Unit -1

Combinatorics

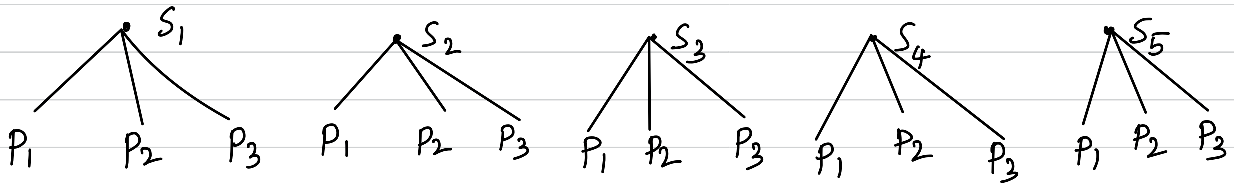
Basic Counting principles

The product rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are N ways to do the first task and for each of these ways of doing the task, there are M ways to do the second task, then there are MN ways to do the procedure.

Ex 1: You have 5 shirts and 3 pants. How many different outfits can you put together?

Soln: we can select a shirt in 5 different ways and for each shirt, we have 3 choices to select a pant.
Therefore, the total no. of different outfits is 5×3=15.



Thus, possible outfits are $\{S_1P_1, S_1P_2, S_1P_3, S_2P_1, S_2P_2, S_3P_3, \ldots, S_5P_3\}$

The sum rule:

If a task can be done either in one of M ways or in one of N ways, where none of the set of M ways is the same as any of the set of N ways, then there are M+N ways to do the task.

Ex 2 : Suppose that either a member of the mathematics faculty or a student who is mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Soln: There are 37 ways to choose a member of the maths faculty.

83 ways to choose a student who is a maths major

Since one cannot be both a student and a faculty member. By sum rule, there are $37+83=120$ possible ways to pick the representative.

Ex 3: The menu at the restaurant is limited: six kinds of rice, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice) miss Preethi went to the shop to get a lunch – either a rice and a hot beverages or a sandwiches and a cold beverages. In how many ways Preethi can purchase her lunch?

Soln: Preethi can get a rice and a hot beverages in 6×2 ways, and a sandwiches and a cold beverages in 8×3 ways

Thus, total no. of ways she can get her lunch is

$$6 \times 2 + 8 \times 3 = 36 \text{ ways.}$$

Definition [words]: Let A be a finite set (alphabet). A word is a finite sequence of elements of A.

Ex: $A = \{1, 2, 3, 4\}$

words of length 3 : 112, 333, 432, 134, ...

Theorem: The no. of words of length k in an alphabet consisting of n letters equals n^k.

Pf: For the 1st letters there are n possibilities

Same for all other letters.

The total no. : $\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{k \text{ times}} = n^k$

In other words, the number of linear ordering (ordered arrangement) of k - element subset of n distinct objects with repetitions is n^k.

It is also called k - permutations with repetition.

Another interpretation

Let $X = \{1, 2, 3, \dots, k\}$

$Y = \{1, 2, 3, \dots, n\}$

The number of maps from X to Y is n^k.

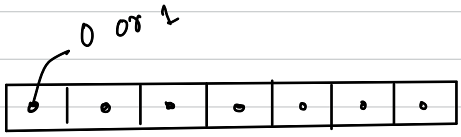
Let $f: X \to Y$, f corresponds to the word $(f(1), f(2), \dots, f(k))$ of length k in n letters where n is $|Y|$. Proof follows from above defn.

Ex 4: how many different bit strings of length seven are there?

Soln: Here $A = \{0, 1\}$. The number of words of length 7 is 2^7.

or

Consider

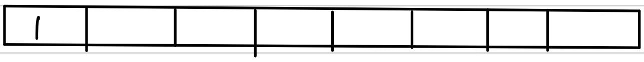


There are 2 possibilities for 1^{st} position and 2 possibilities for all other positions. Thus, total there are 2^7 possibilities.

Ex 5: How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

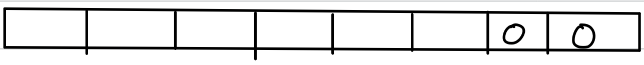
Soln: Consider

Case i:



We fix 1^{st} position by 1, all other positions can be filled in 2^7 ways.

Case ii:



Fix last 2 positions by 00, other can be filled in 2^6 ways

Case iii: | 1 | | | | | | 0 | 0 |

Fix last 2 positions by 0 0 and 1st position by 1, others can be filled in 2^5 ways.

But case iii is in both case i and case ii.

$\therefore$ Total bit strings is $2^7 + 2^6 - 2^5 = 160$

Note: Principle used in the last Ex is called inclusion-exclusion principle, this will be introduced later.

Ex 6: Each user on a computer system has a password, which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Soln: We first find possible passwords of length six.

There are 26 uppercase letters and 10 digits.

No. of passwords of length six that can formed using $26+10=36$ characters are 36^6.

No. of passwords formed using 26 letters are 26^6.

$\therefore$ Possible passwords of length six with atleast one digit are $36^6 - 26^6$.

IIIly possible passwords of length 7 and 8 with at least one digit are $36^7 - 26^7$ and $36^8 - 26^8$ respectively.

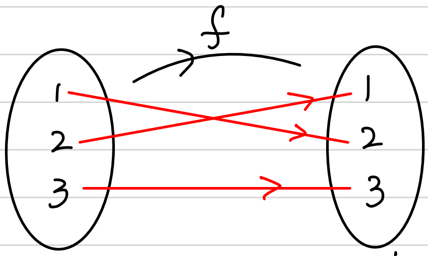
$\therefore$ Total possible passwords are $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8$.

Permutations

A permutation is a bijection from $\{1,2,3,4,\dots,n\}$ (n distinct objects) to itself.

Equivalently, permutation is a linear ordering (ordered arrangement) of n distinct objects.

For example:



Above permutation is 213. III^{ly} other permutations are

123, 132, 231, 312, 321.

Total there are 6 permutations of $\{1,2,3\}$

Thm: The number of permutations of $\{1,2,3,4,\dots,n\}$ is $n!$.

Pf: For the 1^{st} element we have n choices,
2^{nd} element has $(n-1)$ choices,
$\vdots$
k^{th} element has $(n-(k-1))$ choices
$\vdots$
n^{th} element has 1 choice.

$\therefore$ The total no. is $n(n-1)(n-2)\cdots 2\cdot 1 = n!$.

Convention: $0! = 1$.

k-permutation: $(0 \le k \le n)$

A k-permutation is a linear ordering (ordered arrangement) on a k-element subsets of $\{1,2,3,\dots,n\}$.

Ex: $n=3$, $\{1,2,3\}$

1-permutation: 1, 2, 3

2-permutations: 12, 13, 21, 23, 31, 32.

Thm : The number of k - permutations of $\{1, 2, 3, \dots n\}$ is
$$n(n-1)(n-2)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$$

pf: There are n- possibilities for the 1^{st} entry

$(n-1)$ - possibilities for the 2^{nd} entry

$\vdots$

$n-(k-1)$ possibilities for the k^{th} entry

$\therefore$ The total no. of k-permutation is $n(n-1)(n-2)\cdots(n-(k-1))$

$$= \frac{n(n-1)\cdots(n-(k-1))\,(n-k)\,(n-(k+1))\cdots 2\cdot 1}{(n-k)\,(n-(k+1))\cdots 2\cdot 1}$$

$$= \frac{n!}{(n-k)!}$$

It is denoted by ${}^{n}P_{r}$ or $P(n,r)$.

Another interpretation

Let $X = \{1, 2, 3, \dots, k\}$

$Y = \{1, 2, 3, \dots, n\}$ $\quad (k \le n)$

The number of injective maps from X to Y is ${}^{n}P_{r}$

Ex 7 :How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Soln: The number of ways to pick the three prize winners is the number of ordered selection of 3 elements from a set of 100 elements.

The answer is ${}^{100}P_3 = 100 \times 99 \times 98.$

Ex 8: How many permutations of the letters ABCDEFGH contain the string ABC?

Soln: Let ABC be one character. Then characters used in the permutations are ABC, D, E, F, G, H.

There are 6 characters.

Total permutations of these characters are 6!.

For instance, DEABCGHF, ABCHGEFD are two such permutations.

Ex 9: Three cards are chosen one after the other from a 52-card deck. Find the no. of ways this can be done

a) with replacement. b) without replacement.

Soln: a) with replacement

No. of ways three cards are chosen is $52 \times 52 \times 52 = 52^3$.

b) without replacement

This can be done in $52 \times 51 \times 50 = {}^{52}P_3$ ways.

Permutations with indistinguishable objects

The number of different permutations of n objects, where there are

n_1 indistinguishable objects of type 1

n_2 indistinguishable objects of type 2,

$\vdots$

and n_k indistinguishable objects of type k,

where $n = n_1 + n_2 + \cdots + n_k$, is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

Ex 10 : Find the number of possible arragements of the word MASSASAUGA. In how many of these arragements A's are together?

Soln: In the word MASSASAUGA, there are

1 - M 4 - A 3 - S 1 - U 1 - G	The no. of possible arrangements of these letters are $\frac{10!}{4!\,3!}$.

The no. of possible arrangements in which A's are together are

$\frac{7!}{3!} = 840.$ (because we consider AAAA as one single character in arrangements)

Ex 11 : How many arrangements are there of the letters in SOCIOLOGICAL?
(a) In how many of these arrangements A and G are adjacent?
(b) In how many of these arrangements all vowels are adjacent?

Soln: The word SOCIOLOGICAL as

1 − S
3 − O
2 − C
2 − I
2 − L
1 − G
1 − A

No. of arrangements of these letters are $\frac{12!}{3!\,2!\,2!\,2!}$.

a) No. of arrangements in which A and G are adjacent are

$\frac{11!}{3!\,2!\,2!\,2!} \times 2.$ (here we consider AG as single character and AG can be arranged in 2 ways)

b) Vowels in the word are A, I, O.

