$ and $m = 10$

$$m = 10 > 9 = 3n - 6$$
$$\implies m > 3n - 6$$

From Remark 2, K_5 is non planar. □

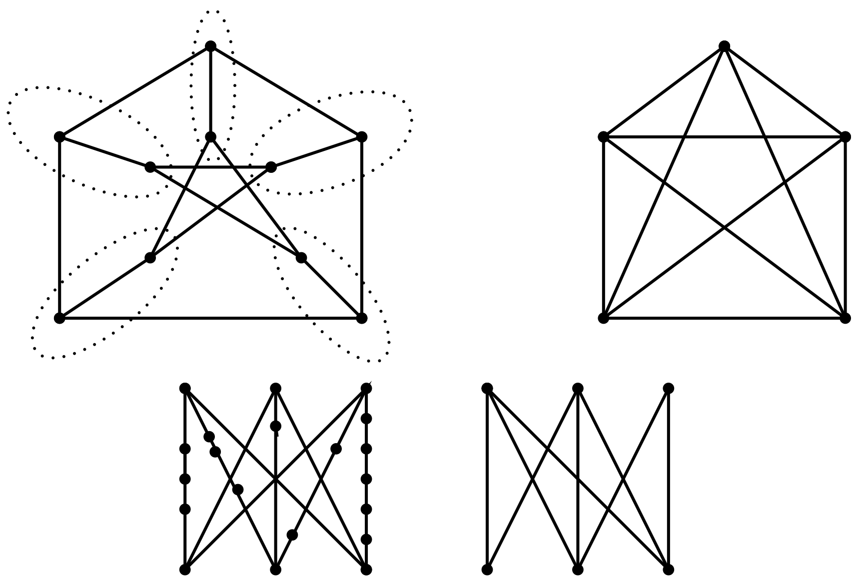
Theorem 19. *The complete bipartite graph $K_{3,3}$ is non planar.*

Proof. In $K_{3,3}$,m $n = 6$, and $m = 9$

$$m = 9 > 8 = 2n - 4$$
$$\implies m > 2n - 4$$

From Remark 2, $K_{3,3}$ is non planar. □

Theorem 20. *[Kuratowski]*
A graph G is planar if and only if it has no subgraphs homeomorphic to either K_5 or $K_{3,3}$.



Detection of Planarity Of A Graph: If a given graph G is planar or non planar is an important problem. We must have some simple and efficient criterion. We take the following simplifying steps:

Elementary Reduction:

Step-1: Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set.

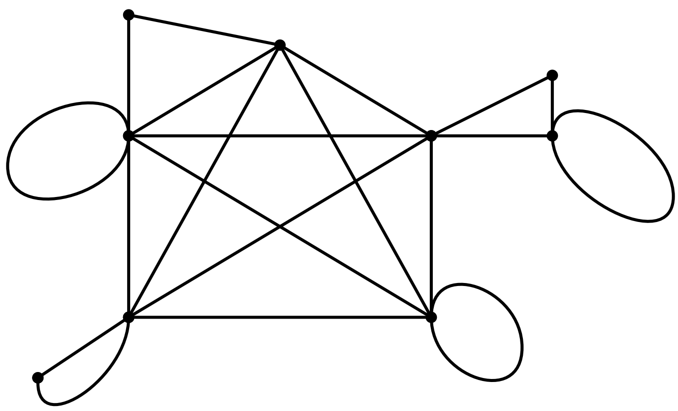
$$G = G1, G2, ...Gk$$

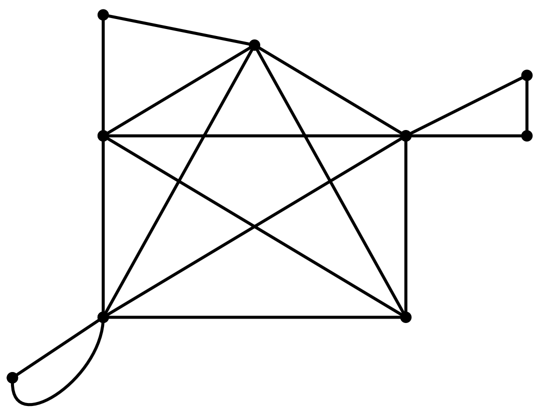
where each Gi is a non separable block of G. Then we have to test each Gi for planarity.

