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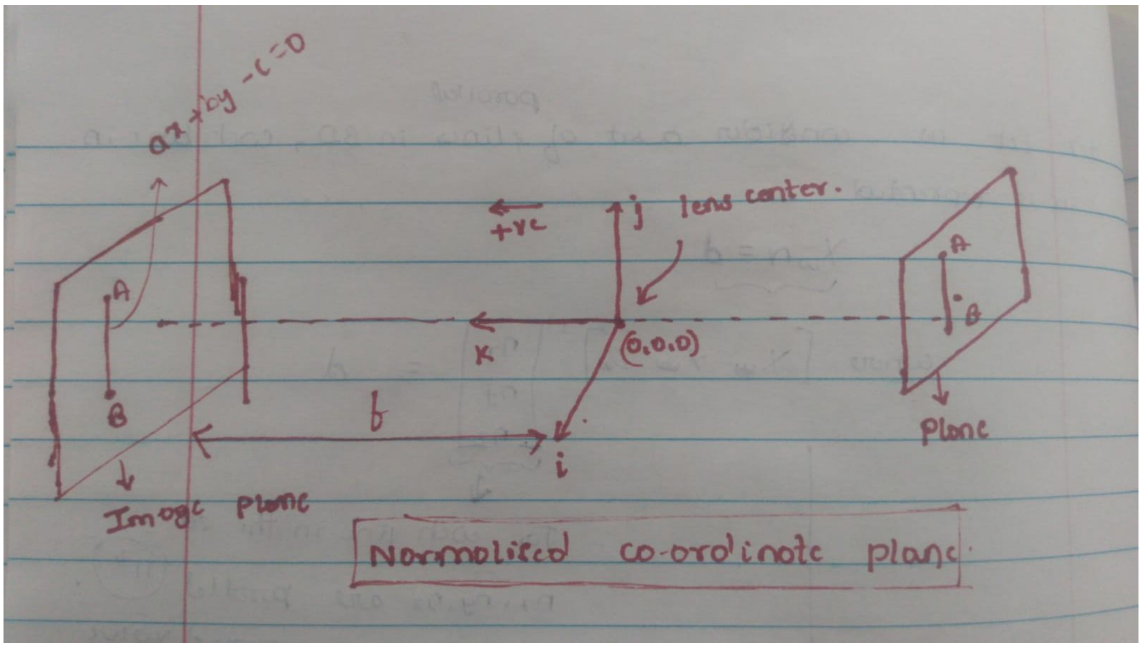
Homework 1

CSCI677 | Fall-2020

1. For this problem, assume that the image coordinates are specified in the normalized coordinate system and 3-D points are specified in the camera coordinate system. Consider a line, l, in the image, given by parameters (a, b, c). We know that the corresponding 3-D line casting this image lies in a plane. Derive the equation of this plane (in the camera coordinate system).

Solution1 Consider the figure below,



Normalized coordinates indicates that origin of the image plane will be at the projection of the principal line and the image plane is parallel to i and j coordinates.

Let us represent the line on the 2D image plane as,

$$L : ax_{im} + by_{im} + c = 0 \tag{1}$$

where, (x,y) are in image coordinates.

Let the plane in 3D is represented as,

$$P : pX_c + qY_c + rZ_c + d = 0$$

We have to derive the equation of this plane given the equation of the line in 2D given in (1).

From perspective projection

$$x_{im} = \frac{fX_c}{Z_c}, y_{im} = \frac{fY_c}{Z_c} \tag{2}$$

where, f is the focal length
Substituting (2) in (1) we get,

$$a.\frac{f.X_c}{Z_c} + b.\frac{f.Y_c}{Z_c} + c = 0$$

$$(af).X_c + (bf)Y_c + c.Z_c = 0 \tag{3}$$

$$\text{Here}, p = af, q = bf, r = c \text{ and } d = 0$$

Therefore, (3) is the final equation of the plane with respect to camera coordinates derived from the line projected on 2D image plane.

