DM - is a study of discrete structures which are abstract mathematical models dealing with discrete objects & their reln b/n them. → sets, rel, graph, FSA. etc.

UNIT - I

Fundamental Principles of Counting.

Rule of Sum:- If a first task can be performed in m ways, while a second task can be performed in n ways & the two tasks cannot be performed simultaneously, then performing either task can be accomplished in m+n ways.

Ex:- A college has 40 books on Sociology & 50 books on anthropology. A student can select among 40+50 = 90 books.

Rule of Product / Principle of Choice:- If a procedure can be broken down into first and second stages, and if there are m possible outcomes for first stage & if, for each of these outcomes, there are n possible outcomes for second stage, then the total procedure can be carried out, in designated order, in mn ways.

Ex: 1) There are 6 men and 8 women for auditioning in a drama club. In how many ways couples can be selected?

Ans: 6 × 8 = 48 ways. (6 way to select a man, 8 ways to sel a woman)

2) There are 2 letters followed by 4 digits in name plates. In how many ways nameplates possible

26 → letters
0-9 → dig

name plates. In how many ways nameplates possible if no letter or digit is repeated?

a) if no

LL DDDD

Ans:- $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3276000$ ways

(b) With repetition of letters & digits.

Ans:- $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$ ways

(c) with repetitions, how many plates have only Vowels & even digits.

Ans:- $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$ ways

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Ex:- Sum & Product

1) There are 6 kinds of muffins, 8 kinds of sandwiches & 5 beverages (hot coffee, hot tea, iced tea, cola & orange juice). In how many ways either a muffin $\times$ & a hot beverage + or a sandwich & $\times$ a cold beverage be selected?

Ans:- $6 \times 2 + 8 \times 3 = 12 + 24 = 36$ ways.

Permutations: are counting linear arrangement of distinct objects. $\Rightarrow {}^nP_r = \frac{n!}{(n-r)!}$

Ex:- i) There are 10 students. 5 are to be chosen & seated in a row for a picture. How many such linear arrangements are possible?

Tot = 10
req = 5

Ans: $10 \times 9 \times 8 \times 7 \times 6 = \frac{10!}{(10-5)!} = \frac{n!}{(n-r)!}$

$= 30240$.

${}^{10}C_5 \quad 5!$

5 people out of 10 — arrangement

<u>Defn</u>:- Given a collection of n distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.

If there are n distinct objects, $a_1, a_2 \ldots a_n$ & r is an integer, $1 \le r \le n$, then no of permutations of size r for n objects is

$$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}$$

<u>Ex</u>:-

1. The no of permutations of the letters in the word COMPUTER is $8!$ (distinct obj)
How many 5 lettered words are possible? $n=8$, $r=5$
<u>Ans</u>:- ${}^8P_5 = \frac{8!}{3!} = 6720.$ $\qquad \frac{8!}{(8-5)!}$
If repetition of letters are allowed, no of 12 letter (not distinct) sequence is $8^{12} = 6.872 \times 10^{10}.$ $\qquad 8 \times 8 \times \ldots 8^{12}$

2. No of arrangement of four letters in BALL $= \frac{4!}{2!} = 12.$ $\qquad \frac{4!}{2!}$

3. " " " all 6 letters in PEPPER $= \frac{6!}{3! \cdot 2!} = 60.$

In general, if there n objects with n_1 of first type, n_2 of second type, ... n_r of r^{th} type, where $n_1 + n_2 + \ldots n_r = n$ then there are $\frac{n!}{n_1! \cdot n_2! \ldots n_r!}$ arrangements of given n objects.

$n_1 \to 4 \quad n_2 \to 3$

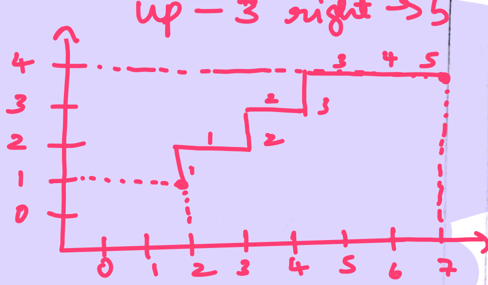
1 a) Arrangement of letters in MASSASAUGA

$$= \frac{10!}{4! \cdot 3!} = 25,200.$$

b) If all A's together $n_2 \to 3$

$$\underbrace{(AAAA)}_{1}\underbrace{MSSSUG}_{6} = 7 \text{ letters}.$$

$$= \frac{7!}{3!} = 840.$$



arrangement

2) No of paths in xy plane from (2,1) to (7,4) where step to be either one unit to right or up. $\frac{8!}{5!\,3!}$

Soln: Overall trip from (2,1) to (7,4) involves $(7-2)=5$ right & $(4-1)=3$ up moves.
Each path involves 5 R & 3 U. ∴ Ans is $\frac{8!}{5! \cdot 3!} = 56$.

(all men & women are not alike so we cannot do MMWWW)

3) No of ways of arranging 3 men & 3 women such that no two men or women seated together

$$= (3 \times 2 \times 1) \times (3 \times 2 \times 1) = 36 \text{ ways}.$$

$$= (3 \times 2 \times 1)^2 = 6^2 = 36.$$

Combinations: The Binomial Theorem

With n distinct objects, each selection or combination of r of these objects, with no reference to order, is r! permutations of size r from n objects. The no of combinations of size r from a collection of n is $C(n,r) = {}^nC_r = \binom{n}{r}$,

$$0 \le r \le n \quad = \frac{n!}{(n-r)!\cdot r!} = \frac{{}^nP_r}{r!} = \frac{P(n,r)}{r!}$$

Ex: (1) There are 20 people. How many 11 members group can be made? ${}^{20}C_{11}$

$n = 20$
$r = 11$

Soln:- $20C_{11} = \frac{20!}{9!\cdot 11!} = 167960.$

$\frac{n!}{(20-11)!\, r!}$

(2) (a) There are 10 questions. No of ways of answering any 7 questions $= \binom{10}{7} = 120$.

n $\quad r \quad nC_r$

${}^5C_3 \times {}^5C_4$

(b) 3 from first 5 questions & 4 from last 5

$= 5C_3 * 5C_4 = 10 * 5 = 50$ ways.

${}^5C_4 \times {}^5C_3$

(c) 4 from first 5 & 3 from last 5 $= 10 \times 5 = 50$.

${}^5C_4 \times {}^5C_3$

(d) 5 from 5 & 2 from last 5 $= \binom{5}{5}\cdot\binom{5}{2} = 1 \times 10 = 10$.

${}^5C_5 \times {}^5C_2$

(e) Student can answer 7 questions where each selection includes at least 3 of first 5 questions

≥ 3

$= (b) + (c) + (d) = 50 + 50 + 10 = 110$

${}^5C_3\, {}^5C_4 + {}^5C_4\, {}^5C_3 + {}^5C_5\, {}^5C_2$

(3) (a) 28 Juniors, 25 seniors.

Selecting 9 from juniors & Seniors = $\binom{53}{9}$ $\qquad 28+25=53$ $\quad {}^{53}C_9$

(b) 2 Junior & 1 best seniors are best, $\quad 53-3=50$

rest can be chosen in $\binom{50}{6}$. $\quad 9-3=6$ $\quad {}^{50}C_6$

(c) 4 juniors & 5 seniors. $\Rightarrow$ ${}^{28}C_4 \times {}^{25}C_5$

$\binom{28}{4}\binom{25}{5}$.

(4) There are 36 girls among which 4 teams of 9 girls to be formed. In how many ways it could be done?

