ALL BRANCHES

ME,CE,EC,EE,CS

Probability and Statistics

Lecture No- 04

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Topics to be Covered

01 Introduction to Random Variables

02 Discrete Random Variables

03 Continuous Random variables

04 Discrete Distributions

Revision

Random Experiment:

Sample Space.

$P(E)$

$E_i \cap E_j = \phi \rightarrow$ Mutually Exclusive Events.

$E_1 \cup E_2 = S \rightarrow$ Mutually Exhaustive Events.

$P(A \cap B) = P(A) \cdot P(B) \rightarrow$ Independent Events.

Addition Theorem $\rightarrow$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability $\rightarrow$ $P(A/B)$

Multiplication Theorem $\rightarrow$ $P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$

$$P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)$$

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A|E_i)}$$ → Bayes' Theorem.

Introduction to Random variables

A Variable that connects outcome of a random Experiment to a real number is called Random Variable.

Ex: 'x' is the Number of times Head appears when a coin is flipped 3 times.

$$X = 0 \quad 1 \quad 2 \quad 3$$

Ex: 'x' is the diameter of a Shaft Picked at random from a lot.

$$x = [149.8 \text{ to } 150.2]$$

Random Variables
- → Discrete Random Variables (Value is obtained by counting)
- → Continuous Random Variables (Value is obtained by Measuring)

Discrete Random Varaiables

The Value of the Random Variable is obtained by counting.

$$\text{Mean} = \bar{x} \text{ (or) } \mu = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \cdots + f_nx_n}{f_1 + f_2 + f_3 + \cdots + f_n = N}$$

$$\boxed{\begin{array}{l} f_1 \to x_1 \\ f_2 \to x_2 \\ f_n \to x_n \end{array}}$$

$$\frac{f_{1c_1}}{N_{c_1}}$$

$$\Rightarrow \bar{x} \text{ (or) } \mu = \frac{f_1}{N}\cdot x_1 + \frac{f_2}{N}\cdot x_2 + \frac{f_3}{N}\cdot x_3 + \cdots + \frac{f_n}{N}\cdot x_n.$$

$$\Sigma f_i = N$$

$$P(x = x_1) = \frac{f_1}{N};\ P(x = x_2) = \frac{f_2}{N};\ \cdots P(x = x_n) = \frac{f_n}{N}.$$

$X : x_1\ x_2\ x_3 \cdots x_n$

→ The value on the token.

$$\Rightarrow \text{Mean} = \bar{x} = \mu = x_1 \cdot P(x=x_1) + x_2 \cdot P(x=x_2) + x_3 \cdot P(x=x_3) + \dots + x_n \cdot P(x=x_n).$$

$$\boxed{\Rightarrow \bar{x} \text{ (or) } \mu = \sum_{i=1}^{n} x_i \cdot P(x=x_i)} \longrightarrow \text{Expected Value of 'x'} = E(x).$$

$$\boxed{\therefore E(x) = \bar{x} = \mu = \sum_{i=1}^{n} x_i \cdot P(x=x_i)}$$

$$\text{Variance} = \sum_{i=1}^{n} \frac{f_i(x_i-\mu)^2}{N} = \frac{1}{N} \cdot \sum_{i=1}^{n} f_i(x_i-\mu)^2$$

$$= \frac{1}{N} \cdot \sum_{i=1}^{n} f_i(x_i^2 + \mu^2 - 2\mu x_i)$$

$$\Rightarrow \sigma^2 = \frac{1}{N}\left\{ \sum_{i=1}^{n} f_i x_i^2 + \sum_{i=1}^{n} f_i \mu^2 - \sum_{i=1}^{n} 2\mu f_i x_i \right\}$$

$$\Rightarrow \sigma^2 = \frac{1}{N} \cdot \left\{ \sum_{i=1}^{n} f_i x_i^2 + \mu^2 \cdot \sum_{i=1}^{n} f_i - 2\mu \cdot \sum_{i=1}^{n} f_i x_i \right\}$$

