UNIVERSITY OF TEXAS ARLINGTON

PHYS 5319-001: Math Methods in Physics III

Random System

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Physics Problems

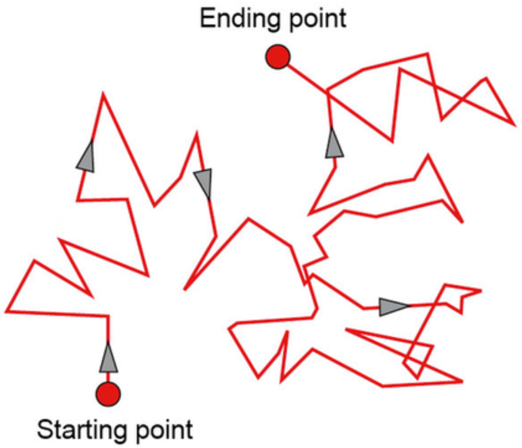
- Deterministic

e.g. Electric Potential at a given charge distribution

$$-\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\rho(\vec{r})$$

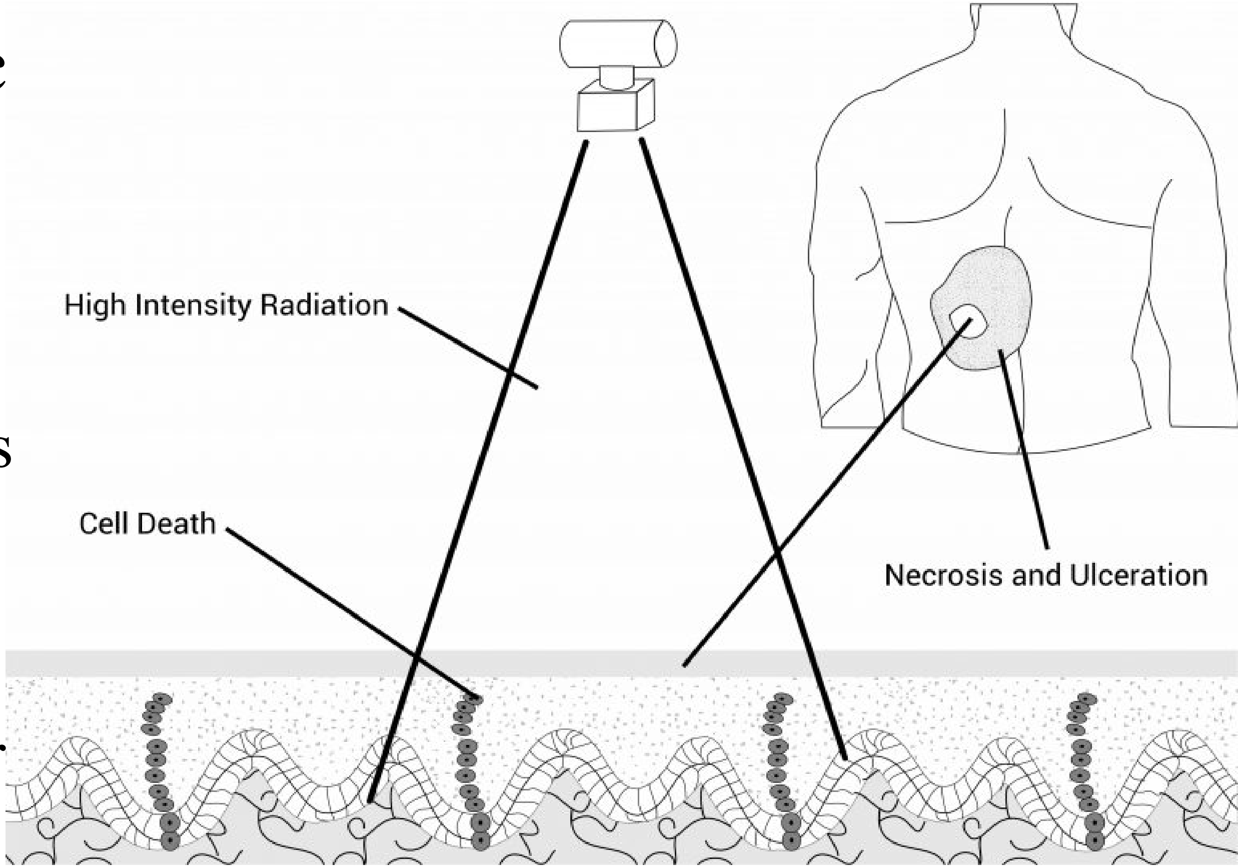
- Stochastic

e.g. Brownian motion



example -- the deterministic and stochastic effect of radiation.