No. of arrangements in which vowels are together are

$\frac{7!}{2!\,2!} \times \frac{6!}{2!\,3!}$ (because vowels are kept together and these vowels can be arranged in $\frac{6!}{2!\,3!}$ ways.)

Ex 12 : In how many different ways can the letters of the word 'LEADING' be arranged in such a way that the vowels always come together?

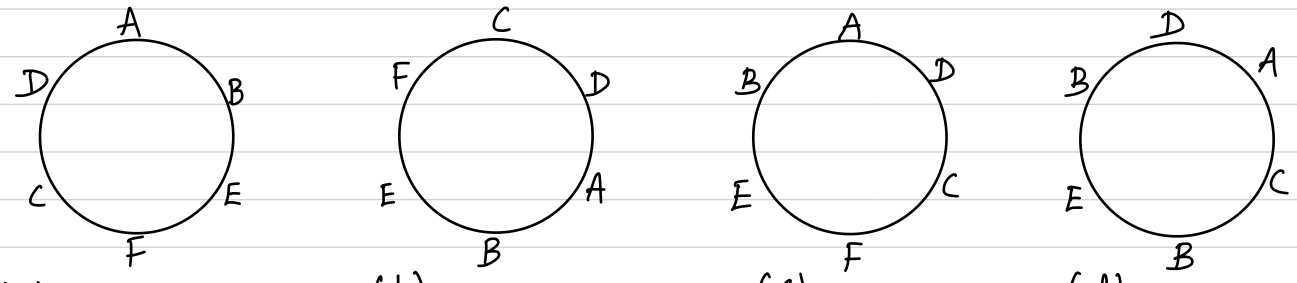
Soln: Vowels in the word LEADING are A, E, I.

∴ The no. of required arrangements are $5! \times 3!$.

Circular permutations

Ex 13: If six people, designated as A, B, C, D, E, F, are seated about a around table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

The following arrangements (a) and (b) are considered identical, whereas (b), (c) and (d) are three distinct arrangements.



(a) (b) (c) (d)

Soln: It is clear that

Each circular arrangement corresponds to 6 linear arrangements

For Ex: The following six linear arrangements –

ABCDEF, FABCDE, EFABCD, DEFABC, CDEFAB, BCDEFA,

– correspond to same circular arrangement.

$\therefore$ $6 \times$ (No. of circular arrangements of A, B, C, D, E, F)

$=$ (No. of linear arrangements of A, B, C, D, E, F)

$= 6!$

$\therefore$ No. of circular arrangement of A, B, C, D, E, F are

$\frac{6!}{6} = 5!$.

In general

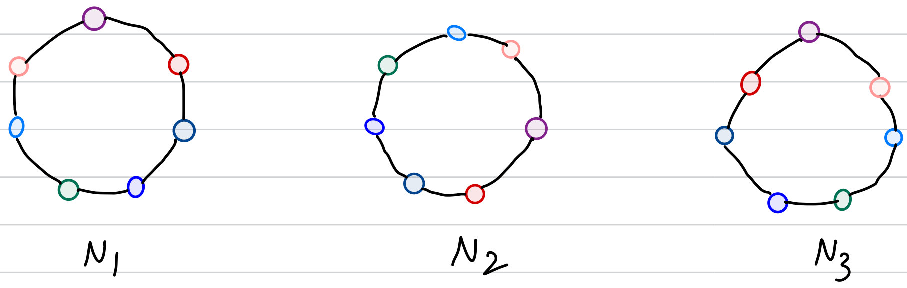
If n people are seated about a round table, then

$\frac{n!}{n} = (n-1)!$ arrangements are possible.

Ex 14: How many necklaces can be formed of 7 distinct beads?

Note: Necklace can be rotated or/and flipped over; all beads are same.

For instance, following arrangements of beads are same



N_2 is obtained by rotating N_1.

N_3 is obtained by flipping N_1.

Circular arrangements of beads can be done in $(7-1)! = 6!$ ways

Two circular arrangements are same, if one is obtained from the other by flip.

$\therefore$ Total no. of necklaces formed $= \dfrac{6!}{2}$.

k- combination

An k-combination of elements of a set is an unordered selection of k elements from the set.
Thus, an k-combination is simply a subset of the set with k elements.

Ex: $S = \{1, 2, 3, 4\}$

2 - combination : 12, 23, 34, 41, 13, 24

3 - combination : 123, 234, 341, 124

The no. of k - combinations of a set with n elements is denoted by ^nC_k or $C(n,k)$ or $\binom{n}{k}$, read as n choose k.

$\binom{n}{k}$ is also called binomial co-efficient.

Theorem: The number of k-combinations of a set with n elements, where n is a nonnegative integer and k is an integer with $0 \le k \le n$ equals

$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

Pf: $P(n,k)$ is ordered arrangement of k-elements from the set of n-elements.
$C(n,k)$ is unordered selection of k-elements from the set of n-elements.

$\therefore \quad P(n,k) = C(n,k) \times k!$ (because ordering of k elements in each k-combination can be done in $r!$ ways)

$$\Rightarrow \quad C(n,k) = \frac{P(n,k)}{k!}$$

or

$$C(n,k) = \frac{n!}{k!\,(n-k)!}.$$

Ex 15: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Soln: five cards from a std deck of 52 cards can be selected in $\binom{52}{5}$ ways.

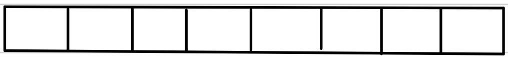
47 cards can be selected in $\binom{52}{47}$ ways.

Note: Once 5 cards are selected from the deck of 52 cards, 47 cards will be automatically selected on the other side.

Therefore, $\binom{n}{k} = \binom{n}{n-k}$.

Ex 16: How many bit strings of length n contain exactly r 1s?

Soln: For instance, let $n=8$, $r=4$.



No. of bit string of length 8 containing exactly 4 1's

= No. of possible selection of 4 positions from 8 positions

$$= \binom{8}{4} = \frac{8!}{4!\,4!}.$$

This can also be analysed as no. of possible arrangements of 8 object where there are

4 indistinguishable objects of type 1 (4 1's)

4 " " type 2 (4 0's)

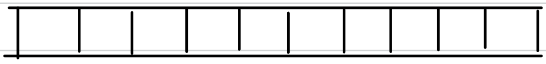
$$= \frac{8!}{4!\,4!}.$$

Note: No. of k-combinations from a set of n objects

=

No. of permutations of n objects where there are k indistinguishable objects of type 1 and n−k indistinguishable objects of type 2.

Ex 17: A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes
(a) are there in total?
(b) contain exactly two heads?
(c) contain at most three tails?
(d) contain the same number of heads and tails?

Soln: Consider 

a) Total possible outcomes = Total no. of ways we can fill this 10 positions by Head (H) or tail (T).

$= 2^{10}$ (one such outcome is HTHTTHTHTT).

b) Total possible outcomes containing exactly two heads = No. of ways of selecting 2 positions of 10 positions.

$= \binom{10}{2} = \frac{10!}{2!\,8!} = 45.$

c) Total possible outcomes containing atmost 3 tails = possible outcomes containing 3-tails

+

possible outcomes containing 2-tails

+

possible outcomes containing 1-tail.

+

possible outcomes containing 0-tails

$= \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0}.$

d) possible outcomes containing same no. of heads and tails $= \binom{10}{5}$.

Ex 18: In how many of the possible arrangements of the word TALLAHASSEE have no adjacent A's?

Soln: The word T A L L A H A S S E E

1 - T 3 - A 2 - L 1 - H 2 - S 2 - E	__ T __ L __ L __ H __ S __ S __ E __ E __ A's can take only blank (__) positions, Since there are 9 blank positions, three can be selected in $\binom{9}{3}$ ways.

And other letters T L L H S S E E can be arrangement in

$$\frac{8!}{2!\,2!\,2!}.$$

By product rule, required possible arrangements are

$$\binom{9}{3} \times \frac{8!}{2!\,2!\,2!}.$$

Ex 19: Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection results in a hand with no clubs?

Soln: There are 13 clubs.

possible ways of selecting 5 cards from 52 cards with no clubs

is same as

no. of ways of selecting 5 cards from $(52-13) = 39$ cards, which is

$$\binom{39}{5}.$$

Ex 20: In how many ways can 12 different books be distributed among four children so that
(a) each child gets three books?
(b) the two oldest children get four books each and the two youngest get two books each?

Soln: There are 12 books and 4 Children (say C_1, C_2, C_3, C_4)

a) Three books can be given to C_1 in $\binom{12}{3}$ ways.

After this event, only 9 books remain. Then 3 books can be given to C_2 in $\binom{9}{3}$ ways.

Later, 3 books can be given to C_3 and C_4 in $\binom{6}{3}$ and $\binom{3}{3}$ ways respectively.

Thus, total no. of ways 12 books can be distributed among 4 children such that each get 3 books are

$$\binom{12}{3} \times \binom{9}{3} \times \binom{6}{3} \times \binom{3}{3}.$$

b) Let C_1 and C_2 be two oldest, and C_3 and C_4 be two youngest.