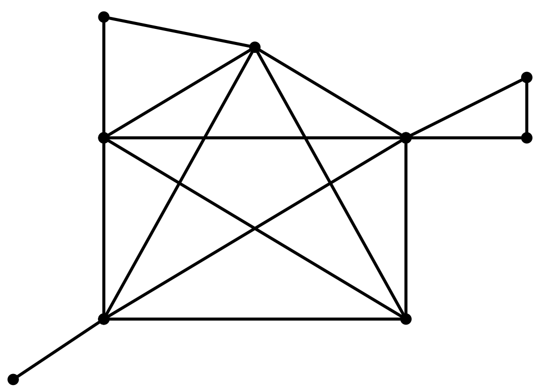
Step 2: Since addition or removal of self-loops does not affect planarity, remove all self-loops.

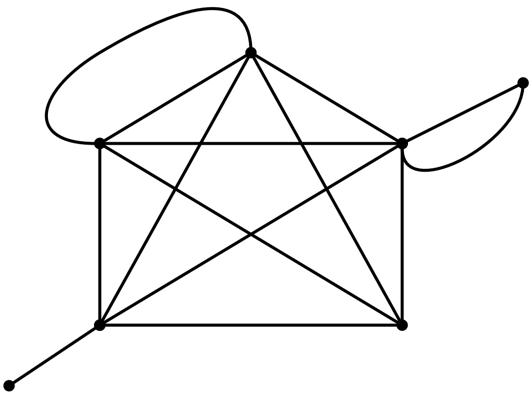
Step 3: Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.

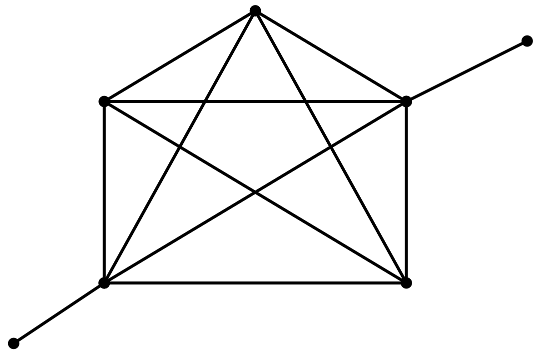
Step 4: Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series. Repeated application of step 3 and 4 will usually reduce a graph drastically.









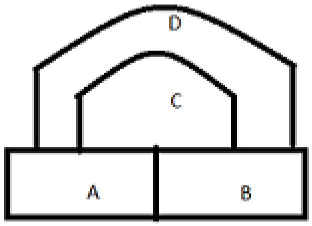


7 Vertex Coloring

Coloring is one of the important branches of graph theory and has attracted the attention of almost all graph theorists, mainly because of the four color theorem. In 1852 Farncis Guthrie (1831-1899), a recent graduate of University College London, observed that the counties of England could be colored with four colors so that neighboring counties were colored differently. Farncis found maps where three color weren't enough but he felt that four color were enough for all maps and he attempted to prove this. He showed his proof to his younger brother Frederick, who was taking class at the time from the well-known Augustus De Morgan.Francis was not completely happy with the proof he had given, however. With Francis's permission, Frederick showed what Francis had written to De Morgan on October 23, 1852. De Morgan was pleased with his and felt it was new. The very same day, De Morgan wrote the following letter to the celebrated mathematician William Rowan Hamilton:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact-an do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figurers with any portion of common

boundary line are differently coloured- four colours may be wanted but not more-the following is his case in which four are wanted.



Definition. *A $k-$Coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are* ***colors****; the vertices have of one color form a* ***color class****. A $k-coloring$ is a* ***proper*** *if adjacent vertices have different labels. A graph is* ***k-colorable*** *if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is $k-colorable$*

We observe that colouring any one of the components in a disconnected graph does not affect the colouring of its other components. Also, parallel edges can be replaced by single edges, since it does not affect the adjacencies of the vertices. Thus, for colouring considerations, we opt only for simple connected graphs.
The following observations are the immediate consequences of the definitions introduced above.

1. A graph is 1-chromatic if and only if it is totally disconnected.
2. A graph having at least one edge is at least 2-chromatic (bichromatic).
3. A graph G having n vertices has $\chi(G) \leq n$.
4. If H is subgraph of a graph G, then $\chi(H) \leq \chi(G)$.
5. $\chi(K_n) = n$ and $\chi(\overline{K_n}) = 1$.
6. $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$.
7. If $G_1, G_2, ..., G_r$ are the components of a disconnected graph G, then $\chi(G) = max\{\chi(G_1), \chi(G_2), ..., \chi$
8. $\chi(K_{m,n}) = 2$

A Characterization of bicolorable (2-colorable) graph was given by Köning

Theorem 21. *A graph is bicolorable if and only if it has no odd cycles(bipartite graph).*

Theorem 22. *For every graph G $\chi(G) \leq 1 + \Delta(G)$*

Theorem 23. *[Brooks Theorem]*
For a connected simple graph G, which is neither complete nor a cycle of odd length we have $\chi(G) \leq \Delta(G)$.