➢ Deterministic effec

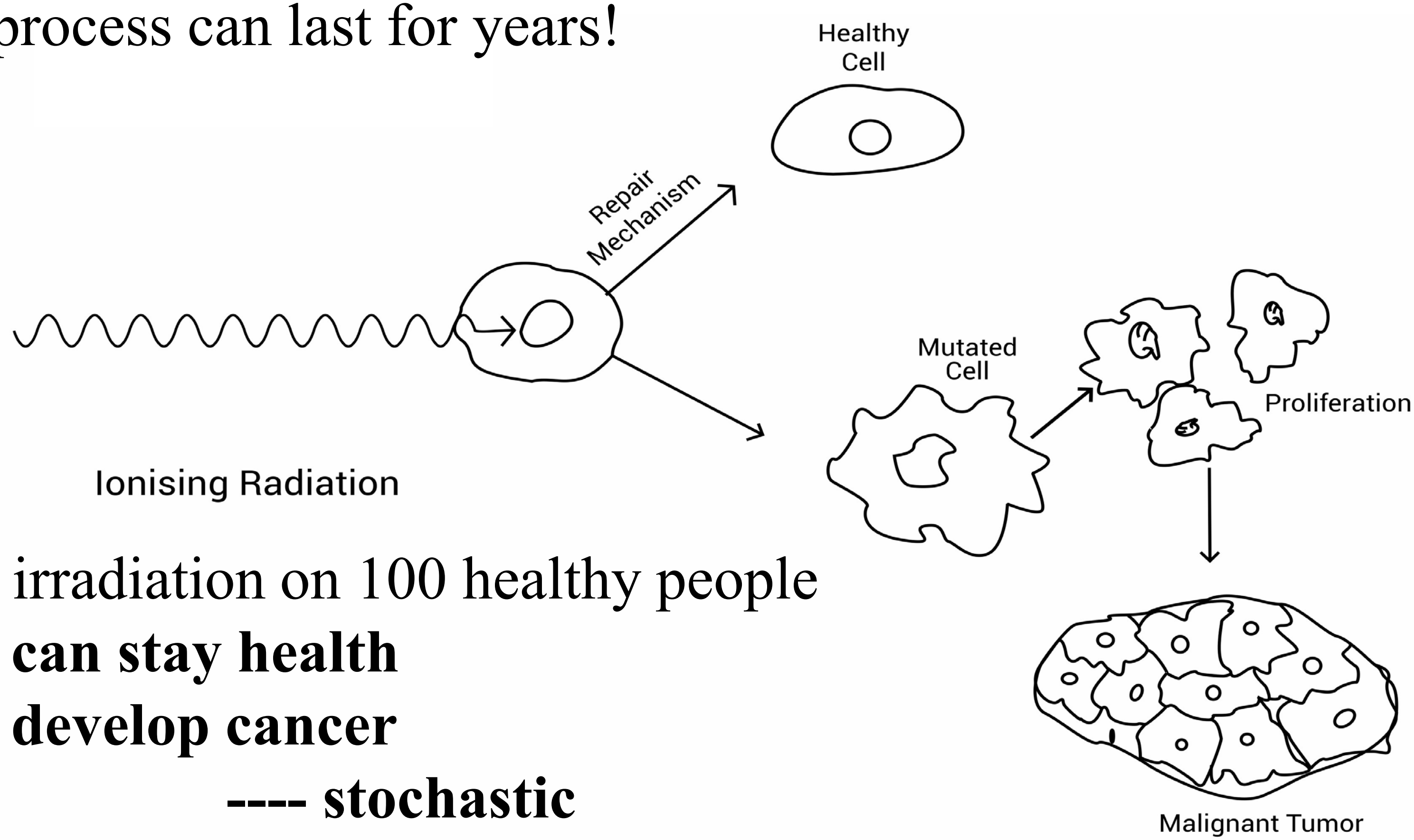


- When a high dose irradiated on one location of the patient’s skin, it can lead to significant cell death, which can manifest as erythema, ulceration or skin necrosis.

- If the same irradiation was delivered on 100 patients, similar skin damage will be observed – **deterministic**!

