UNIT - I

Review of the three laws of motion and vector algebra

In this course on Engineering Mechanics, we shall be learning about mechanical interaction between bodies. That is we will learn how different bodies apply forces on one another and how they then balance to keep each other in equilibrium. That will be done in the first part of the course. So in the first part we will be dealing with STATICS. In the second part we then go to the motion of particles and see how does the motion of particles get affected when a force is applied on them. We will first deal with single particles and will then move on to describe the motion of rigid bodies.

The basis of all solutions to mechanics problems are the Newton's laws of motion in one form or the other. The laws are:

First law: *A body does not change its state of motion unless acted upon by a force.* This law is based on observations but in addition it also defines an *inertial frame* . By definition an inertial frame is that in which a body does not change its state of motion unless acted upon by a force. For example to a very good approximation a frame fixed in a room is an inertial frame for motion of balls/ objects in that room. On the other hand if you are sitting in a train that is accelerating, you will see that objects outside are changing their speed without any apparent force. Then the motion of objects outside is changing without any force. The train is a non-inertial frame.

Second law: The second law is also part definition and part observation. It gives the force in terms of a quantity called the mass and the acceleration of a particle. It says that a force of magnitude F applied on a particle gives it an acceleration a proportional to the force. In other words

$$F = ma , \qquad (1)$$

where$m$ is identified as the inertial mass of the body. So if the same force - applied either by a spring stretched or compressed to the same length - acting on two different particles produces accelerations a_1 and a_2, we can say that

$$m_1 a_1 = m_2 a_2$$

or
$$m_2 = \left(\frac{a_1}{a_2}\right) m_1 \qquad (2)$$

Thus by comparing accelerations of a particle and of a standard mass (unit mass) when the same force is applied on each one them we get the mass of that particle. Thus gives us the definition of mass. It also gives us how to measure the force via the equation F = ma. One Newton (abbreviated as N) of force is that providing an acceleration of $1 m/s^2$ to a standard mass of 1 kg.

If you want to feel how much in 1 Newton , hold your palm horizontally and put a hundred gram weight on it; the force that you feel is about 1N.

Of course you cannot always measure the force applied by accelerating objects. For example if you are pushing a wall, how much force you are applying cannot be measured by observing the acceleration of the wall because the wall is not moving. However once we have adopted a measure of force, we can always measure it by comparing the force applied in some other situation.

In the first part of the course i.e. Statics we consider only equilibrium situations. We will therefore not be looking at F = ma but rather at the balance of different forces applied on a system. In the second part - Dynamics - we will be applying F = ma extensively.

Third Law: Newton's third law states that if a body *A* applies a force F on body *B* , then *B* also applies an equal and opposite force on *A* . (Forces do not cancel such other as they are acting on two different objects)

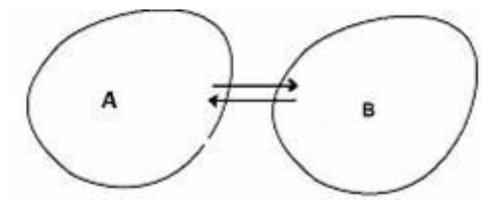


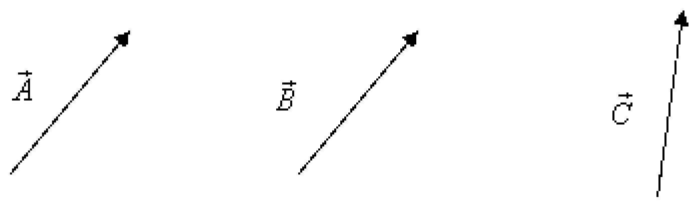
Figure 1

Thus if they start from the position of rest *A* and *B* will tend to move in opposite directions. You may ask: if *A* and*B* are experiencing equal and opposite force, why do they not cancel each other? This is because - as stated above - the forces are acting on two different objects. We shall be using this law a lot both in static as well as in dynamics.

After this preliminary introduction to what we will be doing in the coming lectures, we begin with a review of vectors because the quantities like force, velocities are all vectors and we should therefore know how to work with the vectors. I am sure you have learnt some basic manipulations with vectors in your 12th grade so this lectures is essentially to recapitulate on what you have learnt and also introduce you to one or two new concepts.

You have learnt in the past is that vectors are quantity which have both a magnitude and a direction in contrast to scalar quantities that are specified by their magnitude only. Thus a quantity like force is a vector quantity because when I tell someone that I am applying X-amount of force, by itself it is not meaningful unless I also specify in which direction I am applying this force. Similarly when I ask you where your friend's house is you can't just tell me that it is some 500 meters far. You will also have to tell me that it is 500 meters to the north or 300 meters to the east and four hundred meters to the north from here. Without formally realizing it, you are telling me a about a vector quantity. Thus quantities like displacement, velocity, acceleration, force are vectors. On the other hand the quantities distance, speed and energy are scalar quantities. In the following we discuss the algebra involving vector quantities. We begin with a discussion of the equality of vectors.

Equality of Vectors: Since a vector is defined by the direction and magnitude, two vectors are equal if they have the same magnitude and direction. Thus in figure 2 vector $\vec{A}$ is equal to vector $\vec{B}$ and but not equal to vector $\vec{C}$ although all of them have the same magnitude.



Vectors $\vec{A}$ and $\vec{B}$ are equal to each other but not equal to vector $\vec{C}$

Figure 2

Thus we conclude that any two vectors which have the same magnitude and are parallel to each other are equal. If they are not parallel then they cannot be equal no matter what their magnitude.

In physical situations even two equal vectors may produce different effects depending on where they are located. For example take the force $\vec{F}$ applied on a disc. If applied on the rim it rotates the wheel at a speed different from when it is applied to a point nearer to the center. Thus although it is the same force, applied at different points it produces different effects. On the other hand, imagine a thin rope wrapped on a wheel and being pulled out horizontally from the top. On the rope no matter where the force is applied, the effect is the same. Similarly we may push the wheel by applying the same force at thee end of a stick with same result (see figure 3).

