UNIVERSITY OF TEXAS ARLINGTON

PHYS 5319-001: Math Methods in Physics III

Discrete Fourier Transformation

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Fourier Analysis

- A single-valued period function (w/ period T)

$$y(t+T) = y(t)$$

can be expanded as

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos(n\omega t) + b_n sin(n\omega t)\right)$$

where "true" frequency $\omega = \frac{2\pi}{T}$

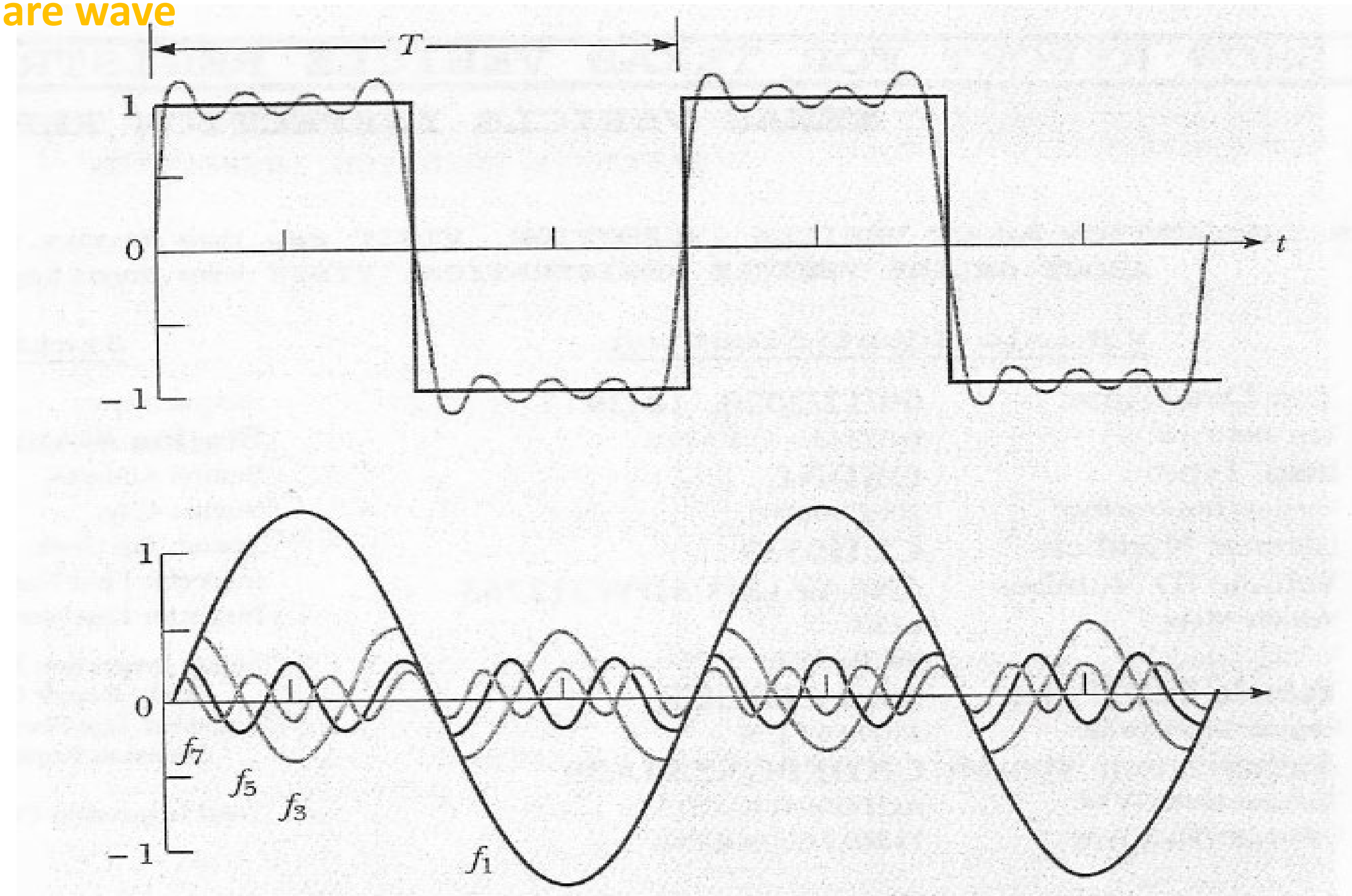
-given a $y(t)$ to find $\{a_n, b_n\}$: $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{T}\int_0^T dt \begin{pmatrix} \cos(n\omega t) \\ \sin(n\omega t) \end{pmatrix} y(t)$

*$a_0 = 2\langle y(t)\rangle$

*for odd , all $\{a_n\}$=0, $b_n = \frac{4}{T}\int_0^{T/2} dt y(t) \sin(n\omega t)$

* for even , all $\{b_n\}$=0, $a_n = \frac{4}{T}\int_0^{T/2} dt y(t) \cos(n\omega t)$

Example: Square wave



Odd function

$$a_0 = \{a_n\} = 0$$

$$b_n = \begin{cases} 0, & n = 2,4,6,8,\ldots \\ \dfrac{4}{n\pi}, & n = 1,3,5,7,\ldots \end{cases}$$

Here we truncated at n=7. The converge is slow.