➢ Stochastic effect

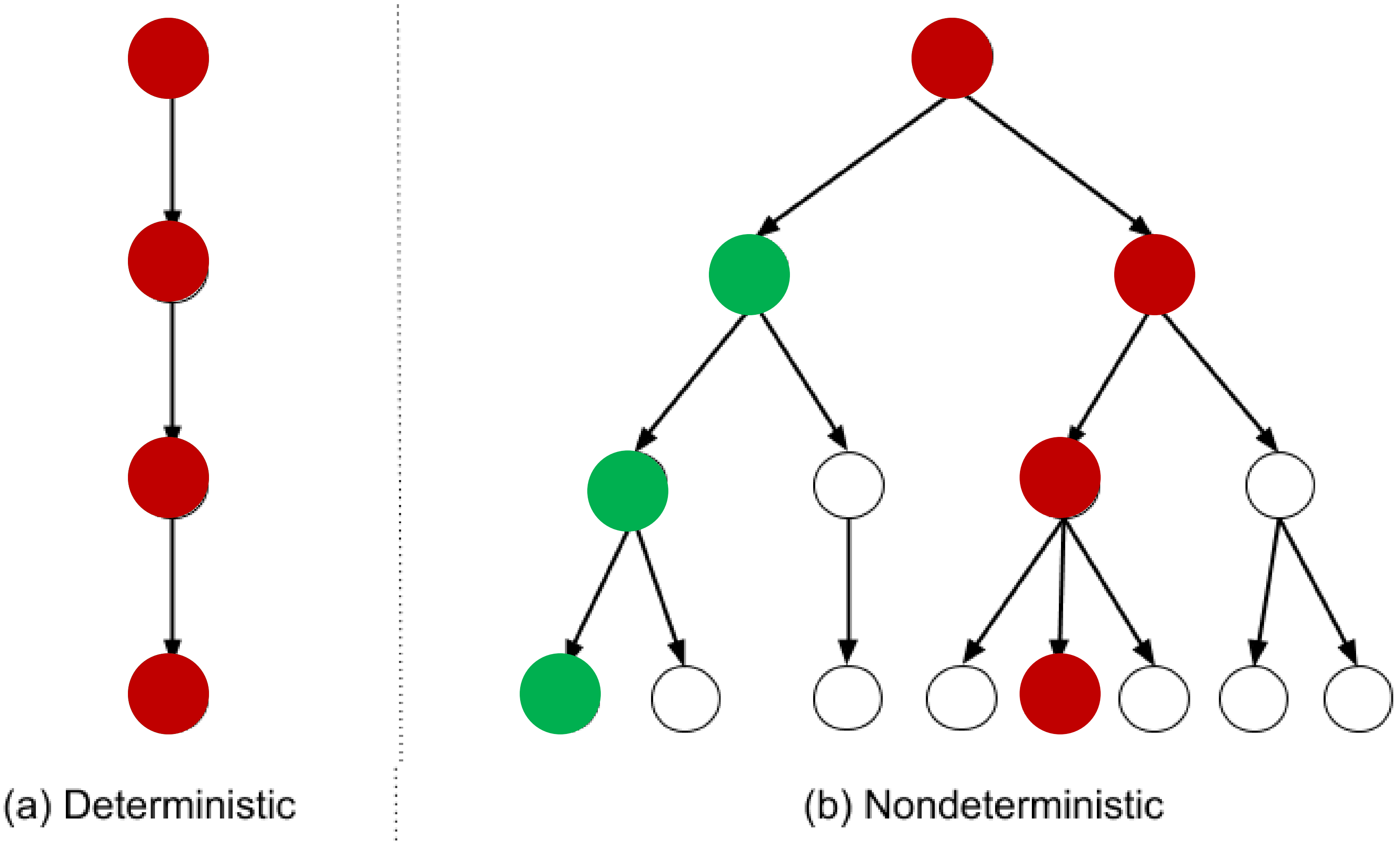
- Low dose radiation can cause cell mutation
  - ✓ Most of the time, these mutated cells can be eliminated by the body's immune system
  - ✓ In rare, this mutation can induce cancer
  - ✓ This process can last for years!



- The same irradiation on 100 healthy people
  - ✓ **Some can stay health**
  - ✓ **Some develop cancer**

---- stochastic

In a chart, these two processes are like:



(a) Deterministic

(b) Nondeterministic

In numerical computation, to simulate a random process, the first thing we need to do is to **generate random numbers**.

- True random numbers:
  - ✓ Generated in non-deterministic ways, not predictable, not repeatable;
  - ✓ Example:
    - ✓ Tossing a coin
    - ✓ Decay times of radioactive materials
    - ✓ Electrical noise from a semiconductor, radio noise
- Pseudo random numbers:
  - ✓ Appear random, but generated in a deterministic, repeatable and predictable manner
  - ✓ Examples: computer random generator
  - ✓ In 1939, first machine to produce 100,000 random digits

Linear congruential generators

It is so far the most common scheme, and it employs an equation:

$$x_{n+1} = (ax_n + b) mod\, m,$$

where

- x_n: integer in $[0, m-1]$
- m, typically is 2^{16} or 2^{32}
- x_n is the n^{th} random integer in the sequence. It can be normalized such that a number between 0 and 1 is returned by the function for use.
- x_0 is the seed. It requires to be given by users in certain programming languages.
- a, b must be carefully chosen.
- Generated random sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$

example

$$x_{n+1} = (ax_n + b) mod\ m$$

For $b=1, a=4, m=9$, start with $x_1 = 3$

$$x_2 = (4 \times 3 + 1) mod\ 9 = rem\left(\frac{13}{9}\right) = 4$$

$$x_3 = (4 \times 4 + 1) mod\ 9 = rem\left(\frac{17}{9}\right) = 8$$

$$x_4 = (4 \times 8 + 1) mod\ 9 = rem\left(\frac{33}{9}\right) = 6$$

When a particular x appears 2nd time $\Rightarrow$ repeat

$x_{5-10} = 7, 2, 0, 1, 5, 3$ and repeat

Sequence length m=9, normalize $\frac{1}{m}$ to (0,1) number

Test shows $a = 1566083941$ or 1812433253 could generate a series of random numbers with

