CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 05

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Recap of Previous Lecture

Topic Asymptotic Notations

Topic Big Oh, Big Omega & Theta Notations

O Ω Θ

Topics to be Covered

Topic Asymptotic Notations

Topic Little Oh, Little Omega

Topic Problem solving

Big Oh (O) $\longrightarrow$ Upper Bound (UB)

Big Omega (Ω) $\longrightarrow$ Lower Bound (LB)

Theta (Θ) $\longrightarrow$ Tight Bound (TB)

1) Big Oh (O) : UB

→ $f(n)$ is $O(g(n)) \implies f(n) \le c * g(n)$, c = const, $c > 0$

Whenever $n \ge n_0$

2) Big Omega (Ω): LB

→ $f(n)$ is $\Omega(g(n))$ iff $f(n) \geq c * g(n)$, $\underline{c > 0}$

whenever $\underline{n \geq n_0}$

3) Theta (Θ): TB ⟹ $f(n)$ is $\Theta(g(n))$ iff

$\boxed{f(n) \text{ is } O(g(n))}$ and $\boxed{f(n) \text{ is } \Omega(g(n))}$

1) Smaller functions are in the order of larger functions.

eg:

$n^2 \leq n^3 \longrightarrow n^2 = O(n^3)$

$n^2 \leq 2^n \longrightarrow n^2 = O(2^n)$

poly expo

2) Larger functions are in the Omega of the smaller functions

3) If the rate of growth of both functions are $\boxed{\text{equal}}$, then they are $\boxed{\text{theta}}$ of each other

Topic : Time Complexity

$f < c \cdot g(n)$ $n > n_0$ $c > 0$

lose bound

Set 1 — Set 2

(a) $O(n^2)$		(b) $\Omega(n^2)$	
$3\log n+8$	$4n^2$	$4n^2$	$4n^3 + 3n^2$
$\boxed{5n + 7}$	$6n^2 + 9$	$6n^2 + 9$	$6n^6 + n^4$
$2n\log n$	$5n^2 + 2n$	$5n^2 + 2n$	$2n^3 + 4n$

$(5n+7) \rightarrow O(n^2)$

$f \geq c * g(n)$

15, 14, 13, 12, 11, 10

$\boxed{5n+7 \rightarrow O(n)}$

$\theta(n^2)$

Set 1 — Set 2

Set 1 only	Set 1 ∩ Set 2	Set 2 only
$3\log n+8$	$4n^2$	$4n^3 + 3n^2$
$5n + 7$	$6n^2 + 9$	$6n^6 + n^4$
$2n\log n$	$5n^2 + 2n$	$2n^3 + 4n$