2. Suppose that we have a right-handed camera coordinate system (Xc, Yc, Zc) associated with origin at the lens center (or the pin hole), as in the examples discussed in class. Suppose that the imaging plane is at a distance of 50 millimeters from the lens center, the imaging surface (a planar patch) is 900 x 900 pixels, each pixel is .05 millimeters in each dimension and that the principal ray intersects the imaging surface in the center. Let the image (or retinal) coordinate system have its origin at the upper left corner of the imaging sensor, the x-axis along the top-row and the y-axis points downward (at an acute angle of 88 degrees to the x-axis). Assume that the x-axis in the image plane is parallel to the $\hat{x}$-axis in the normalized image plane.

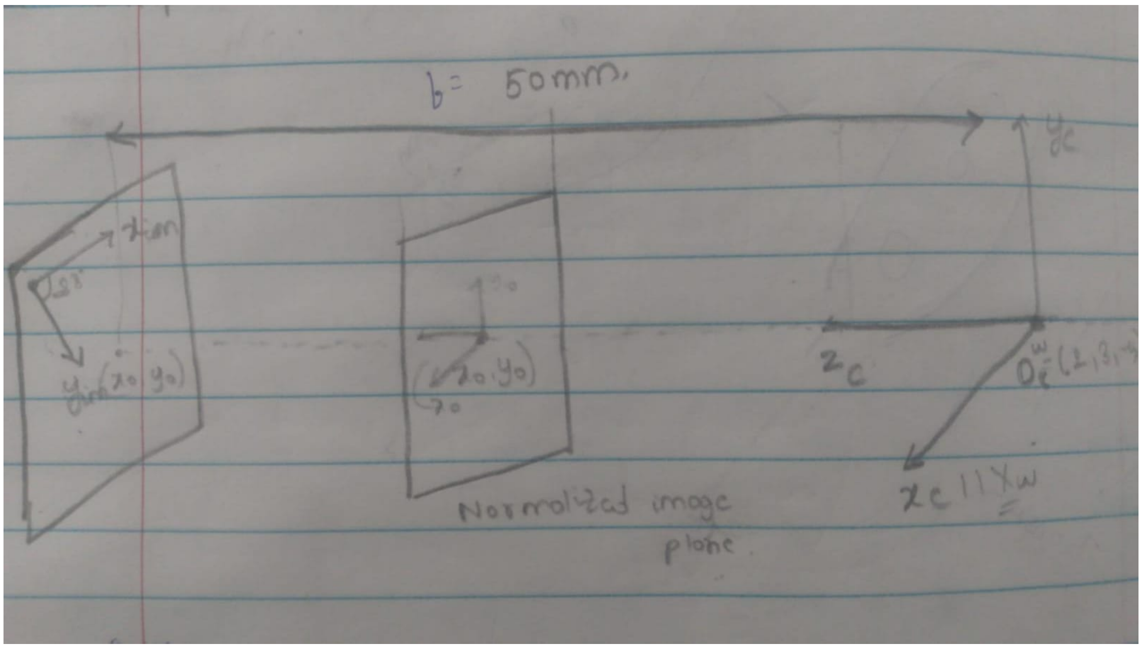
   Now suppose that the camera is placed in a world coordinate system (Ow, Xw, Yw, Zw) such that Oc is at location (2, 3,-5) in the world coordinate system (all distances expressed in meters); Xc is parallel to Xw and then the camera is rotated by 10 degrees about the Xc axis in a clockwise direction (visualize as a person taking a picture with camera pointing down slightly). All questions below correspond to this configuration.

   (a) Compute the final projection matrix, M (as used in equation 1.16 of the FP book, or slide 22 of Lecture 4 notes). Note that M will transform a given point (X, Y, Z,1)T in the world coordinates (expressed in meters) to the image T coordinates, (x, y, 1) , expressed in pixel units (ignore the issue of rounding off pixel coordinates to integers).

   (b) Compute the vanishing point of vertical lines in the world (i.e. lines parallel to the Yw axis) in the image coordinates.

   (c) Consider a set of parallel lines in the horizontal plane (i.e. the Xw, Zw plane). Find the vanishing point, in the image coordinates, of this set of lines in terms of the direction of the lines. Again, you may use any equations given in the book or in class but derive any other equations that you make use of (if any). You are encouraged, but not required, to use projective geometry formulation.

(d) Show that the vanishing points of the horizontal lines lie on a line in the image plane. Derive the equation of this line, also called the horizon line. Again, you may use any equations given in the book or in class but derive any other equations, if any, that you make use of. You are encouraged, but not required, to use projective geometry formulation.