$$\Rightarrow \sigma^2 = \sum_{i=1}^{n} x_i^2 \cdot P(x = x_i) + \frac{\mu^2}{\cancel{N}} \cdot \cancel{\sum_{i=1}^{n} f_i} - 2\mu \cdot \left(\cancelto{\mu}{\frac{\sum_{i=1}^{n} f_i x_i}{N}} \right)$$

$$\Rightarrow \sigma^2 = \sum_{i=1}^{n} x_i^2 \cdot P(x = x_i) + \mu^2 - 2\mu^2 = \sum_{i=1}^{n} x_i^2 \cdot P(x = x_i) - \mu^2 .$$

$$= E(x^2) - (E(x))^2 .$$

$$\therefore \boxed{\sigma^2 = E(x^2) - (E(x))^2 \geq 0}$$

→ Probability Distribution Table:

Let 'x' be a discrete Random Variable with Values $x_1, x_2, x_3, \ldots x_n$. Then the Probability Distribution Table of x is given by

$X =$	x_1	x_2	x_3	------	x_n
$P(x) =$	$P(x=x_1)$	$P(x=x_2)$	$P(x=x_3)$		$P(x=x_n)$

$$\sum_{i=1}^{n} P(x=x_i)=1$$

→ Let 'x' be the Number of Heads appeared when 3 coins are tossed.

X →	0	1	2	3
P(x)	1/8 +	3/8 +	3/8 +	1/8 = 1

Probability distribution Table.

$S=\{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)\}$

Expectation of x = $E(x) = \sum_{i=1}^{n} x_i \cdot P(x=x_i)$

$= 0 \cdot P(x=0) + 1 \cdot P(x=1) + 2 \cdot P(x=2) + 3 \cdot P(x=3)$

$= 0 \cdot \left(\frac{1}{8}\right) + 1 \cdot \left(\frac{3}{8}\right) + 2 \cdot \left(\frac{3}{8}\right) + 3 \cdot \left(\frac{1}{8}\right)$

$= \frac{12}{8} = 1.5 = E(x)$

$\therefore$ Expectation of 'x' = 1.5 //

Variance of 'x' $= \sigma^2 = E(x^2) - (E(x))^2 = \sum_{i=1}^{n} x_i^2 \cdot P(x=x_i) - (E(x))^2$

$$= 0^2 \cdot P(x=0) + 1^2 \cdot P(x=1) + 2^2 \cdot P(x=2) + 3^2 \cdot P(x=3) - (E(x))^2$$

$$= 0 + 1 \cdot \left(\frac{3}{8}\right) + 4 \cdot \left(\frac{3}{8}\right) + 9 \cdot \left(\frac{1}{8}\right) - (1.5)^2$$

$$= 3 - (2.25) = 0.75$$

$\therefore$ Variance of $x = \sigma^2 = 0.75$

$\Rightarrow$ Standard deviation of $x = \sigma = \sqrt{0.75} = 0.866$ //

$$P(x \geq 1) = P(x=1) + P(x=2) + P(x=3)$$

$$= \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}.$$

$\underbrace{\text{Population}}_{\text{Complete units available}}$, $\underbrace{\text{Sample}}$ → Some Part of the total Population.

Population: $\sigma = \sqrt{\frac{1}{n} \cdot \Sigma (x_i - \mu)^2}$.
$(n > 29)$

Sample: $\sigma = \sqrt{\frac{1}{(n-1)} \cdot \Sigma (x_i - \mu)^2}$
→ $(n < 29)$