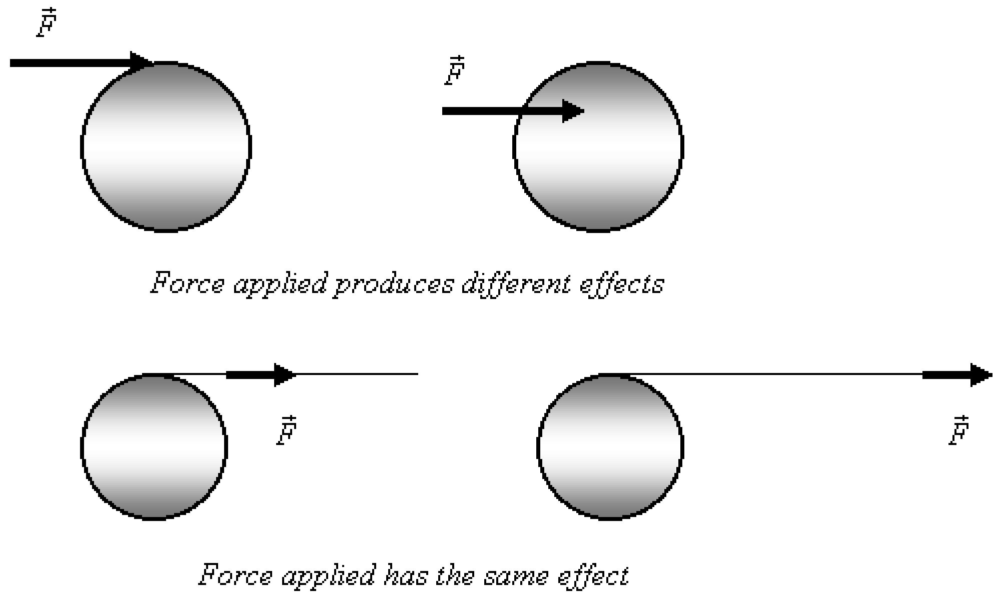
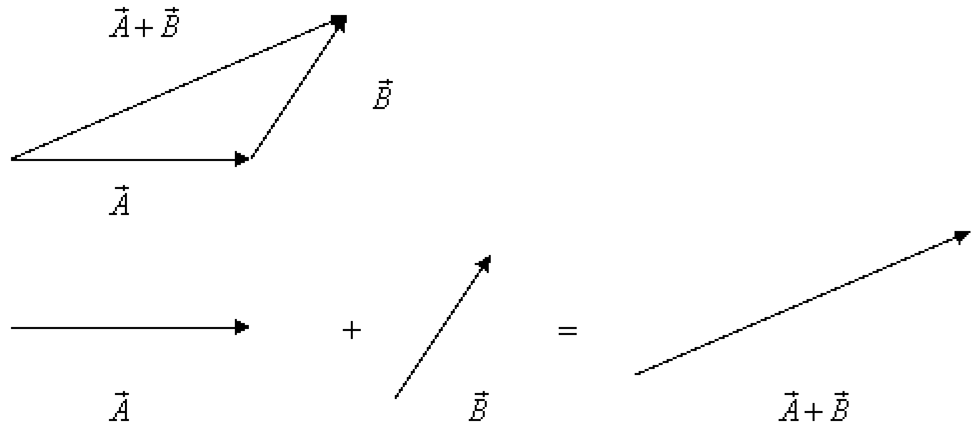


Figure 3

Thus we observe that a force applied anywhere along its line of applications produces the same effect. This is known as transmissibility of force. On the other hand if the same force is applied at a point away from its line of application, the effect produced is different. Thus the transmissibility does not mean that force can be applied anywhere to produce the *same* effect but only at any point on its line of application.

Adding and subtracting two vectors (Graphical Method): When we add two vectors $\vec{A}$ and $\vec{B}$ by graphical method to get $\vec{A}+\vec{B}$, we take vector $\vec{A}$, put the tail of $\vec{B}$ on the head of $\vec{A}$.Then we draw a vector from the tail of $\vec{A}$ to the head of $\vec{B}$. That vector represents the resultant $\vec{A}+\vec{B}$ (Figure 4). I leave it as an exercise for you to show that $\vec{A}+\vec{B}=\vec{B}+\vec{A}$. In other words, show that vector addition is commutative.



Adding two vectors

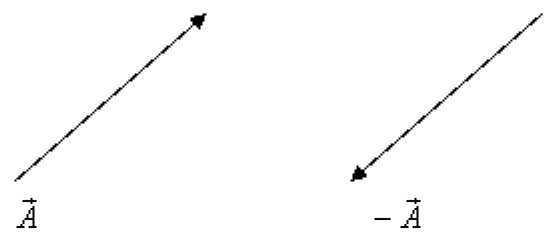
Figure 4

Let us try to understand that it is indeed meaningful to add two vectors like this. Imagine the following situations. Suppose when we hit a ball, we can give it velocity $\vec{B}$. Now imagine a ball is moving with velocity $\vec{A}$ and you hit it an additional velocity $\vec{B}$. From experience you know that the ball will now start moving in a direction different from that of $\vec{A}$. This final direction is the direction of $(\vec{A}+\vec{B})$ and the magnitude of velocity now is going to be given by the length of $(\vec{A}+\vec{B})$.

Now if we add a vector $\vec{A}$ to itself, it is clear from the graphical method that its magnitude is going to be 2 times the magnitude of $\vec{A}$ and the direction is going to remain the same as that of $\vec{A}$. This is equivalent to multiplying the vector $\vec{A}$ by 2. Similarly if 3 vectors are added we get the resultant $3\vec{A}$. So we have now got the idea of multiplying a vector by a number *n* . If simply means: add the vector *n* times and this results in giving a vector in the same direction with a magnitude that *n* times larger.