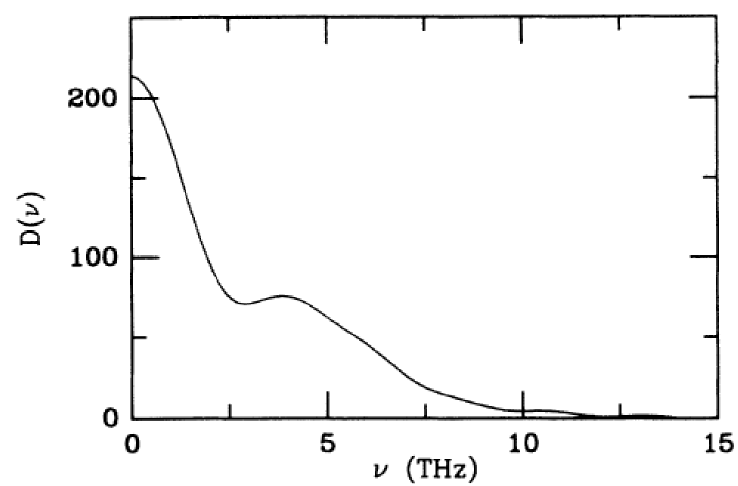
Autocorrelation function

- **Autocorrelation** is the correlation of a signal with a delayed copy of itself as a function of delay. Informally, it is the similarity between observations as a function of the time lag between them. The analysis of autocorrelation is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal obscured by noise, or identifying the missing fundamental frequency in a signal implied by its harmonic frequencies.

e.g. Liquid GaAs, *Phys. Rev. B* 42, 5071 (1990)

$$Z(s) = \frac{< \mathbf{V}(t) \cdot \mathbf{V}(t+s) >}{< \mathbf{V}(t) \cdot \mathbf{V}(t) >} .$$

$$D(\nu) \propto \int_0^\infty Z(t) \cos(2\pi\nu t) dt,$$



In general, the fundamental freq. $\omega_1 = \frac{2\pi}{T}$

Then we have discrete frequency: $\omega_1, 2\omega_1, 3\omega_1, \ldots$

$$with\ \Delta\omega = \omega_1 = \frac{2\pi}{T}$$

For a nonperiodic function, we can imagine $T \to \infty$

Then $\Delta\omega \to 0$ or ω becomes continuous:

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, Y(\omega) e^{i\omega t}$$

For comparison:

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n cos(n\omega t) + b_n sin(n\omega t)\big)$$

$$e^{i\omega t} = cos\omega t + isin\omega t$$

Here *Y(ω)* replace $\{a_n, b_n\}$, called Fourier Transform of $y(t)$

A plot $|Y(\omega)|^2$ of vs. ω Is usually called the power spectrum

Inverse F.T.: $$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, Y(\omega) e^{i\omega t}$$

forward F.T.: $$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, y(t) e^{-i\omega t}$$

Know either of them can get another. Sometimes experiments may well measure *Y(ω)* directly.

Also, note $$\frac{d}{dt} y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, Y(\omega)(i\omega) e^{i\omega t}$$

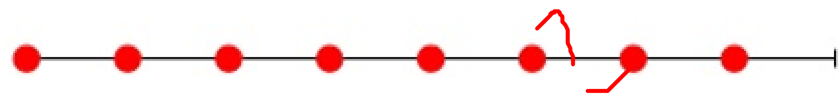
$$\frac{d^2}{dt^2} y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, Y(\omega)(i\omega)^2 e^{i\omega t}$$

Discrete Fourier Transform

multi-grid

For a periodic function

finite $\quad y_k \equiv y(t_k), k = 0,1,2, \ldots N$



fine – coarse

Since *y(t+T)=y(t)* $\Rightarrow y_0 = y_N$

$T = Nh$

$t_k = kh$

Prime freq.: $\omega_1 = \frac{2\pi}{T}$ $\qquad \omega_n = n\omega_1 = n\frac{2\pi}{Nh}, n = 0,1, \ldots N$

