Normality of Toric Rings and Rees Algebras of Pinched Strongly Stable Ideals

Cameron Chandler, Margit Liu, Elizabete Mezinska, Andrew Moore, Shashank Sule, Andrew Tawfeek

Advisor: Gabriel Sosa

Introduction

It is known from the 16th century that the toric ring $K[x^3, x^2y, xy^2, y^3, x^2z, y^2z, xz^2, yz^2, z^3]$ also called the pinched Veronese, is not normal. Sturmfels [5] proved that Veronese-type toric rings are normal. Further results from Conca and De Negri [4] [1] show that toric rings of principal Borel ideals are also normal. Motivated by these inquiries, we set out to investigate toric rings generated by pinchings of principal Borel ideals.

Methods and Notation

Note that these may also be found in [3].

Definition 1. Toric Ring
A *toric ring* is defined as a subring $K[T] \subset K[x_1, \ldots, x_n]$ where T is a set of monomials, i.e $T = \{\mathbf{x_1}^{\mathbf{a_1}}, \ldots, \mathbf{x_s}^{\mathbf{a_s}}\}$. An example of a toric ring is $K[x^2, xy, xz, y^2]$, which is a subring of $K[x, y, z]$.

Definition 2. Normality
Let R be a ring and $\overline{R}$ the set of all rational solutions to monic equations with coefficients in R. Then R is *normal* if and only if $R = \overline{R}$. For instance, $\mathbb{Z}$ is normal because all rational solutions to monic equations with coefficients in $\mathbb{Z}$ are contained in $\mathbb{Z}$. This is one of the fundamental results of algebra: *Gauss' Lemma*.

Definition 3. Borel Moves and Borel Ideals
A *Borel move* is an operation that sends m to a monomial $\frac{x_j m}{x_i}$ where $x_i \mid m$ and $j \leq i$. A monomial ideal B is a *Borel ideal* if B is generated by a set that is closed under Borel moves. Such a set is called a *Borel set*.

Definition 4. Principal Borel Ideals and Pinchings
The *principal Borel ideal* generated by a monomial u is the ideal $\langle B(u) \rangle$ where $B(u)$ is the minimal strongly stable set of monomials containing u. For example, let $u = b^2c$. Then $I = \langle B(b^2c) \rangle = \langle a^3, a^2b, ab^2, b^3, a^2c, abc, b^2c \rangle$. We say that $\langle B(b^2c) \setminus a^2c \rangle = \langle a^3, a^2b, ab^2, b^3, abc, b^2c \rangle$ is the *pinching* of a^2c from $\langle B(b^2c) \rangle$.

Definition 5. Toric Ideal
Let $T = \{m_1, \ldots, m_s\}$ be a set of s-many monomials in $K[x_1 \ldots x_n]$ and $K[T]$ be the associated toric ring. Define ϕ to be the ring homomorphism

$$\phi : K[t_1 \ldots t_s] \to K[T]$$

where $\phi(t_i) = m_i$. The *toric ideal* of $K[T]$ is defined to be $\ker \phi$.

Theorem. ***(Sturmfels)****[5] If the minimal generators of $\mathrm{in}_< \ker \phi$ with respect to some monomial ordering $<$ are all squarefree, then $K[T]$ is normal.*

The proof can be found in [2]. Sturmfels' theorem provides a handy way of discerning the normality of $K[T]$. To put it to effect, we computed (using the computer algebra system, Macaulay2) the initial ideals of the toric ideals of several toric rings of pinched principal Borel ideals and checked them for squarefreeness.

Goal. Find which pinchings of principal Borel ideals form normal toric rings

The Main Result

Before we discuss the main result, we introduce the necessary definitions. Let $\mathcal{H} = \{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be a set of vectors in $\mathbb{Z}^n$.