You may now ask: can I multiply by a negative number? The answer is yes. Let us see what happens, for example, when I multiply a vector $\vec{A}$ by -1. Recall from your school mathematics that multiplying by -1 changes the number to the other side of the number line. Thus the number -2 is two steps to the left of 0 whereas the number 2 is two steps to the right. It is exactly the same with vectors. If $\vec{A}$ represents a vector to the right, $-\vec{A}$ would represent a vector in the direction opposite i.e. to the left. It is now easy to understand what does the vector $-\vec{A}$

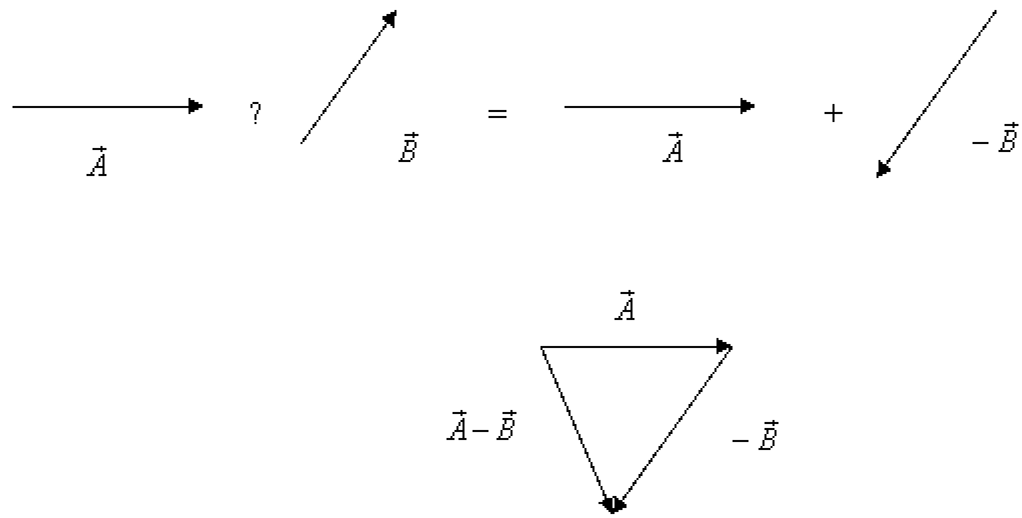
represent? It is a vector of the same magnitude as that of $\vec{A}$ but in the direction opposite to it (Figure 5). Having defined $-\vec{A}$, it is now easy to see what is the vector $-m\vec{A}$? It is a vector of magnitude $m|\vec{A}|$ in the direction opposite to $\vec{A}$.



A vector and its negative

Figure 5

Having defined $-\vec{A}$, it is now straightforward to subtract one vector from the other. To subtract a vector $\vec{B}$ from $\vec{A}$, we simply add $-\vec{B}$ to $\vec{A}$ that is $\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$. Thus to subtract vector $\vec{B}$ from $\vec{A}$ graphically, we add $\vec{A}$ and $-\vec{B}$. This is shown in figure 6.



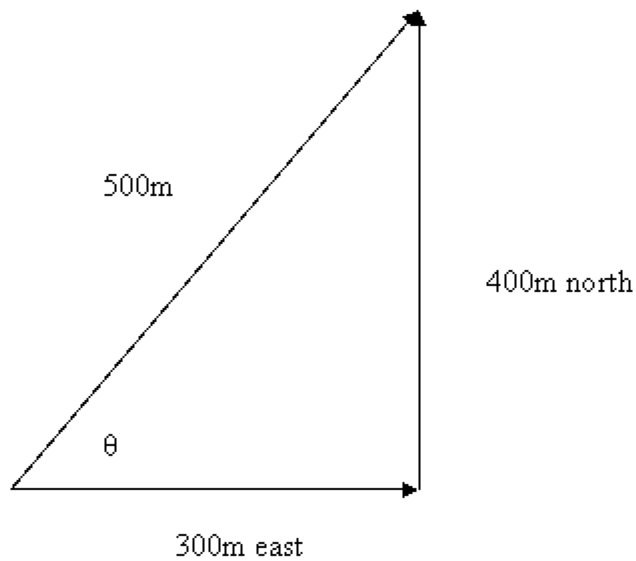
Subtracting vector $\vec{B}$ from vector $\vec{A}$

Figure 6

Again I leave it as an exercise for you to show that $(\vec{A}-\vec{B})$ is not equal to $(\vec{B}-\vec{A})$ but $(\vec{B}-\vec{A})$ = - $(\vec{A}-\vec{B})$. We now solve a couple of examples

Example1: A person walks 300m to the east and 400m to the north to reach his friend's house. What is the total displacement of the person, and what is the total distance traveled by him?

Recall that distance is a scalar quantity. Thus the total distance covered is 700m. Displacement, on the other hand, is a vector quantity so to find the net displacement, we add the two vectors to get a displacement of 500m at an angle $\theta = \tan^{-1}\left(\frac{4}{3}\right)$ from east to north (Figure 7).



Adding displacements of 300m east and 400m north

Figure 7

Example 2 : Two persons are pushing a box so that the net force on the box is 12N to the east If one of the person is applying a force 5N to the north, what is the force applied by the other person.

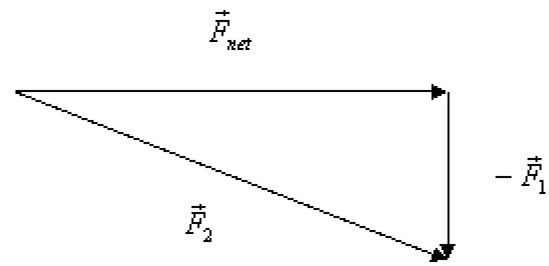
Let the force by person applying 5N be denoted by $\vec{F}_1$and that by the other person by $\vec{F}_2$. We then have

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$$

so that

$$\vec{F}_2 = \vec{F}_{net} - \vec{F}_1$$

Solution for $\vec{F}_2$ is given graphically in figure 8. The force comes out to be 13N at an angle of $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ from east to south.



Finding force applied by a person when the net force and that applied by one of the persons is given.

Figure 8

Although graphical way is nice to visualize vectors in two dimensions, it becomes difficult to work with it in three dimensions, and also when many vectors and many operations with them are involved. So vector algebra is best done by representing them in terms of their components along the x, y & z axes in space. We now discuss how to this is done.