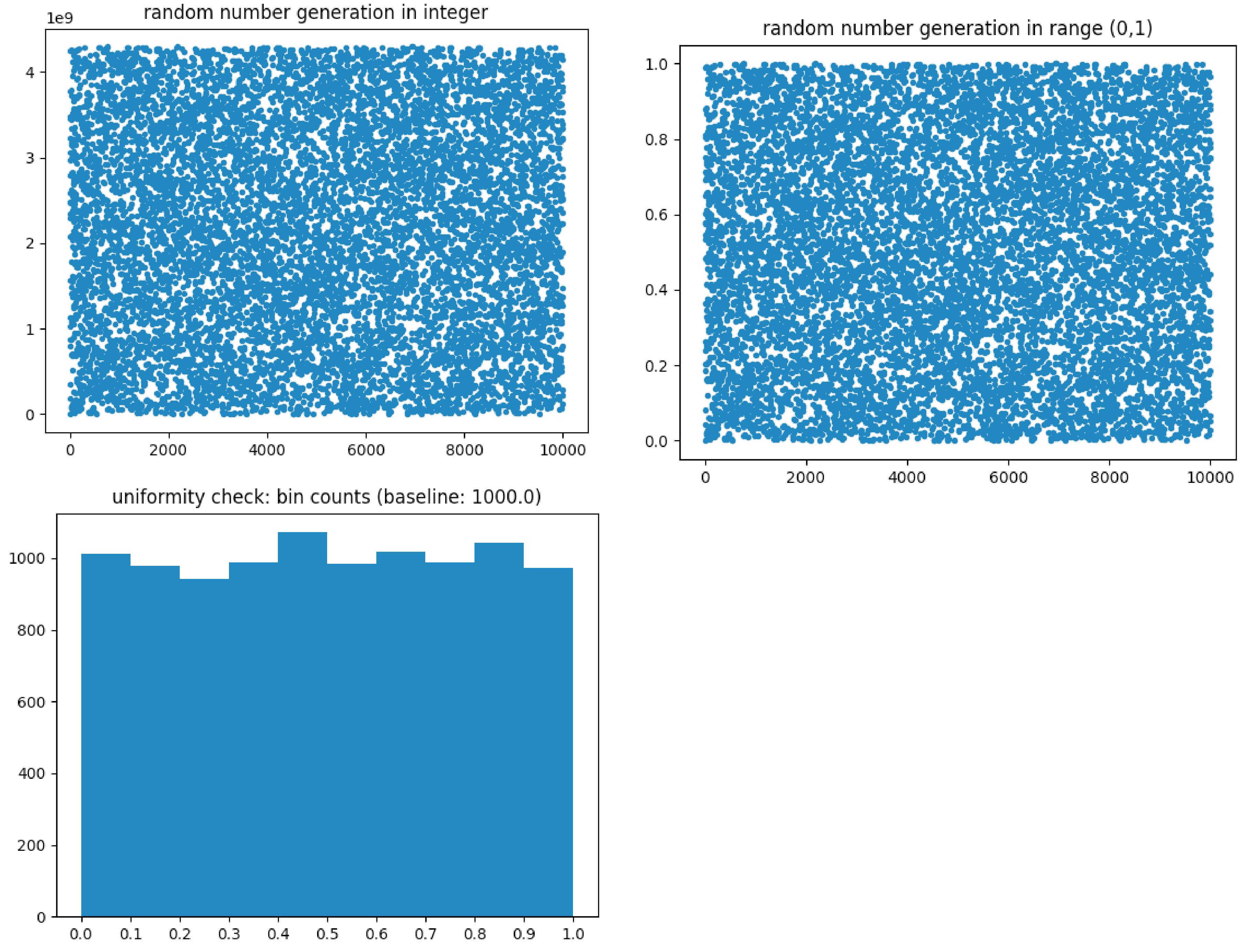
- Long length of period
- High accuracy
- Sparse correlation
- uniform

We could check the generated numbers for

- its uniformity via looking at the bin counts
- Length of period by checking repetition

In general programming, we just call random generator, like rnd to generate the desired random sequence.