$\binom{12}{4} \times \binom{8}{4} \times \binom{4}{2} \times \binom{2}{2}$ ways we can distribute books such that C_1 and C_2 get 4 books each, and C_3 and C_4 get 2 books each.

properties of binomial co-efficient

i) $\binom{n}{0} = \binom{n}{n} = 1$

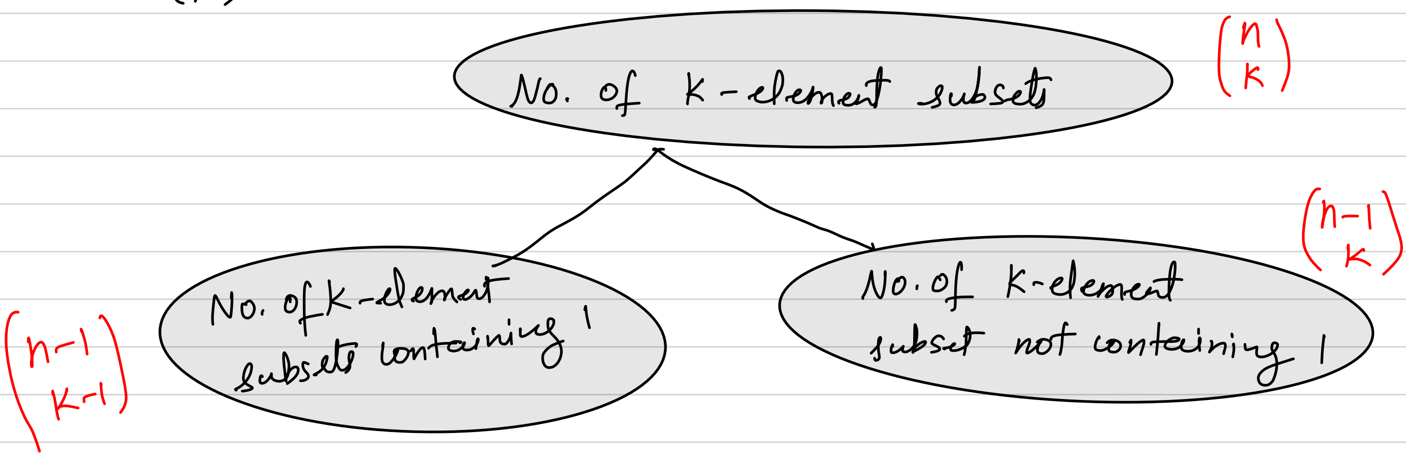
ii) $\binom{n}{1} = \binom{n}{n-1} = n$

iii) $\binom{n}{k} = \binom{n}{n-k}$

iv) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pf of iv) :

$\binom{n}{k} \rightarrow$ No. of k-element subsets of $\{1,2,3,\dots,n\}$



k-element subsets containing 1, Since it already contain 1, we have to choose $(k-1)$ from $(n-1)$ elements that can be done in $\binom{n-1}{k-1}$ ways.

If k-el. subsets donot contain 1, then k have to be chosen in $(n-1)$ elements. The no. of ways this can be done is $\binom{n-1}{k}$ ways.

Thus $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

The Pascal's triangle

1 $\quad 1$

2 $\quad \binom{1}{0}\,1 \qquad 1\,\binom{1}{1}$

3 $\quad \binom{2}{0}\,1 \qquad \binom{2}{1}\,2 \qquad \binom{2}{2}\,1$

4 $\quad \binom{3}{0}\,1 \qquad \binom{3}{1}\,3 \qquad \binom{3}{2}\,3 \qquad \binom{3}{3}\,1$

5 $\quad \binom{4}{0}\,1 \qquad \binom{4}{1}\,4 \qquad \binom{4}{2}\,6 \qquad \binom{4}{3}\,4 \qquad \binom{4}{4}\,1$

6 $\quad \binom{5}{0}\,1 \qquad \binom{5}{1}\,5 \qquad \binom{5}{2}\,10 \qquad \binom{5}{3}\,10 \qquad \binom{5}{4}\,5 \qquad \binom{5}{5}\,1$

From Pascal triangle and induction it is easy to see that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

v) $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

or

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Pf : $\binom{n}{k}$ is the no. of k-el. subsets of $\{1,2,3,\dots,n\}$

$$\sum_{k=0}^{n} \binom{n}{k} = \text{no. of all subsets of } \{1,2,3,\dots,n\}$$

$= \text{no. of maps } \big(\{1,2,3,\dots,n\} \to \{0,1\} \big)$

(elements belong to subset maps to 1
others maps to 0)

$= 2^n.$

Thm (Binomial Thm)

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} a^0 b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k .$$

pf : $(a+b)^n = \underbrace{(a+b)\,(a+b)\,\cdots\,(a+b)}_{n \text{ factors}}$

Compute the co-eff in front of $a^{n-k} b^k$

It is equal to the no. of k-element subsets of $\{1,2,3,\dots,n\}$

$$= \binom{n}{k}$$

Property (v) can be proved using this Thm,

put $a=1$, $b=1$, we get

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} .$$

(vi) put $a=1$, $b=-1$

$$(1-1)^n = 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots (-1)^n \binom{n}{n} .$$

Multinomial theorem

For +ve integers n, t, the co-efficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\, n_2! \cdots n_t!}$$

$0 \le n_i \le n$, $i = 1, 2, 3, \dots t$, and $n_1 + n_2 + \dots + n_t = n$.

Ex 21: Obtain the co-eff of $a^5 b^2$ in the expansion of $(2a-3b)^7$.

Soln: Let $x = 2a$
$y = -3b$

The co-eff of $x^5 y^2$ in the exp $(x+y)^7$ is $\binom{7}{2}$

and $\binom{7}{2} x^5 y^2 = \binom{7}{2} (2a)^5 (-3b)^2$

$= \binom{7}{2} 2^5 (-3)^2 a^5 b^2$

Thus, co-eff of $a^5 b^2$ in the exp of $(2a-3b)^7$ is $\binom{7}{2} 2^5 (-3)^2$

Ex 22: Obtain co-eff of $a^2 b^3 c^2 d^5$ in the exp. of $(a+2b-3c+2d+5)^{16}$.

Soln: Let $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, $x_5 = 5$

Coefficient of $x_1^2 x_2^3 x_3^2 x_4^5 x_5^4$

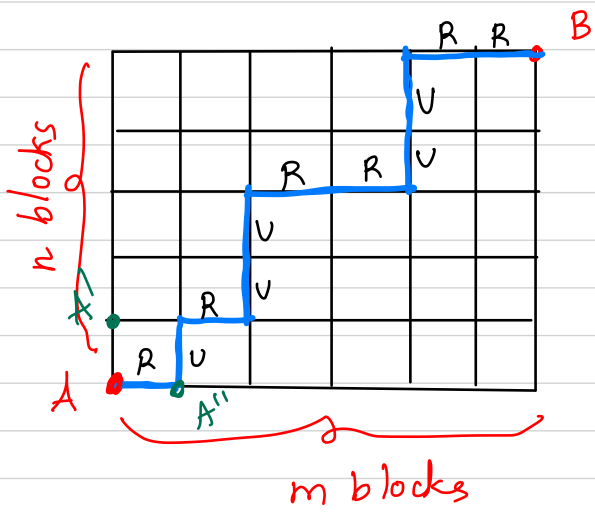
and $\left(\frac{16!}{2!\,3!\,2!\,5!\,4!}\right) x_1^2 x_2^3 x_3^2 x_4^5 x_5^4$

$= \frac{16!}{2!\,3!\,2!\,5!\,4!} a^2\, 2^3 b^3\, (-3)^2 c^2\, 2^5 d^5\, 5^4$

$= \frac{16!\ 8 \cdot 9 \cdot 32 \cdot 5^4}{2!\,3!\,2!\,5!\,4!} a^2 b^3 c^2 d^5$

Coeff. of $a^2 b^3 c^2 d^5$

Cab driver problem



In how many ways can we get from A to B?

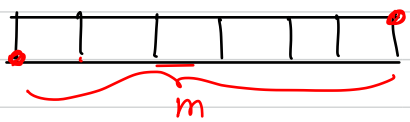
path is made up of individual steps going one unit to the right or one unit upward.

$P_{m,n}$ — The no. of paths from A to B.

$$P_{A\to B} = P_{A'\to B} + P_{A''\to B}$$

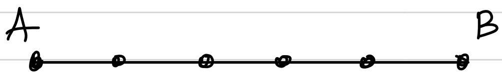
Ex: If $n=1$

$$P_{m,1} = m+1$$



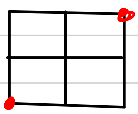
If $n=0$

$$P_{m,0} = 1$$



If $m=n=2$

$$P_{2,2} = 6$$



Soln 1 : Recurrence relation on $P_{m,n}$:

$$P_{m,n} = P_{m,n-1} + P_{m-1,n}.$$

It satisfy recurrence relation as binomial coeff. with same IC's

$$\therefore \quad P_{m,n} = \binom{m+n}{n}$$

We have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

put $n = m+n$, $k=n$

$$\binom{m+n}{n} = \binom{m+n-1}{n-1} + \binom{m+n-1}{n}$$

Soln 2 : Paths $A\to B$ correspond to sequences of R's and U's of length $m+n$ with m R's and n U's.

For instance $\underbrace{\text{RURUURRUURR}}_{m+n}$.

Select n letters. No. of ways of doing this is $\binom{m+n}{n}$.

or

No. of ways this $m+n$ letters can be arranged is

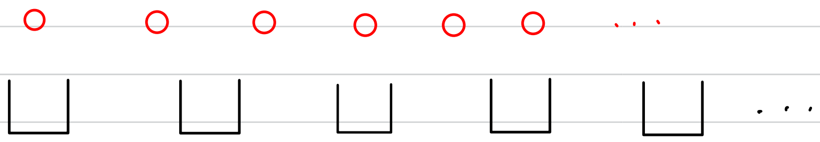
$$\frac{(m+n)!}{n!\ m!} = \binom{m+n}{n}.$$

K- combination with repetition

Consider k balls (identical) and n boxes (distinct).

k balls

n boxes



In how many ways we can put k balls in n boxes?

i) If maximum capacity of each box is 1, the answer is

$$\begin{cases} \binom{n}{k}, & \text{if } k \le n \\ 0, & \text{otherwise.} \end{cases}$$

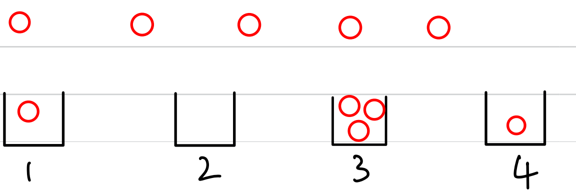
ii) What if there are no restrictions on the capacity of the boxes?

