ALL BRANCHES

ME,CE,EC,EE,CS

Probability and Statistics

Lecture No- 05

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Topics to be Covered

01 Binomial Distribution

02 Poisson Distribution

03 Gaussian Distribution

04 Exponential Distribution

Revision

$$X \rightarrow \begin{cases} \text{Discrete} \\ \text{Continuous} \end{cases}$$

$$E(x) = \sum x_i \cdot P(x = x_i) \quad ; \quad E(x) = \int_{-\infty}^{\infty} x\, P(x)\, dx$$

Probability distribution function.

$$\sigma^2 = E(x^2) - (E(x))^2 \geq 0$$

$$= \sum x_i^2 \cdot P(x = x_i) - \left(\sum x_i \cdot P(x = x_i)\right)^2$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)\, dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)\, dx\right)^2$$

Binomial Distribution

(Bernoulli Trials).

A Trial with only two Possible outcomes.

A Random Variable 'x' is Said to be Binomially distributed if Bernoullis trials are conducted Some finite number of times where one of the outcomes is treated as success and the Probability of Success is 'p', then the number of Success is given by x.

'x' is time Independent.

The Probability of getting 'r' Successes in 'n' Bernoulli trials is given by $P(x=r) = {}^nC_r \cdot p^r \cdot q^{n-r} = {}^nC_r \cdot p^r \cdot (1-p)^{n-r}$

Where $p \rightarrow$ Probability of Success.

$q \rightarrow$ Probability of failure $= 1-p$.

→ Ex: The Probability of getting 2 heads in 3 tosses of a fair coin is ___.

$X \longrightarrow$ Number of Heads.

$P(X=2) = ?$

$S = \{(H,H,H), (H,H,T), (H,T,H), (T,H,H), (H,T,T), (T,H,T), (T,T,H), (T,T,T)\}$

$P(2 \text{ Heads}) = 3/8$

P = Probability of getting a Head = 0.5

$q = 1 - P = 0.5$

$③ \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)$

$$P(x=2) = {}^{3}C_{2} \cdot p^2 \cdot q^1 = {}^{3}C_{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)$$

$$= 3 \cdot \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{3}{8}$$

$$\therefore P(x=2) = \frac{3}{8}$$

$$\text{Mean of } x = E(x) = \sum_{x_i=0}^{n} x_i \cdot P(x=x_i) = \sum_{r=0}^{n} r \cdot P(x=r)$$

$$\Rightarrow \text{Mean} = E(x) = \sum_{r=0}^{n} r \cdot {}^{n}C_{r} \cdot p^{r} \cdot (1-p)^{n-r} = np.$$

$$\text{Variance of } x = E(x^2) - (E(x))^2$$

$$= \sum_{r=0}^{n} r^2 \cdot {}^{n}C_{r} \cdot p^{r} \cdot (1-p)^{n-r} - \left(\sum_{r=0}^{n} r \cdot {}^{n}C_{r} \cdot p^{r} \cdot (1-p)^{n-r}\right)^2$$

$$= np(1-p) = npq.$$

Standard deviation of 'x' $= \sqrt{npq}$.

→ If 'x' is the number of Heads in flipping a coin for '6' times, Calculate Mean, Variance, Standard deviation.

Sol: $p \rightarrow$ Probability of getting a head $= 0.5$

$q \rightarrow$ Probability of getting a Tail $= 0.5$

$\therefore$ Mean $= np = 6(0.5) = 3$

Variance $= npq = 6(0.5)(0.5) = 1.5$

Standard deviation $= \sqrt{npq} = \sqrt{1.5} = 1.224$

→ For the Probability of getting '2' heads in 'n' tosses to be greater than 0.3 find the minimum number of tosses.

$$P(x=2) \geq 0.3$$

$$\Rightarrow {}^nC_2 \cdot p^r \cdot q^{n-2} \geq 0.3$$

$$\Rightarrow {}^nC_2 \cdot \left(\frac{1}{2}\right)^n \geq 0.3$$

$$\Rightarrow \frac{n(n-1)}{2} \geq 0.3(2^n)$$

$$\Rightarrow n(n-1) \geq 0.3(2^{n+1})$$

For $n=2$; $2(1) \geq 0.3(8)$

$$2 \geq 2.4 \times$$

For $n=3$; $3(2) \geq 0.3(2^4)$

$$\Rightarrow 6 \geq 4.8 \checkmark$$

Poisson Distribution

A Variable 'x' is Said to be in Poisson distribution if 'x' is obtained by counting the Number of Successes in which 'x' is time dependent.

