# CS & IT ENGINEERING


## Algorithms

### Analysis of Algorithms

**Lecture No.- 06**

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# Recap of Previous Lecture

- **Topic** — Big Notations
- **Topic** — Problem Solving

# Topics to be Covered

Covered

**Topic** Small Notations

**Topic** Properties of Asymptotic Notations

**Topic** Problem solving

# Topic : Asymptotic

H.W $\sum_{i=1}^{n} \log(i) = \log(n!)$

~~mtd-2~~

$\hookrightarrow \int_{1}^{n} \log x \, dx = \left[x(\log x - 1) + c\right]_{1}^{n}$

$\boxed{n\log n - n + c}$

$\Rightarrow \sum_{i=1}^{n} \log i = \left[n(\log n - 1) + c\right] = \boxed{n\log n - n + c}$

$O(n\log n)$ $\Omega(n\log n)$

$\Theta(n\log n)$

n.w $\boxed{\text{mtd-3}}$ using approximation logic

$$= \log n + \log(n-1) + \log(n-2) \cdots + \log 1 < \log n + \underbrace{\log n \cdots}_{n \text{ times}}$$

UB

$$\log n + \log(n-1) + \log(n-2) \cdots + \log 1 < n \log n$$

$$O(n \log n)$$

for LB :

$$\log n > \log(n/2)$$

$$\log n + \log(n-1) + \log(n-2) \cdots + \log 1 > \underbrace{\log(n/2) + \log(n/2) \cdots}_{n/2 \text{ terms}}$$

LB

$$> \frac{n}{2}\log(n/2)$$

$$\log(n!) > \frac{n}{2}\log(n/2)$$

$$\boxed{\Omega(n\log n)}$$

① $f(n) = \sum_{i=1}^{n} \sqrt{i} = O(\quad)$

$= \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots \sqrt{n}$

$\boxed{\text{using integration}}$

$\int x^n dx = \frac{x^{n+1}}{(n+1)}$

$\rightarrow \int i^{1/2} di = \left[\frac{i^{1/2+1}}{1/2+1}\right]_1^n = \frac{i^{3/2}}{3/2} = \boxed{\left[\frac{2}{3} * i^{3/2}\right]} = \frac{2}{3} * n^{3/2} + c$

$= O(n^{3/2}) = O(n^{(1+1/2)}) = O(n^1 * n^{1/2}) = O(n * \sqrt{n})$

## Topic : Small/Little Notations

$[o, \omega] \rightarrow$ proper Bounds

$\rightarrow$ The bounds provided by Big notations $(O, \Omega)$ may or may not be $\boxed{\text{tight}}$

tight bound $\Leftarrow$ ✓ $\Omega(n)$

✓ $\Omega(\sqrt{n})$

✓ $\Omega(1)$

Lose bound

$f(n) = n \rightarrow \boxed{O(n)}$ ✓ $\rightarrow$ tight bound

$O(n^2)$ ✓

$O(n^3)$ ✓ $\rightarrow$ lose bound

$\vdots$

→ The bounds provided by small/little notations

is always $\boxed{\text{not Asymptotically tight}}$

↳ lose bound

# Topic : Asymptotic Notations

1) Small Oh : (o) : proper Upper Bound

→ $f(n)$ is $o(g(n))$ iff for $\boxed{\text{all}}$ $c > 0$

$$\boxed{f(n) < c * g(n)}$$

whenever $n > n_0$

$n_0 > 0$

$n < 5 * n \rightarrow O(n)$ $\quad f < c \times g(n)$ all $c > 0$

$\boxed{n > 0.1 * n}$

eg: $f(n) = n$

- $O(n^2)$ ✓
- $O(n)$ ✓
- $O(n^2)$ ✓
- $O(n^3)$ ✓
- $O(n \log n)$
- $O(n)$ ✗

eg2

$n^2 + n + 1 \Rightarrow O(n^2)$ ✓

- $O(n^3)$ ✓
- $O(n^2\sqrt{n})$ ✓
- $O(n^2)$ ✗

# Topic : Asymptotic Notations

5) Small omega ($\omega$): proper lower bound

→ $f(n)$ is $\omega(g(n))$ iff for all $\boxed{c>0}$

$$\boxed{f(n) > c * g(n)}$$

whenever, $n > n_0$

$n_0 > 0$

$n_0$

$\boxed{n < 0.001\, n^2}$

Double

$\boxed{n > n_0}$

eg

$f(n) = n$

- $\Omega(n)$ ✓
- $\Omega(1)$ ✓
- $\Omega(\log n)$ ✓
- $\Omega(n^2)$ ✗
- $\boxed{\omega(n)}$ ✗
- $\omega(1)$ ✓
- $\omega(\sqrt{n})$ ✓
- $\omega(1/n)$ ✓