Theorem 24. *[Six Color Theorem]*
For every planar graph G, we have $\chi(G) \leq 6$

Theorem 25. *[Five Color Theorem]*
For every planar graph G, we have $\chi(G) \leq 5$

Theorem 26. *[Four Color Theorem]*
For every planar graph G, we have $\chi(g) \leq 4$.

8 Coloring Enumeration

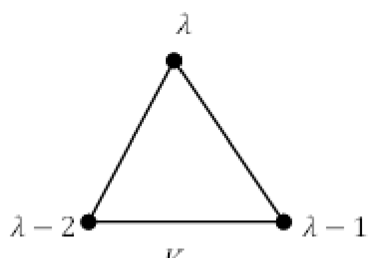
Definition. *Let G be a graph and $\lambda \in \mathbb{N}$. Define be the number of $P(G : \lambda)$ to be the number of proper λ-vertex colorings $c : V(G) \to \{1, 2, 3, ..., \lambda\}$. This property of a graph expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G.*

i.e Let G be a labeled graph. A coloring of G from λ colors is a coloring of G which uses λ or fewer colors. Two colorings of G from λ colors will be considered different is at least one of the labeled vertex is assigned different colors.

1. For each $\lambda < \chi(G)$ we have $P(G : \lambda) = 0$
2. For each $\lambda \geq \chi(G)$ we have $P(G : \lambda) > 0$
3. Indeed the smallest λ for which $P(G : \lambda) > 0$ is the chromatic number of G.

Example 1: There are λ ways of coloring any given vertex of K_3. For a second vertex, any of $\lambda - 1$ colors may be used, while there are $\lambda - 2$ ways of coloring the remaining vertex. Thus

$$P(K_3 : \lambda) = \lambda(\lambda - 1)(\lambda - 2)$$



K_3

This can be generalized to any complete graph

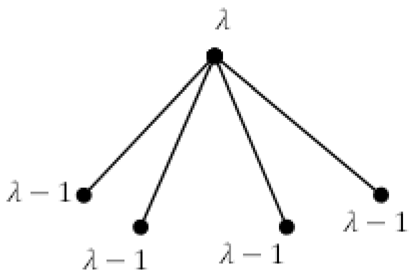
$$P(K_n : \lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

The corresponding polynomial of the totally disconnected graph (Null graph) $\overline{K_n}$ is particularly easy to find since each of its n vertices may be colored independently in any of λ ways

$$P(\overline{K_n} : \lambda) = \lambda^n$$

Example 2: The Central vertex v_0 of $K_{1,4}$ may be colored in any λ ways while each end vertex may be colored in any $\lambda - 1$ ways . Therefore

$$P(K_{1,4} : \lambda) = \lambda(\lambda-1)^4$$



$K_{1,4}$

Certainly, every two isomorphic graphs have the same chromatic polynomial. However, there are often several nonisomorphic graphs with the same chromatic polynomial; in fact, all trees with n vertices have equal chromatic polynomials

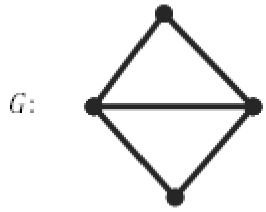
Theorem 27. *A graph G with n vertices is a tree if and only if*

$$P(G,\lambda) = \lambda(\lambda-1)^{n-1}$$

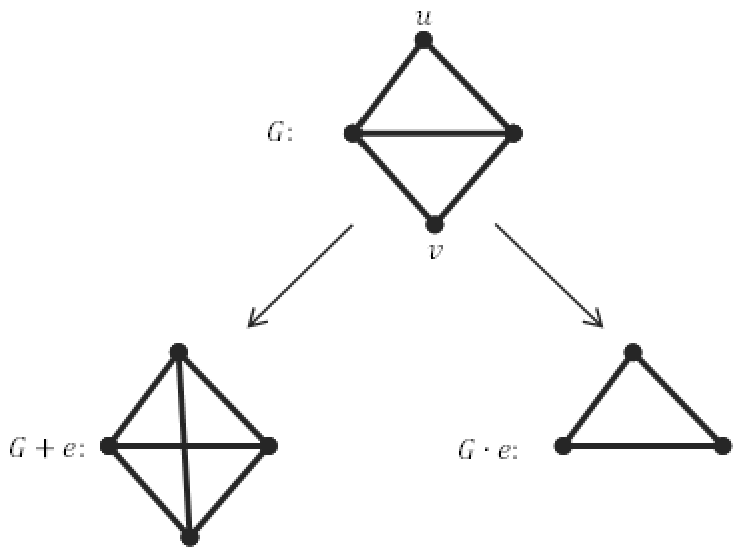
Theorem 28. *Let u and v be two non adjacent vertices in a graph G. Let $G+e$ be a graph obtained by adding an edge between u and v. Let $G \cdot e$ be a simple graph obtained from G by fusing the vertices u and v together and replacing sets of parallel edges with single edge. Then*

$$P(G,\lambda) = P(G+e,\lambda) + P(G \cdot e,\lambda)$$

Example 3: Find the chromatic polynomial of the following graph.