Ex: 1. No. of calls received in a given time.

2. No. of Vehicles Passing a toll gate in a given time.

In Poisson distribution, No. of Successes is not limited.

If 'x' is distributed with Poisson distribution with a mean λ, then the Probability of $x=r$ (i.e 'r' successes) is given by

$$P(x=r) = \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}$$

$$\left.\begin{array}{l} \text{Mean of } X = \lambda \\ \text{Variance of } x = \lambda. \end{array}\right\} \Rightarrow \text{Mean} = \text{Variance} = \lambda.$$

Standard deviation $= \sqrt{\lambda}$.

$$\text{Variance} = E(X^2) - (E(X))^2.$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \cdot P(X=x);$$

$$\Rightarrow E(X^2) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \left\{ \sum_{x=0}^{\infty} \frac{x\,\lambda^x}{(x-1)!} \right\}$$

$$= e^{-\lambda} \left\{ \sum_{x=0}^{\infty} \frac{(x-1+1) \cdot \lambda^x}{(x-1)!} \right\}$$

$$= e^{-\lambda} \cdot \left\{ \sum_{x=0}^{\infty} \frac{(x-1) \cdot \lambda^x}{(x-1)!} + \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!} \right\}$$

$$E(X) = \sum_{x=0}^{\infty} x \cdot P(X=x) = \lambda$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda \cdot \lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda \cdot e^{-\lambda} \left\{ 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right\}$$

$$= \lambda \cdot \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \lambda.$$

$$= e^{-\lambda}\left\{ \sum_{x=2}^{\infty} \frac{\lambda^2 \cdot \lambda^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda \cdot \lambda^{x-1}}{(x-1)!} \right\}$$

$$= e^{-\lambda}\left\{ \lambda^2 \cdot \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \cdot \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right\}$$

$$= \cancel{e^{-\lambda}}\left\{ \lambda^2 \cdot \cancel{e^{\lambda}} + \lambda \cdot \cancel{e^{\lambda}} \right\}$$

$$= \lambda^2 + \lambda .$$

$$\therefore E(x^2) = \lambda^2 + \lambda$$

$$\therefore \sigma^2 = E(x^2) - (E(x))^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

$$\boxed{\therefore \sigma^2 = \text{Variance} = \lambda \Rightarrow \sigma = \sqrt{\lambda}.}$$

$\lambda = \lim_{n \to \infty} nP.$

For large values of 'n', (i.e $n \to \infty$),

Binomial distribution $\longrightarrow$ Poisson distribution.

→ The average number of cars passing a toll gate in an hour is 30. The Probability that atmost 3 cars pass the toll gate in 10 minutes is ____.

Sol:
$$P(X=x) = \frac{e^{-\lambda}\cdot\lambda^{x}}{x!}$$

λ for 10 minutes;

$\frac{\cancel{30}}{\cancel{60}} \times 1\cancel{0}$

$\lambda = 5$

$$P(X\le 3) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

10 minute

$$= \frac{e^{-\lambda}\cdot\lambda^{0}}{0!}+\frac{e^{-\lambda}\cdot\lambda^{1}}{1!}+\frac{e^{-\lambda}\cdot\lambda^{2}}{2!}+\frac{e^{-\lambda}\cdot\lambda^{3}}{3!} = e^{-\lambda}\left\{1+\lambda+\frac{\lambda^2}{2}+\frac{\lambda^3}{6}\right\}$$

$$= \frac{1+5+\frac{25}{2}+\frac{125}{6}}{e^{5}} = 0.265$$

→ If 'x' is a Poisson distributed Variable such that

$P(x=3) = P(x=1) + P(x=2)$ then mean of $x = \lambda = \underline{4.37}$

$\Rightarrow \frac{\cancel{e^{-\lambda}} \cdot \lambda^3}{3!} = \frac{\cancel{e^{-\lambda}} \cdot \lambda}{1!} + \frac{\cancel{e^{-\lambda}} \cdot \lambda^2}{2!}$

