Normality of Toric rings and Rees Algebras of pinched strongly stable ideals

Cameron Chandler, Margit Liu, Elizabete Mezinska, Andrew Moore, Shashank Sule, Andrew Tawfeek

Advisor: Gabriel Sosa

Introduction

It is known from the 16th century that the toric ring $K[x^3, x^2y, xy^2, y^3, x^2z, y^2z, xz^2, yz^2, z^3]$ also called the pinched Veronese, is not normal. Sturmfels [5] proved that Veronese-type toric rings are normal. Further results from Conca and De Negri [4] [1] show that toric rings of principal Borel ideals are also normal. Motivated by these inquiries, we set out to investigate toric rings generated by pinchings of principal Borel ideals.

Methods and Notation

These can also be found in [3]

Definition 1. Toric Ring
A **Toric ring** is defined as a subring $K[T] \subset K[x_1, \ldots, x_n]$ where T is a set of monomials, i.e $T = \{\mathbf{x_1}^{\alpha_1}, \ldots, \mathbf{x_s}^{\alpha_s}\}$. For example, $K[x^2, xy, xz, y^2]$, which is a subring of $K[x, y, z]$, is a toric ring.

Definition 2. Normality
Let R be a ring and $\overline{R}$ the set of all rational solutions to monic equations with coefficients in R. Then R is normal iff $R = \overline{R}$. For example, $\mathbb{Z}$ is normal, because all rational solutions to monic equations with coefficients in $\mathbb{Z}$ are contained in $\mathbb{Z}$. This is known as *Gauss' Lemma* and is one of the fundamental results of abstract algebra.

Definition 3. Borel Moves and Borel Ideals
A *Borel move* is an operation that sends m to a monomial $\frac{x_j m}{x_i}$ where $x_i|m$ and $j \leq i$. A monomial ideal B is a *Borel ideal* if B is generated by a set that is closed under Borel moves. Such a set is called a *Borel Set*

Definition 4. Principal Borel Ideals and Pinchings
A *principal Borel ideal* I, is an ideal such that $I = \langle B(u) \rangle$ where $B(u)$ is the minimal strongly stable set of monomials containing u. For example, let $u = b^2c$. Then $I = \langle B(b^2c) \rangle = \langle a^3, a^2b, ab^2, b^3, a^2c, abc, b^2c \rangle$. We say that $\langle B(b^2c) \setminus a^2c \rangle = \langle a^3, a^2b, ab^2, b^3, abc, b^2c \rangle$ is the *pinching* of a^2c from $\langle B(b^2c) \rangle$

Definition 5. Toric Ideal
Let $T = \{m_1, \ldots, m_s\}$be a set of s monomials in $K[x_1 \ldots x_n]$ and $K[T]$ the associated toric ring. Define ϕ to be the ring homomorphism

$$\phi : K[t_1 \ldots t_m] \mapsto K[T]$$

where

$$\phi(t_i) = m_i$$

Then Ker(ϕ) is called the *Toric Ideal* of $K[T]$.

Theorem. ***(Sturmfels)****[5] If the minimal generators of $in_{<}Ker(\phi)$ with respect to some monomial ordering $<$ are all squarefree, then $K[T]$ is normal.*

The proof can be found in [2]. Sturmfels' theorem provides a handy way of discerning the normality of $K[T]$. To put it to effect, we computed (using the computer algebra system, Macaulay2) the initial ideals of the toric ideals of several toric rings of pinched principal Borel ideals and checked them for squarefreeness.

Goal. To find which pinchings of principal Borel ideals form normal toric rings

The Main Result

Before we discuss the main result, we introduce the necessary definitions.

Definition 6. Affine Semigroup.
Let $\mathcal{H} = \{\alpha_1, \ldots, \alpha_s\}$ be a set of vectors in $\mathbb{Z}^n$. The *affine semigroup* generated by $\mathcal{H}$ is denoted

$$\mathbb{N}\mathcal{H} = \{n_1\alpha_1 + n_2\alpha_2 + \cdots + n_s\alpha_s \mid n_1, n_2, \ldots, n_s \in \mathbb{N}\}$$

Definition 7. Affine Group.
The *affine group* generated by $\mathcal{H}$ is denoted

$$\mathbb{Z}\mathcal{H} = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_s\alpha_s \mid c_1, c_2, \ldots, c_s \in \mathbb{Z}\}$$

Definition 8. Normality.
An affine semigroup $\mathbb{N}\mathcal{H}$ is said to be *normal* if, given $g \in \mathbb{Z}\mathcal{H}$ and $n \in \mathbb{N} \setminus \{0\}$ such that $ng \in \mathbb{N}\mathcal{H}$, then $g \in \mathbb{N}\mathcal{H}$.

A Geometric Interpretation

- An affine semigroup is a subset of a **cone**, which is $\mathbb{R}_+H$. Geometrically, $\mathbb{N}H$ can be understood as the set of lattice points within the cone.