Examples: $a = 1566083941, b = 1, m = 2^{32}$. Sample 10000 points and check its uniformity and repetition.



random number generation in integer
1e9
4
3
2
1
0
0
2000
4000
6000
8000
10000
random number generation in range (0,1)
1.0
0.8
0.6
0.4
0.2
0.0
0
2000
4000
6000
8000
10000
uniformity check: bin counts (baseline: 1000.0)
1000
800
600
400
200
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Correlation Tests

For random number $\{x_n\} \in (0,1)$ center: 0.5

$0 = \langle (x_i - 0.5)(x_j - 0.5) \rangle$

$= \langle x_i x_j \rangle - 0.5 \langle x_i \rangle - 0.5 \langle x_j \rangle + 0.25$

Auto-correlation:

$$C(k) = \frac{1}{N}\sum_{i=1}^{N} x_i x_{i+k} \quad (k = 1,2,\ldots)$$

$$\approx \frac{1}{4} + O\left(\frac{1}{\sqrt{N}}\right)$$

or *k-th* moment:

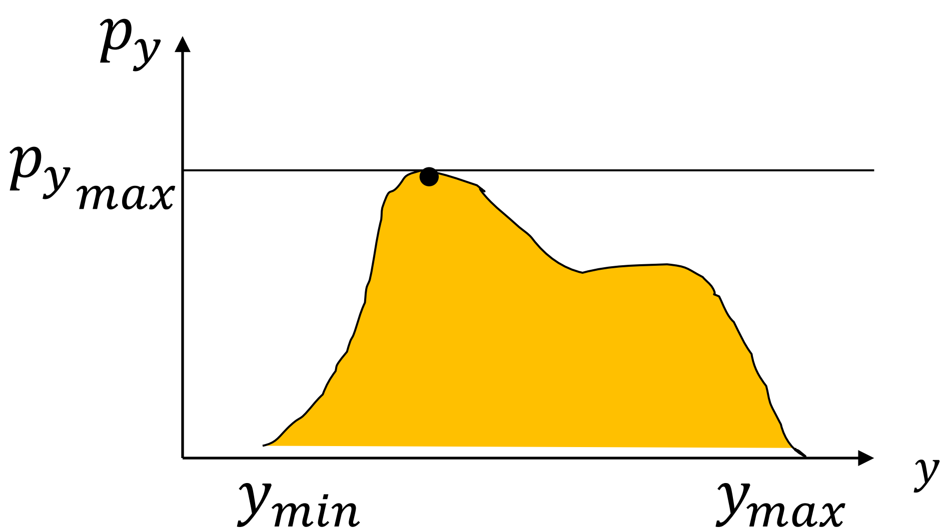
$$\langle x^k \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i^k \approx \int_0^1 dx\, x^k p(x) + O\left(\frac{1}{\sqrt{N}}\right)$$

$$\approx \frac{1}{k+1} + O\left(\frac{1}{\sqrt{N}}\right)$$

If agrees $\Rightarrow$ uniform $\{x_n\}$

Appendix F2. Random number with non-uniform distributions

If we want to generate random numbers in the range of $[y_{min}, y_{max}]$, following the probability distribution of p_y. That is, To generate random numbers in the yellow region.