$\Rightarrow \frac{\lambda^3}{6} = \lambda + \frac{\lambda^2}{2!}$

$\Rightarrow \lambda^2 = 6 + 3\lambda$

$\Rightarrow \lambda^2 - 3\lambda - 6 = 0$

$\lambda = \frac{3 \pm \sqrt{9+4(6)}}{2}$

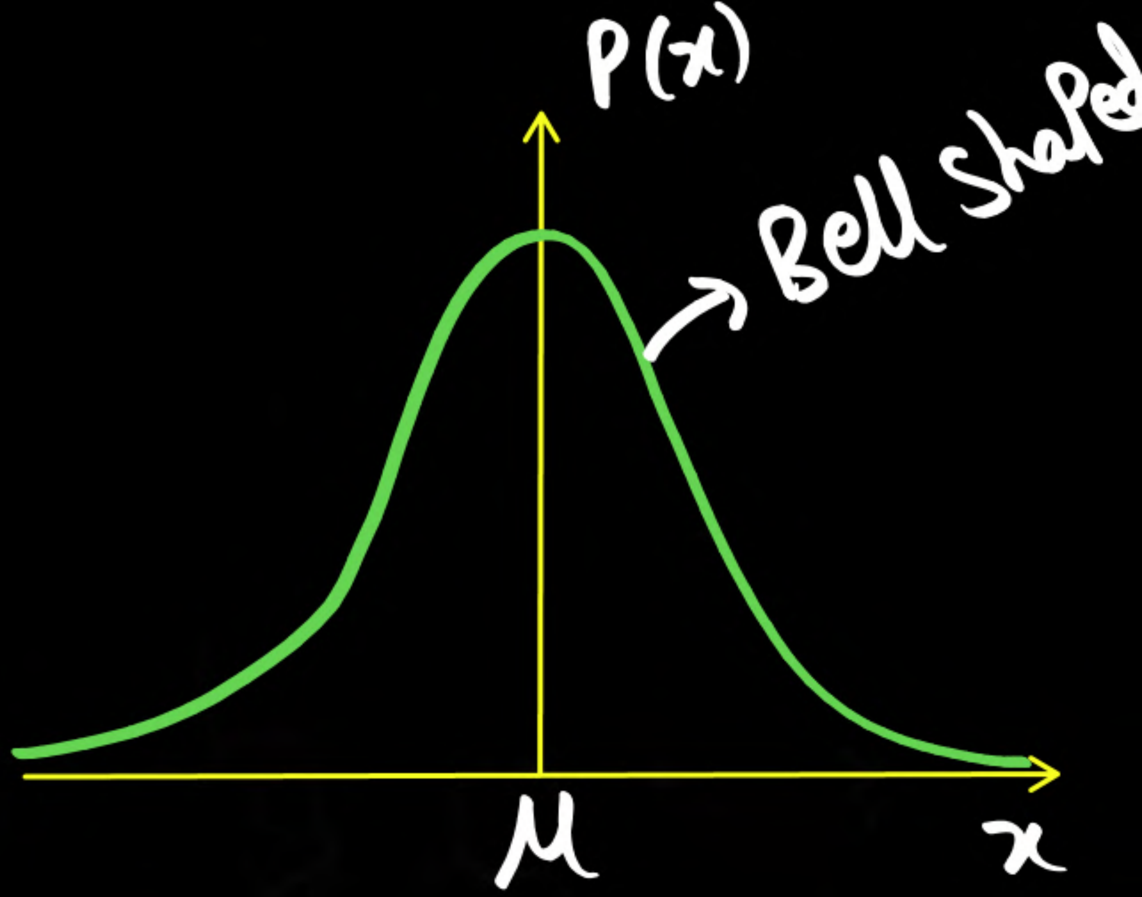
$\Rightarrow \lambda = \frac{3+\sqrt{33}}{2}, \quad \lambda = \frac{3-\sqrt{33}}{2}$ ✗

$\lambda > 0$

$\Rightarrow \lambda = \frac{3+\sqrt{33}}{2} = 4.372$

Gaussian Distribution (or) Normal distribution.