To represent vectors in terms of their x,y and z components, let us first introduce the concept of unit vector. A unit vector $\hat{n}$ in a particular direction is a vector of magnitude '1' in that direction. So a vector in that particular direction can be written as a number times the unit vector $\hat{n}$. Let us denote the unit vector in x-direction as $\hat{i}$, in y-direction as $\hat{j}$ and in z-direction as $\hat{k}$. Now any vector can be described as a sum of three vectors $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ in the directions x, y and z, respectively, in any order (recall that order does not matter because vector sum is commutative). Then a vector

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

Further, using the concept of unit vectors, we can write $\vec{A}_x = A_x\hat{i}$, where A_x is a number. Similarly $\vec{A}_y = A_y\hat{j}$ and $\vec{A}_z = A_z\hat{k}$. So the vector above can be written as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

where A_x, A_y and A_z are known as the x, y, & z components of the vector. For example a vector $\vec{A} = 3\hat{i} + 3\hat{j} + 4\hat{k}$ would look as shown in figure 9.

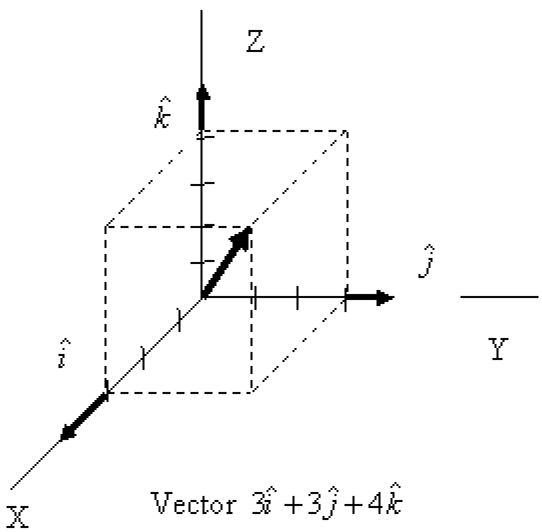


Figure 9

It is clear from figure 9 that the magnitude of the vector $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ is $\left|\vec{A}\right|=\sqrt{A_x^2+A_y^2+A_z^2}$. Now when we add two vector, say $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$, all we have to do is to add their x-components, y-components and the z-components and then combine them to get

$$\vec{A}+\vec{B}=(A_x+B_x)\hat{i}+(A_y+B_y)\hat{j}+(A_z+B_z)\hat{k}$$

Similarly multiplying a vector by a number is same as increasing all its components by the same amount. Thus

$$m\vec{A}=(mA_x)\hat{i}+(mA_y)\hat{j}+(mA_z)\hat{k}$$

How about the multiplying by -1? It just changes the sign of all the components. Putting it all together we see that

$$\vec{A}-\vec{B}=(A_x-B_x)\hat{i}+(A_y-B_y)\hat{j}+(A_z-B_z)\hat{k}$$

Having done the addition and subtraction of two vectors, we now want to look at the product of two vectors. Let us see what all possible products do we get when we multiply components of two vectors. By multiplying all components with one another, we have in all nine numbers shown below:

$$\begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

The question is how do we define the product of two vectors from the nine different numbers obtained above? We will delay the answer for some time and come back to this question after we establish the transformation properties of scalars and vectors. By transformation properties we mean how does a scalar quantity or the components of a vector quantity change when we look at them from a different (rotated) frame?

Let us first look at a scalar quantity. As an example, we take the distance traveled by a person. If we say that the distance covered by a person in going from one place to another is 1000m in one frame, it remains the same irrespective of whether we look at it from the frame (*xy*) or in a frame (*x' y'*) rotates about the z-axis (see figure 10).

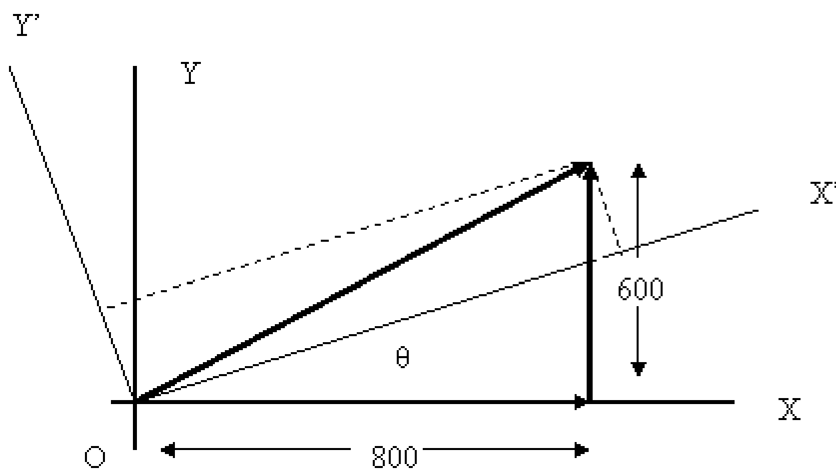


Figure 10

Let us now say that a person moves 800 meter along the x-axis and 600 meters along the y-axis so that his net displacement is a vector of 1000m in magnitude at an angle of from the x-axis as shown in figure 10. The total distance traveled by the person is 1400m. Now let us look at the same situation frame different frame which has its x' & y' axis rotated about the z- axis. Note that the total distance traveled by the person (a scalar quantity) remains the same, 1400m, in both the frames. Further, whereas the magnitude of the displacement & its direction in space remains unchanged, its components along the x' and y' axis, shown by dashed lines in figure 10, are now different. Thus we conclude the scalar quantity remains unchanged when seen from a rotational frame. The component of a given vector are however different in the rotated frame, as demonstrated by the example above. Let us now see how the components in the original frame and the rotated frame are related.

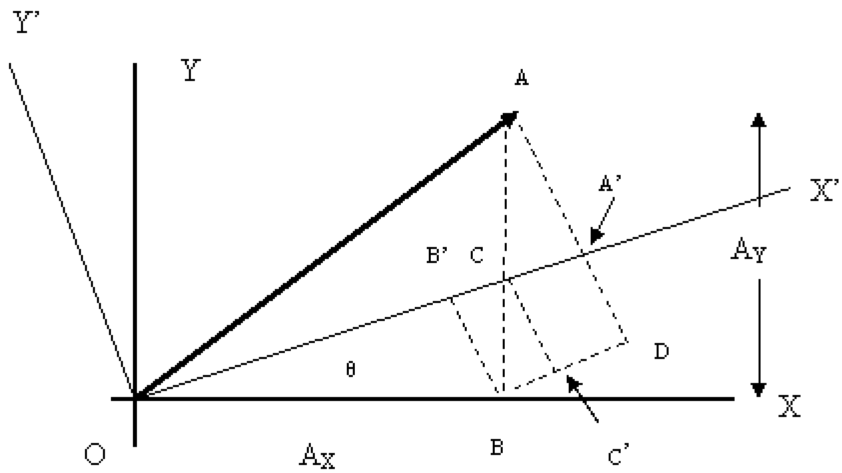


Figure 11

In figure 11, OA is a vector with A_x = OB , A_y = AB, $A_{x'}$ = OA' and $A_{y'}$ = AA'. Using the dashed lines drawn in the figure, we obtain

$$\begin{aligned}A_{x'} = OA' &= OB' + B'A' \\ &= OB\cos\theta + B'C + C'D \\ &= OB\cos\theta + B'C + CA' \\ &= OB\cos\theta + BC\sin\theta + CA\sin\theta \\ &= A_x\cos\theta + (BC+CA)\sin\theta \\ &= A_x\cos\theta + A_y\sin\theta\end{aligned}$$