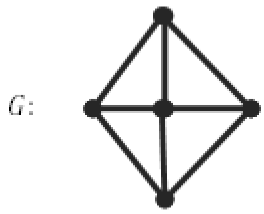
Solution: We obtain



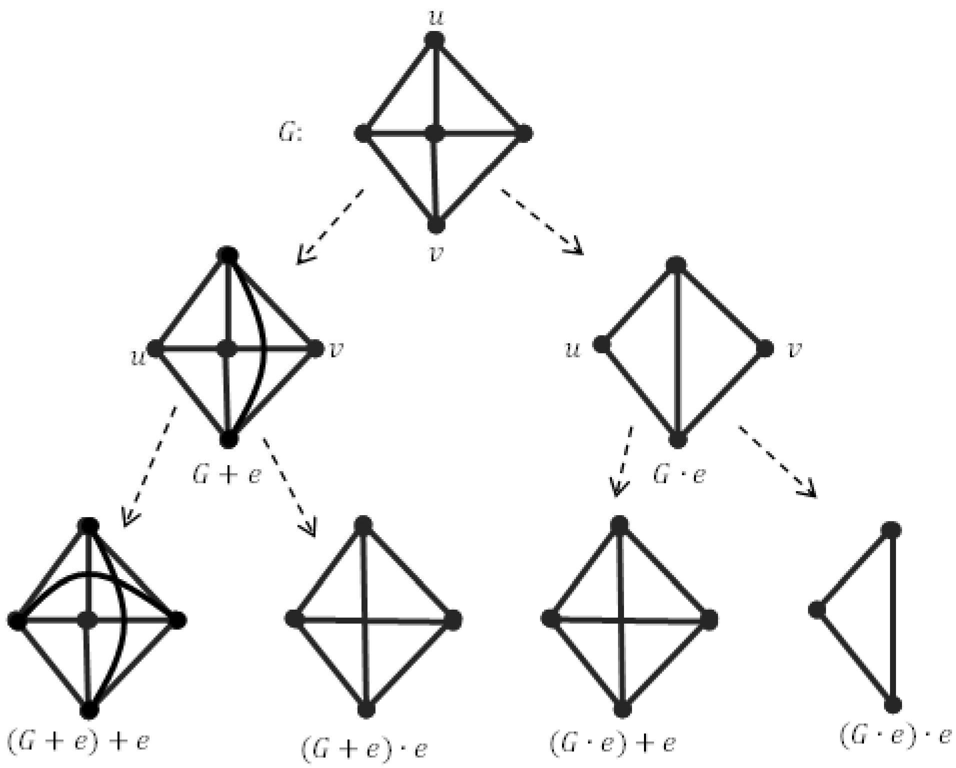
Hence

$$\begin{aligned}
p(G,\lambda) &= P(G+e,\lambda) + P(G\cdot e,\lambda)\\
&= P(K_4,\lambda) + P(K_3,\lambda)\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)\\
&= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)+1]\\
&= \lambda(\lambda-1)(\lambda-2)^2
\end{aligned}$$

Example 4: Find the chromatic polynomial of the following graph.



Solution: We obtain



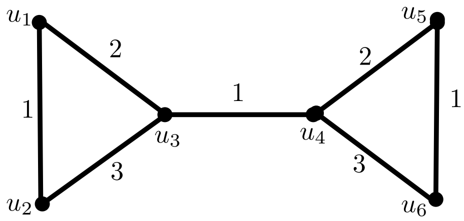
Hence

$$\begin{aligned}
p(G,\lambda) &= P(G+e,\lambda) + P(G\cdot e,\lambda)\\
&= P((G+e)+e,\lambda) + P((G+e)\cdot e,\lambda) + P((G\cdot e)+e,\lambda) + P((G\cdot e)\cdot e,\lambda)\\
&= P(K_5,\lambda) + P(K_4,\lambda) + P(K_4,\lambda) + P(K_3,\lambda)\\
&= P(K_5,\lambda) + 2P(K_4,\lambda) + P(K_3,\lambda)\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)\\
&= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1]\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)
\end{aligned}$$

9 Edge Coloring

Definition. *A $k-$ edge-coloring of G is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are colors; the edges of one color from a color class. A k-edge-coloring is proper if incident* *edges have different labels; that is, if each color class is a matching. A graph is k-edge-colorable if it has a proper k-edge-coloring. The edge-chromatic number (Chromatic index) $\chi^{'}(G)$ of a loopless graph G is the least k such that G is $k-$edge-colorable.*

Example 7: The edge chromatic number of the following graph is three.