Solution-2a Given,

$$S_x = \frac{1}{0.05mm}, S_y = \frac{1}{0.05mm}, f = 50mm$$

Image Plane Dimension (in pixels) = 900 × 900

Angle between the coordinate axes (origin at the upper left corner) = 88°

The final Projection matrix is given by,

$$M_{Final} = M_{intrinsic} \times M_{extrinsic}$$

$M_{intrinsic}->$ Camera coordinates to image coordinates (Perspective Projection)

$M_{extrinsic}->$ World coordinates to camera coordinates (Rigid transformation)

The Intrinsic matrix can be considered as affine transformation

$$x_{im} = \frac{f}{S_x} X_c + x_0$$

$$y_{im} = \frac{f}{S_y} Y_c + y_0$$

where, X_c and Y_c are points in camera coordinates and x_0 and y_0 are translated origin of the image plane

Now since the coordinate system is skewed,

$$x_{im} = \frac{f}{S_x} X_c - \frac{f}{S_x} cot(\phi) + x_0$$

$$y_{im} = \frac{\frac{f}{S_y} Y_c}{sin(\phi)} + y_0$$

In the above equations, the $\frac{f}{S_x}$ and $\frac{f}{S_y}$ are magnification expressed in pixel units.

$$\text{Therefore, } \alpha = \frac{f}{S_x} = \frac{50}{0.05} = 1000 \text{ and } \beta = \frac{50}{0.05} = 1000$$

$(x_0, y_0) = (450, 450)$. Because, the dimension of the image plane is 900×900 and the origin is shifted from center to top left corner.

Combining all these, the intrinsic matrix will be,

$$K = \begin{bmatrix} \alpha & -\alpha cot(\phi) & x_0 \\ 0 & \frac{\beta}{sin(\phi)} & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 1000 & -1000 cot(-88) & 450 \\ 0 & \frac{1000}{sin(-88)} & 450 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{intrinsic} = \begin{bmatrix} 1000 & 34.9207 & 450 & 0 \\ 0 & -1000.6095 & 450 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1}$$

The extrinsic matrix can be considered as rigid transformation with rotation and translation.

$$\text{The camera rotation angle is } (\theta) = 10^\circ$$

Since the X coordinates of camera and world coordinate system are parallel, the rotation is along X-axis. **Since the rotation is clockwise, the sign of the angle becomes negative**

The rotation matrix is given by,

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(-\theta) & -sin(-\theta) \\ 0 & sin(-\theta) & cos(-\theta) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9840 & 0.1736 \\ 0 & -0.1736 & 0.9840 \end{bmatrix}$$

The translation vector can be calculated by the equation,

$$P^C = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} P^W$$

where, P^C and P^W are points in camera and world coordinates respectively.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 0.9840 & 0.1736 & t_y \\ 0 & -0.1736 & 0.9840 & t_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -5 \\ 1 \end{bmatrix}$$

Solving the equation,

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} -2 \\ -2.084 \\ 5.4408 \end{bmatrix}$$

The final extrinsic matrix will be,

$$M_{extrinsic} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0.9840 & 0.1736 & -2.084 \\ 0 & -0.1736 & 0.9840 & 5.4408 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

The final projection matrix from (1) and (2) can be written as,

$$M = M_{intrinsic} \times M_{extrinsic}$$

$$M = \begin{bmatrix} 1000 & -43.7 & 448.9 & 375.6 \\ 0 & -1063.6 & 269.1 & 4533.6 \\ 0 & -0.2 & 1 & 5.4 \end{bmatrix}$$

Solution-2b The vanishing points of vertical lines (parallel to Y axis) is obtained by multiplying the final projection matrix with lines parallel to Y axis

$$V_p = \begin{bmatrix} 1000 & -43.7 & 448.9 & 375.6 \\ 0 & -1063.6 & 269.1 & 4533.6 \\ 0 & -0.2 & 1 & 5.4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$V_p = \begin{bmatrix} -43.7 \\ -1063.6 \\ -0.2 \end{bmatrix} \tag{1}$$