Similarly

$$\begin{aligned}A_{y'} = AA' &= AD - A'D \\ &= AD - B'B \\ &= BA\cos\theta - OB\sin\theta. \\ &= -A_x\sin\theta + A_y\cos\theta\end{aligned}$$

So we learn that if the same vector is observed from a frame obtained by a rotation about the z-axis by an angle θ, its x and y components in the new frame are

$$\begin{aligned}A_x' &= A_x\cos\theta + A_y\sin\theta \\ A_y' &= -A_x\sin\theta + A_y\cos\theta \\ A_z' &= A_z\end{aligned}$$

One can similarly define how components mix when rotation is about the y or the x- axis. Under the y axis rotation

$$A_z' = A_z \cos\theta + A_y \sin\theta$$
$$A_x' = -A_z \sin\theta + A_y \cos\theta$$

And under a rotation about the x-axis

$$A_y' = A_y \cos\theta + A_z \sin\theta$$
$$A_z' = -A_y \sin\theta + A_z \cos\theta$$

Let us summarize the results obtained above:

1. Scalar quantity is specified by a number and that number remains the same in two different frames rotated with respect to each other.
2. A vector quantity is specified by its components along the x, y, and the z axes and when seen from another frame rotated with respect to a given frame, these components change according to the rules derived above.

We are now ready to get back to defining the product of two vectors. Recall that we had a collection of nine quantities:

$$\begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

We are now going to mix these quantities in such a manner that one combination will give a scalar quantity whereas the other one will give us a vector quantity. This then defines the scalar and vector product of two vectors.

Scalar or dot product: Now it is easy to show that $\left(A_x B_x + A_y B_y + A_z B_z\right)$ is a scalar quantity. To show this we calculate this quantity in a rotated frame (rotation could be about the x, y or the z axis) that is obtain $\left(A_x' B_x' + A_y' B_y' + A_z' B_z'\right)$ and show that it is equal to $\left(A_x B_x + A_y B_y + A_z B_z\right)$. As an example we show it for a frame rotated about the z-axis with respect to the other one. In this case

$$A_x' = \cos A_x + \sin\theta A_y \quad A_y' = -\sin\theta A_x + \cos\theta A_y$$
$$B_x' = \cos B_x + \sin\theta B_y \quad B_y' = -\sin B_x + \cos\theta B_y$$
$$A_z' = A_z, B_z' = B_z$$

Therefore we get

$$A_x^{'}B_x^{'} + A_y^{'}B_y^{'} + A_z^{'}B_z^{'} = \left(\cos\theta A_x + \sin\theta A_y\right)\left(\cos\theta B_x + \sin\theta B_y\right) + \left(-\sin\theta A_x + \cos\theta A_y\right)\left(-\sin\theta B_x + \cos\theta B_y\right) + A_z B_z$$

$$= \left(\cos^2\theta + \sin^2\theta\right)\left(A_x B_x\right) + \left(\cos^2\theta + \sin^2\theta\right)\left(A_y B_y\right) + A_z B_z$$

$$= A_x B_x + A_y B_y + A_z B_z$$

One can similarly show it for rotations about other axes, which is left as an exercise. This then leads us to define the scalar product of two vectors $\vec{A}$ and $\vec{B}$ as

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

As shown above this value remain unchanged when view from two different frame-one rotated with respect to the other. Thus it is a scalar quantity and this product is known as the scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$. It is straightforward to see from the definition above that the dot product is commutative that is $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

Scalar product of two vectors can also be written in another form involving the magnitudes of these vectors and the angle between them as

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

where $\left|\vec{A}\right|$ and $\left|\vec{B}\right|$ are the magnitudes of the two vectors, and θ is the angle between them. Notice that although $\left|\vec{A}\right|$ and $\left|\vec{B}\right| > 0, \vec{A}\cdot\vec{B}$ can be negative or positive depending on the angle between them. Further, if two non-zero vectors are perpendicular, $\vec{A}\cdot\vec{B} = 0$. From the formula above, it is also apparent that if we take vector $\vec{B}$ to be a unit vector, the dot product $\vec{A}\cdot\vec{B}$ represents the component of $\vec{A}$ in the direction of $\vec{B}$. Thus the scalar product between two vectors is the product of the magnitude of one vector with the magnitude of the component of the other vector in its direction. Try to see it pictorially yourself. We also write the dot products of the unit vectors along the x, y, and the z axes. These are $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

Vector or cross product: In defining the scalar product above, we have used three out of the nine possible products of the components of two vectors. From the six of these that are left i.e. $A_x B_y, B_x A_y, A_x B_z, B_x A_z, A_y B_z$, and $B_y A_z$, if we define the vector

$$\vec{A}\times\vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

This is known as the vector or cross product of the two vectors. By calling this expression a vector, we implicitly mean that its component transform like those of a vector. Let us again take the example of looking at the components of this quantity from two frames rotated with respect to each other about the z-axis. In that case the x component of the vector product in the rotated frame is

$$\begin{aligned}(\vec{A}\times\vec{B})_{x'} &= A_{y'}B_{z'} - B_{y'}A_{z'} \\ &= \left(-A_x \sin\theta + A_y \cos\theta\right)B_z - A_z(-B_x \sin\theta + B_y \cos\theta) \\ &= (A_yB_z - B_yA_z)\cos\theta + (A_zB_x - B_zA_x)\sin\theta \\ &= (\vec{A}\times\vec{B})_x \cos\theta + (\vec{A}\times\vec{B})_y \sin\theta\end{aligned}$$

and the y component is

$$\begin{aligned}(\vec{A}\times\vec{B})_{y'} &= A_{z'}B_{x'} - B_{z'}A_{x'} \\ &= A_z\left(B_x \cos\theta + B_y \sin\theta\right) - B_z(A_x \cos\theta + A_y \sin\theta) \\ &= -\sin\theta(A_yB_z - B_yA_z) + \cos\theta(A_zB_x - B_zA_x) \\ &= -(\vec{A}\times\vec{B})_x \sin\theta + (\vec{A}\times\vec{B})_y \cos\theta\end{aligned}$$