If this the case, then no. of ways can we put k balls in n boxes is equal to the no. of k-combinations of a set with n objects when repetition of elements is allowed.

For instance, let us consider $k=5$ balls and $n=4$ boxes

5 balls

4 boxes



Let us represent as follows:

$$\circ \;|\; |\; \circ\; \circ\; \circ \;|\; \circ$$

balls

Separator between 2 boxes

Configuration of balls in boxes corresponds to the configuration of bars and balls.

The no. of ways 5 balls and $4-1=3$ bars can be configured is

$$\frac{(5+3)!}{5!\;3!} \quad \text{or} \quad \binom{5+3}{5} \text{ ways}$$

If there are k balls and n boxes, then corresponding k balls and $(n-1)$ bars can be configured in $\binom{n+k-1}{k}$ ways.

Thm: k identical balls can be put into n distinct boxes in $\binom{n+k-1}{k}$ ways.

Equivalently, the number of k-combinations from a set with n elements when repetition of elements is allowed is $\binom{n+k-1}{k}$.

Ex 23 : Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Soln: There are 4 different kinds of cookies. Let $n=4$.

6 cookies can be chosen in $\binom{n+k-1}{k}$ ways

Here $k=6$.

Ans: $\binom{4+6-1}{6} = \binom{9}{6}$

Ex 24: Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \quad \text{where } x_i \geq 0 \text{ for all } 1 \leq i \leq 4.$$

Soln: One soln of this eqn is $x_1 = 2$, $x_2 = 4$, $x_3 = 0$, $x_4 = 1$, this is different from $x_1 = 0$, $x_2 = 4$, $x_3 = 2$, $x_4 = 1$.

A possible interpretation is the no. of ways we can put 7 balls in 4 boxes.

$$\therefore \text{ answer is } \binom{4+7-1}{7} = \binom{10}{7}$$

Remark: The no. of integer solns of the eqn

$$x_1 + x_2 + \cdots + x_n = K, \quad x_i \geq 0, \ 1 \leq i \leq n$$

is $\binom{n+K-1}{K}$.

Ex 25: How many integer solns are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10, \quad x_i \geq 0, \quad 1 \leq i \leq 6$$

Soln: Note that there is a correspondence between the non negative integer solns of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10 \quad \text{———} \ (1)$$

and the integer soln of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 9, \quad x_i \geq 0, \ 1 \leq i \leq 7, \quad \text{———} \ (2)$$

The no. of non negative solns of (2) is

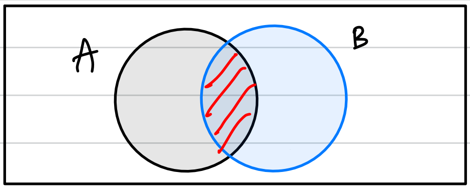
$$\binom{7+9-1}{9} = \binom{15}{9}.$$

$n = 7$
$K = 9$

Principle of inclusion - exclusion

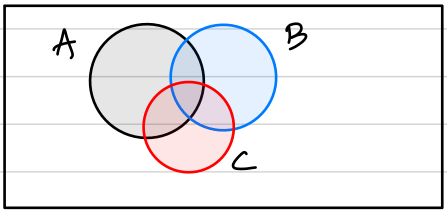
1) Let A and B be two finite sets.

No. of elements in $A \cup B$, i.e $|A \cup B|$ is



$$|A \cup B| = |A| + |B| - |A \cap B|$$

2) Let A, B, C be finite sets



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Thm: Let $A_1, A_2, \ldots, A_n$ be finite subsets of U. Then

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n|$$

Pf: Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$

How many times does it appear in RHS?

Suppose $x \in A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_p$ and $x \notin A_{p+1}, A_{p+2}, \ldots A_n$

This element is counted $\binom{p}{1}$ times by $\sum |A_i|$

It is counted $\binom{p}{2}$ times by $\sum |A_i \cap A_j|$

In general, it is counted $\binom{p}{m}$ times by the summation involving m of the sets A_i.

$\therefore$ x contributes to the RHS with multiplicity

$$p - \binom{p}{2} + \binom{p}{3} - \binom{p}{4} + \cdots + (-1)^{p-1}\binom{p}{p} \quad \text{——} (*)$$

$$(x+y)^p = x^p + \binom{p}{1} x^{p-1} y + \binom{p}{2} x^{p-2} y^2 + \cdots + \binom{p}{p} y^p$$

put $x = 1$, $y = -1$

$$0 = 1 - \binom{p}{1} + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^p \binom{p}{p}$$

$$\begin{aligned}
(*) &= 1 - 1 + p - \binom{p}{2} + \binom{p}{3} - \cdots + (-1)^{p-1}\binom{p}{p} \\
&= 1 - \left(1 - p + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^p \binom{p}{p}\right) \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

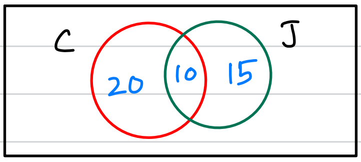
This proves the Thm.

Ex : In a class of 50 college freshmen, 30 are studying C++, 25 are studying Java, and 10 are studying both languages. How many freshmen are (a) not studying C++ (b) studying either computer language? (c) studing C++ but not Java

Soln: Let C be set of students studying C++.
Let J " " JAVA

Given $|C| = 30$, $|J| = 25$ and $|C \cap J| = 10$

Total no. of students $|U| = 50$



a) No. of students not studying C++ is $|\bar{C}| = |U - C|$
$|50 - 30| = 20$

b) No. of students studying either language is

$$|C \cup J| = |C| + |J| - |C \cap J|$$

$$= 30 + 25 - 10 = 45$$

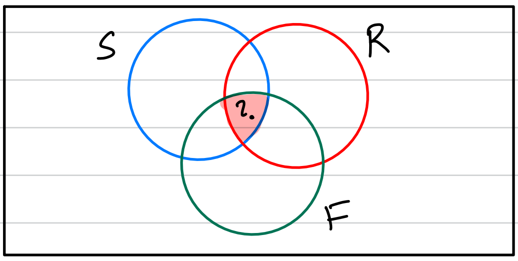
c) Studying C++ but not JAVA is $|C| - |C \cap J| = 20$.

Ex : A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Soln: $|S| = 1232$, $|R| = 114$, $|F| = 879$

$|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$

$|S \cup F \cup R| = 2092$, $|S \cap F \cap R| = ?$



We have

$$|S \cup F \cup R| = |S| + |R| + |F| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$\Rightarrow \quad 2092 = 1232 + 114 + 879 - 103 - 23 - 14 + |S \cap F \cap R|$$

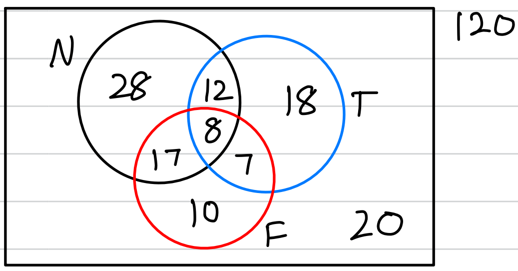
$$\Rightarrow \quad |S \cap F \cap R| = 2092 - 1232 - 114 - 879 + 103 + 23 + 14$$
$$= 7$$

Ex : In a survey of 120 people, it was found that: 65 read Newsweek magazine, 45 read Time, 42 read Fortune, 20 read both Newsweek and Time, 25 read both Newsweek and Fortune, 15 read both Time and Fortune, 8 read all three magazines. (a) Find the number of people who read at least one of the three magazines. (b) Find the number of people who read exactly one magazine.

Soln: $|U| = 120$, $|N| = 65$, $|T| = 45$, $|F| = 42$

$|N \cap T| = 20$, $|N \cap F| = 25$, $|T \cap F| = 15$, $|N \cap T \cap F| = 8$

a) $$|N \cup T \cup F| = |N| + |T| + |F| - |N \cap T| - |N \cap F| - |T \cap F| + |N \cap T \cap F|$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8$$
$$= 100$$



b) No. of people who read exactly one magazine is

$$28 + 18 + 10 = 56$$

Ex : A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?.

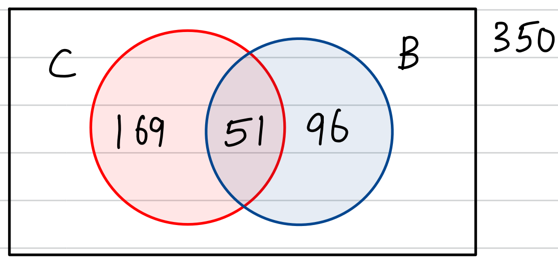
Soln: No. of applications received $|U| = 350$. Let $|C|$ be no. of applicants majored in CS, and $|B|$ be no. of applicants majored in Business

No. of applicants majored neither in CS nor in business is

$$\overline{|C \cup B|} = |U - (C \cup B)|$$

$$= 350 - (220 + 147 - 51)$$

$$= 34$$



Derangements

Scenario

n people leave their umbrella in a stand enter a restaurant for a dinner. After the dinner each of them gets back a random umbrella and leaves the restaurant.

What is the probability that every one will get someone else's umbrella?

Answer: The number of ways the umbrellas can be arranged so that there is no umbrellas in its original position divided by $n!$

A derangement is a permutation of objects with no fixed points (i.e, never maps an object into itself).

Ex: Let us consider 3 objects $\{A, B, C\}$.

The possible permutations are	In these permutations, derangements are
ABC i.e there are	
ACB 6 permutations	BCA
BAC	CAB
BCA	
CAB	There are only 2 derangements.
CBA	

Theorem: The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

pf: We compute the number of permutation with fixed pts.