(b) $\theta(n^2) = O(n^2) \cap \Omega(n^2)$

$f = \underbrace{5n+7}$

$\boxed{O(n)}$ → Closest UB

$O(n^2)$

$O(n^3)$

$O(n^5)$

⋮

20L

10L

5L

3L

100

Topic : Exponentials

For all real a> 0, m, n

$a^0 = 1$

$a^1 = a$

$a^{-1} = 1/a$

$(a^m)^n = a^{mn}$

$(a^m)^n = (a^n)^m$

$a^m * a^n = a^{m+n}$

$a^{-2} = \frac{1}{a^2} = \left(\frac{1}{a}\right)^2$

$(a^m)^n = a^{m*n}$

eg:- $(2^3)^2 = (8)^2 = \boxed{64}$

$(2^3)^2 = 2^{3\times2} = 2^6 = \boxed{64}$

eg: $2^3 * 2^2 = 8*4 = 32$

$= 2^{(3+2)} = 2^5 = 32$

$2^{2\times3} = 2^{3\times2}$

$(2^2)^3 = (2^3)^2$

$= 4^3 = 8^2$

Topic : Analysis of Algorithms

$\log X^y = y \log x$

$\log(xy) = \log x + \log y$

$\log \log n = \log(\log n)$

$a^{\log_b^x} = x^{\log_b^a}$

$a = b^{\log_b^a}$

$\log_b a^n = n.\log_b a$

$$\boxed{\log_b a = \frac{1}{\log_a b}}$$

$\log n = \log_{10}^n$

$\log^k n = (\log n)^k$

$\log\left[\frac{x}{y}\right] = \log x - \log y$

$\log_b^x = \frac{\log_a^x}{\log_a^b}$

$\log_c(ab) = \log_c a + \log_c b$

$\log_b^{(1/a)} = -\log_b a$

$a^{\log_b c} = c^{\log_b a}$

$\log(\log(\log \ldots)) X$

$\log_2(n)^y$

$\log_2(n)$

$a^{(\log_b x)}$

$= x^{(\log_b a)}$

$\log_2 n$

$\boxed{\log x}$

$\to \log_2(x)$

$\log_n(n) = 1$

$b^{\log_b(a)}$

$a^{\boxed{\log_b(b)}}$ $= a^1 = a$

$\boxed{\log_b\left(\frac{1}{a}\right) = -\log_b a}$

$x = a \Rightarrow \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$

$$\log^k n \Rightarrow (\log(n))^k$$

$$\log_b\left(\frac{1}{a}\right) = \cancel{\log_b(1)}^{0} - \log_b(a)$$

$$= 0 - \log_b a$$

$$= \boxed{-\log_b(a)}$$

Topic : Geometric Sum Formula

$$\boxed{\sum_{i=1}^{n} \frac{1}{2^i} = \left(1-\frac{1}{2^n}\right)}$$

1. The geometric sum formula for finite terms is given as:

if r = 1, $S_n = n^*a$ ⇒ [2, 2, 2, 2] $n*2$ $n=5$

if |r| < 1, $S_n = \dfrac{a(1-r^n)}{(1-r)}$ GP: $2, 2^2, 2^3, 2^4, 2^5 \ldots$

if |r| > 1, $S_n = \dfrac{a(r^n-1)}{(r-1)}$

$a=2$, $r = \dfrac{2^2}{2} = 2$

Where

- a is the first term
- r is the common ratio
- n is the number of terms

$$\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} \cdots + \frac{1}{2^n}$$

$$a\left(\frac{1-r^n}{1-r}\right) = \cancel{\frac{1}{2}} \frac{\left(1-\frac{1}{2^n}\right)}{\cancel{1-\frac{1}{2}}} = \left(1-\frac{1}{2^n}\right), \quad r = \frac{1}{2}$$

eg:

$$\sum_{i=1}^{n} 2^i = \frac{2*(2^n-1)}{2-1} = 2(2^n-1)$$

Topic : Geometric Sum Formula

2. The geometric sum formula of infinite terms is given as:

if $\underbrace{|r| < 1}$ ✓ $\boxed{S_\infty = \frac{a}{1-r}}$

if $\underbrace{|r| > 1}$, the series does not converge and it has no sum.

$$\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{(1-x)}} \quad x < 1$$

Topic : Analysis of Algorithms

Arithmetic series (AP)

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + + n = \boxed{\frac{n(n+1)}{2}}$$

eg $\rightarrow$ $\sum_{i=1}^{5} i = 1+2+3+4+5$

$= 15$ ✓

$\Rightarrow \frac{n(n+1)}{2}$

$= \frac{5 \times (5+1)}{2} = \frac{5 \times 6}{2}$

$= 5 \times 3$

$= 15$

Geometric series (GP)

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{(n+1)} - 1}{x - 1} \quad (x \neq 1)$$

$\frac{x^{(n+1)} - 1}{x-1}$

Harmonic series (HP)