Definition 6. Affine Semigroup.
The *affine semigroup* generated by $\mathcal{H}$ is denoted

$$\mathbb{N}\mathcal{H} = \left\{ \sum_{i=1}^{s} a_i \mathbf{v}_i \mid a_i \in \mathbb{N} \text{ for } 1 \leq i \leq s \right\}.$$

Definition 7. Affine Group.
The *affine group* generated by $\mathcal{H}$ is denoted

$$\mathbb{Z}\mathcal{H} = \left\{ \sum_{i=1}^{s} a_i \mathbf{v}_i \mid a_i \in \mathbb{Z} \text{ for } 1 \leq i \leq s \right\}.$$

Definition 8. Normality.
An affine semigroup $\mathbb{N}\mathcal{H}$ is said to be *normal* if, given $g \in \mathbb{Z}\mathcal{H}$ and $n \in \mathbb{N} \setminus \{0\}$ such that $ng \in \mathbb{N}\mathcal{H}$, then $g \in \mathbb{N}\mathcal{H}$.

Definition 9. Cone.
Let $\mathcal{F} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a finite subset of $\mathbb{R}^n$ and let $\mathbb{R}_+$ be the set of nonnegative real numbers. The set

$$\mathbb{R}_+\mathcal{F} = \left\{ \sum_{i=1}^{m} a_i \mathbf{v}_i \mid a_i \in \mathbb{R}_+ \text{ for } 1 \leq i \leq m \right\}$$

is called the *cone* generated by $\mathcal{F}$.

Additionally, note that any finitely generated cone of the form $\mathbb{R}_+\mathcal{F}$, where $\mathcal{F}$ is a finite subset of $\mathbb{Q}^n$ is called **rational**.

A Geometric Interpretation

- Geometrically, $\mathbb{N}\mathcal{H}$ can be understood as the set of lattice points within the cone, as it is merely a subset.

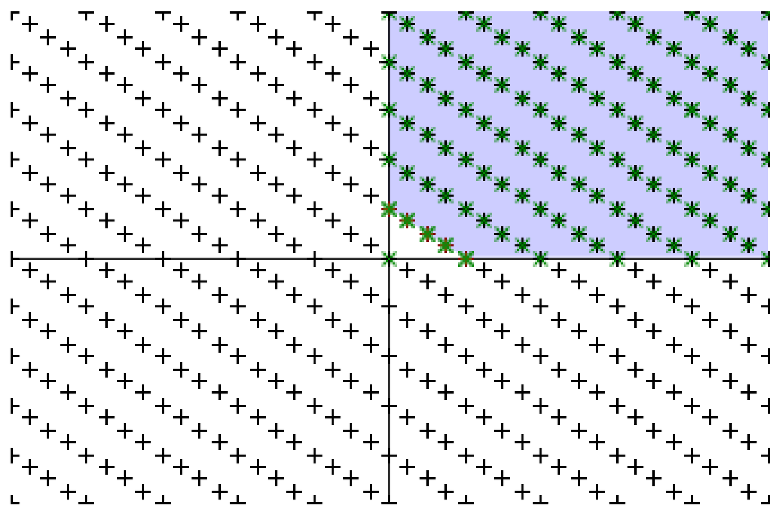

Figure 1: The cone (blue), affine group (+) and affine semigroup (x) generated by $\mathcal{H} = \{(4,0), (3,1), (2,2), (1,3), (0,4)\}$. Note the dashes on the axes are four units apart.

- The generating set of the lattice points $\mathcal{H} = \{(4,0), (3,1), (2,2), (1,3), (0,4)\}$ can be interpreted as generating the toric ring $K[T] = K[x^4, x^3y, x^2y^2, xy^3, y^4]$, where each $\mathbf{a_i} \in \mathcal{H}$ is an exponent vector for a monomial $m_i = \mathbf{x}^{\mathbf{a_i}} \in T$. In fact, this intuition is confirmed by the following theorem:

Theorem. *[2] Let $\mathcal{H} \subset \mathbb{N}^n$ and H the affine semigroup generated by $\mathcal{H}$. Then H is normal if and only if the toric ring $K[H]$ is normal, where K is an algebraically closed field.*

Thus, a normal toric ring is equivalent to the normal affine subgroup. The normality of the affine subgroup can be characterized by the following lemma.