Let $A_i = \{$permutations mapping i to $i\}$

where $i \in \{1, 2, \dots, n\}$

$A_i \cap A_j = \{$ permutations mapping i to i and $j \to j \}$

$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = \{$ permutations mapping $i_1 \to i_1,\ i_2 \to i_2,\ i_k \to i_k \}$

Clearly, $|A_i| = (n-1)!$

$$|A_i \cap A_j| = (n-2)!$$

$$\vdots$$

$$|A_{j_1} \cap A_{j_2} \cap A_{j_3} \cap \cdots \cap A_{j_k}| = (n-k)!$$

The number of derangements of a set with n eles is

$$D_n = n! - |A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n|$$

By inclusion and exclusion principle

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = n(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n-1}\,0!$$

$$= n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\frac{n!}{n!}$$

$$\therefore\ D_n = n! - \left(n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\frac{n!}{n!}\right)$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

$$= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

It is also denoted by $!n$

Note: 1) $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{1}{k!} = e^{-1}$

2) # of derangements, $D_n \approx \dfrac{n!}{e}$

Thus, the probability that everyone gets someone else's umbrella is $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \approx \frac{1}{e}$

Ex1: How many derangements of $\{1,2,3,4,5,6\}$ begin with the integers 1, 2 and 3 in some order?

Soln: Integers 1, 2 and 3 will get one of the places in the first 3 places, and 2, 4 and 6 occupies one of the last 3 places.

Thus, # of derangements $= \left(\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) \times 3!\right)^2$

$= 4$

Derangements are
231564
231645
312564
312645

Ex2: How many derangements of $\{1,2,3,4,5,6\}$ end with the integers 1, 2 and 3, in some order.

Soln: Integers 1, 2 and 3 occupies one of last 3 places, and 4, 5 and 6 occupies one of first 3 places.

$\therefore$ # of derangements $= 3! \times 3! = 36$

Ex3: [Cave problem]: There are 10 couples, each with a single child. The 10 children get lost in a dark cave. Each couple goes into the cave and pulls out one child. How many ways are there in which at least one child get paired with their parents? What is the probability of that occuring?

Soln: The number of ways of NO child getting paired correcty is $D_{10} = !10$.

Thus, the number ways for at least one child to get paired is $10! - !10 = 10! - 10! \sum_{k=0}^{10} (-1)^k \frac{1}{k!}$

$$\approx 10! \left(1 - \frac{1}{e}\right)$$

$$= 2293839$$

The probability is $\frac{10! - !10}{10!} \approx 1 - \frac{1}{e} = 0.63212056$

i.e there is about 63.2% chance of at least one child being matched up with their parents.

Sequence: It is a real / complex valued function on the positive integers.

We use the notation a_n to denote the image of the integer n.

We call a_n n^{th} term of the sequence (a_n).

Ex: Consider the sequence (a_n), where $a_n = \frac{1}{n}$.

The list of the terms of the sequence,

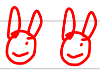
$a_1, a_2, a_3, \dots$ are

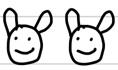
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

Linear recurrence relation

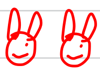
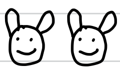
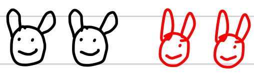
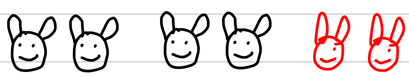
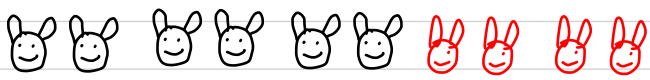
Fibonacci's problem about rabbits

Liber Abaci
1202 (Leonardo from Pisa, known as Fibonacci)

Notation:  – baby rabbits

 – adult rabbits

Assume that after one month baby rabbits will be fully grown up and every other month adult rabbits give birth to baby rabbits.

month	rabbits	No. of rabbit pairs
1		1
2		1
3		2
4		3
5		5
⋮		

Question: How many pairs of rabbits will you have after a year?

Let F_n be the no. of pairs of rabbits during n^{th} month.

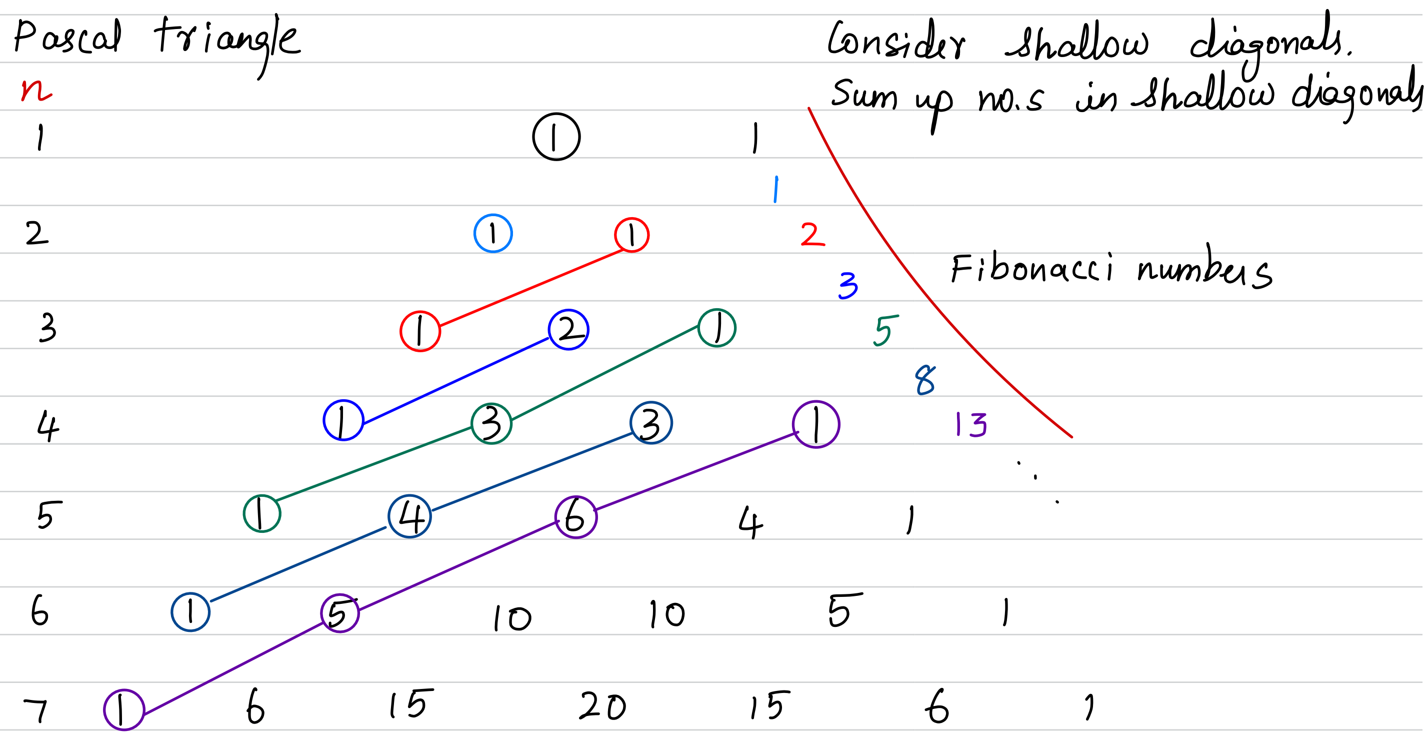
Then $\underbrace{F_n = F_{n-1} + F_{n-2}}_{\text{Recurrence relation for } F_n}$, $\underbrace{F_1 = 1, \quad F_2 = 1}_{\text{Initial conditions}}$

List of terms of the sequence F_n is

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \underline{144} = F_{12}, \cdots$

This sequence is called Fibonacci sequence.

Fibonacci numbers and Pascal triangle



Here $F_2 = \binom{1}{0}$, $F_3 = \binom{2}{0} + \binom{1}{1}$

$F_4 = \binom{3}{0} + \binom{2}{1}$, $F_5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2}$

$F_6 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}$, $F_7 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$

In general, $F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$

Moreover, $F_n = F_{n-1} + F_{n-2}$ $\left(\because \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \right)$

and $F_1 = F_2 = 1$

Ex : Find the no. of bit string of length n that do not have two consecutive 0's

Soln: Let a_n denote the number of bit strings of length n that donot have consecutive 0's.

No. of bit string of length n with no consecutive 0's.
=
No. of such bit string ending with 0.
+
No. of such bit string ending with 1.

The bit strings of length n ending with 1 that donot have consecutive 0's
= The bit strings of length $(n-1)$ with no consecutive 0's.

The bit strings of length n ending with 0 that donot have consecutive 0's
= The bit strings of length $(n-2)$ with no consecutive 0's.
(because $(n-1)^{st}$ bit must be 1).

$\therefore \quad a_n = a_{n-1} + a_{n-2}$

For instance, when $n=4$, the bits with no. consecutive 0's are

0101	Consider the strings that end with 1, that is Equal to no. of bit strings of length 3.
0110	010
0111	011
1010	101
1011	110
1101	111
1110	
1111	

Consider the strings that end with 0, that is equal to strings of length 2.

01
10
11

Linear Recurrence relation

Defn: A linear homogenous recurrence relation of degree k with constant coefficients is a recurrance relation of the form

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad \text{——} \quad (*)}$$

- linear because $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ appear in seperate terms and to the 1st power.

- Homogeous because no terms occur that are not multiple of the a_j's.

- degree k because a_n is expressed in terms of the previous k terms of the sequence

- Coefficients $c_1, c_2, \ldots, c_k$ are constant

Remember: A sequence defined by a linear recurrence relation is uniquely determined by its 1st k terms $a_0, a_1, a_2, \ldots, a_{k-1}$.