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n$$

In general, Dominance Relation

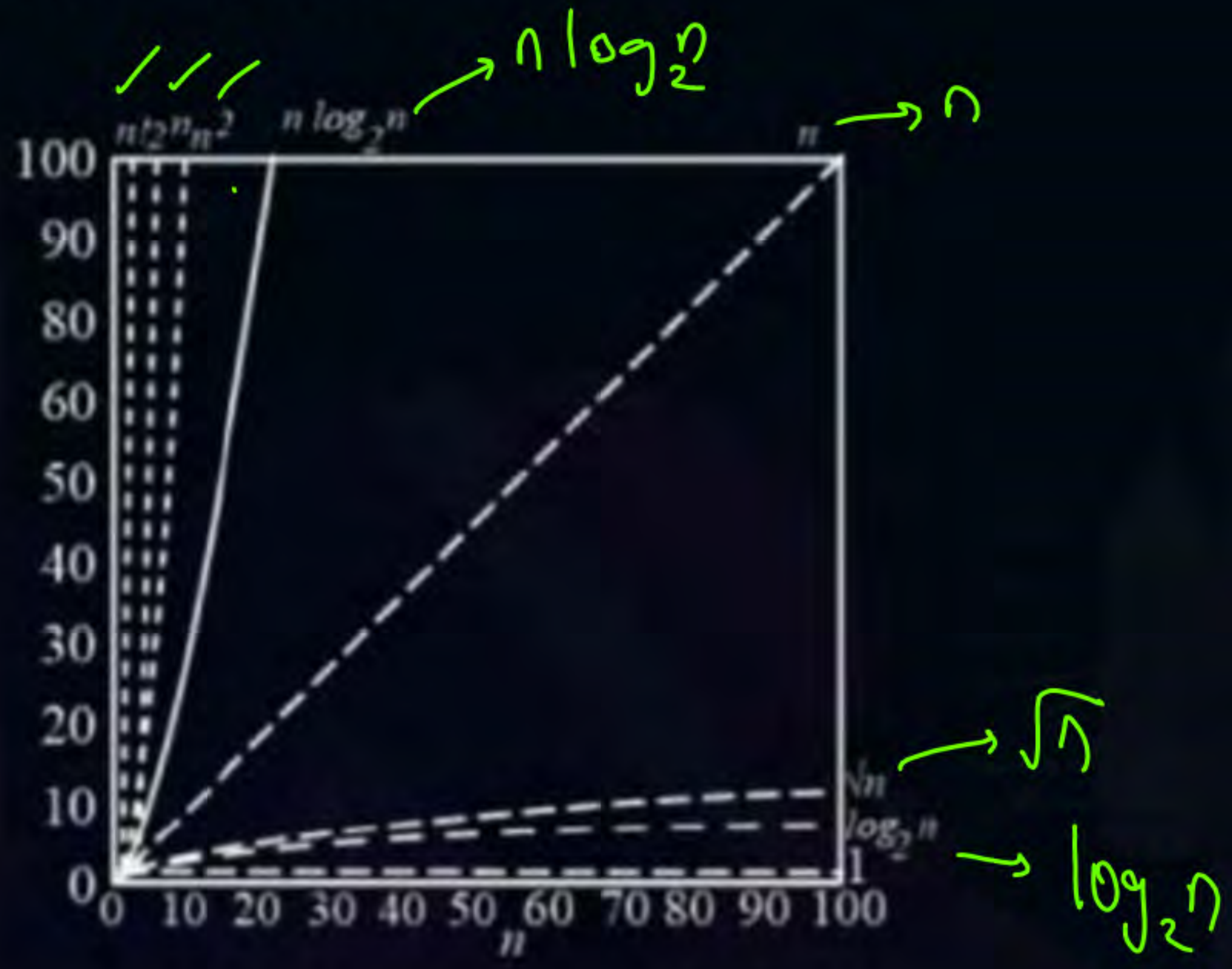
$$c < \log n < \sqrt{n} < n < n\log(n) < n^2 < n^3 < n^4 \ldots < 2^n < 4^n < \ldots < n! < n^n < (n^n)^n \ldots$$

$$\boxed{\text{Constants} < \text{logarithm} < \text{Polynomial} < \text{Exponential}}$$

Topic : Analysis of Algorithms

Dominance Relation

Constant < logarithms < poly < exponential



Practice Problems :-

1) $f(n) = \sum_{i=1}^{n} 1$

2) $f(n) = \sum_{i=1}^{n} i$

3) $f(n) = \sum_{i=1}^{n} n$

4) $\sum_{i=1}^{n} i^2$

5) $\sum_{i=1}^{n} i^3$

6) $\sum_{i=1}^{n} 2^i$

7) $\sum_{i=1}^{n} \left(\frac{1}{2}\right)^i$

8) $\sum_{i=1}^{n} i * 2^i$

1) $\sum_{i=1}^{n} 1 = \underbrace{1+1+1 \cdots\cdots +1}_{n \text{ times}} = n*1 = n$

$f = \sum_{i=1}^{n} 1 \rightarrow n = O(n)$

2) $\sum_{i=1}^{n} i = 1+2+3+4 \cdots\cdots + n$

$= \boxed{\frac{n(n+1)}{2}}$

$\frac{n(n+1)}{2} = \frac{n^2+n}{2}$

$= \frac{n^2}{2} + \frac{n}{2}$

$= O(n^2)$

3) $\sum_{i=1}^{n} n = n * \sum_{i=1}^{n} 1 = n * n = n^2 = O(n^2)$

4) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}$

$= \frac{(n^2+n)(2n+1)}{6} = \frac{n^2(2n+1) + n(2n+1)}{6}$

$= O(n^3)$

5) $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 \cdots\cdots n^3$

$= \left[\frac{n(n+1)}{2}\right]^2 = \left(\frac{n^2+n}{2}\right)^2 \Rightarrow O(n^4)$

$2 \times 2^n - 2$

$O(2^n)$

$\underbrace{2 \times 2^n}_{f} \leq 4 \times 2^n$

$c \times g$

6) $\sum_{i=1}^{n} (2)^i = 2^1 + 2^2 + 2^3 + \cdots + 2^n = \frac{a(r^n - 1)}{r-1}$

$a = 2$

$r = 2$

$= \frac{2 \times (2^n - 1)}{2-1} = \boxed{2(2^n - 1)} = 2 * 2^n - 2 = O(2^n)$

7) $\sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} \cdots + \frac{1}{2^n}$

$\underline{\underline{GP}} = a = 1/2$

$\frac{1}{2^n} \longrightarrow \underline{\underline{decr}}$

$r = 1/2 \quad < 1$

$\Rightarrow \frac{a(1-r^n)}{1-r} = \frac{\cancel{\frac{1}{2}} \times \left(1 - \left(\frac{1}{2}\right)^n\right)}{\cancel{1 - 1/2}}$