(1) is in homogeneous coordinates, so we have to convert it to normal coordinates

$$p_{vanish} = \begin{bmatrix} \frac{-43.7}{-0.2} \\ \frac{-1063.6}{-0.2} \end{bmatrix} = \begin{bmatrix} 218.5 \\ 5318 \end{bmatrix} = \begin{bmatrix} 219 \\ 5318 \end{bmatrix} \text{(Rounding off)}$$

Solution-2c Similar to the previous question, we can formulate the set of lines in the horizontal plane (X,Z) by $[i, 0, k, 0]^T$ where, $\vec{r} = [i, 0, k]$ is the direction of the line such that $||r||^2 = 1$

The Vanishing points are given by

$$V_p = \begin{bmatrix} 1000 & -43.7 & 448.9 & 375.6 \\ 0 & -1063.6 & 269.1 & 4533.6 \\ 0 & -0.2 & 1 & 5.4 \end{bmatrix} \times \begin{bmatrix} i \\ 0 \\ k \\ 0 \end{bmatrix}$$

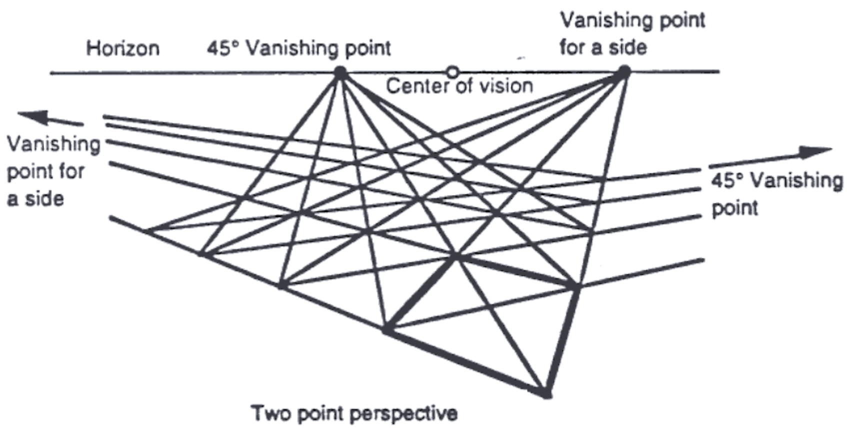
$$V_p = \begin{bmatrix} 1000i + 448.9k \\ 269.1k \\ k \end{bmatrix} \tag{1}$$

Converting the homogeneous coordinates to normal coordinates, we get

$$p_{vanish} = \begin{bmatrix} (1000\frac{i}{k} + 448.9 \\ 269.1 \end{bmatrix} \tag{2}$$

Thus (2) represents the vanishing points in terms of the direction of the lines.

Solution-2d Consider X-Z plane (as in previous problem) as a super-set of parallel lines. Each set of parallel line is going to intersect at infinity. **Horizon line (Vanishing line)** is the line passing through such points at infinity. Below figure shows two sets of lines intersecting at infinity (The lines are not derived from the same plane though).



[Image Credits: Google Images]

Let us consider a set of parallel lines in 3D and each line in 3D is represented by the equation,

$$\begin{bmatrix} X & Y & Z \end{bmatrix}^T \times \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \end{bmatrix} = d \tag{1}$$

(1) represents the general equation of the set of parallel lines. The normal vector of each line will be same or scaled values i.e the direction of normal vector of each line will be the same.

Using Perspective projection, if we project these set of lines to the image plane, we get,

$$x_{im} = \frac{X\lambda}{Z}$$
$$y_{im} = \frac{Y\lambda}{Z}$$

where, λ is the focal length.

$$\begin{bmatrix} x_{im} \\ y_{im} \\ \lambda \end{bmatrix} = \lambda . \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix}$$

If we take dot product of normal vector on both sides.

$$\begin{bmatrix} x_{im} \\ y_{im} \\ \lambda \end{bmatrix} . \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \end{bmatrix} = \lambda . \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix} . \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \end{bmatrix}$$

As $Z- > \infty$ RHS will become zero.

Therefore, the equation of the horizon line is given by,

$$x_{im}\eta_x + y\eta_{im} + \lambda\eta_z = 0 \tag{2}$$

.

All the vanishing points from the set of parallel lines, will lie on (2) which is horizon line (because it is X-Z plane)