Ans:- $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!\cancel{27!}} \cdot \frac{\cancel{27!}}{9!\cdot\cancel{18!}} \cdot \frac{\cancel{18!}}{9!\cancel{9!}} \cdot \frac{\cancel{9!}}{9!}$

${}^{36}C_9 \times {}^{27}C_9 \times {}^{18}C_9 \times {}^{9}C_9$ $\quad = \frac{36!}{(9!)(9!)(9!)(9!)} = 2.145\times10^{19}$ ways.

It is like S_1, S_2 - - - . S_{36} to be allotted to one of 4 teams. So it is like no of arrangements of 36 letters comprising 9 each of A, B, C, D teams.

(5) (a) No of arrangements of letters in TALLAHASSEE

$= \frac{11!}{3!\,2!\,2!\,2!} = 831,600$ ways

(b) No of arrangements with no adjacent A's

$= \frac{8!}{2!\,2!\,2!}$

Without A's, there are 8 letters which can be arranged in $\frac{8!}{2!\cdot 2!\, 2!} = 5040$.

$\uparrow$ E E $\uparrow$ S $\uparrow$ T $\uparrow$ L $\uparrow$ L $\uparrow$ S $\uparrow$ H $\uparrow$ — 9 locations

Possible locations for A's.

Three of these 9 locations can be selected in $\binom{9}{3} = 84$ ways. ${}^{9}C_3$

By product rule Ans is $5040 \times 84 = 423{,}360$.

A A A A A A

${}^{9}C_3 \times 5040$

6. $\Sigma = \{0,1,2\}$.

No. of strings of length $2 = |\Sigma^2| = 3 \times 3 = 3^2 = 9$.

$|\Sigma^3| = 3^3 = 27$.

$|\Sigma^n| = 3^n$

If $x = x_1 x_2 x_3 \ldots x_n$, $wt(x) = x_1 + x_2 + \ldots + x_n$.

Ex:- $wt(12) = 1+2 = 3$, $wt(22) = 2+2 = 4$.

$wt(101) = 1+0+1 = 2$. $wt(210) = 3$

$|\Sigma^{10}| = 3^{10} \rightarrow$ how many have even weight?

Ans:- When no of 1's in string is even. There are 6 cases. 1) 0's & 1's = 2^{10}

2) 2 1's = $\binom{10}{2}$ ways $\cdot\, 2^8$ ways

1's — 0's & 2's.

3) 4 1's = $\binom{10}{4} \cdot 2^6$

4) 6 1's $\binom{10}{6} 2^4$

5) 8 1's $\binom{10}{8} 2^2$

6) 10 1's $\binom{10}{10}$

$\therefore$ No of strings of length 10 that have even weight

$$= 2^{10} + \binom{10}{2} 2^8 + \binom{10}{4} 2^6 + \binom{10}{6} 2^4 + \binom{10}{8} 2^2 + \binom{10}{10}.$$

$$= \sum_{n=0}^{5} \binom{10}{2n} 2^{10-2n}$$

Thm1:- Binomial Thm:

If x and y are variables and n is a +ve integer, then

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \dots + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n} x^n y^0 = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Ex:- $n=4$. Coefficient of x^2y^2 in expansion of product $(x+y)(x+y)(x+y)(x+y)$ is the no of ways in which we can select two x's from four x's.

factor1 2 3 4

We select x from factor 1 & 2, y from factor 3 & 4.

Factors selected for x		selected for y	
1)	1, 2	1)	3, 4
2)	1, 3	2)	2, 4
3)	1, 4	3)	2, 3
4)	2, 3	4)	1, 4
5)	2, 4	5)	1, 3
6)	3, 4	6)	1, 2

Consequently, coefficient of x^2y^2 in expansion of $(x+y)^4$ is $\binom{4}{2}=6$, the no of ways to select two distinct objects from a collection of 4 distinct objects.

Proof:- In expansion of

$$\underbrace{(x+y)(x+y)}_{\text{factor } 2} \cdots \underbrace{(x+y)}_{n}$$

coefficient of $x^k y^{n-k}$, $0 \le k \le n$, is no of different ways in which we can select k x's from n variable factors. Total no of such selections of size k from collection of size n is $C(n,k)=\binom{n}{k}$.

$\binom{n}{k}$ is called as a binomial coefficient.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}$$

$$^7C_5 \times x^5 \times y^{7-5} \Rightarrow {}^7C_5 \times x^5 \times y^2$$
$$\Rightarrow {}^7C_5 \times 1 \times 1\, x^5 y^2$$

Ex:-

1) Coefficient of x^5y^2 in expansion of $(x+y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21\, x^5y^2$

$k = 5$
$n = 7$

2) Coefficient of a^5b^2 in expansion of $(2a-3b)^7$ $k=5, n=7$

$^7C_5\, (2a)^5 \times (-3b)^{7-5} \Rightarrow {}^7C_5 (2a)^5 (-3b)^2$

Put $x = 2a \quad y = -3b$

Coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$.

and $\binom{7}{5} \cdot x^5 y^2 = \binom{7}{5} (2a)^5 \cdot (-3b)^2$

$= \binom{7}{5} \cdot 2^5 \cdot (-3)^2\, a^5 b^2$

$= \binom{7}{5} \cdot 32 \cdot 9\, a^5b^2 = 6048\, a^5b^2$

$21 \times 2^5 \times -3^2$
$\Rightarrow 6048$

Corollary:

For integer $n>0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots (-1)^n \binom{n}{n} = 0$.

Proof: a) Put $x=1$ $y=1$ in binomial thm.

b) Put $x=1$, $y=-1$ in " " .

Thm 2:- Multinomial Thm:

For +ve integers n, t, Coefficient of $x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$ in expansion of $(x_1 + x_2 + \dots x_t)^n$ is

$$\frac{n!}{n_1!\, n_2! \dots n_t!}$$

where each n_i is an integer $0 \leq n_i \leq n$, $1 \leq i \leq t$ & $n_1 + n_2 + \dots + n_t = n$.

$$\binom{n}{n_1, n_2 \dots n_t}$$ is called multinomial coefficient.

Ex:- 1) In expansion of $(x+y+z)^7$ a) coefficient of $x^2y^2z^3$ is

$$\binom{7}{2,2,3} = \frac{7!}{2!.2!\,3!} = 210.$$

b) Coefficient of xyz^5 is $\binom{7}{1,1,5} = \frac{7!}{5!} = 42$.

c) Coefficient of x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!\cdot 4!} = 35$

2) In expansion of $(a+2b-3c+2d+5)^{16}$, coefficient of $a^2b^3c^2d^5$.

Let $v = a$, $w = 2b$, $x = -3c$, $y = 2d$, $z = 5$.

$(v+w+x+y+z)^{16}$, coefficient of $v^2w^3x^2y^5z^4$ is

$$\binom{16}{2,3,2,5,4} = 302{,}702{,}400.$$

~~But $\binom{16}{2,3,2,5,4} a^2(2b)^3(-3c)^2(2d)^5\, 5^4$~~

~~$= \binom{16}{2,3,2,5,4} \cdot (2^3 (-3)^2\, 2^5\, 5^4)(a^2 b^3 c^2 d^5)$~~

~~$= \binom{16}{2,3,2,5,4} (2^8 \cdot 3^2 \cdot 5^4)\ (a^2 b^3 c^2 d^5)$.~~

Ex: 1.3

(2) Selecting 5 magazines from 12 $= \binom{12}{5} = \frac{12!}{5!\,7!}$

$$= \frac{12 \times 11 \times \cancel{10} \times \cancel{9} \times \cancel{8}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} = \underline{\underline{792}}.$$

(5) (a) No of permutations of size 3 from m, r, a, f, t.