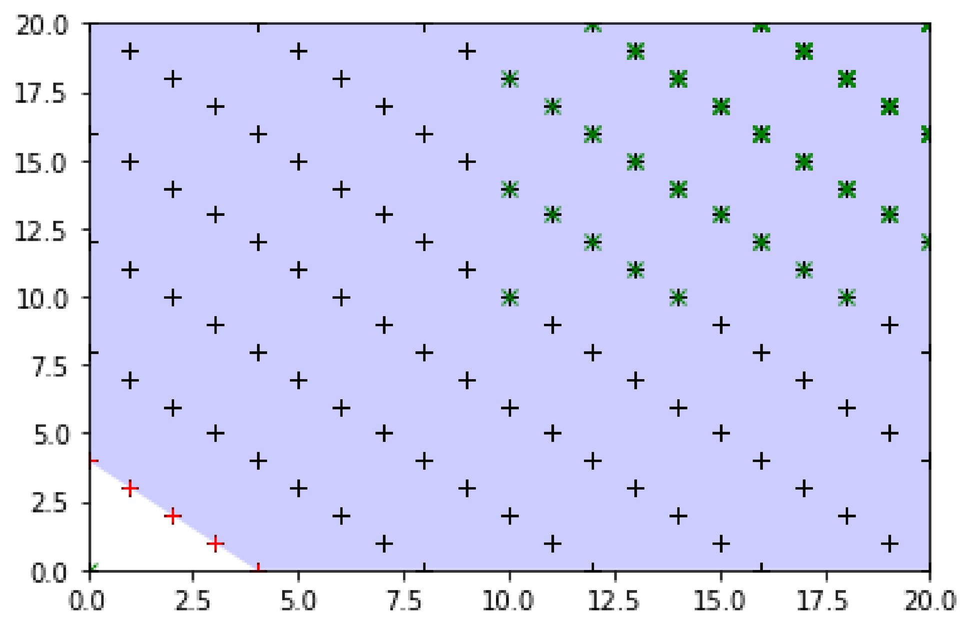

Figure 1: The cone (blue), affine group (+) and affine semigroup (x) generated by $\{(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)\}$

- The generating set of the lattice points $\mathcal{H} = \{(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)\}$ can be interpreted as generating the toric ring $K[T] = K[x^4, x^3y, x^2y^2, xy^3, y^4]$, where each $\mathbf{a_i} \in \mathcal{H}$ is an exponent vector for a monomial $m_i = \mathbf{x}^{\mathbf{a_i}} \in T$. In fact, this intuition is confirmed by the following theorem:

  Theorem. *[2] Let $\mathcal{H} \subset \mathbb{N}^n$ and H the affine semigroup generated by $\mathcal{H}$. Then H is normal iff the toric ring $K[H]$ is normal, where K is an algebraically closed field.*

  Thus, a normal toric ring is equivalent to the normal affine subgroup. The normality of the affine subgroup can be characterised by **Gordan's Lemma**.

  Theorem. ***Gordan's Lemma*** *[2]*
  1. If H is a normal affine semigroup then $H = \mathbb{Z}H \cap \mathbb{R}_+H$
  2. If C is a cone in $\mathbb{R}_+{}^n$ then $H = \mathbb{Z}^n \cap C$ is an affine semigroup

Theorem 1. *Let $K[B(u)] \subset K[x_1, \ldots, x_n]$ be a toric ring generated by the Borel set of a monomial $u \in K[x_1, \ldots, x_n]$. Then $\forall\, m_i = \mathbf{x}^{\mathbf{a_i}} \in B(u)$, $K[B(u) \setminus m_i]$ is normal iff a_i is a corner in the n-dimensional polygon formed by the exponent vectors of monomials in $B(u)$ in $\mathbb{R}^n$*

Proof. From the contrapositive of part 1 of , we see that if $H \neq \mathbb{Z}H \cap \mathbb{R}_+H$, then H is not an affine semigroup. Let $\mathcal{H}$ be the set of exponent vectors on the monomials of $B(u)$. Let m_k with exponent vector $\mathbf{a}_k$ be pinched from $\mathcal{H}$. Let $\mathcal{H}' = \mathcal{H} \backslash \mathbf{a}_k$ and H' the corresponding affine semigroup.

Suppose $\mathbf{a}_k$ is a non-corner. Then since $B(u)$ is homogenous, the vectors in $\mathcal{H}'$ are coplanar, so $\mathbf{a}_k \in \mathbb{Z}H' \cap \mathbb{R}_+H'$ and $\mathbf{a}_k \notin H$. Thus, H' is non-normal.