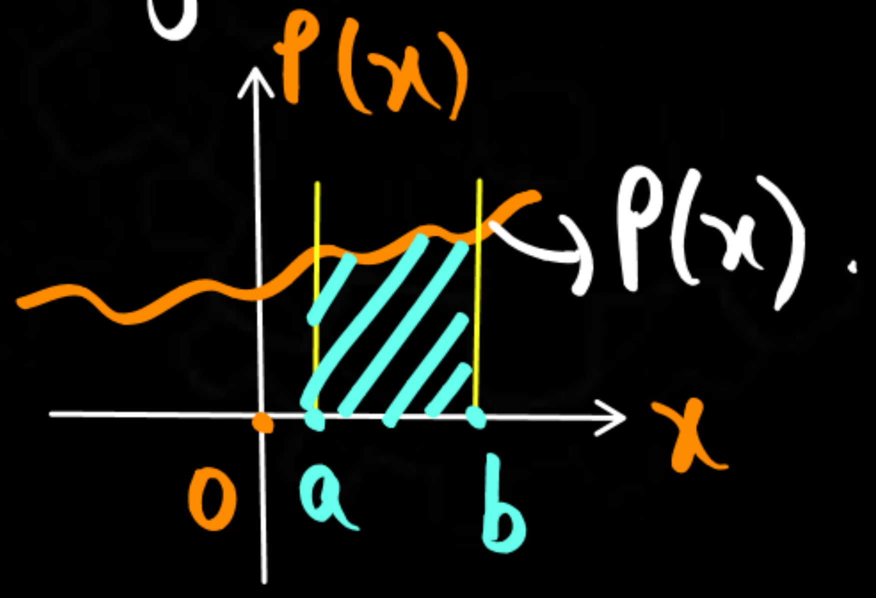
Continuous Random Variables

The Value of the Random Variable is obtained by Measuring.

Probability Density (or) Distribution function (Pdf):

A continuous function $P(x)$ is defined as Probability density function of a random Variable 'x' is given by

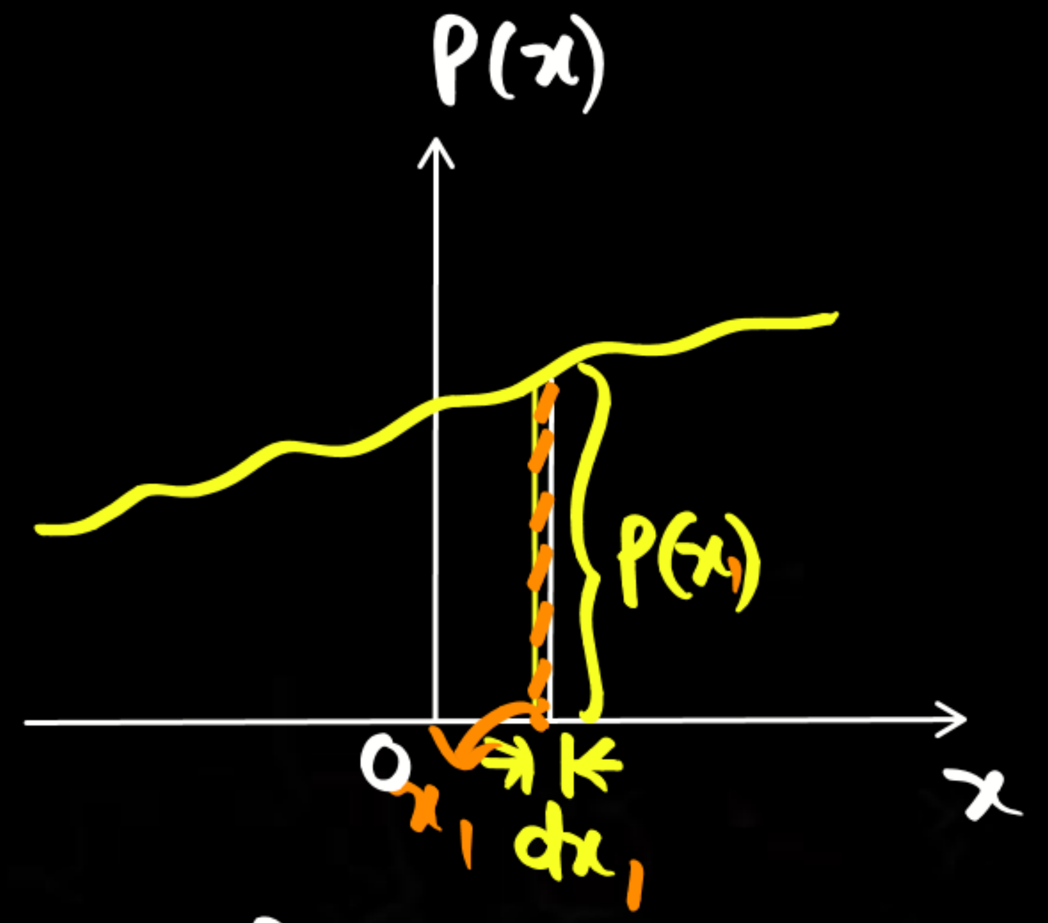
$$P(a \leq x \leq b) = \int_a^b P(x)\,dx \, ;$$