$= P(5,3) = \frac{5!}{2!} = 5 \times 4 \times 3 = \underline{\underline{60}}$.

$n=5$, $r=3$ $\frac{5!}{2!} = \underline{\underline{60}}$

(b) No. of combinations of size 3 from m, r, a, f & t

$= \binom{5}{3} = \frac{5!}{2! \cdot 3!} = \frac{5 \times \cancel{4}}{\cancel{2}} = 10$.

$n=6$, $r=3$ $\frac{6!}{2!\,3!} = \underline{\underline{10}}$

$\{m,r,a\}, \{r,a,f\}, \{a,f,t\}\ \{f,t,m\}, \{m,a,f\}, \{m,a,t\}$

$\{r,f,t\}, \{r,t,m\}, \{r,a,t\}, \{f,m,r\}$

⑦ In A committee of 12 is to be selected from 10 men & 10 women. In how many ways selection can be carried out if

$n = 12$

$r_1 = 10$ men

$r_2 = 10$ women

(a) no restriction $\quad {}^{20}C_{12}$

Ans:- $\binom{20}{12}$

${}^{10}C_6 \times {}^{10}C_6$

(b) There must be 6 men & 6 women.

Ans: $\binom{10}{6}\binom{10}{6}$

${}^{10}C_2 \times {}^{10}C_{10} + {}^{10}C_4 \times {}^{10}C_8 + {}^{10}C_6 \times {}^{10}C_6 + {}^{10}C_8 \times {}^{10}C_4 + {}^{10}C_{10} \times {}^{10}C_2$

2, 4, 6, 8, 10

(c) There must be even no of women.

women, men

Ans:- $\binom{10}{2}\cdot\binom{10}{10} + \binom{10}{4}\binom{10}{8} + \binom{10}{6}\binom{10}{6} + \binom{10}{8}\binom{10}{4}$

$+ \binom{10}{10}\cdot\binom{10}{2} = 2\cdot\left[\binom{10}{2}\binom{10}{10}\right]^2 + 2\left[\binom{10}{4}\binom{10}{8}\right]^2 + \binom{10}{6}\binom{10}{6}$

$= \sum_{i=1}^{5} \binom{10}{12-2i}\binom{10}{2i}$.

${}^{10}C_{10} \times {}^{10}C_2 + {}^{10}C_9 \times {}^{10}C_3 + {}^{10}C_8 \times {}^{10}C_4 + {}^{10}C_7 \times {}^{10}C_5$

(d) There must be more women than men.

10, 9, 8, 7

men, women

Ans:- $\binom{10}{2}\binom{10}{10} + \binom{10}{3}\binom{10}{9} + \binom{10}{4}\binom{10}{8} + \binom{10}{5}\binom{10}{7}$

$= \sum_{i=7}^{10} \binom{10}{i}\cdot\binom{10}{12-i}$

8, 9, 10

${}^{10}C_8 \times {}^{10}C_4 + {}^{10}C_9 \times {}^{10}C_3 + {}^{10}C_{10} \times {}^{10}C_2$

(e) There must be atleast 8 men.

≥ 8

men, women

Ans:- $\binom{10}{8}\binom{10}{4} + \binom{10}{9}\binom{10}{3} + \binom{10}{10}\binom{10}{2} = \sum_{i=2}^{4}\binom{10}{i}\binom{10}{12-i}$

(9) How many bytes contain (1 byte = 8 bits)

$^8C_2$ (a) exactly two 1's $= \binom{8}{2} = 28 \because \frac{8!}{6!\,2!} = \frac{\cancel{8}^{4}\times 7}{\cancel{2}} = 28.$

$8C_4$ (b) exactly four 1's $= \binom{8}{4} = \frac{8!}{4!\,4!} = \frac{\cancel{8}\times 7\times \cancel{6}^{2}\times 5}{\cancel{4}\times\cancel{3}\times\cancel{2}} = \underline{70}.$

$8C_6 + 8C_7 + 8C_8$

(c) atleast 6 1's $= \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = \frac{8!}{2!\,6!} + \frac{8!}{7!} + \frac{8!}{8!}$

≥ 6

$= \frac{\cancel{8}^{4}\times 7}{\cancel{2}} + 8 + 1 = 28 + 9 = \underline{37}.$

$8C_6$ (d) exactly six 1's $= \binom{8}{6} = \underline{28}.$

(10) Pick 5 person team from 12 players. $^{12}C_5$

(a) $= \binom{12}{5} = \frac{12!}{5!\,7!} = \frac{\cancel{12}\times 11\times \cancel{10}\times 9\times 8}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}} = \underline{792}.$

(b) No of selections which include weakest & strongest players?

Ans:- Let x be strongest & y be weakest players in 12 players team which will be there in selection. From Remaining 10 players, selecting 3 players can be done in $\binom{10}{3} = \frac{10!}{3!\cdot 7!}$

$= \frac{10\times\cancel{9}^{3}\times\cancel{8}^{4}}{\cancel{3}\times\cancel{2}} = 120.$

$(1)\cdot 120 = 120$ ways

(weak & strong) (remaining 3)

x → weakest
y → strongest

12 − 2 = 10
5 − 2 = 3

$^{10}C_3$

(11) A student has to answer 7 out of 10 questions. In how many ways he can select questions if

$^{10}C_7$ $r=7$ $n=10$

(a) No restrictions

Ans $\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10\times \overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{\cancel{3}\times\cancel{2}} = \underline{\underline{120}}$

(b) He must answer first two

Ans: $^{8}C_5$ $\binom{1}{}$ $\binom{8}{5}$ $= \frac{8!}{5!\,3!} = \frac{8\times7\times\cancel{6}}{\cancel{3\times2}} = \underline{56}$ $\times\ ^{8}C_5$

first two remaining 5

$10-2=8$

$7-2=5$

(c) He must answer atleast 4 of first 6 ($>$)

Ans: $\binom{6}{4}\binom{4}{3} + \binom{6}{5}\binom{4}{2} + \binom{6}{6}\binom{4}{1}$

4 of first 6 5 of first 6 6 of first 6

$^{6}C_4 \times {}^{4}C_3 +$

$^{6}C_5 \times {}^{4}C_2 +$

$^{6}C_6 \times {}^{4}C_1$

$= \frac{6!}{4!\,2!}\ \frac{4!}{3!\,1!} + \frac{6!}{5!}\ \frac{4!}{2!\,2!} + \frac{\cancel{6!}}{\cancel{6!}}\ \frac{4!}{3!}$

$= \frac{6\times5\times\overset{2}{\cancel{4}}\times3\times2\times1}{\cancel{2}\ \ \cancel{3!}} + 6\ \frac{\overset{2}{\cancel{4}}\times3}{\cancel{2}} + 4$

$= 60 + 36 + 4 = \underline{\underline{100}}$

(12) How many ways 12 diff books be distributed among 4 children so that $^{12}C_3 \times {}^{9}C_3 \times {}^{6}C_3 \times {}^{3}C_3$

a) each child gets 3 books?

~~Ans: $\binom{12}{3}$ $\frac{12!}{3!\,3!\,3!\,3!} = \frac{12\times11\times10\times9\times8\times7\times6\times5\times4}{3\times2\times3\times2\times3\times2} = 369{,}600$~~

(13) How many arrangements of letters in MISSISSIPPI have no consecutive S's?