For a graph G and any vertex $u \in V(G)$, all edges with u as an end vertex are adjacent and hence must receive different colors in a proper edge coloring of G. Hence, we note the obvious lower bound for the edge chromatic number of G

$$\chi^{'}(G) \geq \Delta(G),$$

the maximum degree in G.
Edge chromatic number of some basic graphs:

1. $$\chi^{'}(K_n) = \begin{cases} n & if\, n\, is\, odd, \\ n-1 & if\, n\, is\, even,. \end{cases}$$

2. $$\chi^{'}(C_n) = \begin{cases} 2 & if\, n\, is\, even, \\ 3 & if\, n\, is\, odd. \end{cases}$$

3. $$\chi^{'}(p_n) = 2$$

Theorem 29. *For a bipartite graph G, we have*

$$\chi^{'}(G) = \Delta(G).$$

From this theorem we obtain the next corollary.

Theorem 30. *For the complete bipartite graph $K_{m,n}$, we have*

$$\chi^{'}(K_{m,n}) = max(\{m, n\}).$$

The following Theorem gives us as tight bound for $\chi^{'}(G)$ as we can hope for when G is a simple graph. It was proved by Vizing in 1964 and independently by Gupta in 1966, although the latter proof never was published excepted as an abstract. It is usually referred to as Vizing's Theorem.

Theorem 31. *If G is a simple graph, then*

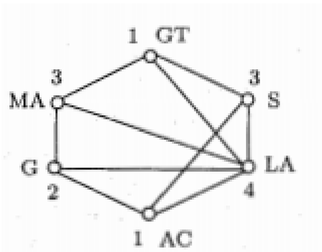
$$\Delta(G) \leq \chi^{'}(G) \leq \Delta(G) + 1$$

10 Scheduling problems

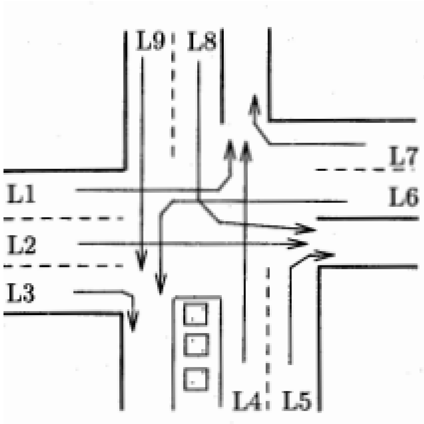
Example 5.1: The mathematics department of a certain college plans to schedule the classes Graph Theory (GT), Statistics (s), Liner Algebra (LA), Advanced Calculus (AC), Geometry (G), and Modern Algebra (MA) this summer. Ten students (see below) have indicated the courses they plan to take. With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period.

Anden: LA, S	Brynn: MA, LA, G
Chase: MA, G, LA	Denise: G, LA, AC
Everett: AC, LA, S	Francois: G, AC
Greg: GT, MA, LA	Harper: LA, GT, S
Irene: AC, S, LA	Jennie: GT, S

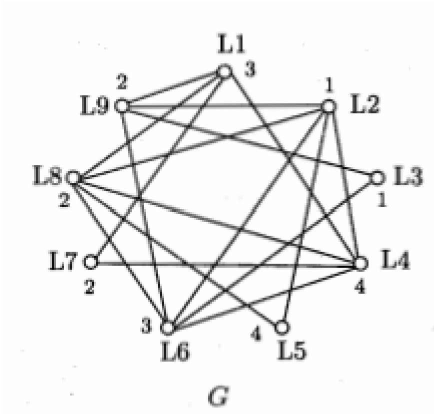
Solution: First, we construct a graph H whose vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking classes in these two subjects(see Figure). The minimum number of time periods is $\chi(H) = 4$.