The Probability density function of a Gaussian (or) Normal distributed Variable with mean as 'μ' and Variance (Standard deviation$)^2$ as σ^2, is given by.



$$P(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

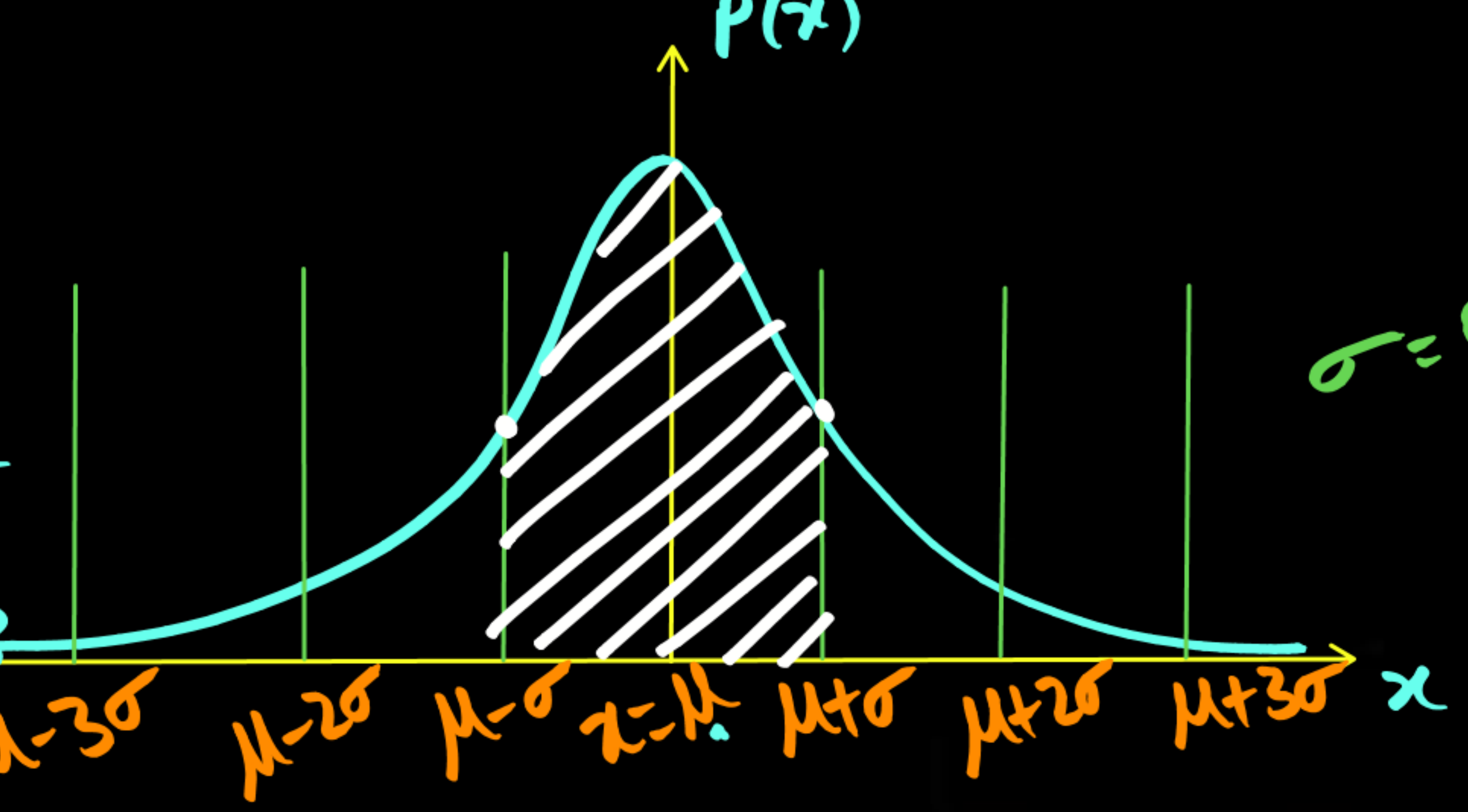
$$\boxed{\text{Mean} = \text{Median} = \text{Mode} = \mu}$$