## * Properties of ASN :-

→ Anology between ASN & Real nos:

$f \to a$

$g \to b$

Let a, b be Real nos, & f, g → +ve functions

$\boxed{f \le c \cdot x g(n)}$

1) If $f(n)$ is $O(g(n)) \Longrightarrow a \le b$
2) If $f(n)$ is $\Omega(g(n)) \Longrightarrow a \ge b$
3) If $f(n)$ is $\Theta(g(n)) \Longrightarrow a = b$
4) If $f(n)$ is $o(g(n)) \longrightarrow a < b$
5) If $f(n)$ is $\omega(g(n)) \longrightarrow a > b$

[MCQ]

#Q. $f(n) = \sum_{i=1}^{n} i^3 = x$ choice for x.

I. $\theta(n^4)$ II. $\theta(n^5)$ ✗

III. $O(n^5)$ IV. $\Omega(n^3)$

A. I, II, III

B. II, III, IV

C. I, II, III, IV

D. I, III, IV

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{n^2(n+1)^2}{4} = \frac{n^2(n^2+2n+1)}{4}$$

$$= \frac{n^4+2n^3+n^2}{4}$$

$\boxed{O(n^4)}$ $\Omega(n^4)$

$O(n^5)$ $\Omega(n^3)$ ~~$\Omega(n^5)$~~

$\theta(n^4)$

# Topic : General Properties of Big Oh Notation

Let d(n), e(n), f(n), and g(n) be functions mapping nonnegative integers to non-negative reals. Then

eg:- $n \longrightarrow O(n)$

$10*n \longrightarrow O(n)$

1. If d(n) is O(f(n)), then $\boxed{a d(n) \text{ is } O(f(n))}$, for any constant a > 0   $a*d(n)$
2. If d(n) is O(f(n)) and e(n) is O(g(n)), then d(n) + e(n) is O(f(n) + g(n)).
3. If d(n) is O(f(n)) and e(n) is O(g(n)), then d(n)e(n) is o(f(n)g(n))
4. If d(n) is O(f(n)) and f(n) is O(g(n)), then d(n) is O(g(n)). → Transitive   $a \le b$, $b \le c$ → $a \le c$
5. If f(n) is a polynomial of degree d (that is, $f(n) = (a_0 + a_1 n + \ldots + a_d n^d)$ )then f(n) is $O(n^d)$. → $f = 1 + n^2 + n^3 \longrightarrow O(n^3)$
6. $n^x$ is $O(a^n)$ for any fixed x > 0 and a > 1 →   $n^x$ → Polynomial   $a^n$ → expo   poly < expo
7. $\log n^x$ is O(log n) for any fixed x > 0
8. $\log^x n$ is $O(n^y)$ for any fixed constants x > 0 and y > 0   → $(\log n)^x$

2) eg:- $d(n) = n^3 \longrightarrow O(n^3)$

$e(n) = n^4 \longrightarrow O(n^4) \longrightarrow O(\underline{f(n)+g(n)})$

$d(n)+e(n) \Rightarrow n^3+n^4 \longrightarrow O(\underline{n^3+n^4}) \longrightarrow O(n^4)$

$\longrightarrow O(\underline{\underline{max}}(f(n), g(n))) \longrightarrow O(max(n^3, n^4))$

$= \underline{O(n^4)}$ ✓

7) $\log(\underline{\underline{n^x}}) \longrightarrow \underline{\underline{x}} * \log(n) \longrightarrow \underline{O(\log n)}$

3) eg:- $d(n) = n^3 \longrightarrow O(n^3)$

$e(n) = n^4 \longrightarrow O(n^4)$

$d(n) * e(n) = n^3 \times n^4 \longrightarrow O(f(n) * g(n))$

$= n^{4+3} = n^7$

$O(n^7)$

$\log(n^x) \neq (\log n)^x$

$\log(n) * \log(n) \ldots$

$(\log n)^x = O(n^y)$

(log < poly)