<u>Ans:-</u> without S's, $\uparrow M \uparrow I \uparrow I \uparrow I \uparrow P \uparrow P \uparrow I \uparrow$ 8 locations

$\boxed{{}^8C_4 \times 105}$

<u>No</u> of ways of arranging letters without S is

$$\frac{7!}{4!\cdot 2!} = \frac{7\times6\times5\times4!}{2} = \underline{\underline{105}} \text{ ways.}$$

4-S's can be arranged in $\binom{8}{4} = \frac{8!}{4!4!} = \frac{8\times7\times6\times5}{3\times2\times4} = 70$ ways.

By product rule ans $= 105\times70 = \underline{\underline{7350}}$ ways.

(14) A gym coach must select 11 seniors to play. If he can make his selection in 12,376 ways, how many seniors are eligible to play?

$r = 11$
$n = ?$

<u>Ans:-</u> Let <u>no</u> of seniors be n.

Selecting 11 out of n is $\binom{n}{11} = 12376$. $nC_r = 12376$

ie $\frac{n!}{(n-11)!\cdot 11!} = 12376$. $\Rightarrow \underline{\underline{n=17}}$.

$$\frac{n!}{(n-11)!\cdot 11!} = 12376$$

if $n = 12, 13, 14\ldots$ (trial & error method)

$n \cdot \frac{2n+1}{6}$

16) a) $\sum_{i=1}^{6} (i^2+1) = 2+5+10+17+26+37$

$= \underline{\underline{97}}$

b) $\sum_{j=-2}^{2} (j^3-1) = -9+8-1+0+7 = \underline{\underline{-5}}$.

Combinations with repetition.

When repetitions are allowed, for n distinct objects, an arrangement of size r can be obtained in n^r ways, $r \geq 0$.

No of combinations of n objects taken r at a time with repetition is $C(n+r-1, r) = (n+r-1)C_r = \binom{n+r-1}{r}$.

diff types of donuts

Ex-1) There are 20 kinds of donuts. There are atleast a dozen of each kind. We can select a dozen donuts in $\binom{20+12-1}{12} = \binom{31}{12}$ ways. ($n=20$, $r=12$)

$n=20$
$r=12$
${}^{20+12-1}C_{12}$

2) President P has 4 vice presidents B, G, M, O. $n=4$
He has to distribute among them \$1000 check, where each check will be a multiple of \$100.

a) Allowing one or more vice presidents get nothing, P makes selection of size 10 from $r=10$

$n=4$

a collection of 4 with repetition $= \binom{10+4-1}{10} = \underline{286}$.

$^{n+r-1}C_r = {}^{10+4-1}C_{10} = {}^{13}C_{10}$

(b) if each vice president has to receive at least \$100,
P has to make selection of size 6 out of $^{4+6-1}C_6$ ($r=6$)
collection of size 4. $= \binom{6+4-1}{6} = \binom{9}{6} = 84$. $= {}^9C_6 = 84$
$n=4$

(c) If each vice President gets atleast \$100 &
O as executive vice president gets atleast \$500,
then no of ways to distribute =

$$\underbrace{\binom{3+2-1}{2}}_{\text{O gets exactly 500}} + \underbrace{\binom{3+1-1}{1}}_{\text{O gets 600}} + \underbrace{\binom{3+0-1}{0}}_{\text{O gets 700}} = \underline{10} = \binom{4+2-1}{2} = \underline{\binom{5}{2}}.$$

dist objects

(3) In how many ways we can distribute 7 apples & 6 oranges among 4 children ($n=4$) so that each receives atleast one apple?

Ans:- If Each child gets one apple, there are ($n=4$, $r=3$) $\binom{4+3-1}{3} = \binom{6}{3}$
$= 20$ ways to distribute remaining 3 apples
& ($n=4$, $r=6$) $\binom{4+6-1}{6} = 84$ ways to distribute 6 oranges among 4 children.

Oranges : $n=4$, $r=6$

$^{4+6-1}C_6 = \underline{84}$

Ans $= 20 \times 84 = \underline{1680 \text{ ways}}$.

each child gets one apple $\Rightarrow$ "must" $\Rightarrow 7-4=\underline{3}$ $\rightarrow$ 20×84

$n=4, r=3 \Rightarrow {}^{4+3-1}C_3 \Rightarrow \underline{20}$

~~No of integer solns of~~

~~$x_1+x_2+x_3+\dots x_n = r \quad x_i \geq 0, \quad 1 \leq i \leq n.$~~

~~$\equiv$ No of selections with repetition of size r from a collection of size n~~

~~$\equiv$ No of ways r identical objects be distributed among n distinct containers.~~

~~$= \binom{n+r-1}{r}$.~~

Ex:-

1. In how many ways one can distribute 10 identical white marbles among 6 distinct containers?

$n=6$
$r=10$

Ans:- $r=10 \quad n=6$. (or) $x_1+x_2+\dots x_6=10$

$\binom{6+10-1}{10} = \binom{15}{10} = \underline{\underline{3003}}$ $\quad {}^{6+10-1}C_{10} = \underline{\underline{3003}}$

$\Downarrow$

There are 3003 non-ve solns to $x_1+x_2+\dots x_6=10$

How many solns are there $x_1+x_2+\dots+x_6 < 10$? $\quad 10 \to 9$, $+x_7$

How many solns are there $x_1+x_2+\dots+x_6+x_7=10$, $\quad 0 \leq x_i, \quad 1 \leq i \leq 6,$ $0 < x_7$.

$= x_1+x_2+\dots+x_6+x_7=10$

$= y_1+y_2\dots y_6+y_7=9, \quad y_i=x_i, \quad 1 \leq i \leq 6, \quad y_7=x_7-1$

$= \binom{7+9-1}{9} = \underline{\underline{5005}}$. $\quad \binom{(n+1)+(r-1)-1}{r-1} = \binom{n+r-1}{r-1}$

${}^{n+r-1}C_r = {}^{7+9-1}C_9 = {}^{15}C_9 = \underline{\underline{5005}}$

3) ~~Different ways to write 4 as a sum of +ve integers (Composition of 4)~~

~~$= 4;\ 3+1;\ 2+2;\ 2+1+1;\ 1+1+1+1$~~

~~[Partitions for 4]~~

4) ~~Composition of 7~~

~~$= 7,\ 6+1;\ 1+6,\ 5+2;\ 1+2+4;\ 2+4+1;$~~

~~$3+2+1+1$~~

~~$$= \binom{6}{6}+\binom{6}{5}+\binom{6}{4}+\binom{6}{3}+\binom{6}{2}+\binom{6}{1}+\binom{6}{0} = \sum_{k=0}^{6}\binom{6}{k} = 2^6$$~~

~~In general, for each +ve integer m, there are $\sum_{k=0}^{m-1}\binom{m-1}{k} = 2^{m-1}$ compositions.~~

---

5)
```
for i=1 to 20
  for j=1 to i
    for k=1 to j
      print (i*j+k)
```
$\Rightarrow n = 20$

$r = \text{size} = 3$

${}^{20+3-1}C_3 \Rightarrow {}^{22}C_3$

How many times, print is executed? $\Rightarrow 1540$

$1 \le k \le j \le i \le 20$.

ie selection of a, b, c $(a \le b \le c)$ of size 3, with repetitions allowed, from $1, 2 \dots 20$

$$= \binom{20+3-1}{3} = \binom{22}{3} = 1540 \text{ times}.$$

In general, if there had been $r(\geq 1)$ for loops, print would have been executed $\binom{n+r-1}{r}$ times where n is outerloop upper bound.