6-σ limits:

$$P(\mu-\sigma \le x \le \mu+\sigma) = 0.6828$$

$$P(\mu-2\sigma \le x \le \mu+2\sigma) = 0.9577$$

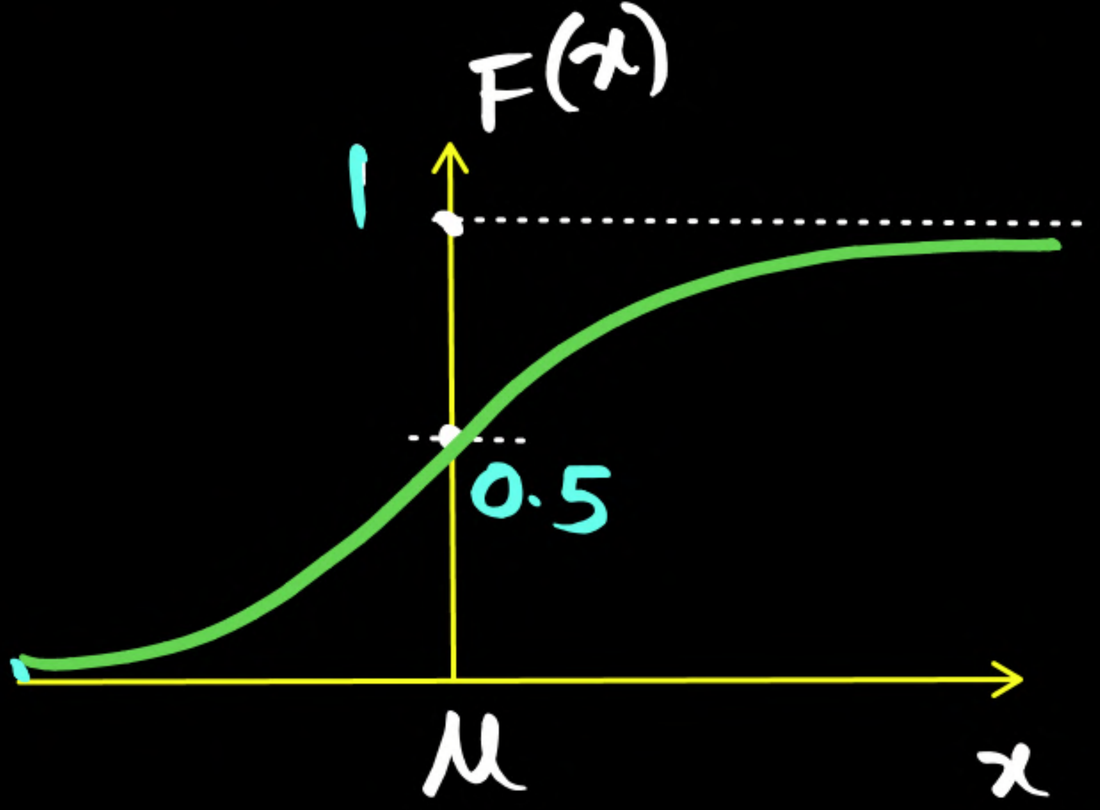
$$P(\mu-3\sigma \le x \le \mu+3\sigma) = 0.9973$$