# Topic : Discrete Properties of ASN

Properties

$a \le b$
$b \le c \rightarrow a \le c$

| | $O$ | $\Omega$ | $\Theta$ | $o$ | $\omega$ |
|---|---|---|---|---|---|
| Reflexive | ✓ | ✓ | ✓ | ✗ | ✗ |
| Symmetric | ✗ | ✗ | ✓ | ✗ | ✗ |
| Transitive | ✓ | ✓ | ✓ | ✓ | ✓ |
| Transpose Symmetry | ✓ | ✓ | ✗ | ✓ | ✓ |

$n = O(n^2)$
$n^2 = O(n)$

if $f(n)$ is $O(g(n))$
then $g(n)$ is $\Omega(f(n))$

$a \le b \Rightarrow b \ge a$

if $f(n)$ is $o(g(n))$
then $g(n)$ is $\omega(f(n))$

★ <u>Tricotomy Property</u> :

→ if 'f' & 'g' are two +ve functions

1) $f \leq_A g \rightarrow O, o$

2) $f \geq_A g \rightarrow \Omega, \omega$

3) $f =_A g \rightarrow \Theta$

for any 2 real nos (a, b)

1) $a < b$ ✓

2) $a > b$

3) $a = b$

Conclusion from all the Discussion:

→ ASN's does NOT satisfy Tricotomy property

(Q) Does ASN follow/obey Tricotonomy property always?

T/F ?

eg:- 1) $f(n) = n, \; g(n) = n^2$

$f \stackrel{?}{=} O(g) \stackrel{?}{=} o(g) \rightarrow f \le g$

2) $f(n) = n, \; g(n) = 1/n \longrightarrow f \geq g$

$f \geq g$

$f = \Omega(g)$

$f = \omega(g)$

Q3 : $f(n) = n^2 + 10$, $g(n) = 10n^2 + 8$

$\downarrow$ $O(n^2)$ $\qquad$ $\downarrow$ $O(n^2)$ $\longrightarrow$ $\underline{\underline{\Theta}}(n^2)$

$$f(n) \underset{A}{=} g(n)$$

eg4:- $f(n)=n \quad , \quad g(n) = n^{(1+\sin x)}$

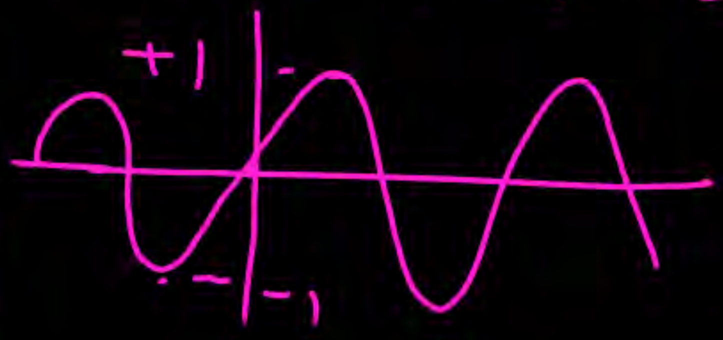
Case 1:- $\sin x \longrightarrow +1 \Rightarrow f(n)=n,\ g(n)=n^{1+1}=n^2$

$(f<g)$

Case 2:- $\sin x \longrightarrow -1 \Rightarrow f(n)=n,\ g(n)=n^{1-1}=n^0$

$(f>g)$

$\sin x \Rightarrow [+1, -1]$



# Topic : Asymptotic Notations & Apriorist Analysis

State True / False → H.W

1. $100n.\log n = O(n.\log n)$
2. $2^{n+1} = O(2^n)$ :
3. $2^{2n} = O(2^n)$
4. $0 < x < y$ then $n^x = O(n^y)$:
5. $(n+k)^m \neq \theta(n^m)$ $(k, m) > 0$
6. $\sqrt{\log n} = 0(\log\log n)$
7. $\log(n)$ is $\Omega(1/n)$
8. $2^{n2}$ is $0(n!)$
9. $n^2$ is $O(2^{2\log n})$

10. $a^n \neq O(n^x)$, $a > 1$, $x > 0$
11. $2^{\log_2 n2}$ is $O(n^2)$

$\boxed{2^{(\log_2 n^2)}}$

# 2 mins Summary

**Topic** — Small Notations ✓ $o, \omega$

**Topic** — Properties

**Topic** — Problem Solving ✓ + HW

# THANK - YOU