6) Counter=0
for $i=1$ to n
  for $j=1$ to i
    counter++

size = 2
$n = ?$
$^{n+2-1}C_2 = {}^{n+1}C_2$

for loop is executed $\binom{n+2-1}{2} = \binom{n+1}{2}$ times.

$\equiv$ $i=1$ $j=1$ to 1
$i=2$ $j=1$ to 2
$i=3$ $j=1$ to 3.
⋮
$i=n$ $j=1$ to n.

$= 1+2+3+\cdots n$

$= \sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}.$

Ex:- 1.4

1) How many ways 10 identical dimes be distributed among 5 children if

$r = 10$ $5 = n$

a) No restrictions.

Ans:- $n=5$ $r=10$

$= \binom{n+r-1}{r} = \binom{14}{10} = \frac{14!}{10! \cdot 4!} = \frac{\cancel{14}^{7} \times 13 \times \cancel{12} \times 11}{\cancel{4 \times 3} \times \cancel{2}} = 1001$

(b) each child gets atleast one dime?

Ans:- Give each of 5 child one dime, remaining 5 dimes be distributed among 5 children in $\binom{5+5-1}{5} = \binom{9}{5} = \frac{9!}{5!\cdot 4!}$

$10-1=9=r$

$$= \frac{\cancel{9}^{3}\times\cancel{8}\times7\times6}{\cancel{4}\times\cancel{3}\times\cancel{2}} = 126 \text{ ways}.$$

$n=5$

(c) oldest child gets atleast two dimes.

$10-2=8=r$

Ans:- one child gets 2.

$n=5$

Remaining 8 dimes to be distributed among 5 children = $r=8$ $n=5$.

$$\binom{8+5-1}{8} = \binom{12}{8}.$$

(2) In how many ways can 15 (identical) candy bars be distributed among 5 children so that youngest child gets only one or two of them.

$5-1=4$

Ans:- Youngest gets one.

Remaining 14 bars be distributed among 4 children $= \binom{14+4-1}{14} = \binom{17}{14}$.

$15-1=14=r$

$n=4$

Youngest gets two.

Remaining 13 bars among 4 people

$$= \binom{13+4-1}{13} = \binom{16}{13}.$$

$15-2=13=r$

$5-1=4=n$

Ans $= \binom{17}{14} + \binom{16}{13}$ //.

3) No of ways 20 coins be selected from 4 large containers filled with pennies, nickels, dimes & quarters (Each container filled with only one type). (identical)

Ans:- $r = 20$ $n = 4$ $r = 20$ $n = 4$

$$\binom{4+20-1}{20} = \binom{23}{20} =$$

4) 31 flavors of ice creams. Order a dozen of ice cream. $n=31$ $r=12$ $r = 12$ $n = 31$

a) do not want same flavor ${}^{31}C_{12}$

$$= \binom{31}{12}$$

b) flavor may be ordered as many as 12 times. $r = 12$ $n = 31$

Here $r=12$ $n=31$.

$$\binom{n+r-1}{r} = \binom{31+12-1}{12} = \binom{42}{12}$$

c) flavor may be ordered no more than 11 times.

$$\sum_{r=1}^{11} \binom{n+r-1}{r} \text{ with } n=31. = \sum_{r=1}^{11} \binom{30+r}{r}$$

7) No of integer Solns of $x_1+x_2+x_3+x_4=32$, where

a) $x_i \geq 0$, $1 \leq i \leq 4$. $r=32$ $n=4$ $\binom{n+r-1}{r}$

Ans:- $\binom{32+4-1}{32} = \binom{35}{32}$. $r = 32$ $n = 4$

${}^{32+4-1}C_{32}$

$\Rightarrow {}^{35}C_{32}$

$32 - 4 = 28 = r$

$^{4+28-1}C_{28} \Rightarrow {}^{31}C_{28}$

(b) $x_i > 0$ → all are 1

$1 \leq i \leq 4$.

No of +ve integer solns for

$x_1 + x_2 + x_3 + x_4 = 32$

$= \binom{32-1}{32-4} = \binom{31}{28}$

(c) 3. Find no. of non negative solutions of following:

a) $x_1+x_2+x_3+x_4+x_5=40$ $\Rightarrow n=5, r=40$ $\;{}^{5+40-1}C_{40} = {}^{44}C_{40}$

Ans: Here n=5, r=40 so, $\binom{n+r-1}{r}=\binom{5+40-1}{40}=\binom{44}{40}$

b) $x_1+x_2+x_3+x_4+x_5 \leq 40$ $\;+x_6$ $\;n=6, r=40 \Rightarrow {}^{6+40-1}C_{40} = {}^{45}C_{40}$

Ans: Here n=5, r=40 so, $\binom{n+r}{n} = \binom{45}{5}$

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c) $x_1+x_2+x_3+x_4+x_5 = 40$ with $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 \geq 5$ $\;1+2+3+4+5 = 15$

Ans: Given equation becomes, $x_1+x_2+x_3+x_4+x_5$ = 40-15=25

Here n=5, r=25 so, $\binom{n+r-1}{r} = \binom{29}{4}$ $\;n=5, r=25$

d) $x_1+x_2+x_3+x_4+x_5 = 40$ with $x_1 \leq 20$

Ans: Total solutions are $\binom{44}{4}$ *and solutions* with $x \geq 20$ are $\binom{24}{4}$.

So required ans is $\binom{44}{4}$ - $\binom{24}{4}$ $\;x_1 \geq 21$ $\;40-21=19$ $\;{}^{5+19-1}C_{19} = 8855$

$\Rightarrow 135751 - 8855 = 126896$

Q) $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

if $x_1 \geq 2$, $x_2 > 2$, $x_3 \geq 4$, $x_4 = 3$, $x_5 > 2$

(2, 3, 4, 3, 3)

$\Rightarrow x_1 + x_2 + x_3 + \cancel{x_4} + x_5 = 20 - (2+3+4+3+3)$

$= 20 - 15 = 5$

$\Rightarrow n = \cancel{4}, r = 5$ $\;{}^{5+4-1}C_5 = {}^{9}C_5 = 56$

${}^{5+5}C_5 = {}^{10}C_5$

${}^{n+r}C_n$

(c) $x_1, x_2 \geq 5$, $x_3, x_4 \geq 7$.

~~$= 7+5-1 = 11$.~~

~~$x_1 + x_2 + x_3 + x_4 = 32$~~

~~$\geq 10 + x_3 + x_4 = 32$.~~

~~$x_3 + x_4 \leq 22$.~~

~~$\therefore x_3, x_4 \geq 7$,~~

~~$x_3 + x_4 \geq 14$.~~

~~$32 - 14 = 8$.~~

$\Rightarrow x_1 \geq 5$, $x_2 \geq 5$, $x_3 \geq 7$, $x_4 \geq 7$

$\Rightarrow 5 + 5 + 7 + 7 = 24$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 32 - 24 = 8$

$\Rightarrow r = 8$, $n = 4$

$\Rightarrow {}^{4+r-1}C_8 \Rightarrow \boxed{{}^{11}C_8}$

(all are ≥ 8) $\;32 - 32 = 0$ $\;r = 0$, $n = 4$ $\;{}^{4+0-1}C_0 \Rightarrow {}^{3}C_0 \Rightarrow 1$

(d) $x_i \geq 8$ $\;1 \leq i \leq 4$.

$x_i = 8$ is soln. Ans is 1.

$1 \leq i \leq 4$.

