Sobolev Orthogonal Polynomials on the Sierpinski Gasket

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The Sierpinski Gasket

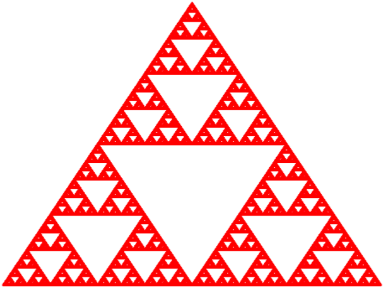

Figure: The Sierpinski Gasket

Let $V_0 = \{q_0, q_1, q_2\} \in \mathbb{R}^2$ and $F_i(x) = \frac{1}{2}(x + q_i)$ for $i = 0, 1, 2$. Then

$$SG = \bigcup_{i=0}^{2} F_i(SG)$$

The Sierpinski Gasket

We work on the finite graph approximation $V_m = \bigcup_{|w|=m} F_w(V_0)$

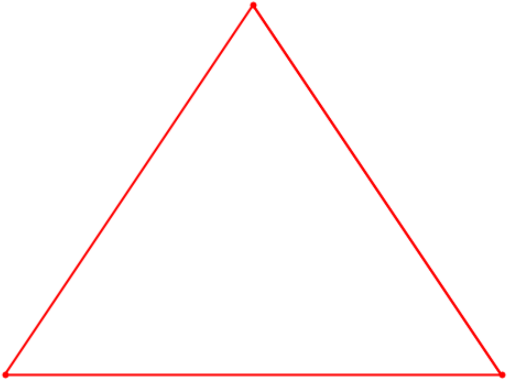

Figure: V_0

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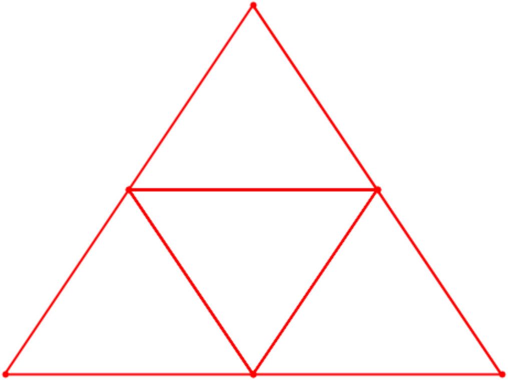

Figure: V_1

The Sierpinski Gasket

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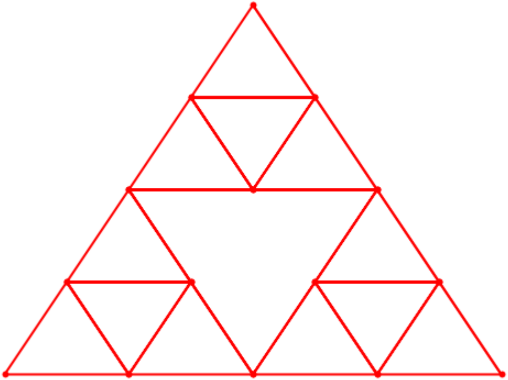

Figure: V_2

The Sierpinski Gasket

We work on the finite graph approximation $V_m = \bigcup_{|w|=m} F_w(V_0)$

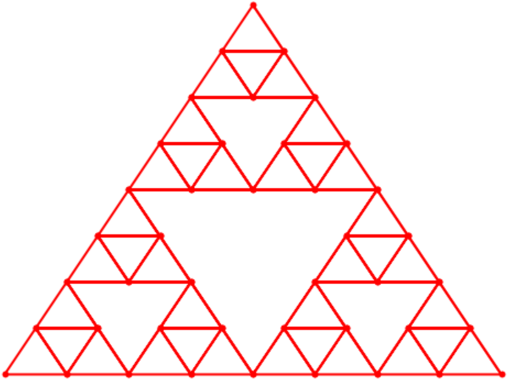

Figure: V_3

The Sierpinski Gasket

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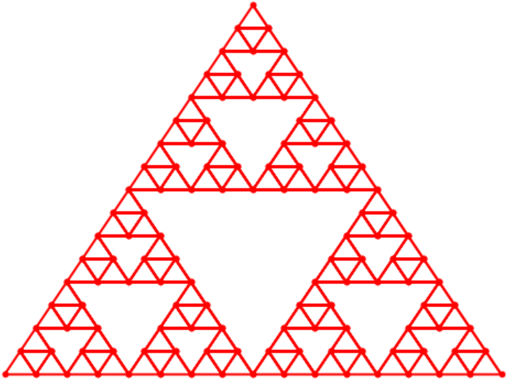