Theorem. ***Gordan's Lemma*** *[2]*

1. *If H is a normal affine semigroup generated by $\mathcal{H} \subset \mathbb{Z}^n$, then $H = \mathbb{Z}\mathcal{H} \cap \mathbb{R}_+\mathcal{H}$.*
2. *If C is a finitely generated rational cone in $\mathbb{R}^n$, then $H = \mathbb{Z}^n \cap C$ is a normal affine subgroup.*

Theorem 1. *Let $K[B(u)] \subset K[x_1, \ldots, x_n]$ be a toric ring generated by the Borel set of a monomial $u \in K[x_1, \ldots, x_n]$. Then for all $m_i = \mathbf{x}^{\mathbf{a_i}} \in B(u)$, $K[B(u) \setminus m_i]$ is normal if and only if a_i is a corner in the n-dimensional polygon formed by the exponent vectors of monomials in $B(u)$ in $\mathbb{R}^n$.*

Proof. Let $\mathcal{H}$ be the set of exponent vectors on the monomials of $B(u)$, where u is an p-degree monomial. Let m_k with exponent vector $\mathbf{a}_k$ be pinched from $\mathcal{H}$. Let $\mathcal{H}' = \mathcal{H} \backslash \{\mathbf{a}_k\}$ and H' the corresponding affine semigroup.

Suppose $\mathbf{a}_k$ is a non-corner. Then since $B(u)$ is homogenous, the vectors in $\mathcal{H}$' lie in the same $(n-1)$-dimensional space, so $\mathbf{a}_k \in \mathbb{Z}\mathcal{H}' \cap \mathbb{R}_+\mathcal{H}'$ and $\mathbf{a}_k \notin \mathcal{H}'$. By the contrapositive of the first part of Gordon's Lemma, H' is non-normal.

Suppose $\mathbf{a}_k$ was a corner point. If we take away $\mathbf{a}_k$ from $\mathcal{H}$, $\mathcal{H}'$ will have a new set of corner points. It then follows that $\mathbb{R}_+\mathcal{H}' \cap \mathbb{Z}^n = H'$, and from the second part of Gordan's Lemma, it clear that H' is normal. □

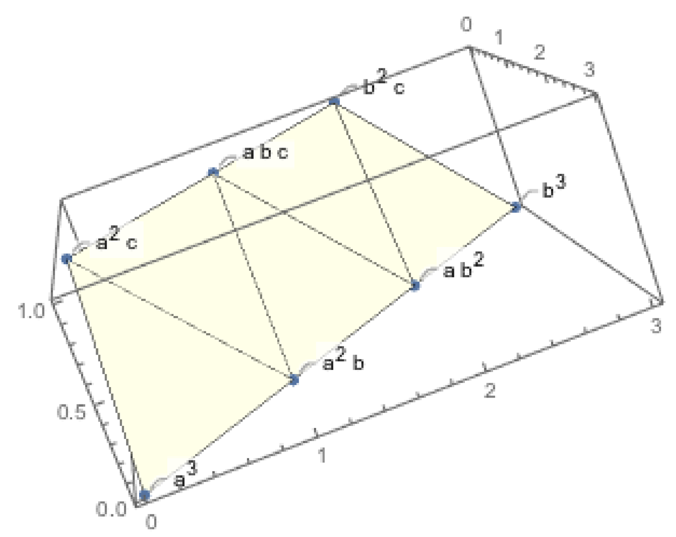

Figure 2: The lattice points of $B(b^2c)$

The corners in $B(b^2c)$ are a^3, b^3, a^2c, and b^2c. These when pinched result in a new polygon. The non-corners such as abc "puncture" the polygon, resulting in its non-normality.

An extension to Rees Algebras

Definition 10. Rees Algebra
The *Rees Algebra* of a Borel Ideal $\langle B(u) \rangle$ is the polynomial subring, and more precisely, the R-subalgebra,

$$\mathcal{R} = K[x_1, \ldots, x_n][B(u)t] \subset K[x_1, \ldots, x_n][t]$$

Analogous to the toric ideal and toric map, we can define a *Rees ideal* and a *Rees Map*. The following theorem concerns the normality of the Rees algebra:

Theorem 2. *Let $m_i \in B(u)$ be a monomial. Then the toric ring $K[B(u) \setminus m_i]$ is normal if and only if the Rees algebra of $B(u) \setminus m_i$ is normal. Thus, those pinchings of a principal Borel ideal which lead to normal toric rings also lead to normal Rees algebras and vice versa.*

The proof follows from the correspondence between corners and non-corners of the affine semigroups associated with pinchings of the Rees Algebra, $(t = 1)$, and the toric ring $(t = 0)$.