Now suppose $\mathbf{a}_k$ was a corner point. Then, since the vectors in $\mathcal{H}$ are coplanar, the cone $\mathbb{R}_+H$ is just the cone formed by the corners of the polygon. If we take away $\mathbf{a}_k$ from $\mathcal{H}$, $\mathcal{H}'$ will have a new set of corner points. Now we can find a new cone, namely the one formed by the new set of corner points whose intersection with $\mathbb{Z}^n$ gives us H'. From the second part of Gordan's lemma, H' is normal. □

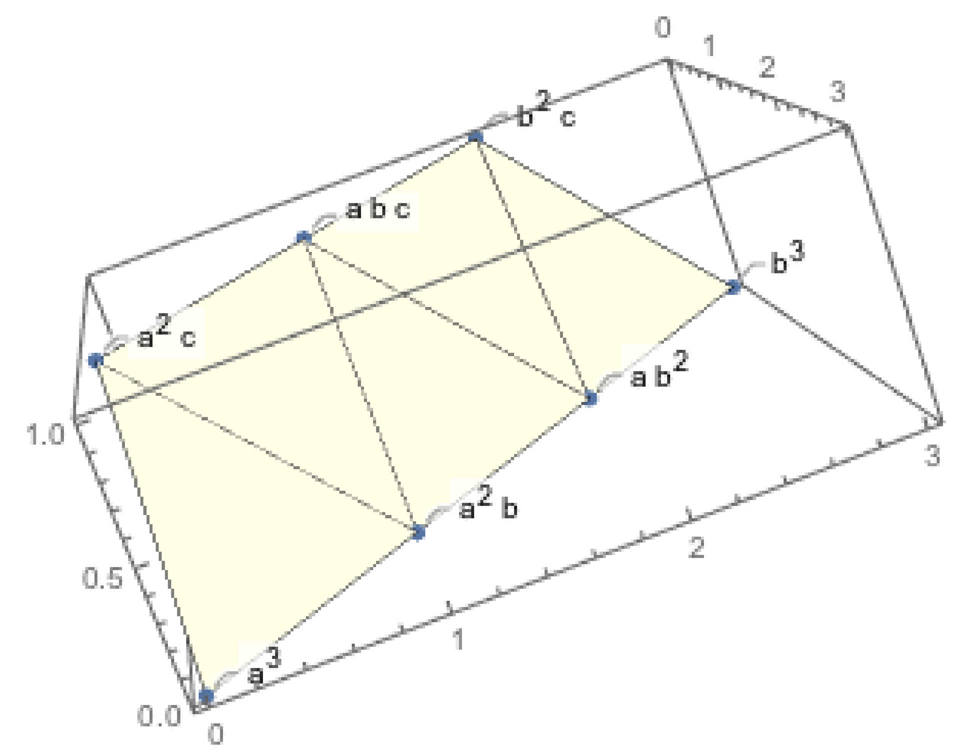

Figure 2: The lattice points of $B(b^2c)$

The corners in $B(b^2c)$ are a^3, b^3, a^2c, and b^2c. These when pinched result in a new polygon. The non-corners such as abc "puncture" the polygon, resulting in its non-normality.

An extension to Rees Algebras

Definition 9. Rees Algebra
The *Rees Algebra* of a Borel Ideal $\langle B(u) \rangle$ is the polynomial subring, and more precisely, the R-subalgebra,

$$\mathcal{R} = K[x_1, \ldots, x_n][B(u)t] \subset K[x_1, \ldots, x_n][t]$$

Analogous to the toric ideal and toric map, we can define a *Rees ideal* an a *Rees Map.* The following theorem concerns the normality of the Rees algebra:

Theorem 2. *Let $m_i \in B(u)$ be a monomial. Then the toric ring $K[B(u) \setminus m_i]$ is normal iff the Rees algebra of $B(u) \setminus m_i$ is normal. Thus, those pinchings of a principal Borel ideal which lead to normal toric rings also lead to normal rees algebras and vice versa.*

Proof. Note that the Rees map is just a toric map between rings in $n+1$ indeterminates. We intend to know which pinchings create "punctures" in the affine semigroup of the Rees Algebra, which is just a toric ring in $n + 1$ variables. But since we only pinch those exponent vectors with $t = 1$, the problem reduces to determining which points are corners and non-corners in the $t = 1$ plane. But the $t = 1$ plane is all the points which correspond to the generators of the affine semigroup associated with the toric ring. Thus, the corners and non-corners are identical between toric rings and Rees algebras. As a result, normality of the Rees algebra of a pinched principal borel is equivalent to the normality of its toric ring. □

Further Directions

Our further work involves developing results for toric rings of two Borel ideals, and studying the Koszulness of toric rings and Rees algebras of pinched principal Borel ideals.

Conjecture. *The rev-lex earliest non-normal, homogenous of degree s, two-Borel ideal $B(u, v)$, where $u, v \in K[x_1, \ldots, x_n]$ is $B(x_1^{s-2}x_3^2, x_2^s)$*

Conjecture. *Let $B(u, v)$ be a homogenous non-normal two-Borel ideal. Then $\forall\, m : m \prec_{revlex} u$, define a chain of monomials $u = m_0 \to m_1 \to \ldots \to m_l = m$ where $\forall\, i : 1 \leq i \leq l, m_i = \frac{x_{k+1}}{x_k} m_{i-1}$ s.t $x_k \,|\, m$. If $\forall\, i : 1 \leq i \leq l, m_i$ and v are coprime, then $B(m, v)$ is also non-normal.*

Question. *Which pinched principal Borel ideals have toric rings that are Koszul?* This question remains open, and we have currently been trying to identify patterns in the data on Koszulness.

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