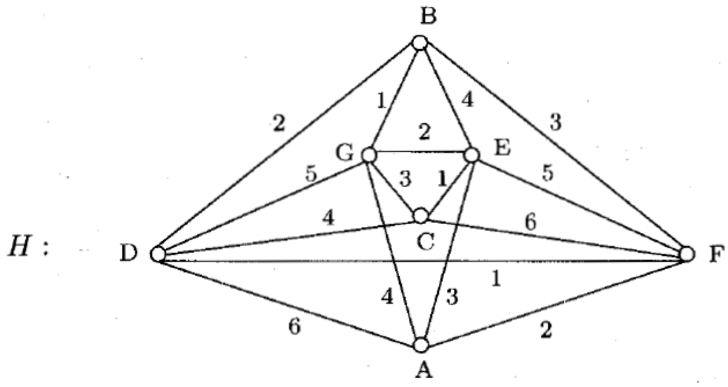
Example 5.2: The following figure shows the traffic lanes $L1, L2, ..., L9$, at the intersection of two busy streets. A traffic light is located at this intersection. During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection. What is the minimum number of phase needed for the traffic light so that all cars may proceed trough the intersection?



Solution: Construct a graph G to model this situation, where $V(G) = \{L1, L2, ..., L9\}$ and two vertices (lanes) are joined by a edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is a possibility of an accident. Answering this question require determining the chromatic number of the graph. First notice that $\langle\{L2, L4, L6, L8\}\rangle \cong K_4$. Since there exists a 4-coloring of G, as indicated in the graph, therefore $\chi(G) = 4$.



Example 5.3: Alvin (A) has invited three married couples to his summer house for a week: Bob(B) and Carrie (C) Hanson, David (D) and Edith (E) Irwin, and Frank (F) and Gena (G) Jackson. Since all six guest enjoy playing tennis match against every other guest except his/her spouse. In addition, Alvin is to play a match against each of David, Edith, Frank, and Gena. If no one is to play two match on the same day, what is a schedule of matches over the smallest number of days.
Solution: First, we construct a graph H whose vertices are the people at Alvin's summer house, so $V(G) = \{A, B, C, D, E, F, G\}$, and two vertices of H are adjacent if the two vertices (people) are to play a tennis match. (The graph H is shown in the below graph). To answer the question, we determine the edge chromatic number of H. The edge chromatic number of the graph H is $\chi^{'}(H) = 6$.

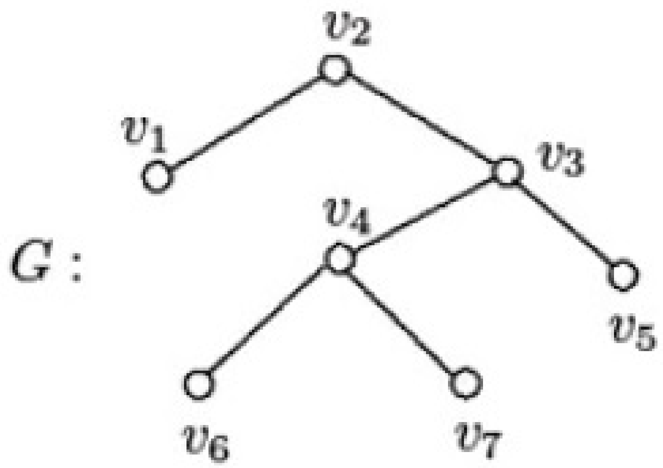


The above graph gives a 6-edge coloring of H, which provides a schedule of matches.

Day 1:	Bob-Gena, Carrie-Edith, David-Frank
Day 2:	Alvin-Frank, Bob-David, Edith-Gena
Day 3:	Alvin-Edith, Bob-Frank, Carrie-Gena
Day 4:	Alvin-Gena, Bob-Edith, Carrie-David
Day 5:	David-Gena, Edith-Frank
Day 6:	Alvin-David, Carrie-Frank

11 Tree

Definition. *A graph G is called* ***acyclic*** *if it has no cycles. A* ***tree*** *is an acyclic connected graph.*



Theorem 32. *Every tree of order n has size $n-1$.*

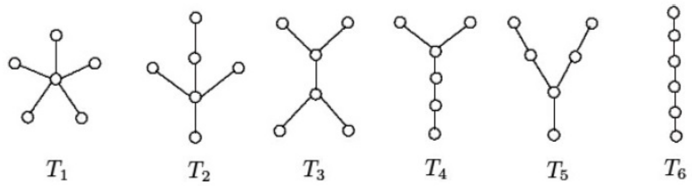
Proof. There is only one tree of order 1, namely K_1, which has size 0. Thus, the result is true for $n = 1$. Assume for a positive integer k that the size of every tree of order k is $k-1$. Let T be a tree of order $k+1$. Every non-trivial tree T contains at least two end vertices. Let v be one of them. Then $T' = T - v$ is a tree of order k. By the induction hypothesis, the size of T' is $m = k-1$. Since T has exactly one more edge than T', the size of T is $m+1 = (k-1)+1 = (k+1)-1$, as desired. □