(e) $x_i \geq -2$ becomes

$x_1 + x_2 + x_3 + x_4 = 32 \Rightarrow x_3 + x_4 \leq 36$.

$\geq -4 + x_3 + x_4 = 32$

(all are ≥ -2) $\Rightarrow 32 + 8 = 40 = r$

${}^{4+40-1}C_{40} \Rightarrow {}^{43}C_{40}$

25) Find coefficients of

a) xyz^2 in $(x+y+z)^4$.

Ans:- Let $a=x$, $b=y$ $c=z$.

$(a+b+c)^4$.

coefficient of abc^2 is $\binom{4}{1,1,2} = \frac{4!}{2!\,1!\,1!} = \frac{\overset{2}{\cancel{4}}\times 3 \times 2}{\cancel{2}} = 6\times 2 = 12$

b) xyz^2 in $(w+x+y+z)^4$.

$= \binom{4}{0,1,1,2} = 12$.

c) xyz^2 in $(2x-y-z)^4$.

$a=2x$ $b=-y$ $c=-z$.

$(a+b+c)^4$ abc^2 coefficient $= \binom{4}{1,1,2} = 12$.

$(12)\cdot(2x)(-y)\cdot(-z)^2 = 12\cdot 2\, x\,(-1)\,y\cdot z^2$

$= (-24)\,xyz^2$.

d) xyz^{-2} in $(x-2y+3z^{-1})^4$.

$a=x$, $b=-2y$ $c=3z^{-1}$ $(a+b+c)^4$

coefficient of $abc^2 = \binom{4}{1,1,2} = 12$.

$(12)\cdot x\,(-2y)\cdot(3z^{-1})^2 = 12\cdot x\cdot y\cdot(-18)\cdot z^{-2} = (-216)\cdot xyz^{-2}$.

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

$a=2w,\ b=-x,\ c=3y,\ d=-2z$

$(a+b+c+d)^8$

coefficient of $a^3b^2cd^2 = \binom{8}{3,2,1,2} = \frac{8!}{3!2!2!}$

$= \frac{8\times7\times6\times5\times4}{\cancel{2\times2}} = 1680$

$1680\cdot(2w)^3(-x)^2\cdot(3y)(-2z)^2$

$= 1680\cdot8\cdot3\cdot4\ w^3x^2yz^2 = \underline{161280\ w^3x^2yz^2}$.

2) Find coefficient of $w^2x^2y^2z^2$ in expansion of

a) $(w+x+y+z+1)^{10}$.

Soln:- $a=w\quad b=x\quad c=y\quad d=z\quad e=1$

$(a+b+c+d+e)^{10}$. Coefficient of $a^2b^2c^2d^2e^2$

$= \binom{10}{2,2,2,2,2} = \frac{10!}{2\cdot2\cdot2\cdot2\cdot2} = \frac{10\times9\times\cancel{8}^{4}\times7\times6\times5\times4\times3\times2}{\cancel{2\cdot2\cdot2\cdot2\cdot2}}$

$= 226800.$

$= 226800\cdot w^2\cdot x^2\cdot y^2\cdot z^2\cdot 1^2 = \underline{226800\ w^2x^2y^2z^2}$.

14) Find coefficient of $v^2 w^4 x z$ in $(3u+2w+x+y+z)^8$.

abcdef
uvwxyz

Ans:- Let ~~u=a, v=b~~

$a=3u$, ~~b=2w~~, $c=2w$, $d=x$, $e=y$, $f=z$.

then $(a+\cancel{b}+c+d+e+f)^8$

coefficient of ~~$v^2 w^4$~~ $a^2 c^4 d\, f = \binom{8}{2,4,1,1} = \frac{8!}{2!\cdot 4!}$

$= \frac{\overset{4}{\cancel{8}}\times 7\times 6\times 5}{\cancel{2}} = \underline{840}$

$\Rightarrow \binom{8}{2,4,1,1} \cdot 3^2\, 2^4 . 1 . 1 \cdot v^2 w^4 x z = (840 \cdot 9 \cdot 16)\, v^2 w^4 x z$

$= \underline{(120,960)}\, v^2 w^4 x z$.

Set Theory

Set is a collection of well defined collection of objects.

Ex:- $A = \{1,2,3,4\}$. $2 \in A$, $6 \notin A$.

$= \{x \mid x$ is an integer & $1 \le x \le 4\}$.

Sets can be finite or infinite.

The no of elements in a finite set is called cardinality or size of set.

In above example, $|A| = 4$.

Dfn:- If C, D are sets from universe $\mathcal{U}$, C is a subset of D and is written as $C \subseteq D$ or $D \supseteq C$ if every element of C is also an element of D.

If in addition, D contains an element that is not in C, then C is called a proper subset of D and is denoted by $C \subset D$ or $D \supset C$.

If $C \subseteq D$, then

$\forall x\,[x \in C \Rightarrow x \in D]$ & viceversa.

When ~~for C⊆D~~ C and D are finite,

$C \subseteq D \Rightarrow |C| \leq |D|$ & $C \subset D \Rightarrow |C| < |D|$.

If variable name starts with a single letter followed by at most 5 chars (letters or digits), then

$|U| = 26 + 26 \cdot 36 + 26 \cdot 36^2 + 26 \cdot 36^3 + 26 \cdot 36^4 + 26 \cdot 36^5$.

$= 26 \cdot \sum_{i=0}^{5} 36^i$.

If variable starts with I, J, K, L, M, N then

$|U| = 6 \cdot \sum_{i=0}^{5} 36^i$.

$U = \{1,2,3,4,5\}$. $A = \{1,2\}$ $B = \{x \mid x^2 \in U\}$

then $B = \{1,2\}$ Here A and B are equal.

Here $A \subseteq B$ & $B \subseteq A$.

Defn:- For a given universe U, sets C and D are equal when $C \subseteq D$ & $D \subseteq C$ & is written as $C = D$.

$\{1,2,3\} = \{3,2,1\} = \{2,2,1,3\}$.

$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.

$A \neq B \Leftrightarrow \neg(A \subseteq B \wedge B \subseteq A) \Leftrightarrow \neg(A \subseteq B) \vee \neg(B \subseteq A)$

$\Leftrightarrow A \nsubseteq B$ or $B \nsubseteq A$

$C \subset D \Leftrightarrow C \subseteq D \wedge C \neq D$.

$U = \{1, 2, 3, x, y, \{1,2\}, \{1,2,3\}\}$

$|U| = 7$.

a) If $A = \{1,2,3\}$ $|A| = 3$. Then
$A \subseteq U$, $A \subset U$, $\{A\} \subseteq U$, $A \in U$,
$\{A\} \subset U$, but $\{A\} \notin U$.

<u>Thm</u>:- Let $A, B, C \subseteq U$,

a) If $A \subset B$ & $B \subseteq C$ then $A \subseteq C$.
b) If $A \subseteq B$ & $B \subseteq C$ then $A \subset C$.
c) If $A \subseteq B$ & $B \subset C$ then $A \subset C$.
D) If $A \subset B$ & $B \subset C$ then $A \subset C$.

Defn:- Null set or empty set, is the unique set containing no elements. & is denoted by ϕ or $\{\}$

$|\phi| = 0$. but $\{0\} \neq \phi$.
Also $\phi \neq \{\phi\}$ because $\{\phi\}$ is a set with one element, namely null set.

Defn:- If A is a set from universe U, the Powerset of A is denoted by $P(A)$ is the collection of all subsets of A.