And, recurrence relation (*) along with k IC's is called recursive definition of the sequence a_n.

Ex: 1) $F_n = F_{n-1} + F_{n-2}$ is Linear homo. RR of deg 2

2) $P_n = (1.11) P_{n-1}$ " deg 1.

3) $a_n = a_{n-1} + a_{n-2}^2$, is not linear.

4) $H_n = 2H_{n-1} + 1$, is Linear but not homogenous

5) $b_n = c\, b_{n-6}$ Linear homo. RR of deg 6

Solution of (*)

A sequence $a_0, a_1, a_2, \ldots$ is said to be soln of (*) if it satisfies (*).

Note: If $a_0, a_1, a_2, \ldots$ and $a_0', a_1', a_2', \ldots$ satisfy the relatn (*), then

$a_0+a_0', \; a_1+a_1', \; a_2+a_2', \ldots$ also satisfies (*).

and so does $ca_0, ca_1, ca_2, \ldots$.

Method to Solve (*)

when $K=1$,

$$a_n = c\, a_{n-1}$$

$$\Rightarrow \quad a_1 = c\, a_0$$

$$a_2 = c\, a_1 = c^2 a_0$$

$$a_3 = c a_2 = c^3 a_0$$

$$\vdots$$

$$\therefore \quad a_n = c^n a_0 \quad \text{— geometric progression.}$$

Assume that $a_n = r^n$ is a soln of (*).

Then, this has to satisfy (*).

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + c_3 r^{n-3} + \cdots + c_K r^{n-K}$$

$\div$ both sides by r^{n-K}

$$\frac{r^n}{r^{n-K}} = c_1 \frac{r^{n-1}}{r^{n-K}} + c_2 \frac{r^{n-2}}{r^{n-K}} + \cdots + c_K \frac{r^{n-K}}{r^{n-K}}$$

$$\Rightarrow \quad r^K = c_1 r^{K-1} + c_2 r^{K-2} + \cdots + c_K$$

Then r is a root of the characteristic eqn

$$\boxed{t^K = c_1 t^{K-1} + c_2 t^{K-2} + \cdots + c_K.}$$

Consider the recurrence relation of degree 2.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \text{———} \quad (**)$$

characteristic eqn

$$t^2 = c_1 t + c_2 \quad \text{———} \quad (\#)$$

i) Suppose that λ and μ are roots of $(\#)$ $(\lambda \neq \mu)$

Solns of $(**)$ are

$$1, \lambda, \lambda^2, \lambda^3, \ldots \quad \text{and} \quad 1, \mu, \mu^2, \mu^3, \ldots$$

General soln: $a_n = C\lambda^n + D\mu^n$

ii) If $\lambda = \mu$ (say r), then soln are

$$1, \lambda, \lambda^2, \lambda^3, \ldots \quad ((\lambda^n))$$

and

$$0, \lambda, 2\lambda^2, 3\lambda^3, \ldots \quad ((n\lambda^n))$$

General soln is

$$a_n = C\lambda^n + D n\lambda^n$$

Thm: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ be an recurrence relatn of degree k.

Suppose the characteristic equation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

has t distinct roots $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so $m_i \geq 1$ and $m_1 + m_2 + \cdots + m_t = k$.

Then a sequence $\{a_n\}$ is a soln of the recurrence relation iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2 + \cdots + \alpha_{1,m_1-1} n^{m_1-1})\lambda_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1})\lambda_2^n$$
$$+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1})\lambda_t^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Ex: Suppose 2, 2, 2, 5, 5 and 9 are roots of the characteristic eqn of a linear recurrence relation. What is the form of general soln

Soln: Independent solns of the recurrence relation are

$$2^n,\ n2^n,\ n^2 2^n,\ 5^n,\ n5^n \text{ and } 9^n.$$

General soln

$$A_1 2^n + A_2 n2^n + A_3 n^2 2^n + A_4 5^n + A_5 n5^n + A_6 9^n$$

Ex1: Find the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Soln: characteristic eqn.

$$t^2 = t + 2$$

$$\Rightarrow t^2 - t - 2 = 0$$

$$\Rightarrow t = \frac{1 \pm \sqrt{1+8}}{2}$$

$$= \frac{1 \pm 3}{2}$$

$$\therefore t = 2, -1.$$

If $ax^2 + bx + c = 0$,

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sequence $\{a_n\}$ is the soln iff

$$a_n = A_1 2^n + A_2 (-1)^n;$$

A_1 and A_2 are some constant.

From IC:

$$a_0 = 2 = A_1 + A_2$$
$$a_1 = 7 = 2A_1 - A_2$$

$$\Rightarrow 9 = 3A_1 \quad \text{or} \quad A_1 = 3$$

and $A_2 = -1$

$\therefore$ The soln of the recurrence relation with IC's is

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Ex2: Find an explicit formula for the Fibonacci no.s.

Soln: Fibonacci sequence F_n satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{and IC's are} \quad F_0 = 0,\ F_1 = 1$$

Characteristic eqn.

$$t^2 = t + 1$$

$$\Rightarrow \quad t^2 - t - 1 = 0$$

$$t = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\Rightarrow \quad t = \frac{1 \pm \sqrt{5}}{2}$$

i.e $\quad t = \dfrac{1+\sqrt{5}}{2},\ \dfrac{1-\sqrt{5}}{2}$

$\therefore$ Fibonacci no.s are given by

$$F_n = A_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$ A_1 and A_2 are constants.

IC's are $F_0 = 0$, $F_1 = 1$, we have

$$F_0 = A_1 + A_2 = 0 \quad \text{—— } (1)$$

$$F_1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad \text{—— } (2)$$

$\boxed{A_1 = -A_2}$ (from (1))

Sub in (2), $\quad A_1\left(\dfrac{1+\sqrt{5}}{2}\right) - A_1\left(\dfrac{1-\sqrt{5}}{2}\right) = 1$

$$\Rightarrow \quad A_1(\sqrt{5}) = 1 \quad \text{or} \quad \boxed{A_1 = \frac{1}{\sqrt{5}}}$$

$\therefore$ The Fibonacci no.s are given by

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Ex 3: Solve $a_n = 6a_{n-1} - 9a_{n-2}$ IC's, $a_0 = 1$ $a_1 = 6$.

Soln: Char. eqn

$$t^2 = 6t - 9$$

$$\Rightarrow t - 6t + 9 = 0$$

$$\Rightarrow (t-3)^2 = 0$$

$$\therefore t = 3, 3$$

If $\{a_n\}$ is a seq that satisfy given recurrence relatn,

$$a_n = A_1 3^n + A_2 n 3^n \quad \text{——— (1)}$$

IC's are $a_0 = 1$, $a_1 = 6$

Sub in (1), $a_0 = 1 = A_1$

$$a_1 = 6 = A_1 3 + A_2 3$$

$$\Rightarrow A_2 = 1$$

$\therefore$ The soln to the recurrence relation with IC's is

$$a_n = 3^n + n3^n.$$

Ex 4: Solve: $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$,

IC's $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.

Soln: Characteristic eqn

$$t^3 = 6t^2 - 11t + 6$$

$$\Rightarrow t^3 - 6t^2 + 11t - 6 = 0 \quad \text{——— (1)}$$

$t=1$ satisfies ①.

We use synthetic division method

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

① $\Rightarrow$

$$(t-1)\left(t^2-5t+6\right)=0$$

Consider $t^2-5t+6=0$,

$$t=2,\ 3$$

$\therefore$ roots of ① are $1, 2, 3$

Soln: $a_n = A_1 1^n + A_2 2^n + A_3 3^n$.

Use IC's to find constants A_1, A_2 and A_3.

$a_0 = 2 = A_1 + A_2 + A_3$

$a_1 = 5 = A_1 + 2A_2 + 3A_3$

$a_2 = 15 = A_1 + 4A_2 + 9A_3$

Solving three eqns for A_1, A_2 and A_3, we get

$A_1=1$, $A_2=-1$, and $A_3=2$.

Soln to the given recurrence relation with IC's is

$$a_n = 1 + (-1)\,2^n + 2\cdot 3^n$$

Recurrence relation with complex roots:

Ex5: Solve $a_n = 2a_{n-1} - 2a_{n-2}$; $a_0=1$, $a_1=2$

Soln: Characteristic Eqn

$$t^2 = 2t-2 \Rightarrow t^2-2t+2=0$$

Roots are $1 \pm i$

Solns are $(1+i)^n$ and $(1-i)^n$

Convert $1+i$ to polar form, we get

Magnitude, $r = \sqrt{1^2+1^2} = \sqrt{2}$

Argument, $\theta = \tan^{-1}\frac{1}{1} = \pi/4$

So $(1+i) = \sqrt{2}\, e^{i\pi/4}$

Thus, $(1+i)^n = (\sqrt{2})^n e^{in\pi/4} = (\sqrt{2})^n \left(\cos(n\pi/4) + i \sin(n\pi/4)\right)$

||ly $(1-i)^n = (\sqrt{2})^n \left(\cos(n\pi/4) - i \sin(n\pi/4)\right)$

If $(1+i)^n$ satisfy the given recurrence relation, then $(\sqrt{2})^n \cos(n\pi/4)$ and $(\sqrt{2})^n \sin(n\pi/4)$ also satisfy.

Thus, GS is

$$a_n = (\sqrt{2})^n \left(A_1 \cos(n\pi/4) + A_2 \sin(n\pi/4)\right)$$

$a_0 = 1 \Rightarrow 1 = A_1$

$a_1 = 2 \Rightarrow 2 = \sqrt{2}\left(\cos \pi/4 + A_2 \sin \pi/4\right)$

$\Rightarrow 2 = 1 + A_2 \Rightarrow A_2 = 1$

Thus, $a_n = (\sqrt{2})^n \left(\cos(n\pi/4) + \sin(n\pi/4)\right)$

In general,

Suppose $(\alpha \pm i\beta)$ are Characteristic roots of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. Then

General soln is $a_n = r^n \left(A_1 \cos n\theta + A_2 \sin n\theta\right)$

where $r = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$.