Theorem 33. *A graph G is a tree if and only if every two vertices of G are connected by a unique path.*

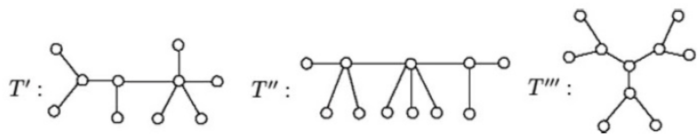
Proof. First, let G be a tree. Then G is connected by definition. Thus, every two vertices of G are connected by a path. Assume, to the contrary, that there are two vertices of G that are connected by two distinct paths. Then a cycle is produced from some or all of the edges of these two paths. This is a contradiction.
For the converse, suppose that every two distinct vertices of G are connected by a unique path. Certainly then, G is connected. Assume, to the contrary, that G has a cycle C. Let u and v be two distinct vertices of C. Then C determines two distinct $u-v$ paths, producing a contradiction. Thus G is acyclic and so G is a tree. □

Theorem 34. *Every edge in a tree is a bridge.*

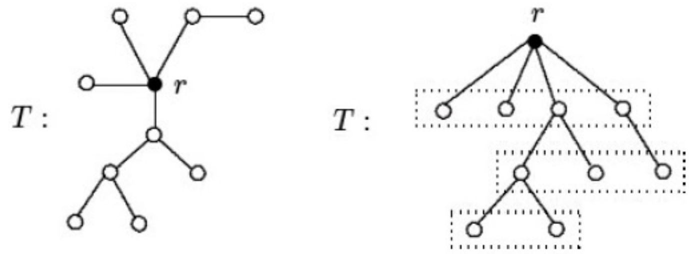
The below figure shows all six trees of order 6.



The tree $T_1 = K_{1,5}$ is a **star** and $T_6 = P_6$ is a **path**. The number of end-vertices in the trees of the above figure ranges from 2 to 5. A tree containing exactly two end vertices (which are necessarily adjacent) is called a **double star**. The trees T_2 and T_3 in the above figure are double stars.
A **caterpillar** is a tree of order 3 or more, the removal of whose end-vertices produces a path called the **spine** of the caterpillar. Thus every path and star (of order at least 3) and every double star is a caterpillar, as is every tree shown in the above figure. The trees T' and T'' of the below figure are also caterpillars but T''' is not.



Sometimes it is convenient to select a vertex of a tree T under discussion and designate this vertex as the **root** of T. The tree T then becomes a **rooted tree**. Often the rooted tree T is drawn with the root r at the top and the other vertices of T drawn below, in levels, according to their distances from r. An example is given in the below figure.



Acyclic graphs are also referred to as **forests**. Therefore, each component of a forest is a tree. One fact that distinguishes trees from forests is that a tree is required to be connected, while a forest is not required to be connected. Since a tree is connected, every two vertices in a tree are connected by a path.

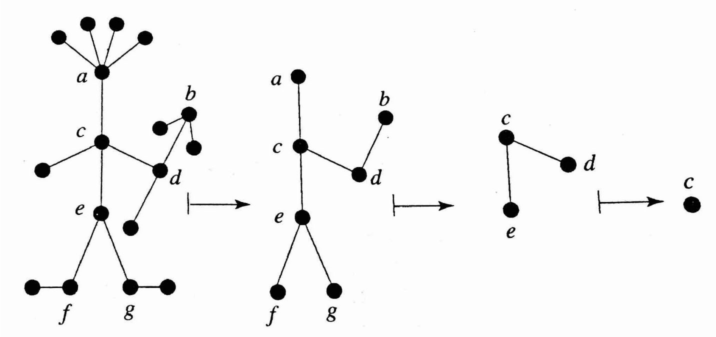
Problem: The degrees of the vertices of a certain tree T of order 13 are 1, 2, and 5. If T has exactly three vertices of degree 2, how many end-vertices does it have?

Ans: $x \times 1 + 3 \times 2 + (10 - x) \times 5 = 2 \times (13 - 1) \implies x = 8$ Hence the tree 8 end vertices.

Theorem 35. *Every forest of order n with k components has size $n - k$.*

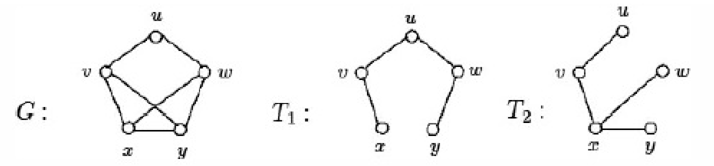
For a vertex v in a connected graph G, the **eccentricity** $e(v)$ of v is the distance between v and a vertex farthest from v in G. The **center** of G is a vertex having minimum eccentricity.