Thus we see that the components of the vector product defined above do indeed transform like those of a vector. We leave it as an exercise to show that when the other frame is obtained by rotating about the x and the y axes also, the transformation of the components is like that of a vector. This is known as the vector or the cross product of vectors $\vec{A}$ and $\vec{B}$. It can also be written in the form of a determinant as

$$\vec{A}\times\vec{B} = \begin{pmatrix}\hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{pmatrix}_S$$

Notice that this is the only contribution that transforms in this manner. For example

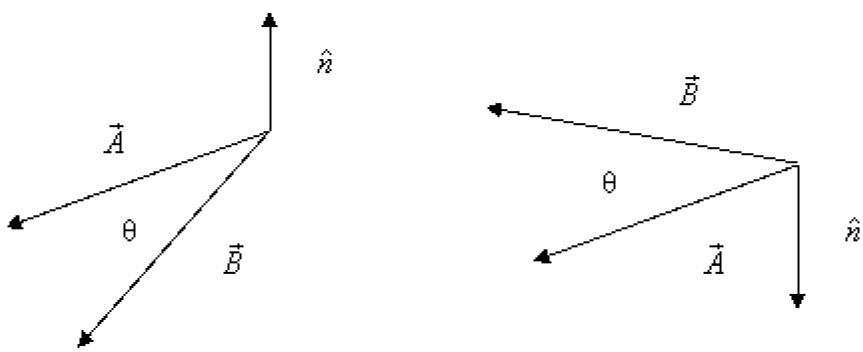
$$(A_yB_z + A_zB_y)\hat{\mathrm{i}} + \hat{\mathrm{j}}(A_zB_x + A_xB_z) + \hat{\mathrm{k}}(A_xB_y + A_yB_x)$$

does not transform like a vector; I leave it as an exercise for you to show. So this cannot form a vector.

Now if we take the dot product of $\vec{A}$ or $\vec{B}$ with $\vec{A}\times\vec{B}$, the result is zero as is easy to see. This implies that the vector product of two vectors is perpendicular to both of them. As such an alternate expression for the vector product of $\vec{A}$ and $\vec{B}$ is

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector in the direction perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ in such a way that if the fingers of the right hand turn from $\vec{A}$ to $\vec{B}$ through the smaller of the angle between them, the thumb gives the direction of in direction of $\hat{n}$. It is also clear from this expression that the vector product of two non-zero vectors will vanish if the vectors are parallel i.e. the angle between them is zero.

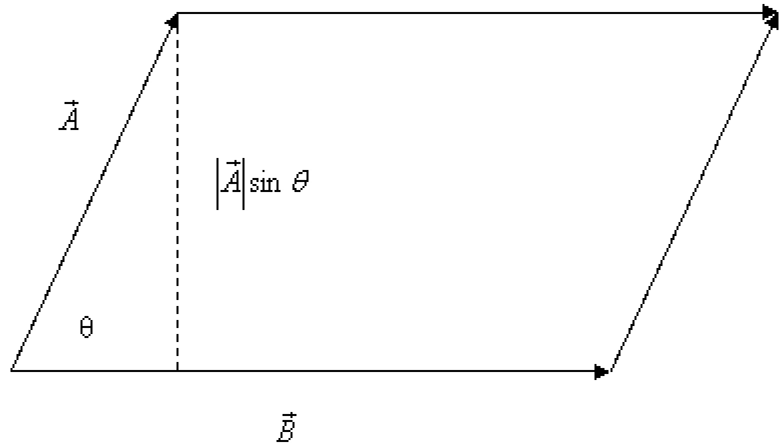


Direction of $\vec{A}\times\vec{B}$ for two different orientations of $\vec{A}$ and $\vec{B}$

Figure 12

The vector product between two vectors is not commutative in that $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ but rather $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

Geometric interpretation of cross product : The magnitude of the cross-product $(\vec{A}\times\vec{B})$, which is equal to $|\vec{A}||\vec{B}|\sin\theta$, is the area of a parallelogram formed by vectors $\vec{A}$ & $\vec{B}$. This is shown in figure 13.



Area of a parallelogram formed by two vectors is equal to the magnitude of their cross product

Figure 13

Derivative of a vector: After reviewing the vector algebra, we would now like to introduce you to the idea of differentiating a vector quantity. Here we take a vector $\vec{A}(t)$ as depending on one parameter, say time *t* , and evaluate the derivative $\frac{d\vec{A}(t)}{dt}$. This is similar to what we do for a regular function. We evaluate the vector $\vec{A}(t+\Delta t)$ at time *(t+ Δt)* , subtract $\vec{A}(t)$ from it, divide the difference $\Delta\vec{A}(t)$ by *Δt*and then take the limit *Δ t → 0* . This is shown in figure 14. Thus

$$\frac{d\vec{A}(t)}{dt} = \lim \Delta t \to 0 \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

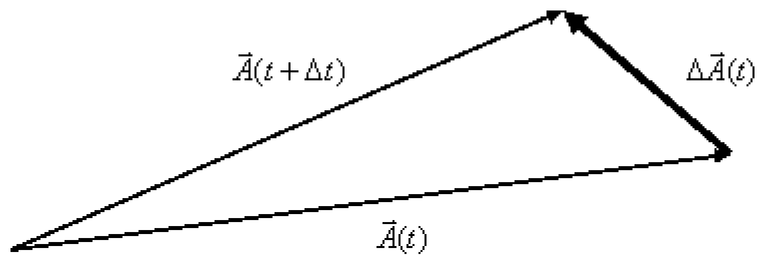


Figure 14

The derivative is easily understood if we think in terms of its derivatives. If we write a vector as

$$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$$

then the derivative of the vector is given as

$$\frac{d\vec{A}(t)}{dt} = \frac{dA_x(t)}{dt}\hat{i} + \frac{dA_y(t)}{dt}\hat{j} + \frac{dA_z(t)}{dt}\hat{k}$$

Notice that only the components are differentiated, because the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ are fixed in space and therefore do not change with time. Later when we learn about polar coordinates, we will encounter unit vectors which also change with time. In that case when taking derivative of a vector, the components as well as the unit vectors both have to be differentiated.