and $\displaystyle\int_{-\infty}^{\infty} P(x)\,dx = 1$

Mean of Continuous Random Variable:

The Mean of a continuous Random Variable is given by



$$E(x) = \mu = \bar{x} = \int_{-\infty}^{\infty} x \cdot P(x)\, dx$$

Ex: let $P(x) = \begin{cases} e^{-x} ; & x > 0 \\ 0 ; & x \le 0 \end{cases}$

$$\int_{-\infty}^{\infty} P(x)\, dx = 1$$

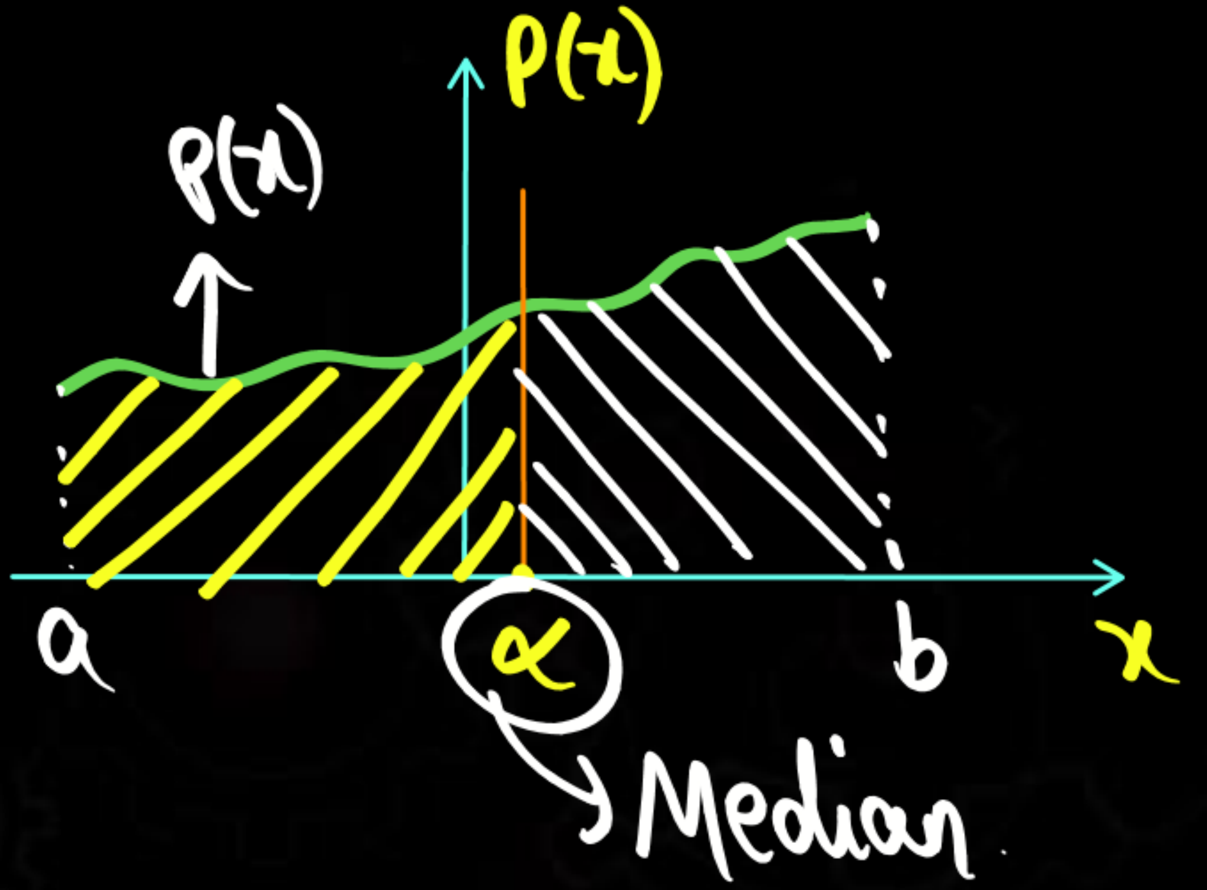
$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot P(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{\infty} x \cdot e^{-x}\, dx$$