Linear non homogenous recurrence relations with constant coefficients

A recurrence relation is of the form

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).} \quad \text{— } (\$)$$

where $c_1, c_2, \dots, c_k$ are real no.s and $F(n) \neq 0$ (a fn of n)

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad \text{———— } (₹)$$

is called associated homogenous recurrence relation.

Thm: General soln of linear non homogenous relation $(\$)$ is of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $a_n^{(h)}$ is a soln of the associated homogenous recurrence relation, and

$a_n^{(p)}$ is a particular soln of the non homogenous recurrence relation $(\$)$.

Pf: Because $\{a_n^{(p)}\}$ is a particular soln of $(\$)$, we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n)$$

Now suppose $\{b_n\}$ is a second soln of $(\$)$, Then

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n)$$

Subtracting 1st eqn from the 2nd,

$$b_n - a_n^{(p)} = c_1\left(b_{n-1} - a_{n-1}^{(p)}\right) + c_2\left(b_{n-2} - a_{n-2}^{(p)}\right) + \cdots$$
$$+ \; c_k\left(b_{n-k} - a_{n-k}^{(p)}\right)$$

It follows that $\{b_n - a_n^{(p)}\}$ satisfies $(₹)$, associated homo. recurrence relatn.

If $b_n - a_n^{(p)} = a_n^{(h)} \Rightarrow b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

Thm: Suppose

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \text{ is}$$

linear non homogenous recurrence relation, and

$$F_n = (b_0 + b_1 n + b_2 n^2 + \cdots + b_t n^t) s^n$$

where $b_0, b_1, \ldots, b_t$ and s are real no.s.

i) When s is not a root of the characteristic eqn of the associated linear homo. recurrence relatn, then particular soln is of the form

$$(p_0 + p_1 n + p_2 n + \cdots + p_t n^t) s^n$$

ii) When s is a root of the characteristic eqn of multiplicity m, $a_n^{(p)}$ is of the form

$$n^m (p_0 + p_1 n + \cdots + p_t n^t) s^n.$$

Ex: If the recurrence relation is

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

Associated homogenous recurrence relation is

$$a_n = 6a_{n-1} - 9a_{n-2}$$

Char. Eqn

$$r^2 - 6r + 9 = 0$$

root are $r = 3, 3$

i) when $F(n) = 3^n$,

$$a_n = 6a_{n-1} - 9a_{n-2} + 3^n$$

$\therefore$ particular soln is of the form

$$n^2 p_0 3^n$$ ($\because$ 3 is the root of the char. eqn with multiplicity 2)

ii) when $F(n) = n^2 2^n$, particular soln is of the form

$$(P_0 + P_1 n + P_2 n^2)\, 2^n$$

iii) when $F(n) = (n^2+1)\, 3^n$, particular soln is of the form

$$n^2 \left(P_0 + P_1 n + P_2 n^2 \right) 3^n$$

Ex: Solve

$$a_n = 3a_{n-1} + 2n \quad , \quad \text{IC is } a_0 = 3.$$

Soln: General soln of the recurrence relatn

$$a_n = a_n^{(p)} + a_n^{(h)}.$$

To find $a_n^{(h)}$

Consider the associated homogenous recurrence relation,

$$a_n = 3a_{n-1}$$

Char. eqn

$$t = 3$$

$$\therefore \boxed{a_n^{(h)} = c\, 3^n.}$$

To find $a_n^{(p)}$.

Here $F(n) = 2n$, therefore $a_n^{(p)}$ is of the form

$$a_n^{(p)} = (P_0 + P_1 n)$$

Then eqn, $a_n = 3a_{n-1} + 2n$ becomes

$$P_0 + P_1 n = 3\left(P_0 + P_1 (n-1) \right) + 2n$$

$$\Rightarrow P_0 + P_1 n = 3P_0 + 3P_1 n - 3P_1 + 2n$$

$$\Rightarrow (P_1 - 3P_1 - 2)n + (P_0 - 3P_0 + 3P_1) = 0$$

$$\Rightarrow -2P_1 - 2 = 0 \ , \quad -2P_0 + 3P_1 = 0$$

Solve for p_0 and p_1

$$p_1 = -1, \qquad p_0 = -3/2$$

$$\therefore \quad \boxed{a_n^{(p)} = -3/2 - n}$$

Thus,

$$a_n = c\,3^n - 3/2 - n$$

Use IC $a_0 = 3$ to find c.

$$a_0 = 3 = c - 3/2$$

$$\text{or} \quad c = 3 + 3/2 = 9/2$$

$\therefore$ Soln of the given recurrence relation with IC is

$$\boxed{a_n = \frac{9}{2} \cdot 3^n - 3/2 - n}$$

Ex: Find all solns of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

Soln: Let a_n be all solns of the given recurrence relan.

Then

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

To find $a_n^{(h)}$.

Consider associated homo. recurrence relan,

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Char. eqn is

$$t^2 = 5t - 6$$

$$\text{or} \quad t^2 - 5t + 6 = 0$$

$$\text{roots are} \quad t = 2, 3$$

$$\therefore \quad \boxed{a_n^{(h)} = C\, 2^n + D\, 3^n.}$$

To find $a_n^{(p)}$

Here $F(n) = 7^n$. $\therefore$ $a_n^{(p)}$ is of the form,

$$a_n^{(p)} = P_0\, 7^n,$$

it follows that

$$a_n^{(p)} = 5\, a_{n-1}^{(p)} - 6\, a_{n-2}^{(p)} + 7^n$$

or $$P_0\, 7^n = 5\, P_0\, 7^{n-1} - 6\, P_0\, 7^{n-2} + 7^n$$

$\div$ by 7^{n-2},

$$49\, P_0 = 5 \cdot 7\, P_0 - 6\, P_0 + 49$$

$$\Rightarrow (49 - 35 + 6)\, P_0 = 49$$

or $$P_0 = \frac{49}{20}$$

$\therefore$ All solns are of the form

$$\boxed{a_n = C\, 2^n + D\, 3^n + \frac{49}{20}\, 7^n}$$

Ex: i) Find all solns of the recurrence relation

$$a_n = 4\, a_{n-1} - 4\, a_{n-2} + (n+1)\, 2^n.$$

ii) Find the soln of the recurrence relan in (i) with IC $a_0 = 0$ and $a_1 = 1$.

Soln: i) Let a_n be all solns of the recurrence relan,

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

To find $a_n^{(h)}$

Consider associated Homo. recurrence relation,

$$a_n = 4a_{n-1} - 4a_{n-2}.$$

Char Eqn.

$$t^2 = 4t - 4$$

or $\quad t^2 - 4t + 4 = 0$

roots are $\quad t = 2, 2.$

Hence, $\boxed{a_n^{(h)} = C.2^n + D.n2^n}$ (C and D are constants)

<u>To find $a_n^{(p)}$</u>,

Here $\quad F_n = (n+1)2^n. \quad \therefore a_n^{(p)}$ is of the form,

$$a_n^{(p)} = n^2(P_0 + P_1 n)2^n$$

It follows that

$$a_n^{(p)} = 4a_{n-1}^{(p)} - 4a_{n-2}^{(p)} + (n+1)2^n$$

$$\Rightarrow \quad n^2(P_0 + P_1 n)2^n = 4(n-1)^2(P_0 + P_1(n-1))2^{n-1}$$
$$- 4(n-2)^2(P_0 + P_1(n-2))2^{n-2}$$
$$+ (n+1)2^n$$

$$\Rightarrow n^2(P_0 + P_1 n)2^n = 4(n^2 - 2n + 1)(P_0 + P_1 n - P_1)2^{n-1}$$
$$- 4(n^2 - 4n + 4)(P_0 + nP_1 - 2P_1)2^{n-2}$$
$$+ n2^n + 2^n$$

$\div$ by 2^{n-2}

$$\Rightarrow (\cancel{P_0 n^2} + \cancel{P_1 n^3})4 = 8(\cancel{P_0 n^2} + \cancel{P_1 n^3} - \cancel{P_1 n^2} - 2P_0 n - \cancel{2P_1 n^2} + 2P_1 n$$
$$+ P_0 + P_1 n - P_1)$$
$$- 4(\cancel{P_0 n^2} + \cancel{P_1 n^3} - \cancel{2P_1 n^2} - 4P_0 n - \cancel{4P_1 n^2} + 8P_1 n$$
$$+ 4P_0 + 4P_1 n - 8P_1)$$
$$+ 4n + 4$$

$$\Rightarrow \quad 0 = 8(-2P_0 + 3P_1)n + 8(P_0 - P_1) - 4(-4P_0 + 12P_1)n$$
$$- 4(4P_0 - 8P_1) + 4n + 4$$

$$\Rightarrow \left(-\cancel{16P_0}+24P_1+\cancel{16P_0}-48P_1+4\right)n+\left(8P_0-8P_1-16P_0+32P_1+4\right)=0$$

$$\Rightarrow \quad -24P_1+4=0, \qquad -8P_0+24P_1+4=0$$

$$\Rightarrow \quad P_1=\frac{1}{6}, \qquad P_0=\frac{24\left(\frac{1}{6}\right)+4}{8}=1$$

Thus, $\boxed{a_n^{(p)}=n^2\left(1+\frac{1}{6}n\right)2^n}$

General soln is

$$\boxed{a_n=C\,2^n+D\,n\,2^n+n^2\left(1+\tfrac{1}{6}n\right)2^n}$$

We use IC's to find C and D,

Given IC's $a_0=0$ and $a_1=1$

put $n=0$ in G.S

$$\boxed{0=C}$$

put $n=1$ in G.S

$$1=C.2+D.2+\left(1+\tfrac{1}{6}\right)2$$

$$\Rightarrow \quad 1=2D+\tfrac{7}{3}$$

$$\text{or} \quad D=\left(1-\tfrac{7}{3}\right)/2=\frac{-4}{6}=-\tfrac{2}{3}$$

$\therefore$

$$\boxed{a_n=-\tfrac{2}{3}\,n\,2^n+n^2\left(1+\frac{1}{6}n\right)2^n}$$

Generating functions (g.fs)

Defn: Let $(a_0, a_1, a_2, \dots)$ be a sequence of no.s $(a_i \in \mathbb{R})$.
The g.f of $(a_0, a_1, a_2, \dots)$ is the following formal power series

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots := \sum_{n=0}^{\infty} a_n x^n$$

x is a formal variable, we donot assign values.

