CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 04

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Recap of Previous Lecture

Topic Asymptotic Notations: Big Oh, Omega and Theta

Topic Properties of ASN

Topic

Topic

Topic

Topics to be Covered

Topics Problem Solving with ASN

Topic: Adding Functions

The sum of two functions is governed by the dominant one, namely:

$$O\left(f(n)\right) + O\left(g(n)\right) \rightarrow O\left(\max\left(f(n), g\,(n)\right)\right)$$

$$\Omega f(n)) + \Omega(g(n)) \rightarrow \Omega(\max(f(n), g\,(n)))$$

$$\ominus\left(f(n)\right) + \ominus\left(g(n)\right) \rightarrow \ominus\left(\max\left(f(n), g(n)\right)\right)$$

Topic: Adding Functions

$$O(f(n)) * O(g(n)) \rightarrow O(f(n) * g(n))$$

$$\Omega(f(n)) * \Omega(g(n)) \rightarrow \Omega(\mathrm{f(n)} * \mathrm{g(n)})$$

$$\ominus(f(n)) * \ominus(g(n)) \rightarrow \ominus(f(n) * g(n))$$

Topic: Analysis of Algorithms

(i) $f(n) = \boxed{\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}}$

$O(1)$

$S_n = \frac{a\left(1 - \frac{1}{r^n}\right)}{1 - r}$

$= \frac{\cancel{\frac{1}{2}}\left(1 - \frac{1}{2^n}\right)}{\cancel{1 - \frac{1}{2}}}$

$= \left(1 - \frac{1}{2^n}\right)$

Topic: Analysis of Algorithms

(ii) $f(n) = \sum_{i=1}^{n} \log(i) = [\log 1 + \log 2 + \log 3 + \cdots + \log n]$

$= \log n + \log n-1 + \log n-2 + \cdots + \log 1 = \log(n \cdot n-1 \cdot n-2 \cdots 1)$

$= \log n!$

1. $n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$

$$\log n! = \log\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n\right)$$

$$= \log\sqrt{2\pi} + \log\sqrt{n} + \log n^n - \log e^n$$

$$= \boxed{c + \frac{1}{2}\log n + n * \log n - n \cdot c_1}$$

$O(n \cdot \log n)$ $\Omega(n \cdot \log n)$

$$\boxed{\therefore \log n! = \Theta(n \cdot \log n)}$$

II. $\log n! = \sum_{i=1}^{n} \log i \sim \int_{x=1}^{n} \log x \cdot dx = \Big[x \cdot (\log x - 1) + c \Big]_1^n$

$= n(\log n - 1) + c$

$= \boxed{n \log n - n + c}$

$O(n \cdot \log n)$ $\Omega(n \log n)$

$$\boxed{\therefore \log n! = \Theta(n \cdot \log n)}$$

III.

$$f(n) = \sum_{i=1}^{n} \log i = \left[\log n + \log (n-1) + \dots + \log 1\right]$$

U.B : $f(n) < \log n + \log n + \dots + \log n < (n \cdot \log n)$

$\therefore O(n \cdot \log n)$

L.B : $f(n) > \left(\log \frac{n}{2} + \log \frac{n}{2} + \dots + \log n/2\right)$ ($n/2$ Times)

$> \left[\frac{n}{2} * \log (n/2)\right]$

$\Omega(n * \log n)$

$$\boxed{\log n! = \Theta(n \cdot \log n)}$$

I : $\boxed{\begin{array}{l} n! \text{ is } o(n^n) \\ \qquad\quad O(n^n) \\ \hline \text{It is NOT } \Theta(n^n) \end{array}}$

II. $\boxed{\log n! \text{ is } \Theta(n \cdot \log n)}$

Topic : Asymptotic Notations & Apriori Analysis

State True / False

1. $100n.\log n = O(n.\log n)$: T
2. $2^{n+1} = O(2^n)$ $= 2 \cdot 2^n =$ T
3. $2^{2n} = O(2^n)$: F $\quad 2^{2n} = (2^2)^n = 4^n$
4. $0<x<y$ then $n^x = O(n^y)$: T
5. $(n+k)^m \neq \theta(n^m)$ $(k, m) > 0$: F
6. $\sqrt{\log n} = O(\log\log n)$: F
7. $\log(n)$ is Ω $(1/n)$: T
8. 2^{n^2} *is* $O(n!)$: F
9. n^2 is $O(2^{2\log n})$: T
10. $a^n \neq O(n^x)$, $a > 1$, $x > 0$: T
11. $2^{\log_2 n^2}$ is $O(n^2)$: T