Figure: V_4

Calculus on the Sierpinski Gasket

- Let $u : SG \mapsto \mathbb{R}$. Then the **Laplacian** Δ_μ is defined as

$$\Delta_\mu u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x)$$

where $\Delta_m u(x) = \sum_{y \underset{m}{\sim} x} (u(x) - u(y))$

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- $G : SG \times SG \mapsto \mathbb{R}$ is called **Green's Function** where

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- $\partial_n u(q_i)$ is the **normal derivative** of u at q_i where
$$\partial_n u(q_i) = \lim_{m \to \infty} \left(\frac{5}{3}\right)^m (2u(q_i) - u(F_{i+1}^m q_i) - u(F_{i-1}^m q_i))$$

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- $\partial_T u(q_i)$ is the **tangential derivative** of u at q_i where
$$\partial_T u(q_i) = \lim_{m\to\infty} 5^m (u(F_{i+1}^m q_i) - u(F_{i-1}^m q_i))$$

Polynomials on SG

- Let $f : SG \mapsto \mathbb{R}$. Then f is a j-degree polynomial if and only if $\Delta^{j+1} f = 0$ and $\Delta^j f \neq 0$, i.e f is j-harmonic but not $(j-1)$-harmonic.

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- The space of polynomials with degree $\leq j$ is denoted $\mathcal{H}_j$. $\mathrm{Dim}(\mathcal{H}_j) = 3j + 3$
- We introduce the following basis $\{P_{jk}\}$ where

$$\begin{aligned}
\Delta^n P_{jk}(q_0) &= \delta_{nj}\delta_{k1} \\
\Delta^n \partial_n P_{jk}(q_0) &= \delta_{nj}\delta_{k2} \\
\Delta^n \partial_T P_{jk}(q_0) &= \delta_{nj}\delta_{k3}
\end{aligned}$$