$= \left(1 - \frac{1}{2^n}\right) = \boxed{O(1)}$ ✓

H.W

8)

$$\sum_{i=1}^{n} i*2^i = 1\times 2^1 + 2\times 2^2 + 3\times 2^3 \cdots + n\times 2^n$$

$$= (n-1)*2^{(n+1)} + 2$$

$$= \left[n*2^{(n+1)} - 2^{(n+1)} + 2 \right]$$

$$= O(n*2^n)$$

$$\boxed{2^{n+1} \Rightarrow 2^1 * 2^n = 2*2^n = O(2^n)}$$

$n+2^n \longrightarrow O(2^n)$

$\boxed{n*2^n} \longrightarrow O(n\times 2^n)$

9) $f(n) = \prod_{i=1}^{n} (1)$

10) $f(n) = \prod_{i=1}^{n} (i) \Longrightarrow$ (Sterling's Approximation)

11) $f(n) = \sum_{i=1}^{n} \log(i)$

Soln :-

9) $f(n) = \prod_{i=1}^{n} 1 = \underbrace{1 \times 1 \times 1 \cdots 1}_{n \text{ times}}$

$= \underline{1}$

$= \underline{\underline{O(1)}}$

10) $f(n) = \prod_{i=1}^{n} i$

$= 1 \times 2 \times 3 \times 4 \, \ldots \ldots \, n$

$= n!$ (n factorial)

$$n! = 1 \times 2 \times 3 \cdots * n$$

$$= \underbrace{n * (n-1) * (n-2) \cdots 1}$$

$$\boxed{n! \quad vs \quad n^{(n)}}$$

eg:- $n = 5$

$5 \times 4 \times 3 \times 2 \times 1$

$= 5!$

$$n * \underline{(n-1)} < n * \underline{n}$$

$$\underbrace{n \times (n-1) \times (n-2) \cdots * 1}_{n!} < \underbrace{n \times n \times n \cdots n}_{n \text{ times}}$$

$$n! < 1 * n^{(n)} \Rightarrow \boxed{n! = O(n^n)}$$

2nd $\longrightarrow$ Using Stirling's Approximation

$$n! \sim \sqrt{2\pi n} * \left(\frac{n}{e}\right)^n \quad \text{vs} \quad n^n$$

$$\sqrt{2\pi} * \sqrt{n} * \left(\frac{n}{e}\right)^n \qquad n^n$$

$$\Rightarrow C * \sqrt{n} * \frac{\cancel{n^n}}{e^n} \qquad \cancel{n^n}$$

$$\boxed{C * \sqrt{n} < e^n}$$

Is $n! \Rightarrow \Omega(n^n)$?

But

$$\boxed{\begin{aligned} n! &= O(n^n) \\ n! &\neq \Omega(n^n) \end{aligned}}$$

$$n! = n \times (n-1) \times (n-2) \cdots \times 1 \ \cancel{\geq} \ \underbrace{n \times n \times \cdots n}_{n \text{ times}}$$

$$\boxed{n! \neq \Omega(n^n)}$$

11) $f = \sum_{i=1}^{n} \log(i) = \log(1) + \log(2) + \dots \log(n)$

$\log(a * b) = \log a + \log b$

$= \log(1 \times 2 \times 3 \dots \times n)$

$= \boxed{\log(n!)}$ ✓

$$\boxed{\log(n!) = \Theta(n \log n)}$$

Ap!: Sterling's Approximation

$$n! \sim \sqrt{2\pi n} \times \left(\frac{n}{e}\right)^n$$

$\boxed{f \geq c \cdot g(n)}$

$$\log(n!) = \log\left(\sqrt{2\pi n} * \left(\frac{n}{e}\right)^n\right)$$

$\sqrt{n} = (n)^{1/2}$

$$= \log(\sqrt{2\pi}) + \log(\sqrt{n}) + \log\left(\left(\frac{n}{e}\right)^n\right)$$

$$= \log(\sqrt{2\pi}) + \frac{1}{2}\log(n) + \left[\log(n)^n - \log(e^n)\right]$$

$$= \boxed{\log(\sqrt{2\pi})} + \frac{1}{2}\log n + n\log n \boxed{- n\log e}$$

$\Rightarrow \log(n!)$

$O(n \log n)$

$\Omega(n \log n)$

① $n! = O(n^n)$

HW1

But $n! \neq \Omega(n^n)$

HW2

more approaches

② $\log(n!) = O(n \log n)$

$\Omega(n \log n)$ $\rightarrow$ $\Theta(n \log n)$

2 mins Summary

Topic Revision

Topic Problem Solving

THANK - YOU