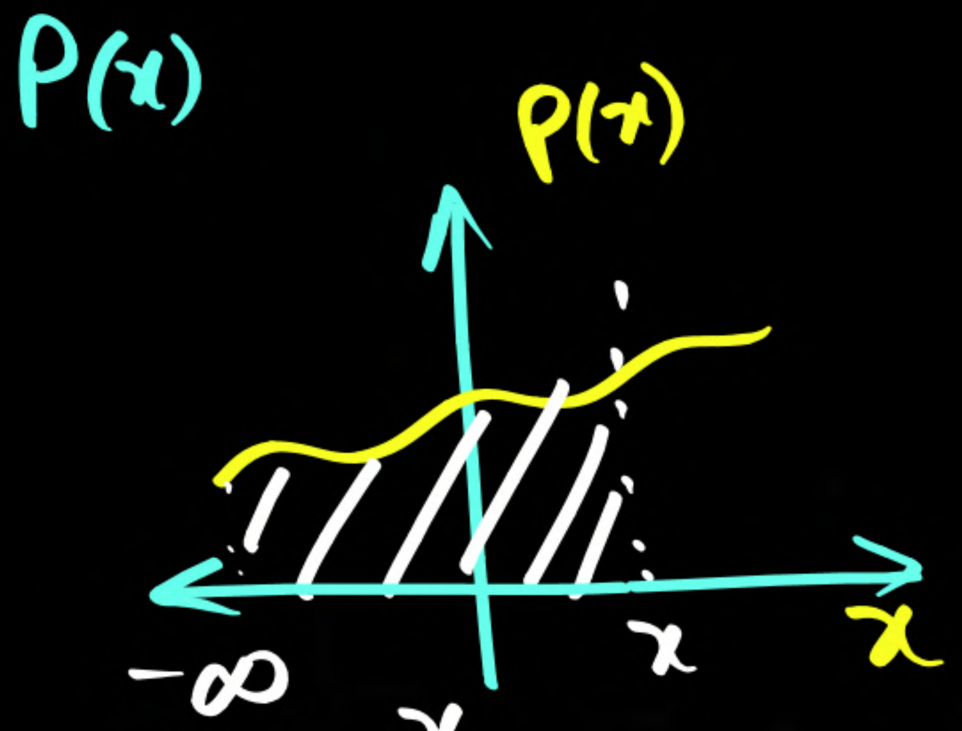


$\sigma = 0.01$

$x=\mu-\sigma$, $x=\mu+\sigma$ are Points of inflection of Gaussian distribution.

Cumulative distribution function:





$$P(X \le x) = \int_{-\infty}^{x} P(x)\,dx = F(x).$$

$$\Rightarrow \frac{d}{dx}(F(x)) = P(x)$$

$$\int_{-\infty}^{x} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$\longrightarrow$ 'S' shaped function

Newton-Leibnitz Rule:

$$\frac{d}{dx}\left(\int_{\cancel{\phi(x)} \to \text{const.}}^{\psi(x)} f(x)\cdot dx\right) = f(\psi(x))\cdot\psi'(x) - f(\phi(x))\cdot\cancel{\phi'(x)}^{\,0}$$

$\Rightarrow$ Differentiation of cumulative distribution function is equal to Probability distribution function.

→ Standard Normal distribution:

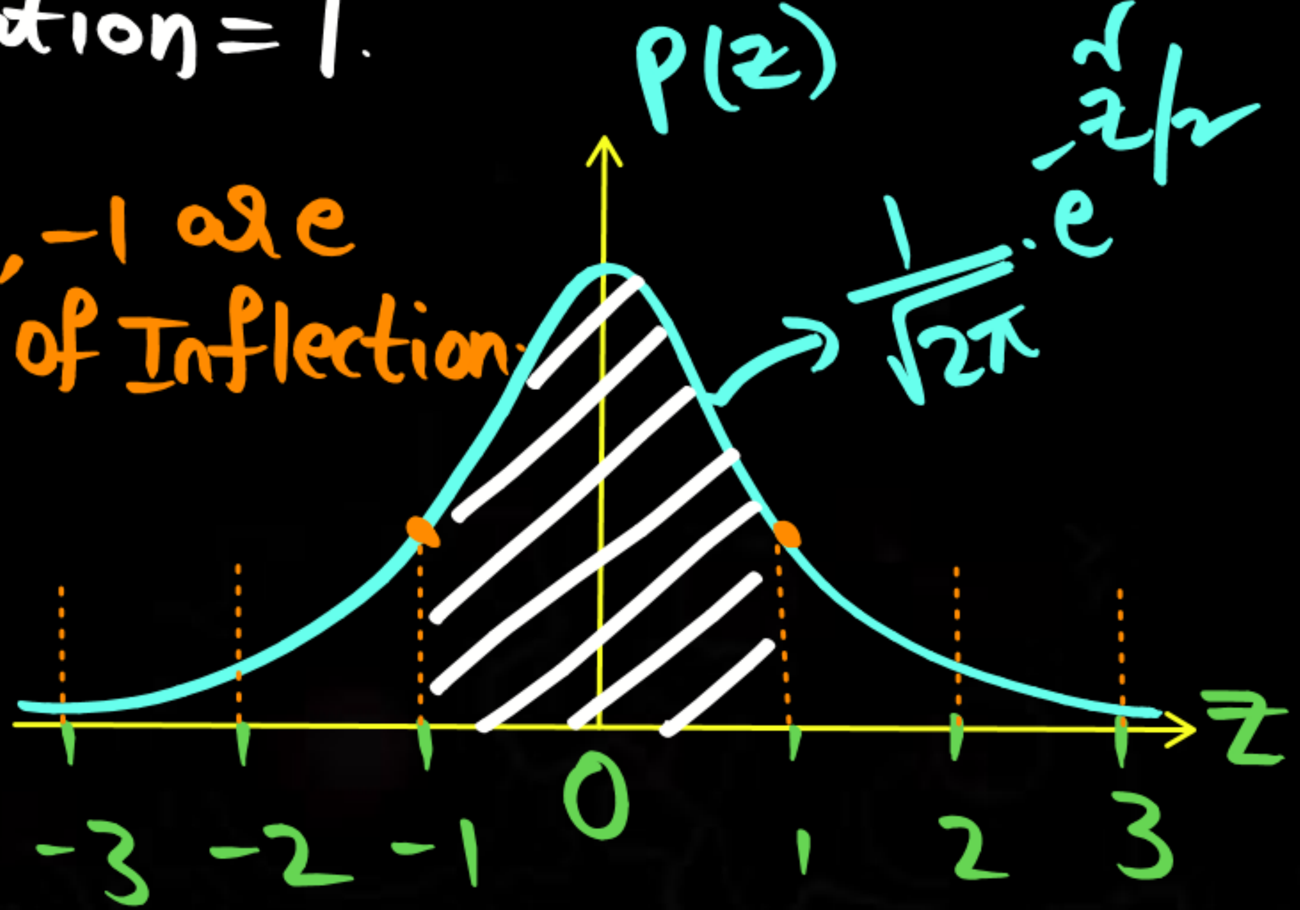
A standard Normal variable 'z' is defined with Mean $= 0$ and $\sigma =$ standard deviation $= 1$.

$$z = \frac{x-\mu}{\sigma} = \frac{x-0}{1} = x.$$

$$P(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow P(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}$$

$z = 1, -1$ are Points of Inflection



$P(-1 \le z \le 1) = 0.6828$

$P(-2 \le z \le 2) = 0.9577$; $P(-3 \le z \le 3) = 0.997$

$P(-1 \le z \le 1) = 2 \times P(0 < z < 1)$

$= 2 \times 0.3414$

$= 0.6828.$

$P(0 < z < 0.8) =$

$$\int_0^{0.8} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} dz =$$

(Gaussian Tables)

$P(0 < z < 1.45) = 0.4115$

$P(-0.72 < z < 0.31)$

$= P(-0.72 < z < 0) + P(0 < z < 0.31)$

$= P(0 < z < 0.72) + P(0 < z < 0.31) = 0.2642 + 0.1217$

$= \underline{\quad}.$

$$\lambda = np$$
$$= 2000\cdot(0.001)$$
$$= 2. \quad \therefore \lambda = 2$$

$$P(x\geq 2) = 1-\{P(x=0)+P(x=1)\}$$

$$= 1-\left\{\frac{e^{-\lambda}\cdot\lambda^{0}}{0!}+\frac{e^{-\lambda}\cdot\lambda^{1}}{1!}\right\}$$

$$= 1-e^{-\lambda}\{1+\lambda\} = 1-\left(\frac{1+\lambda}{e^{\lambda}}\right) = 1-\frac{3}{e^{2}} = 0.594$$

$$P(x\geq 2) = 1-\{P(x=0)+P(x=1)\}$$

$$= 1-\left\{{}^{2000}C_0\cdot(0.001)^{0}\cdot(0.999)^{2000} + {}^{2000}C_1\cdot(0.001)^{1}\cdot(0.999)^{1999}\right\}$$

$$= 1-0.999^{1999}\left\{0.999+2\cancel{000}\left(\frac{1}{\cancel{1000}}\right)\right\}$$

$$= 0.594$$

Thank You!

GW Soldiers