$$= \int_{0}^{\infty} e^{-x} \cdot x\, dx = \Gamma 2 = 1! = 1$$

Median of a Continuous Random Variable:

The Value of 'x' (Say α) is median of continuous Random Variable 'x' if the line $x=\alpha$ divides the complete area under $P(x)$ to two equal halfs.

$$\int_{-\infty}^{\alpha} P(x)\,dx = \int_{\alpha}^{\infty} P(x)\,dx.$$



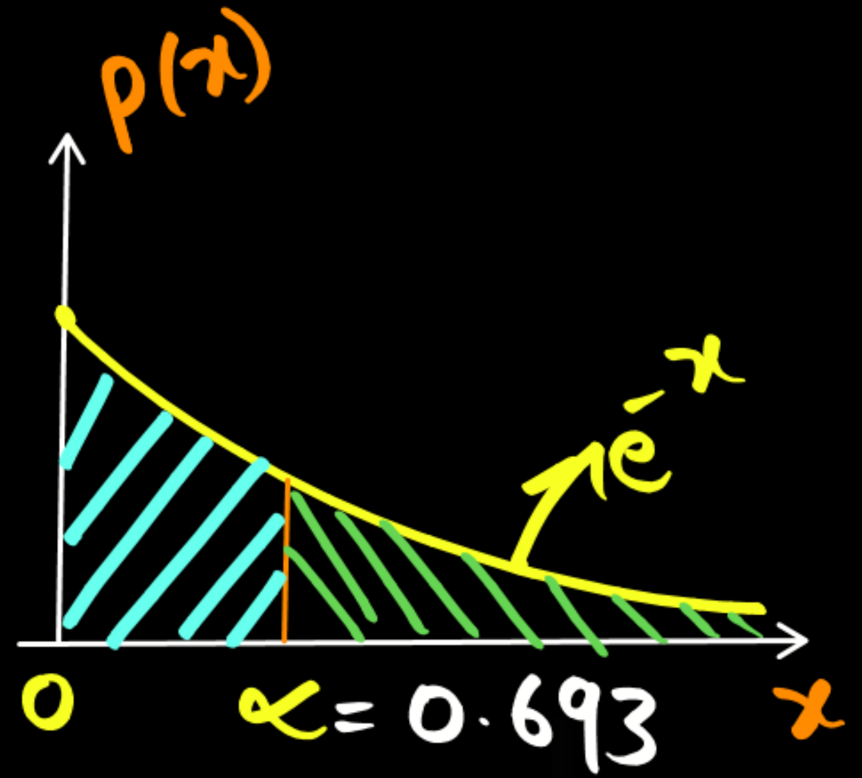
Let $P(x) = \begin{cases} e^{-x}; & x > 0 \\ 0; & x \le 0 \end{cases}$; Median = ?

$$\int_{-\infty}^{\alpha} P(x)\,dx = \int_{\alpha}^{\infty} P(x)\,dx.$$

$$\Rightarrow \int_{0}^{\alpha} e^{-x}\,dx = \int_{\alpha}^{\infty} e^{-x}\,dx.$$

$$\Rightarrow \left.\frac{e^{-x}}{\cancel{-1}}\right|_0^{\alpha} = \left.\frac{e^{-x}}{\cancel{-1}}\right|_{\alpha}^{\infty} \Rightarrow e^{-\alpha} - 1 = 0 - e^{-\alpha} \Rightarrow 2 \cdot e^{-\alpha} = 1$$