This is known as the **monomial basis**

Polynomials on SG: The monomial basis

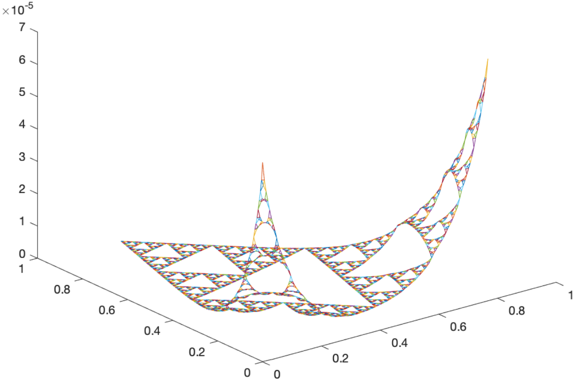

Figure: P_{31}

Polynomials on SG: The monomial basis

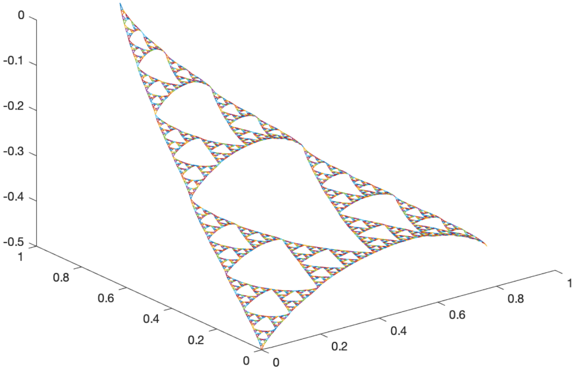

Figure: P_{02}

Polynomials on SG: The monomial basis

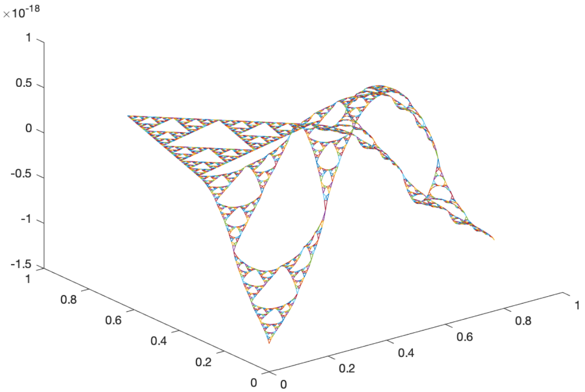

Figure: P_{82}

Polynomials on SG: The monomial basis

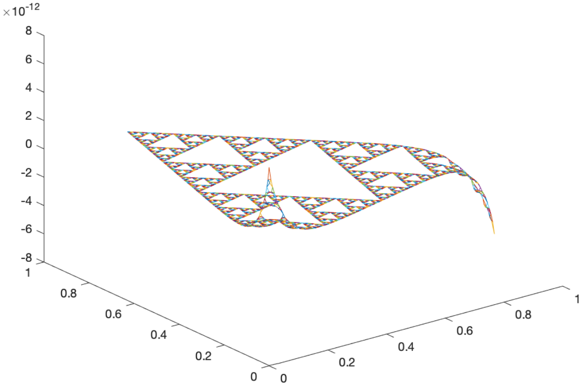

Figure: P_{53}

The Sobolev inner product

- We adopt the following inner product on $\mathcal{H}_j$:

$$\langle f, g\rangle_{H_1} = \int_{SG} (fg + \lambda \Delta f \Delta g)\, d\mu$$

- This is termed the H^1, or the **Sobolev inner product**.

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Question: What are the orthogonal polynomials with respect to the Sobolev inner product? What are their properties? Can we replicate the theory of Sobolev orthogonal polynomials on $[-1, 1]$?

Main Results

Theorem

The monic orthogonal polynomials with respect to $k = 2, 3$ families satisfy a 3-term recurrence and a differential equation:

$$S_{n+2} - a_n S_{n+1} - b_n S_n = f_{n+2}$$

$$S_{n+2} - a_n S_{n+1} - b_n S_n = p_{n+2} - \alpha_n p_{n+1} - \beta_n p_n$$

$$S_n(x) - \lambda \Delta^2 S_n(x) = A_n \Delta p_{n+1}(x) + B_n \Delta p_n(x) + C_{n-1} \Delta p_{n-1}(x)$$

Here $f_{n+1} = -\int_{SG} G(x,y) p_n d\mu(y)$ *where* p_n *is the* **nth Legendre polynomial**, *the monic orthogonal polynomial corresponding to the* L_2 *inner product.*

Main Results

Theorem

Fix $k = 2, 3$ and let S_n denote the corresponding H^1 orthogonal polynomials. Then we have the following estimates in terms of the L^2 orthogonal polynomials and G:

$$\|G\|_{L^2}\|p_{n-1}\|_{L^2} \geq \|S_n\|_{L^2} \geq \|p_n\|_{L^2} \tag{1}$$

$$\|G\|_{L^2}^2\|p_{n-1}\|_{L^2}^2 + \lambda\|p_{n-1}\|_{L^2}^2 \geq \|S_n\|_H^2 \geq \|p_n\|_{L^2}^2 + \lambda\|p_{n-1}\|_{L^2}^2 \tag{2}$$

$$\|\Delta S_n\|_{L^2}^2 \leq \lambda^{-1}\|G\|_{L^2}^2\|p_{n-1}\|_{L^2}^2 + \|p_{n-1}\|_{L^2}^2 \tag{3}$$

$$\|S_n\|_{L^\infty} \leq C(1 + \lambda^{-\frac{1}{2}})\|p_{n-1}\|_{L^2} \tag{4}$$