Ex:- $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

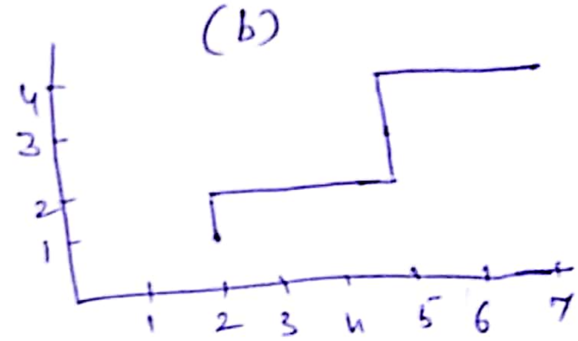
In general, for any finite set A, with $|A| = n \geq 0$, $|P(A)| = 2^n$. For any $0 \leq k \leq n$, there are $\binom{n}{k}$ subsets of size k.

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} \cdots \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n, n \geq 0.$$

Ex:-

(a)

(b)



Staircase path from (2,1) to (7,4)

R, U, R, R, U, R, R, U
1 2 3 4 5 6 7 8

U, R, R, R, U, U, R, R
1 2 3 4 5 6 7 8

In fig(a), path has three U's located in positions, 2, 5, 8. It determines three element subset $\{2, 5, 8\}$. Illy fig(b) determines $\{1, 5, 6\}$. If subset $\{1, 3, 7\}$ rep's path U, R, U, R, R, R, U, R.

Here no of paths is equal to no of subsets of A $\{1, 2, 3 \ldots 8\}$ where $|A| = 3$. There are

$\binom{8}{5} = \binom{8}{3} = \frac{8!}{5!\,3!} = 56$ such paths.

Ex:- There are 2^6 ways to write 7 as a sum of one or more +ve integers, where the order of summands is relevant.

Composition of 7 can be written as

$1+1+1+1+1+1+1$.

Here there are 7 summands & each is 1. and six plus signs.

For set $\{1,2,3,4,5,6\}$ there are 2^6 subsets.

Subset $\{1,4,6\}$ reps.

$(1+1)+1+(1+1)+(1+1)$ indicates $2+1+2+2$

Composition of 7	Subset of $\{1,2,3,4,5,6\}$
1) $1+1+1+1+1+1+1$	ϕ
2) $1+2+1+1+1+1$	$\{2\}$
3) $1+1+3+1+1$	$\{3,4\}$
4) $2+3+2$	$\{1,3,4,6\}$
5) $4+3$	$\{1,2,3,5,6\}$
6) 7	$\{1,2,3,4,5,6\}$.

For integers n, r with $n \geq r \geq 1$,

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Ex:- No of non -ve integer solutions of inequality

$x_1+x_2+ \cdots +x_6 < 10$

For each k, $0 \leq k \leq 9$, no of solns to $x_1+x_2+\cdots x_6 = k$ is $\binom{6+k-1}{k} = \binom{5+k}{k}$. So no of non negative integer solns to $x_1+x_2+\dots x_6<10$ is & $0\leq k\leq 9$

$$\binom{5}{0}+\binom{6}{1}+\binom{7}{2}+\binom{8}{3}+\dots\binom{14}{9}$$

$\because \binom{5}{0}=1$ & $\binom{6}{0}=1$

$\binom{5}{0}$ is replaced by $\binom{6}{0}$.

$$=\underbrace{\left[\binom{6}{0}+\binom{6}{1}\right]}+\binom{7}{2}+\dots\binom{14}{9}.$$

$\boxed{\binom{n+1}{r}=\binom{n}{r}+\binom{n}{r-1}}$

$$=\left[\binom{7}{1}+\binom{7}{2}\right]+\dots\binom{14}{9}=\left[\binom{8}{2}+\binom{8}{3}\right]+\dots\binom{14}{9}$$

$$=\dots=\binom{14}{8}+\binom{14}{9}=\binom{15}{9}=5005.$$

Ex 2) Pascal $\triangle^{le}$.

$n=0$ $\quad\binom{0}{0}$

$n=1$ $\quad\binom{1}{0}\ \binom{1}{1}$

$n=2$ $\quad\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$

$n=3$ $\quad\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$

$n=4$ $\quad\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$

$n=5$ $\quad\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$

<u>Note</u>:- For real nos, a,b with $a<b$,

$[a,b]=\{x\in\mathbb{R}\mid a\leq x\leq b\}$. closed interval.

$(a,b)=\{x\in\mathbb{R}\mid a<x<b\}$. open interval

$[a,b)=\{x\in\mathbb{R}\mid a\leq x<b\}$ } half open intervals.

$(a,b]=\{x\in\mathbb{R}\mid a<x\leq b\}$.

Operations on Sets:-

1. $A \cup B = \{x \mid x \in A \vee x \in B\}$
2. $A \cap B = \{x \mid x \in A \wedge x \in B\}$
3. Symmetric Diff $A \Delta B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$
4. $\bar{A} = \{x \mid x \in U \wedge x \notin A\}$.

Defn: Two sets are disjoint when they don't have common elements.

ie A, B are disjoint when $A \cap B = \phi$.

Thm:- If $A, B \subseteq U$, then A, B are disjoint when $A \cup B = A \Delta B$.

5. Relative complement of A in B is
$B - A. = \{x \mid x \in B \wedge x \notin A\}$.

When $A \subseteq B$,
$A \cup B = B$, $A \cap B = A$, $\bar{B} \subseteq \bar{A}$ & viceversa.

A	B	$A \cup B$	$A \cap B$	$\bar{A}$
0	0	0	0	1
0	1	1	0	1
1	0	1	0	0
1	1	1	1	0

$\overline{A \cup B} = \bar{A} \cap \bar{B}$ → DeMorgan's law.

$\overline{A - B} = \overline{A \cap \bar{B}}$
$= \bar{A} \cup B$.

$|A\cup B| = |A| + |B| - |A\cap B|$.

$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| +$
$|A\cap B\cap C|$.

$|\overline{A\cup B\cup C}| = |\bar{A}\cap\bar{B}\cap\bar{C}| = |U| - |A\cup B\cup C|$

$= |U| - |A| - |B| - |C| + |A\cap B| + |A\cap C| + |B\cap C| - |A\cap B\cap C|$.

Ex:- A students plays one game among X, Y, Z. In how many ways he can play one game each day so that he plays each of three types at least once during a given week.

Soln:- Here ~~|U| =~~ U is all arrangements of size 5 taken from set of 3 games with repetitions. Let A - subset of all sequences of 5 games played during week without playing X. |||ly B - without playing Y. C - without play Z.

$|U| = 3^5$, $|A| = |B| = |C| = 2^5$. $|A\cap B| = |B\cap C| =$
$|A\cap C| = 1^5 = 1$, $|A\cap B\cap C| = 0$.

$\therefore\ |\bar{A}\cap\bar{B}\cap\bar{C}| = 3^5 - 2^5 - 2^5 - 2^5 + 1 + 1 + 1 - 0$
$= 3^5 - 3\cdot 2^5 + 3\cdot 1 = \underline{150}$.

∴ There are 150 ways student can select his daily games during a week & play each type at least once.

The Principle of Inclusion & Exclusion

Let S be set with $|S| = N$. & let $c_1, c_2 \ldots c_t$ be a collection of conditions or properties satisfied by some or all of elements of S.