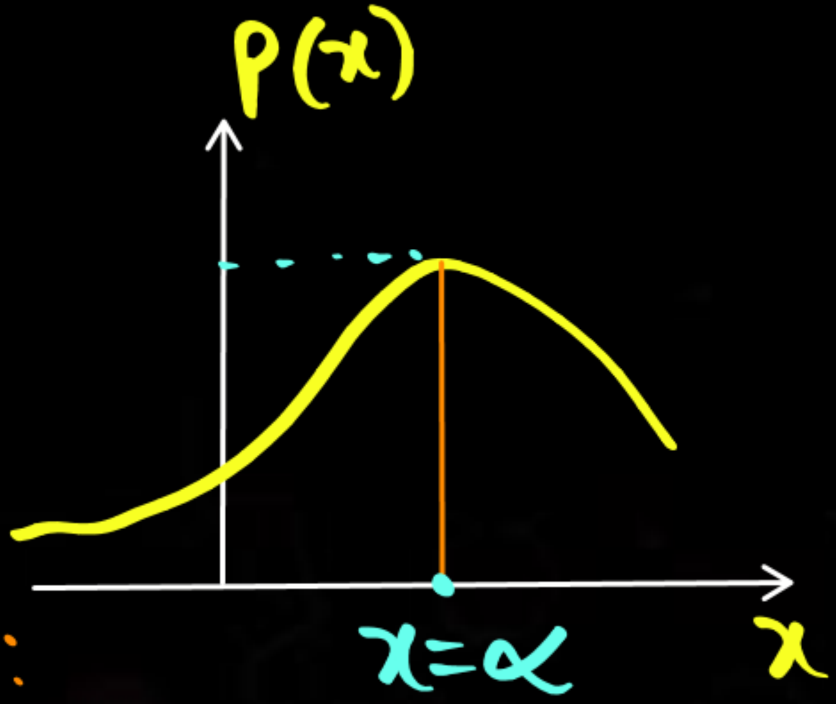
$$\Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow \alpha = -\ln\left(\frac{1}{2}\right) = 0.693.$$



$\therefore$ Median = 0.693 //.

Mode of a Continuous Random Variable:

A Value 'α' is Said to be Mode of a continuous Random Variable if $P(x)$ is maximum at $x=\alpha$.



Variance of a Continuous Random Variable:

$$\sigma^2 = E(x^2) - (E(x))^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)\,dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)\,dx\right)^2.$$

$\rightarrow$ Let $P(x) = \begin{cases} e^{-x}; & x \geq 0 \\ 0; & x < 0 \end{cases}$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)\,dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)\,dx\right)^2$$

$$\int_0^{\infty} e^{-x} \cdot x^{n-1}\,dx = \Gamma n$$
$$= (n-1)!$$

$$\Rightarrow \sigma^2 = \int_0^{\infty} e^{-x} \cdot x^2\,dx - \left(\int_0^{\infty} e^{-x} \cdot x\,dx\right)^2$$