Main Results

Theorem

$S_n(x;\lambda)$ converges to f_n uniformly in x as $\lambda \to \infty$. Consequently $\Delta S_n \to p_{n-1}$ uniformly as $\lambda \to \infty$. Also,

$$\lambda(S_n(\lambda) - f_n) \to -\frac{\langle f_n, f_{n-1}\rangle_{L^2}}{\|p_{n-2}\|_{L^2}^2} f_{n-1} - \frac{\|p_{n-1}\|_{L^2}^2}{\|p_{n-3}\|_{L^2}^2} f_{n-2}$$

uniformly in x as $\lambda \to \infty$

Main Results

$[-1,1]$	SG
$\int_{-1}^{1} fg + \lambda f'g'\,dx$	$\int_{SG} fg + \lambda \Delta f \Delta g\,dx$
$S_n - S_{n-2} = a_n(p_n - p_{n-2})$	$S_{n+2} - a_n S_{n+1} - b_n S_n = p_{n+2} - \alpha_n p_{n+1} - \beta_n p_n$
$S_n - \lambda S_n'' = A_n P'_{n+1} + B_n P'_{n-1}$	$S_n(x) - \lambda \Delta^2 S_n(x) = A_n \Delta p_{n+1}(x) + B_n \Delta p_n(x) + C_{n-1} \Delta p_{n-1}(x)$

Table: Sobolev orthogonal polynomials on $[-1,1]$ vs SG

Main Results

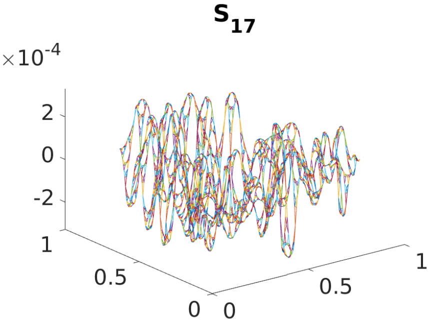

Figure: Sobolev orthogonal polynomials

An Application to Interpolation and Quadrature on SG

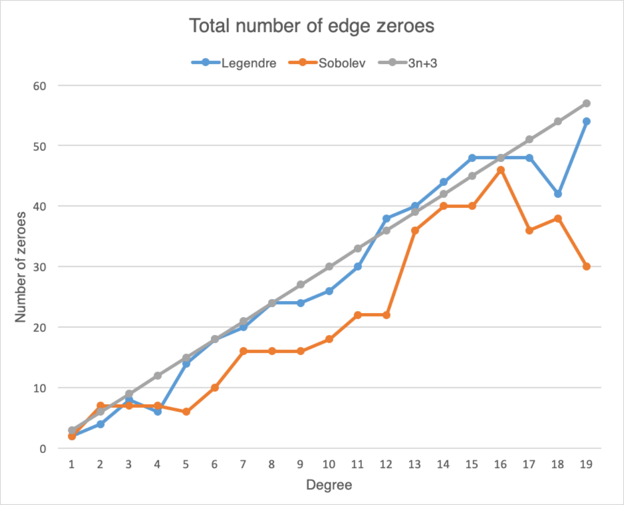

Figure: Edge zeroes of Anti-symmetric Sobolev and Legendre polynomials

An Application to Interpolation and Quadrature on SG

- For which points $x_1, \ldots, x_{3n+3}$ is the matrix

$$M_n = \begin{bmatrix} P_{0,1}(x_1) & \ldots & P_{n,3}(x_1) \\ \vdots & \ddots & \vdots \\ P_{0,1}(x_{3n+3}) & \ldots & P_{n,3}(x_{3n+3}) \end{bmatrix}$$

  invertible?
- Clearly not any set of $3n+3$ points since we could take the $3n+3$ zeroes of the polynomials found previously.

An Application to Interpolation and Quadrature on SG

Theorem

Let $I_n \subseteq SG$ where for any subset $N \subseteq I$ such that $|N| = 3n+3$, M_n is invertible on N. Then I_n has empty interior for every $n \in \mathbb{N}$.

Theorem

For any $n \geq 0$, take $x_i = F_0^{(i-1)}(q_1)$ for $1 \leq i \leq 2n+2$, and $x_i = F_0^{(i-2n-3)}(q_2)$ for $2n+3 \leq i \leq 3n+3$. Then M_n is invertible.

Theorem

Let I_n^m be any quadrature rule which perfectly integrates n-degree polynomial splines at level m. Then

$$\left| I_n^m(f) - \int_{SG} f \right| \leq c_1(n) 5^{-(n+1)m} \|\Delta^{(n+1)} f\|_\infty$$

Further Questions

- Characterize those sets for which M_n is non-invertible.
- Fix $k = 1$. Prove that for every $n \geq 0$, $\partial_n f_n(q_0) \neq 0$.

References