Some times we can treat $G(x)$ as a fn. of some values of x and study convergence.

For instance $G(x) = 1 + x + x^2 + \cdots$

converges when $|x| < 1$.

Ex: $a_n = n^n$, g.f $G(x) = \sum_{n=0}^{\infty} n^n x^n$ does not converges for $x \neq 0$, but it is still a formal power series.

Ex 1: Suppose $(a_0, a_1, a_2, \dots) = (1, 1, 1, \dots)$, then

g.f is $G(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

$$= \frac{1}{1-x}, \quad |x| < 1$$

Ex 2: If $(a_0, a_1, a_2, \dots) = (1, \lambda, \lambda^2, \lambda^3, \dots)$,

$$G(x) = 1 + \lambda x + \lambda^2 x^2 + \cdots$$

$$= \frac{1}{1 - \lambda x}, \quad |\lambda x| < 1$$

Ex 3: Let $(a_0, a_1, a_2, \dots a_m)$ be the m^{th} row of the pascal Δ^{le}.

$a_n = \binom{m}{n}$; $a_n = 0$, $n > m$,

$$G(x) = \binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + \cdots + \binom{m}{m} x^m$$

$$= (1+x)^m.$$

Ex: G.f for the seq 1, 1, 1, 1, 1, 1 is

Soln: $G(x) = 1 + x + x^2 + x^3 + x^4 + x^5$

$$= \frac{x^6 - 1}{x - 1}$$

Addition and multiplication of formal power series.

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $\quad B(x) = \sum_{n=0}^{\infty} b_n x^n$.

Then

$$A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$A(x) \cdot B(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \times (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} c_n x^n, \quad \text{where } c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

That is $A(x) \cdot B(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n$

Example:

Let $A(x) = \frac{1}{1-x}$; $\quad B(x) = \frac{1}{1-x}$

We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \longrightarrow (1)$$

$$\therefore A(x) \cdot B(x) = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = (1+x+x^2+\cdots)(1+x+x^2+\cdots)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} 1 \right) x^n$$

$$= \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + \cdots$$

This can also be obtained by differentiating (1).

Differentiate (1) on B.S, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

Inverse

Let $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Defn: $B(x) = (A(x))^{-1}$ is the inverse of $A(x)$, if

$A(x) \cdot B(x) = 1$

Ex: $A(x) = 1 + x + x^2 + \cdots$

It's inverse is $(A(x))^{-1} = 1 - x$

Since
$$(1+x+x^2+\cdots)(1-x) = \begin{array}{l} 1 + x + x^2 + \cdots \\ \quad - x - x^2 - \cdots \end{array}$$
$$= 1$$

So, $(1+x+x^2+\cdots) = \dfrac{1}{1-x}$

Thm: $A(x)$ has an inverse iff $a_0 \neq 0$,

where $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

pf: Suppose that $B(x) = (A(x))^{-1} = b_0 + b_1 x + b_2 x^2 + \cdots$,

$$A(x) \cdot B(x) = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n = 1$$

or

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots = 1$$

$$\Rightarrow a_0 b_0 = 1, \qquad \sum_{j=0}^{k} a_j b_{k-j} = 0 \quad \text{for } k \geq 1$$

if $a_0 \neq 0$

$$b_0 = \frac{1}{a_0}, \quad b_1 = \frac{a_1 b_0}{a_0}, \quad b_2 = \frac{-a_1 b_1 - a_2 b_0}{a_0}, \ldots$$

we can find all b_i's and they are uniquely determined.

Binomial theorem for negative integer exponents

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k. \qquad (n > 0)$$

Thm. can be proved by differentiating ① and using mathematical induction.

Combinatorial pf.

$$\frac{1}{(1-x)^n} = (1 + x + x^2 + \cdots)^n$$

$$= \underbrace{(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n \text{ times}}$$

$$= \sum_{d_1=0}^{\infty} x^{d_1} \cdot \sum_{d_2=0}^{\infty} x^{d_2} \cdots \sum_{d_n=0}^{\infty} x^{d_n}$$

$$= \sum_{d_1, d_2, \ldots d_n = 0}^{\infty} x^{d_1+d_2+\cdots+d_n}$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

where a_k is the no. of ways to present k as $d_1+d_2+d_3+\cdots+d_n$

and it is $\binom{n+k-1}{k}$

Thus, $$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

Using g.f to solve recurrence relation

Ex: Solve the recurrence relation $a_k = 3a_{k-1}$, $k \geq 1$ and IC $a_0 = 2$.

Soln: Let $G(x)$ be the g f for the sequence $\{a_n\}$

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= \sum_{k=0}^{\infty} a_k x^k$$

Consider

$$a_k = 3a_{k-1}, \quad k \geq 1$$

multiply B.S by x^k

$$\Rightarrow \quad a_k x^k = 3 a_{k-1} x^k, \quad k \geq 1$$

$$a_1 x = 3 a_0 x$$

$$a_2 x^2 = 3 a_1 x^2$$

$$a_3 x^3 = 3 a_2 x^3$$

$$\vdots$$

$$a_1 x + a_2 x^2 + \cdots = 3 a_0 x + 3 a_1 x^2 + \cdots$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k x^k = 3 \sum_{k=1}^{\infty} a_{k-1} x^k$$

$$\sum_{k=0}^{\infty} a_k x^k - a_0 = 3 \sum_{k=0}^{\infty} a_k x^{k+1}$$

$$= 3 x \sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow G(x) - a_0 = 3x\, G(x)$$

$$\Rightarrow G(x)(1-3x) = a_0 \qquad (a_0 = 2)$$

$$\Rightarrow G(x) = \frac{2}{1-3x}$$

$$= 2 \sum_{k=0}^{\infty} (3x)^k$$

$$\frac{1}{1-\lambda x} = 1 + \lambda x + \lambda^2 x^2 + \cdots = \sum_{n=0}^{\infty} (\lambda x)^n$$

$$= 2 \sum_{k=0}^{\infty} 3^k x^k$$

But $$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

Thus, $\boxed{a_k = 2 \cdot 3^k}$, $k \geq 0$

Useful generating functions

	a_k	$G(x)$
1)	$\binom{n}{k}$	$\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n = (1+x)^n$
2)	1	$1 + x + x^2 + \cdots = \frac{1}{1-x}$
3)	λ^k	$1 + \lambda x + \lambda x^2 + \cdots = \frac{1}{1-\lambda x}$
4)	$k+1$	$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$
5)	$\binom{n+k-1}{k}$	$\binom{n-1}{0} + \binom{n}{1} x + \binom{n+1}{2} x^2 + \cdots = \frac{1}{(1-x)^n}$
6)	$\frac{1}{k!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$
7)	$\frac{(-1)^{k+1}}{k}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots = \log(1+x)$

Ex: Solve the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1} \quad ; \quad a_0 = 1$$

Soln: Let $G(x)$ be the g.f of the sequence $\{a_k\}$.

Then $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$$= \sum_{n=0}^{\infty} a_n x^n.$$

Given $a_n = 8a_{n-1} + 10^{n-1}$; $n \geq 1$

multiply B.S by x^n.

$$a_n x^n = 8 a_{n-1} x^n + 10^{n-1} x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 8 a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$$

(Replace n by $n+1$) (Replace n by $n+1$)

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - a_0 = \sum_{n=0}^{\infty} 8 a_n x^{n+1} + \sum_{n=0}^{\infty} 10^n x^{n+1}$$

$$\Rightarrow G(x) - a_0 = 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n$$

$$\left(G(x) = \sum_{n=0}^{\infty} a_n x^n \right)$$

$$\Rightarrow G(x) - 1 = 8x\, G(x) + x \frac{1}{1-10x}$$

$$\left(\sum_{n=0}^{\infty} 10^n x^n = 1 + 10x + 10^2 x^2 + \dots = \frac{1}{1-10x} \right)$$

$$\Rightarrow G(x) - 8x\, G(x) = 1 + \frac{x}{1-10x}$$

$$\Rightarrow G(x)(1-8x) = \frac{1-9x}{1-10x}$$

$$\Rightarrow G(x) = \frac{1-9x}{(1-8x)(1-10x)}, \quad \text{This is g.f of the seq. } \{a_n\}.$$

(*)

Using partial fraction we see that

$$\frac{1-9x}{(1-8x)(1-10x)} = \frac{A}{1-8x} + \frac{B}{1-10x}$$

$$\Rightarrow \frac{1-9x}{\cancel{(1-8x)(1-10x)}} = \frac{A(1-10x) + B(1-8x)}{\cancel{(1-8x)(1-10x)}}$$

$$\Rightarrow 1-9x = A - 10Ax + B - 8Bx$$

Comparing co-eff of x^1 : $-9 = -10A - 8B$

Comparing co-eff of x^0 : $1 = A + B$

Solving, we get $A = 1/2$, $B = 1/2$

$$\therefore \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

Sub in $(*)$

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}(8x)^n + \sum_{n=0}^{\infty}(10x)^n\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(8^n + 10^n)\,x^n$$

But, $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\boxed{\therefore\ a_n = \frac{1}{2}(8^n + 10^n)}$$