$\sqrt{\log n} \qquad \log\log n$

$\log(\log n)^{1/2} \qquad \log\log\log n$

$\frac{1}{2} * \log\log n \quad > \quad ''$

$2^{n^2} \;\; v \;\; n^n$

$\log 2^{n^2} \qquad n \log n$

$n^2 \;\; vs \;\; n \log n$

$>$

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Topic : Asymptotic Comparisons

Which fn is the order of the other fn (O)

1. $f(n) = n, g(n) = \log n$

$\therefore g(n) \text{ is } O(f(n))$

2. $f(n) = n^2\log n, g(n) = n.\log^{10}n$

PYQ

$\cancel{(n.\log n)}\, n \qquad \cancel{(n.\log n)}.(\log n)^9$

$\therefore g(n) \text{ is } O(f(n))$

$n \qquad (\log n)^9$

$\log n > 9.\log\log n$

Topic : Asymptotic Comparisons

03. $f(n) = n^3, \; 0 < n \le 10{,}000$

*

$= n, \; n > 10{,}000$

$g(n) = n, \; 0 < n \le 100$

$= n^3, \; n > 100$

f is $O(g)$ ✓

for what Min value of $n > k$, that $f = O(g)$

$$f(n) = n^3 \quad , \; n > 100, \le 10{,}000$$
$$= n \quad , \; n > 10000$$

$$g(n) = n^3 \quad , \; n > 100, \le 10{,}000$$
$$= n^3 \quad , \; n > 10000$$

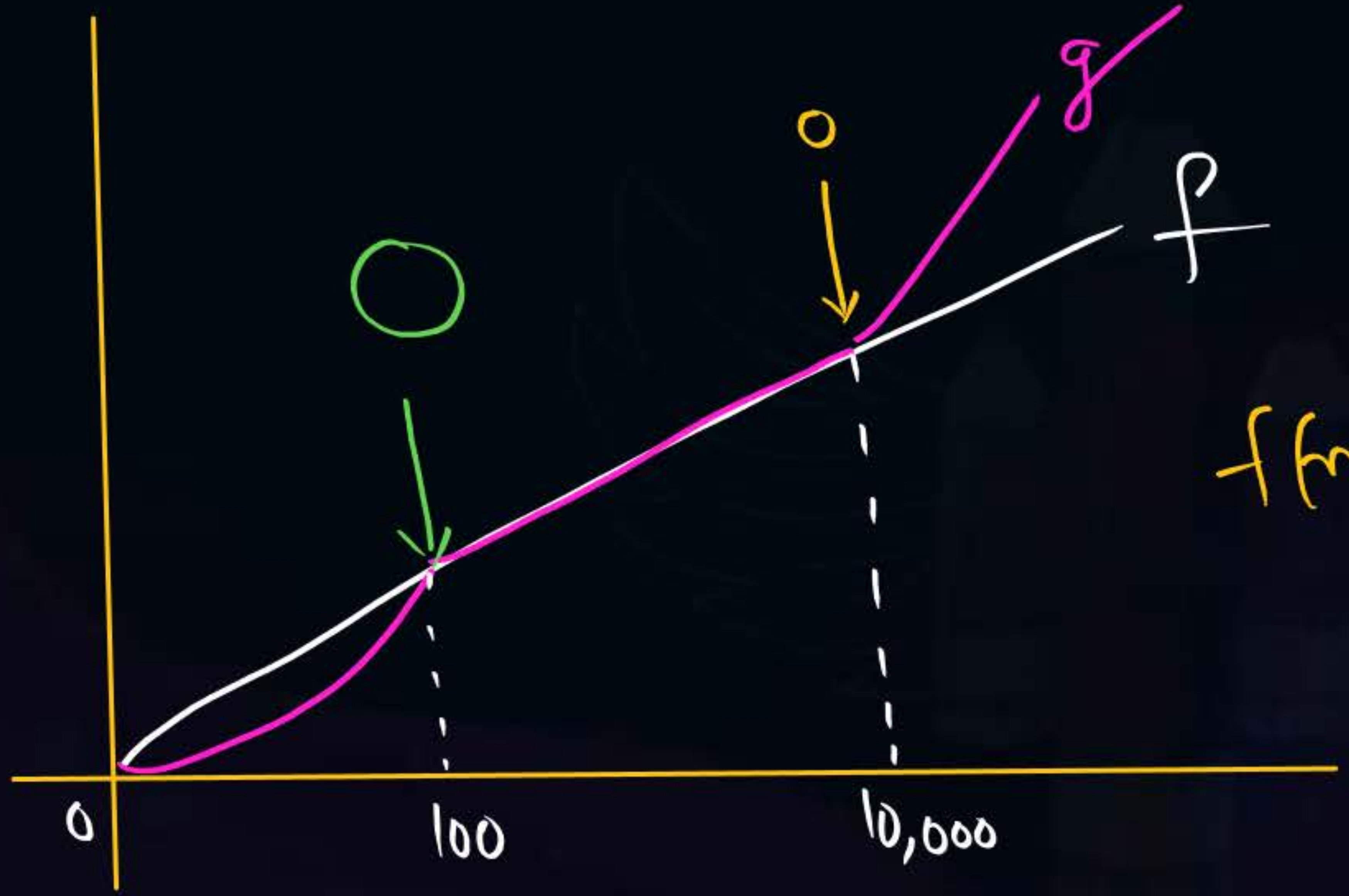
$f(n)$ is $O(g(n))$

$n > 100$

Is 'f' $o(g)$?

✓

$f(n) = o(g(n))$



Slide 10

04. Two Packages are available for processing a Data Base having 10^x records. Package A takes a times of $10.n.\log_{10} n$ while package B takes a time of $0.0001n^2$ for processing 'n' records. Determine the smallest integer x for which Package 'A' outperforms Package 'B'.

$A = O(B)$; 'A' is better than 'B'

(A) : $10 \cdot n \cdot \log_{10} n$

(B) : $0.0001n^2$

$n = 10^x$ records

DB

If $x = 2$

$n = 10^2 = 100$

$A : 10 * 100 * \log_2 10^2$

: 2000 unit

$B : 10^{-4} \times (10^2)^2$

: 1 unit

$x = 3$

$n = 1000$

$A : 10 \cdot 1000 \cdot 3$

: 30,000

$B : 10^{-4} \times (10^3)^2$

: 100 unit

$n = 10^x$ $\quad$ A: $10 * n * \log_{10} n$; B: $0.0001 n^2$

$$A < B$$