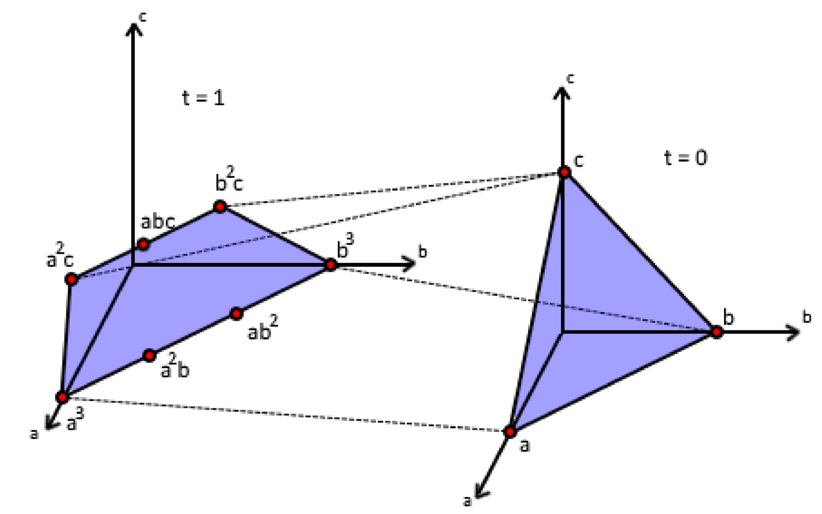

Figure 3: Correspondence mapping for $B(b^2c)$.

Further Directions

Our further work involves developing results for toric rings of two Borel ideals, and studying the Koszulness of toric rings and Rees algebras of pinched principal Borel ideals.

Conjecture. *The rev-lex earliest non-normal, homogenous of degree s, two-Borel ideal $B(u, v)$, where $u, v \in K[x_1, \ldots, x_n]$ is $B(x_1^{s-2}x_3^2, x_2^s)$*

Conjecture. *Let $B(u, v)$ be a homogenous non-normal two-Borel ideal. Then for all $m : m \prec_{revlex} u$, define a chain of monomials $u = m_0 \to m_1 \to \ldots \to m_l = m$ where for all $i : 1 \leq i \leq l, m_i = \frac{x_{k+1}}{x_k} m_{i-1}$ s.t $x_k | m$. If for all $i : 1 \leq i \leq l, m_i$ and v are coprime, then $B(m, v)$ is also non-normal.*

Question. *Which pinched principal Borel ideals have toric rings that are Koszul?*

This question remains open, and we have currently been trying to identify patterns in the data on Koszulness.

Acknowledgements

We sincerely thank the SURF and Gregory S. Call Fellowships for supporting our research and Prof. Sosa for his patience, guidance, and for introducing the research problem. We also extend our heartfelt gratitude to the computer algebra system Macaulay2 and to Andy Anderson for allowing us access and help with the Amherst Computing Cluster.

References

[1] Winfried Bruns and Aldo Conca. *Linear Resolutions of Powers and Products*. Singularities and Computer Algebra, pages 47–69, 03 2017.

[2] V. Ene and J. Herzog. *Gröbner Bases in Commutative Algebra*. Graduate studies in mathematics. American Mathematical Society.

[3] Michael DiPasquale, Christopher A. Francisco, Jeffrey Mermin, Jay Schweig, and Gabriel Sosa. The Rees algebra of a two-Borel ideal is Koszul. To appear on the *Proceeding of the American Mathematical Society 2018*, 2017.

[4] Emanuela De Negri. *Toric rings generated by special stable sets of monomial*. Mathematische Nachrichten, 203:31–45, 1999.

[5] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.