Considering the eccentricity of the vertices of a tree, it is evident that the vertices having the largest eccentricities are among the leaves, since each path of maximal length connects two leaves. In particular, a leaf is never a center of a tree on three or more vertices, and centers must therefore be among the internal vertices. In the below figure, we see that the center must be among the vertices labeled a, b, c, d, e, f or g. By removing all the leaves from the original tree, the eccentricity of each internal vertex of a tree will be reduced by one, and hence the center of the original tree is the vertex of the pruned tree with minimum eccentricity. Again in the below figure we see that the center of the original tree must therefore be among the non leaves of the pruned tree, namely among c, d or e. This process can be continued until we are left with either a single vertex or two adjacent vertices. In the below figure we are indeed left with a single vertex c, which is therefore the unique center of this tree.



Theorem 36. *There are one or more centers in every tree; in the later case they are adjacent.*

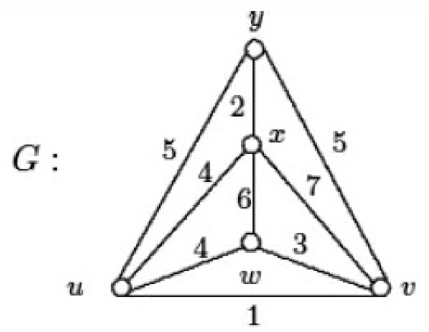
We have just described two ways of producing trees T that are subgraphs of a given connected graph G such that $V(T) = V(G)$. Also it is a spanning subgraph of G.
In the below figure T_1 and T_2 represent spanning trees of the graph G.

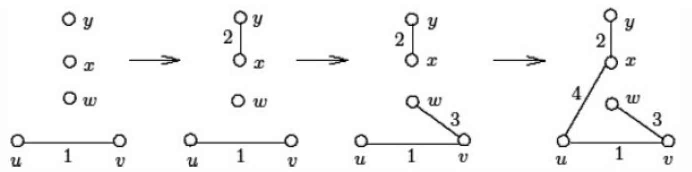


Let G be a connected graph each of whose edges is assigned a number (called the cost or weight of the edge). We denote the weight of an edge e of G by $w(e)$. Recall that such a graph is called a weighted graph. For each subgraph H of G, the weight $w(H)$ of H is defined as the sum of the weights of its edges, that is,

$$w(H) = \sum_{e \in E(H)} w(e).$$

We seek a spanning tree of G whose weight is minimum among all spanning trees of G. Such a spanning tree is called a **minimum spanning tree**. The problem of finding a minimum spanning tree in a connected weighted graph is called the **Minimum Spanning Tree Problem**.
Kruskal's Algorithm: For a connected weighted graph G, a spanning tree T of G is constructed as follows: For the first edge e_1 of T, we select any edge of G of minimum weight and for the second edge e_2 of T, we select any remaining edge of G of minimum weight. For the third edge e_3 of T, we choose any remaining edge of G of minimum weight that does not produce a cycle with the previously selected edges. We continue in this manner until a spanning tree is produced.
The following figure shows how a spanning tree of a connected weighted graph is constructed using Kruskal's Algorithm.



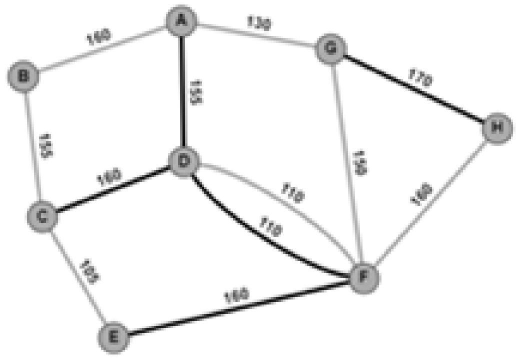


Problem 37. *Eight cities A, B, C, D, E, F, G, and H are required to be connected by a new railway network. The possible tracks and the cost involved to lay them (in crores of rupees) are summarized in the following table:*

Between	*Cost*	*Between*	*Cost*
A and B	*160*	*D and F*	*110*
A and D	*155*	*E and F*	*160*
A and G	*130*	*F and G*	*150*
B and C	*155*	*F and H*	*160*
C and D	*160*	*G and H*	*170*
C and E	*105*	*D and F*	*110*

(i) Draw the weighted graph which represents the new railway network.
(ii) Further determine a railway network of minimal cost that connects all these cities using Kruskal's algorithm. Also mention the minimum cost.

Solution:



AB, AG, BC, CE, GF, HF, FD
$160 + 130 + 155 + 105 + 150 + 160 + 110 = 970$