$$10 * n * \log_{10} n < 10^{-4} \times n^2$$

$$10 * \cancel{10}^{\cancel{x}} * x < 10^{-4} \times \cancel{10}^{\cancel{2}x}$$

$$\boxed{x < \frac{10^x}{10^5}}$$

$$\boxed{x = 6}$$

$$6 < \frac{10^6}{10^5}$$

Topic : Asymptotic Comparisons

Inc/Dcr order

0.4 Arrange the functions in increasing order of rates of growth.

$e = 2.7$

1. $\underset{P}{n^2}; \underset{P}{n.\log n}; \underset{P}{n\sqrt{n}}; \underset{E}{e^n}; \underset{P}{n}; \underset{E}{2^n}; \underset{P}{(1/n)}$

$$\left(\frac{1}{n} < n < n.\log n < n * \sqrt{n} < n^2\right) < 2^n < e^n$$

2. $\underset{E}{2^n}; \underset{P}{n^{3/2}}; \underset{P}{n\log n}; \underset{E}{n^{\log n}}$:

$$\boxed{n\log n < n^{3/2} < n^{\log n} < 2^n}$$

$2^n > n^{\log n}$

2^n vs $n^{\log n}$

$n.\log_2 2$ $\quad (\log n)^2$

$n > (\log n)^2$

3. $\underset{P}{n^{(1/3)}}; \underset{E}{e^n}; \underset{P}{n^{7/4}}; \underset{P}{n\log^9 n}; \underset{E}{1.001^n}$

$$\boxed{n^{1/3} < n.\log^9 n < n^{7/4} < (1.001)^n < e^n}$$

Q.5 Consider the following functions from positive integers to real numbers:

$$10, \sqrt{n}, n, \log_2 n, \frac{100}{n}.$$

The CORRECT arrangement of the above functions in increasing order of asymptotic complexity is:

(a) $\log_2 n, \frac{100}{n}, 10, \sqrt{n}, n$

(b) $\frac{100}{n}, 10, \log_2 n, \sqrt{n}, n$

(c) $10, \frac{100}{n}, \sqrt{n}, \log_2 n, n$

(d) $\frac{100}{n}, \log_2 n, 10, \sqrt{n}, n$

$\frac{100}{n} < 10 < \log n < \sqrt{n} < n$

Topic : Asymptotic Comparisons

06. Which of the following is **TRUE**? (M.S.Q)

$f(n)$ is $O(g(n))$
$g(n)$ is NOT $O(f(n))$ } $f < g$

$g(n)$ is $O(h(n))$
$h(n)$ is $O(g(n))$ } $(g = h)$

$$\boxed{f < (g = h)}$$

(a) $f(n)$ is $O(h(n))$ ✓
(b) $f(n) + h(n)$ is $O(g(n)+h(n))$ ✓
(c) $h(n) \neq O(f(n))$ ✓
(d) $f(n) \cdot g(n) \neq O(g(n)) \cdot h(n))$ ✗