Using the definition above, it is easy to show that in differentiating the product of two vectors, the usual chain rule can be applied. This gives

$$\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}$$

and

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

This pretty much sums up our introduction to vectors. I leave this lecture by giving you three exercises.

1. Show that $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{A}\times\vec{C})$ and that $\vec{A}\cdot(\vec{B}\times\vec{C})$ is the volume of a parallelepiped formed by $\vec{A}, \vec{B}\ \&\ \vec{C}$.
2. Show that $\vec{A}\cdot(\vec{B}\times\vec{C})$ can also be written as the determinant

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

1. Show that if the magnitude of a vector quantity $\vec{A}(t)$ is a fixed, its derivative with respect to *t* will be perpendicular to it. Can you think of an everyday example of this?

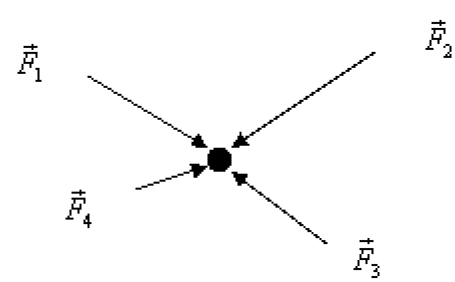
Equilibrium of bodies I

In the previous lecture, we discussed three laws of motion and reviewed some basic aspects of vector algebra. We are now going to apply these to understand equilibrium of bodies. In the static part when we say that a body is in equilibrium, what we mean is that the body is not moving at all even though there may be forces acting on it. (In general equilibrium means that there is no acceleration i.e., the body is moving with constant velocity but in this special case we take this constant to be zero).

Let us start by observing what all can a force do to a body? One obvious thing it does is to accelerate a body. So if we take a point particle P and apply a force on it, it will accelerate. Thus if we want its acceleration to be zero, the sum of all forces applied on it must vanish. This is the condition for equilibrium of a point particle. So for a point particle the equilibrium condition is

$$\sum_i \vec{F}_i$$

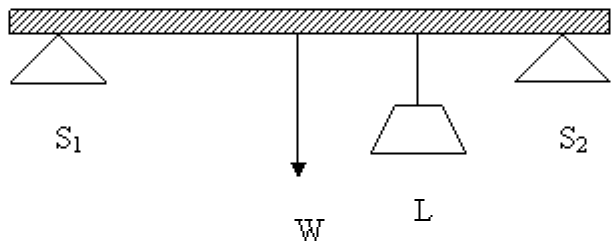
where $\vec{F}_i; i = 1,2,3....$ are the forces applied on the point particle (see figure 13)



A particle in equilibrium under four forces

Figure 1

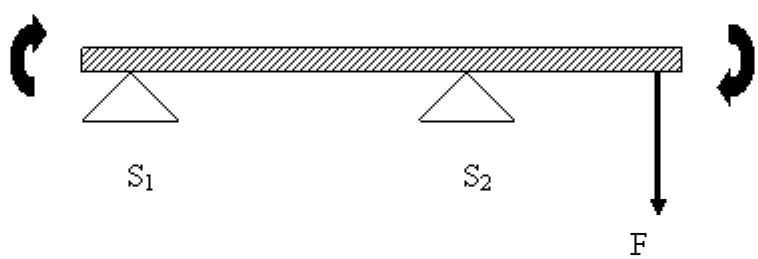
That is all there is to the equilibrium of a point particle. But in engineering problems we deal not with point particles but with extended objects. An example is a beam holding a load as shown in figure 2. The beam is equilibrium under its own weight *W* , the load *L* and the forces that the supports $S_1$and S_2apply on it.



A beam of weight W on supports S_1 and S_2 and holding a load L

Figure 2

To consider equilibrium of such extended bodies, we need to see the other effects that a force produces on them. In these bodies, in addition to providing acceleration to the body, an applied force has two more effects. One it tends to rotate the body and two it deforms the body. Thus a beam put on two supports S_1 and S_2 tends to rotate clockwise about S_2 when a force F is applied downwards (figure 3).



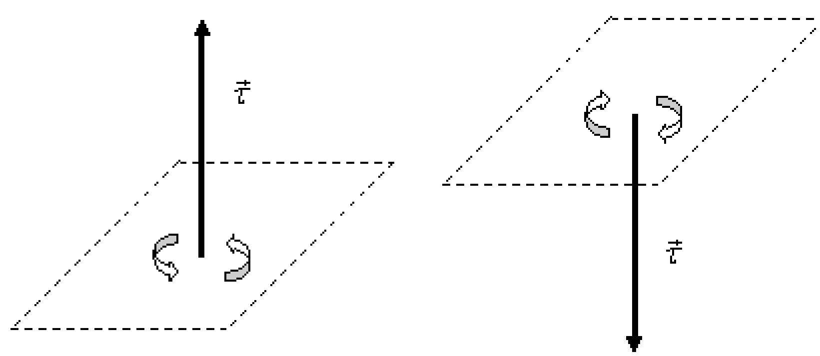
A beam on supports S_1 and S_2 tends to rotate clockwise under the force F