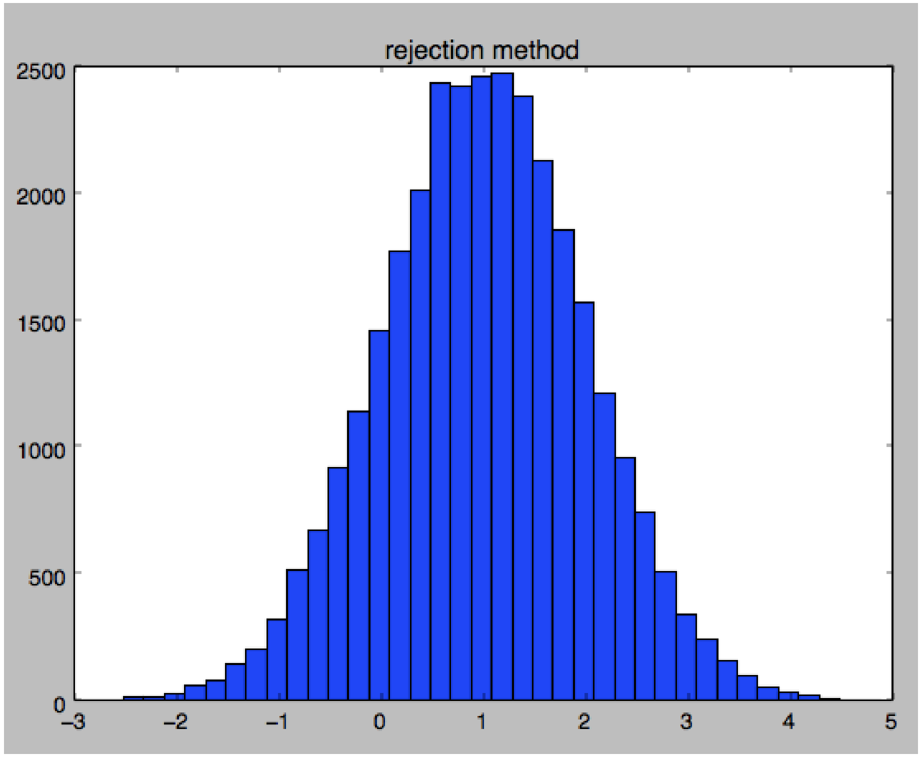
Rejection method

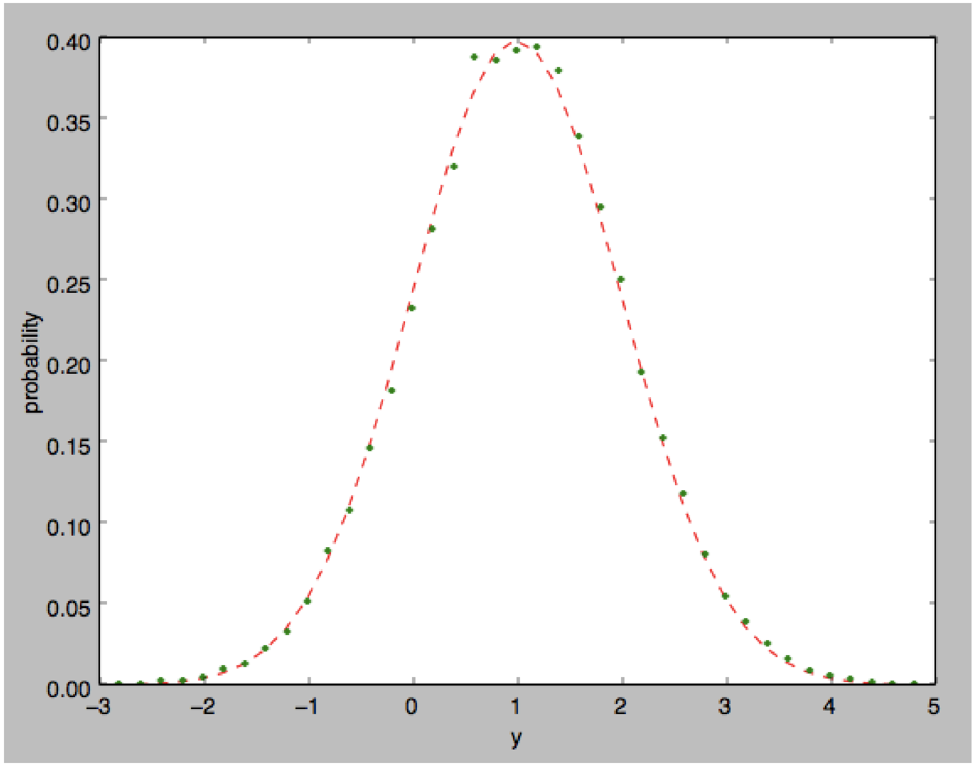
1. Generate a sequence of random numbers $\mathrm{Y} = \{y_1, y_2, y_3, \ldots\}$ distributed uniformly in the range of interest $[y_{min}, y_{max}]$, using the random number generator (e.g. $rand$).
2. Start with y_1, evaluate $p_y(y_1)$.
   1) Generate a random number $p_{test} \in (0, p_{y_{max}})$;
   2) If $p_{test} > p_y(y_1)$, reject y_1; otherwise, keep it.
3. Repeat step 2 for all y_i in Y.

Rejection method to sample random numbers following Gaussian distribution:

$$P_y = B\exp(-(y - y_c)^2/(2\sigma^2))$$

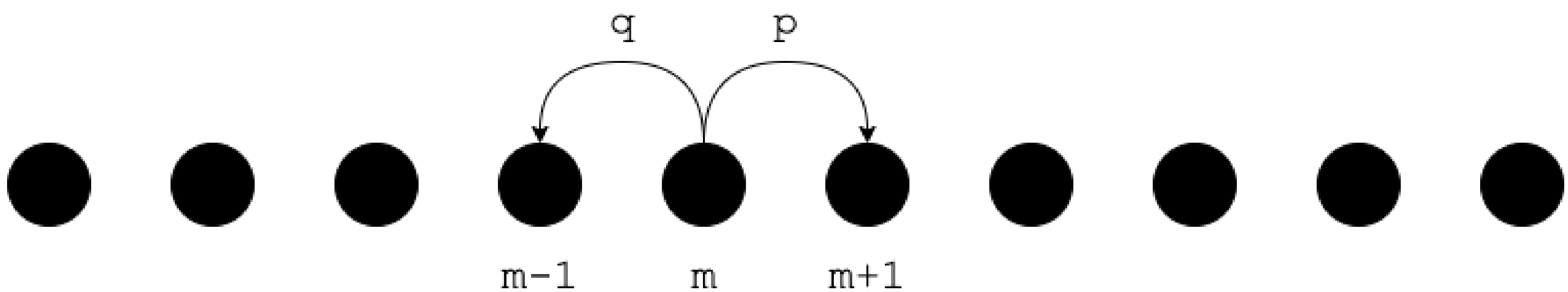
Let $y_c = 1$, $\sigma = 1$ and $B = \frac{1}{\sqrt{2\pi\sigma^2}}$, sampling 100000 points.