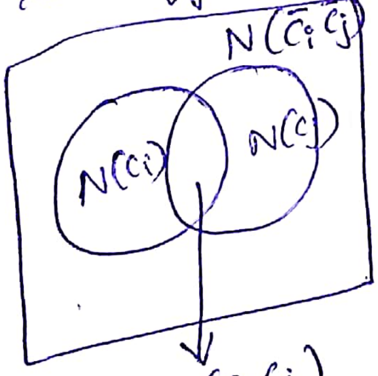
For all $1 \le i \le t$, $N(c_i)$ – no of elements in S that satisfy c_i.

For all $i, j \in \{1, 2 \ldots t\}$ where $i \neq j$, $N(c_i c_j)$ – no of elements in S that satisfy both c_i, c_j.

$N(c_i c_j)$ does not count elements of S that satisfy only $c_i c_j$.

$N(\bar{c}_i) = N - N(c_i)$.

$N(\bar{c}_i \bar{c}_j)$ = no of elements in S that does not satisfy either of c_i or c_j. $\neq N(\overline{c_i c_j})$.



$$\therefore N(\bar{c}_i \bar{c}_j) = N - [N(c_i) + N(c_j)] + N(c_i c_j)$$

$$N(\bar{c}_i \bar{c}_j \bar{c}_k) = N - [N(c_i) + N(c_j) + N(c_k)] + [N(c_i c_j) + N(c_j c_k) + N(c_i c_k)] - N(c_i c_j c_k).$$

Thm: Principle of Inclusion & Exclusion:-

Consider a set S, with $|S| = N$ & Conditions c_i, $1 \le i \le t$, satisfied by some of elements of S. The no of elements of S that satisfy none of the conditions C_i, $1 \le i \le t$, is denoted by $\bar{N} = N(\bar{C_1}\bar{C_2}\ldots\bar{C_t})$ where

$$\bar{N} = N - [N(C_1) + N(C_2) + \ldots N(C_t)] + [N(C_1C_2) + N(C_2C_3) + \ldots + N(C_tC_1) + N(C_1C_3) + \ldots] - [N(C_1C_2C_3) + N(C_1C_2C_4) + \ldots N(C_1C_2C_5) + N(C_1C_3C_4) + \ldots + N(C_{t-2}C_{t-1}C_t)] + \ldots \ldots + (-1)^t N(C_1C_2C_3\ldots C_t).$$

or

$$\bar{N} = N - \sum_{1 \le i \le t} N(C_i) + \sum_{1 \le i < j \le t} N(C_iC_j) - \sum_{1 \le i < j < k \le t} N(C_iC_jC_k) + \ldots + (-1)^t N(C_1C_2C_3\ldots C_t).$$

Corollary:-

No of elements in S that satisfy atleast one of conditions C_i, $1 \le i \le t$ is given by $N(C_1 \text{ or } C_2 \ldots \text{ or } C_t) = N - \bar{N}$.

1. Determine the no of +ve integers n, $1 \leq n \leq 100$ & n is not divisible by 2, 3 or 5.

Soln:- $S = \{1, 2, \ldots \ldots 100\}$. & $N = |S| = 100$.

For $n \in S$, n satisfies

a) C_1 if n is divisible by 2

b) C_2 ... 3

c) C_3 ..... 5.

Ans is $N(\bar{C}_1 \bar{C}_2 \bar{C}_3)$.

$N(C_1) = \lfloor 100/2 \rfloor = 50$.

$N(C_2) = \lfloor 100/3 \rfloor = 33$

$N(C_3) = \lfloor 100/5 \rfloor = 20$.

$N(C_1 C_2) = \lfloor 100/6 \rfloor = 16$. {ie elements which are divisible by both 2, & 3 = divisible by lcm(2,3)=6}.

$N(C_1 C_3) = \lfloor 100/10 \rfloor = 10$

$N(C_2 C_3) = \lfloor 100/15 \rfloor = 6$

$N(C_1 C_2 C_3) = \lfloor 100/30 \rfloor = 3$.

$N(\bar{C}_1 \bar{C}_2 \bar{C}_3) = S_0 - S_1 + S_2 - S_3 = N - [N(C_1) + N(C_2) + N(C_3)]$
$+ [N(C_1 C_2) + N(C_2 C_3) + N(C_1 C_3)] - N(C_1 C_2 C_3)$

$= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3$

$= 100 - 103 + 32 - 3 = 132 - 106 = \underline{26}$.

2) Non -ve integer soln to $x_1+x_2+x_3+x_4=18$ with $x_i \leq 7$, $1 \leq i \leq 4$.

<u>Soln:</u> S is set of solns to $x_1+x_2+x_3+x_4=18$

$0 \leq x_i$, $1 \leq i \leq 4$.

$|S| = \binom{4+18-1}{18} = \binom{21}{18} = 1330$ $\quad$ $x_1+x_2+\dots x_n=k$ solns is $\binom{n+k-1}{k}$.

x_1, x_2, x_3, x_4 satisfy cond c_i, $1 \leq i \leq 4$, if $x_i > 7$ Then Ans is $N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4)$.

$N(c_1) = N(c_2) = N(c_3) = N(c_4)$.

<u>To find $N(c_1)$</u> $\quad$ ie c_i is $x_1 > 7$ or $x_i \geq 8$.

$x_1+x_2+x_3+x_4=10$.

$\therefore N(c_1) = \binom{4+10-1}{10} = \binom{13}{10}$.

$S_1 = \binom{4}{1} \cdot \binom{13}{10}$.

$|||^{ly}$ $N(c_1 c_2)$ is no of integer solns of $x_1+x_2+x_3+x_4=2$ where $x_i \geq 0$. $1 \leq i \leq 4$.

So $N(c_1 c_2) = \binom{4+2-1}{2} = \binom{5}{2}$. & $S_2 = \binom{4}{2}\binom{5}{2}$.

$\therefore N(c_i c_j c_k) = 0$ for every selection of 3 conds.

& $N(c_1 c_2 c_3 c_4) = 0$. we have

$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = N - S_1 + S_2 - S_3 + S_4 = \binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\cdot\binom{5}{2} - 0 + 0 = 246$.

$\therefore$ Out of $\binom{4+18-1}{18} = \binom{21}{18} = 1330$ non -ve integer solns of $x_1+x_2+x_3+x_4=18$, only 246 of them satisfy $x_i \leq 7$ for each $1 \leq i \leq 4$.

3. No. of onto funs from A to B, $|A|=m, |B|=n$ is $\sum_{i=0}^{n} (-1)^i \binom{n}{n-i} (n-i)^m$.

4. In how many ways can 26 letters of alphabets be permuted so that none of patterns car, dog, pun or byte occurs?

Soln:- S = set of all perm. of 26 letters $= 26!$

$|S| = 26!$

$1 \le i \le 4$, S satisfy c_i if permutation contain car, dog, pun, byte resp.

$N(c_1) = 24!$ ∵ car, b, d, e, f .... z can be permuted in 24!

$N(c_2) = N(c_3) = 24!$

$N(c_4) = 23!$

For $N(c_1 c_2)$ we deal with 22 symbols, which are permuted in 22! ways.

$N(c_1 c_3) = N(c_2 c_3) = 22!$

$N(c_i c_4) = 21!$ $i \ne 4$.

$26 - 9 = 17 + 3 = 20$.

Also $N(c_1 c_2 c_3) = 20!$

$N(c_i c_j c_4) = 19!$ $1 \le i < j \le 3$.

$N(c_1 c_2 c_3 c_4) = 17!$

$26 - 13 = 13 + 4 = 17$

So req. soln is $N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = 26! - [3 \cdot 24! + 23!] + [3 \cdot 22! + 3 \cdot 21!] - [20! + 3 \cdot 19!] + 17!$