$$\Rightarrow \sigma^2 = 2! - (1!)^2 = 2 - 1 = 1$$

$$\Rightarrow \sigma^2 = 1 \Rightarrow \sigma = 1$$

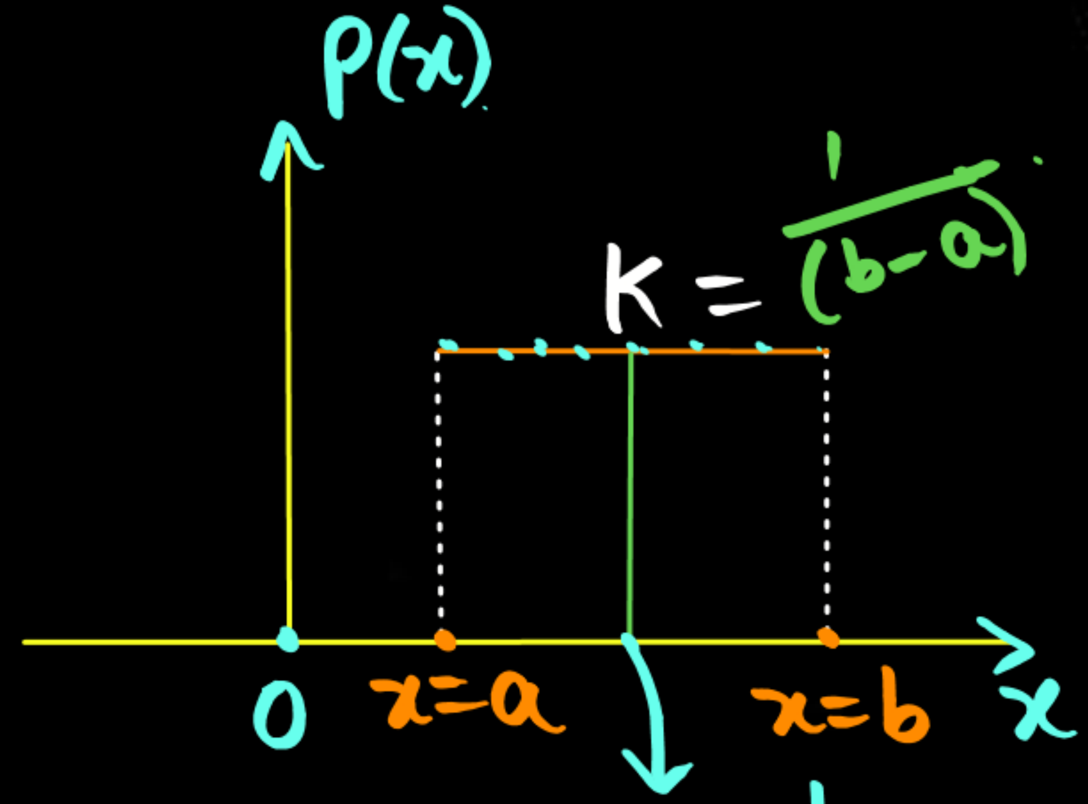
→ uniformly distributed Random Variable:

$$P(x) = K \; \{a \leq x \leq b\}$$

Since Total Probability $= 1$

$$\Rightarrow \int_a^b K \cdot dx = 1 \Rightarrow K = \frac{1}{(b-a)}$$

$$\boxed{\therefore P(x) = \frac{1}{(b-a)}}$$



$x = \frac{a+b}{2}$ is Median

This distribution has infinite Number of modes.

$$\sigma^2 = E(x^2) - (E(x))^2.$$

$$\Rightarrow \sigma^2 = \int_a^b x^2 \cdot P(x)\,dx - \left(\int_a^b x \cdot P(x)\,dx\right)^2$$

$$\Rightarrow \sigma^2 = \int_a^b x^2 \cdot K \cdot dx - \left(\int_a^b x \cdot K \cdot dx\right)^2$$

$$\Rightarrow \sigma^2 = K \cdot \left(\frac{b^3 - a^3}{3}\right) - K^2 \cdot \left(\frac{b^2 - a^2}{2}\right)^2.$$

$$\Rightarrow \sigma^2 = \frac{1}{\cancel{(b-a)}} \cdot \frac{1}{3}\left(\cancel{(b-a)}(b^2+ab+a^2)\right) - \frac{1}{\cancel{(b-a)^2}} \cdot \frac{1}{4} \cdot \cancel{(b-a)^2} \cdot (b+a)^2$$

$$\Rightarrow \sigma^2 = \frac{a^2+ab+b^2}{3} - \frac{(a^2+2ab+b^2)}{4}$$

$$\Rightarrow \sigma^2 = \frac{a^2 - 2ab + b^2}{12}$$

$$\boxed{\Rightarrow \sigma^2 = \frac{(b-a)^2}{12}}$$

$$\Rightarrow \sigma = \frac{1}{\sqrt{12}}(b-a)$$

Thank You!

GW Soldiers