Higher frequency contribution is missing $\Rightarrow$ more rapid change

FT $\quad Y(\omega_n) = \frac{1}{T}\int_0^T e^{-i\omega_n t} y(t)dt \quad \approx \frac{1}{T}\sum_{k=1}^{N} y(t_k)e^{-i\omega_n t_k}h \quad = \frac{1}{N}\sum_{k=1}^{N} y_k e^{-i2\pi kn/N}$

usually: $\qquad Y_n \equiv \sum_{k=1}^{N} y_k\, e^{-2\pi ikn/N}$

FT^{-1} $\quad y_k \equiv \frac{1}{N}\sum_{n=1}^{N} Y_n\, e^{2\pi ikn/N}$

DFT recap

$$Y_n \equiv \sum_{k=1}^{N} y_k\, e^{-2\pi ikn/N} \qquad y_k \equiv \frac{1}{N}\sum_{n=1}^{N} Y_n\, e^{2\pi ikn/N}$$

Periodic: $Y_{n+N} = Y_n$ $\quad y_{k+N} = y_k$

N independent input $\Rightarrow$ *N* independent output

Let $Z \equiv e^{2\pi i/N}$ $\quad e^{2\pi ikn/N} = Z^{kn}$ $\quad$ *Z* could be pre-calculated

Cost of DFT: for each $Y_n \equiv \sum_{k=1}^{N} y_k * Z^{kn}$, about *O(N)*

But *n=1,2,…,N*. So we have *N* Y_n's to calculate, the total operation ~ $O(N^2)$

An algorithm called Fast Fourier Transformation (FFT) will reduce it to $O(N\log_2 N)$

Similarly, in 1-d real space, we have a periodic function $f(x+L) = f(x)$

Discrete Fourier Transform (DFT)

$$f_i, i = 1,2, \dots N$$

FT

$$g(k) = \frac{1}{L}\int_0^L f(x)e^{ikx}dx$$

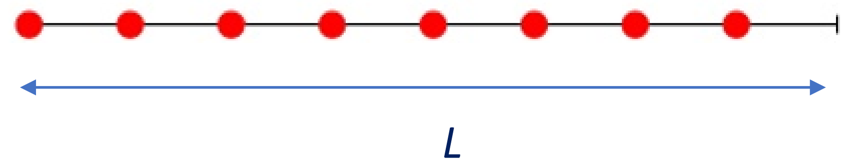
$$= \frac{1}{N}\sum_{j=1}^{N} f_j e^{ikx_i}$$

FT^{-1}

$$f(x) = \sum_{l=1}^{N} g_l e^{-ik_l x}$$

N discrete points:

$$x = 0, \frac{L}{N}, \frac{2L}{N}, \dots, \frac{L}{N}(N-1)$$



Since $f(x+L)=f(x) \Rightarrow e^{ikL}=1$,

$$kL = 2\pi n, n = 0,1,2, \dots, N-1$$

We have N discrete k-points:

$$k = \frac{2\pi n}{L}, n = 0,1,2, \dots, N-1$$

Fast Fourier Transformation

- Idea by Gauss (1886); developed by Cooley & Tukey (1965)
- "the most important numerical algorithm of our lifetime“

For N=2^m^,

$$y_n = \sum_{k=0}^{N-1} Y_k \, e^{2\pi i k n/N}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} Y_{2k} e^{i2\pi 2kn/N} + \sum_{k=0}^{\frac{N}{2}-1} Y_{2k+1} e^{i2\pi(2k+1)n/N}$$

Split the FT into odd/even terms

$$y_n = \sum_{k=0}^{N-1} Y_k \, e^{2\pi ikn/N}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} Y_{2k} e^{i2\pi 2kn/N} + \sum_{k=0}^{\frac{N}{2}-1} Y_{2k+1} e^{i2\pi(2k+1)n/N}$$

$$= x_n + z_n e^{i2\pi n/N}$$

where $x_n = \sum_{k=0}^{\frac{N}{2}-1} Y_{2k} e^{i2\pi kn/\left(\frac{N}{2}\right)}$, $z_n = \sum_{k=0}^{\frac{N}{2}-1} Y_{2k+1} e^{i2\pi kn/\left(\frac{N}{2}\right)}$

Let $w = e^{i2\pi/N}$, then $y_n = x_n + w^n z_n$

x_n and z_n has period of *N/2*

This is very significant!

$$y_n = x_n + w^n z_n$$

x_n and z_n has period of *N/2*

i.e. $x_0 = x_{N/2}, z_0 = z_{N/2}$

Can be done recursively: $N \rightarrow \frac{N}{2} \rightarrow \frac{N}{4} \rightarrow \cdots \rightarrow 2$

Total cost *O(Nlog₂N)*