Topic : Asymptotic Comparisons

#Q. $f(n)=2^n$; $g(n) = n^n$

$2^n < n^n$

A. $f(n) = O(g(n))$

B. $f(n) = \Omega(g(n))$

C. $f(n) = \theta(g(n))$

D. None of these

Topic : Asymptotic Comparisons

#Q. $f(n) = n.2^n$; $g(n) = 4^n$

A. $f(n) = O(g(n))$

B. $f(n) = \Omega(g(n))$

C. $f(n) = \theta(g(n))$

D. None of these

$n \cdot 2^n \qquad 4^n$

$\log_2(n \cdot 2^n) \qquad \log_2 4^n$

$$\boxed{[\log_2 n + n] \quad vs \quad 2n \\ < (n+n)}$$

Topic : Asymptotic Comparisons

#Q. $f(n) = n^2 . \log n; \; g(n) = n^{100}$

H/W

A. $f(n) = O(g(n))$

B. $f(n) = \Omega(g(n))$

C. $f(n) = \theta(g(n))$

D. None of these

Topic : Asymptotic Comparisons

$\log_2 n$ $\log_{10} n$

#Q. $f(n) = \log_2^{n}$; $g(n) = \log_{10}^{n}$

H/W

A. $f(n) = 0\ (g(n))$

B. $f(n) = \Omega(g(n))$

C. $f(n) = \theta\ (g(n))$

D. None of these

Topic : Asymptotic Comparisons

#Q. $f(n) = 2^n; g(n) = n^{\sqrt{n}}$

H/W

A. $f(n) = O(g(n))$

B. $f(n) = \Omega(g(n))$

C. $f(n) = \theta(g(n))$

D. None of these

Topic : Asymptotic Comparisons

#Q. $f(n) = n^{\log_2^n}$; $g(n) = n^{\log_{10}^n}$

H/W

A. f(n) = O (g(n))

B. f(n) = Ω(g(n))

C. f(n) = θ (g(n))

D. None of these

Topic : Arrange in increasing order:

#Q. $\log n;\ \log_n^{10};\ \log\ \log n; (\log\ \log n)^{10}$

H/W

Topic : Arrange in increasing order:

#Q. 2^{2^n} ; $n!$; 4^n ; 2^n

H/W

Topic : Arrange in increasing order:

#Q. $2^{\log n}; (\log n)^2; \sqrt{\log n}\ ;\ \log\ \log n$

H/W

Topic : Asymptotic Comparisons

Q) Which one of the following statements is TRUE for all positive functions f(n)?

(GATE-22)

(a) $f(n^2) = \Theta(f(n)^2)$, when f(n) is a polynomial $\quad n^x$

(b) $f(n^2) = o(f(n)^2)$ ×

(c) $f(n^2) = O(f(n)^2)$, when f(n) is an exponential function ×

(d) $f(n^2) = \cancel{\Omega(f(n^2))}$

$= \Omega((f(n))^2)$ ×

$f(n) = n^2$

$= n^4$

$(n^2)^2 = n^4$

$f(n) = 2^n$

$f(n^2) = 2^{n^2}$

$(f(n))^2 = (2^n)^2$

$= 2^{2n}$

$= 4^n$

$= 2^{n^2} \quad 2^{2n}$

$n^2 > 2n$

$f(n) = \log n$

$f(n^2) = \log n^2$

$= 2 \cdot \log n$

$= (\log n + \log n)$

$(f(n))^2 = (\log n)^2$

$= (\log n) * (\log n)$

$< (\log n) * (\log n)$

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H/w Q: Given $f(n)$ & $g(n)$ as +ve fn's

also that $f(n) = O(g(n))$

a) Is $f(n) = O(f(n)^2)$?

b) Is $2^{f(n)} = O(2^{g(n)})$?

Q) Let $W(n)$, $A(n)$ denote respectively the worst case and Average case running Time of an Algorithm on I/P size 'n'. Which of the following is always TRUE?

Always True a) $A(n) = O(W(n))$

b) $A(n) = \Omega(W(n))$ (Sometimes True)

c) $A(n) = \Theta(W(n))$ Sometimes

d) $A(n) = o(W(n))$ (Sometimes True)

$$\boxed{B(n) \le A(n) \le W(n)}$$

$\rightarrow A(n) = O(W(n))$

Always be false

$$A(n) = \omega(W(n))$$

THANK - YOU