Now we could simulate a random process with our random generators.

Example: random walker



A walker walks along x direction. For each step, he has a probability q to go towards negative x direction, and a probability $p = 1 - q$ to go positive.

Suppose at step 0, he is at the origin, i.e. $P_0 = 0$. What would be his position at step n? Suppose the step size is 1 and $p = \mathrm{q} = 0.5$.

At step 1, there is 0.5 probability for $P_1 = -1 = P_0 - 1$,
and 0.5 probability for $P_1 = 1 = P_0 + 1$.

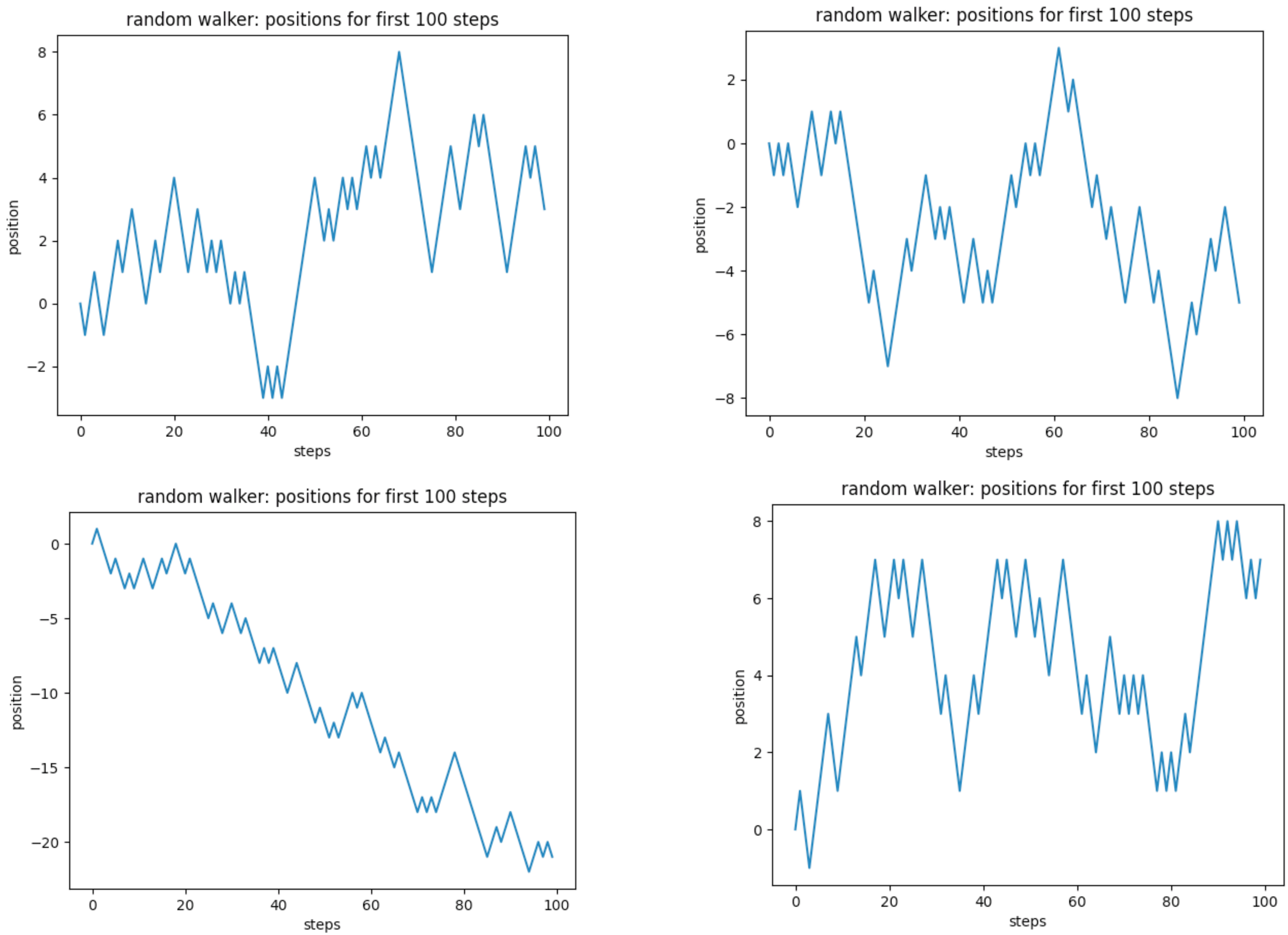
More generally, at step *i+1*, there is 0.5 probability for $P_{i+1} = P_i - 1$,
and 0.5 probability for $P_{i+1} = P_i + 1$.

Pseudo code:

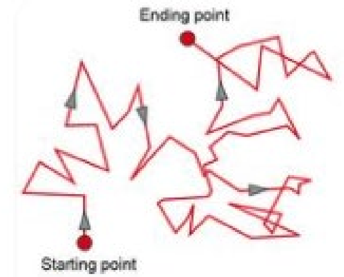
Main:

- Define input variables with given values, $p = 0.5, q = 0.5, stepsize = 1$;
- Define total step N=100, a list of positions P with length N. P[0]=0
- Compute P[1, …, N-1] based on the subroutine;
- Visualize results;
- Subroutine:
  - ✓ For each i in range [1,N)
    sample a random number α,
    if $\alpha < p$, $P[i] = P[i-1] - 1$; if $\alpha > p$, $P[i] = P[i-1] + 1$;

Random walker: stochastic system, different outputs with same initial conditions in different runs.



Brownian motion (1827)



- Pullen grains in water observed by microscope
- Similar processes: liquid atoms, electron transport thru metals, perculation
- 2-d model: N displacements $\{\Delta x_i, \Delta y_i\}$, $i=1,2,\ldots,N$

The radial distribution R from the starting point after N steps:

$$R^2 = (\Delta x_1 + \Delta x_2 + \cdots + \Delta x_N)^2 + (\Delta y_1 + \Delta y_2 + \cdots + \Delta y_N)^2$$
$$= (\Delta x_1^2 + \Delta x_2^2 + \cdots + \Delta x_N^2) + (\Delta y_1^2 + \Delta y_2^2 + \cdots + \Delta y_N^2)$$
$$+2(\Delta x_1 \Delta x_2 + \Delta x_1 \Delta x_3 + \cdots + \Delta y_1 \Delta y_2 + \cdots + \Delta y_{N-1} \Delta y_N)$$

The cross terms canceled each other when N is large:

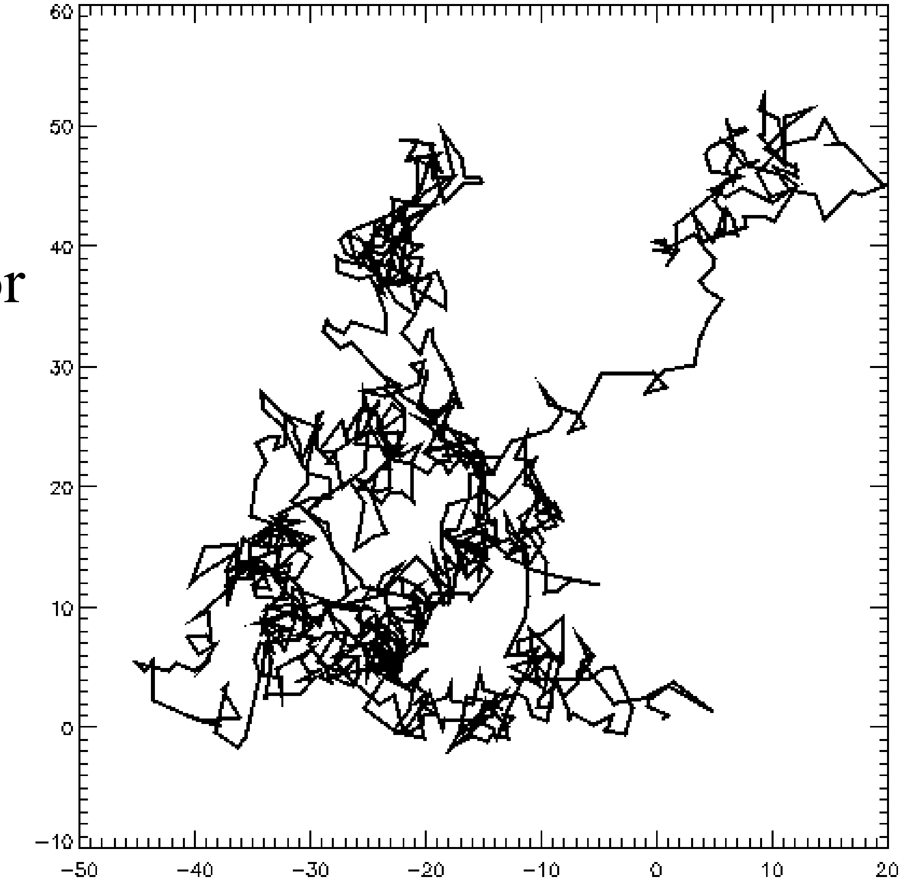
$$R^2 \approx (\Delta x_1^2 + \Delta x_2^2 + \cdots + \Delta x_N^2) + (\Delta y_1^2 + \Delta y_2^2 + \cdots + \Delta y_N^2)$$

So, $R \approx \sqrt{N} r_{rms}$ where $r_{rms} = \sqrt{\overline{(\Delta x^2 + \Delta y^2)}}$

More generally,

$$\langle x_n^2 \rangle = qDt_n$$

D is the diffusion constant. q=2,4,6 for 1-d,2-d,3-d, respectively.



Random walks and diffusion

The cream-in-your-coffee problem.

- Randomness
  - random walks = diffusion?