Figure 3

The strength or ability of a force $\vec{F}$ to rotate the body about a point *O* is given by the torque $\vec{\tau}$ generated by it. The torque is defined as the vector product of the displacement vector $\vec{r}_O$ from *O* to the point where the force is applied. Thus

$$\vec{\tau} = \vec{r}_O \times \vec{F}$$

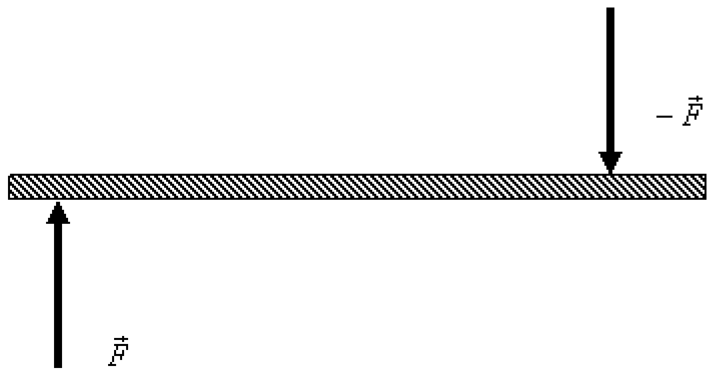
This is also known as the moment of the force. Thus in figure 3 above, the torque about S_2 will be given by the distance from the support times the force and its direction will be into the plane of the paper. From the way that the torque is defined, the torque in a given direction tends to rotate the body on which it is applied in the plane perpendicular to the direction of the torque. Further, the direction of rotation is obtained by aligning the thumb of one's right hand with the direction of the torque; the fingers then show the way that the body tends to rotate (see figure 4). Notice that the torque due to a force will vanish if the force $\vec{F}$ is parallel to $\vec{r}$.



Sense of rotation for a given direction of torque

Figure 4

We now make a subtle point about the tendency of force to rotate a body. It is that even if the net force applied on a body is zero, the torque generated by them may vanish i.e. the forces will not give any acceleration to the body but would tend to rotate it. For example if we apply equal and opposite forces at two ends of a rod, as shown in figure 5, the net force is zero but the rod still has a tendency to rotate. So in considering equilibrium of bodies, we not only have to make sure that the net force is zero but can also that the net torque is also zero.



The net force on the rod is zero but the torque is not

Figure 5

A third possibility of the action by a force, which we have ignored above, and which is highly explicit in the case of a mass on top of a spring, is that the force also deforms bodies. Thus in the case of a beam under a force, the beam may deform in various ways: it may get compressed, it may get elongated or may bend. A load on top of a spring obviously deforms it by a large amount. In the first case we assume the deformation to be small and therefore negligible i.e., we assume that the internal forces are so strong that they adjust so that there is no deformation by the applied external force. This is known as treating the body as a rigid body. In this course, we are going to assume that all bodies are rigid. So the third kind of action is not considered at all.

So now focus strictly on the equilibrium of rigid bodies: As stated, we are going to assume that internal forces are so great that the body does not deform. The only conditions for equilibrium in them are:

(1) The body should not accelerate/ should not move which, as discussed earlier, is ensured if $\sum_i \vec{F}_i = 0$ that is the sum of all forces acting on it must be zero no matter at what points on the body they are applied. For example consider the beam in figure 2. Let the forces applied by the supports S_1 and S_2 be F_1 and F_2, respectively. Then for equilibrium, it is required that

$$\vec{F}_1 + \vec{F}_2 + \vec{W} + \vec{L} = 0$$

Assuming the direction towards the top of the page to be y-direction, this translates to

$$F_1\hat{j} + F_2\hat{j} - W\hat{j} - L\hat{j} = 0 \text{ or } F_1 + F_2 - W - L = 0$$

The condition is sufficient to make sure that the net force on the rod is zero. But as we learned earlier, and also our everyday experience tells us that even a zero net force can give rise to a turning of the rod. So F_1 and F_2 must be applied at such points that the net torque on the beam is also zero. This is given below as the second rule for equilibrium.

(2) Summation of moment of forces about any point in the body is zero i.e. $\sum_i \vec{\tau}_{iO} = 0$, where $\vec{\tau}_{iO}$ is the torque due to the force $\vec{F}_i$ about point *O*. One may ask at this point whether $\sum_i \vec{\tau}_{iO} = 0$ should be taken about many different points or is it sufficient to take it about any one convenient point. The answer is that any one convenient point is sufficient because if condition (1) above is satisfied, i.e. net force on the body is zero then the torque as is independent of point about which it is taken. We will prove it later.

These two conditions are both necessary and sufficient condition for equilibrium. That is all we need to do to achieve equilibrium so in principle solving for equilibrium is quite easy and what we should learn is how to apply these condition efficiently in different engineering situations. We are therefore going to spend time on these topics individually.

We start with a few simple examples:

Example 1: A person is holding a 100N weight (that is roughly a 10kg mass) by a light weight (negligible mass) rod AB. The rod is 1.5m long and weight is hanging at a distance of 1m from the end A, which is on a table (see figure 6). How much force should the person apply to hold the weight?

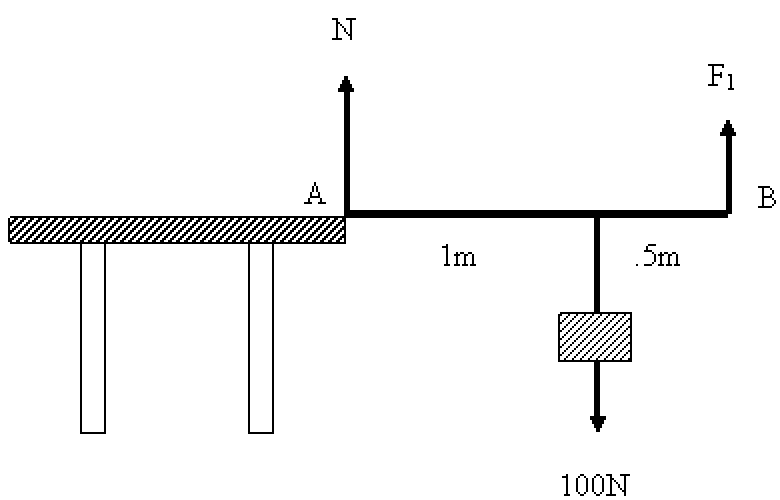


Figure 6

Let the normal reaction of the table on the rod be N and the force by the point be F_1. Then the two equilibrium conditions give

$$\sum \vec{F} = 0 \Rightarrow (F_1 + N - 100)\hat{j} = 0 \Rightarrow F_1 + N = 100 \qquad (1)$$

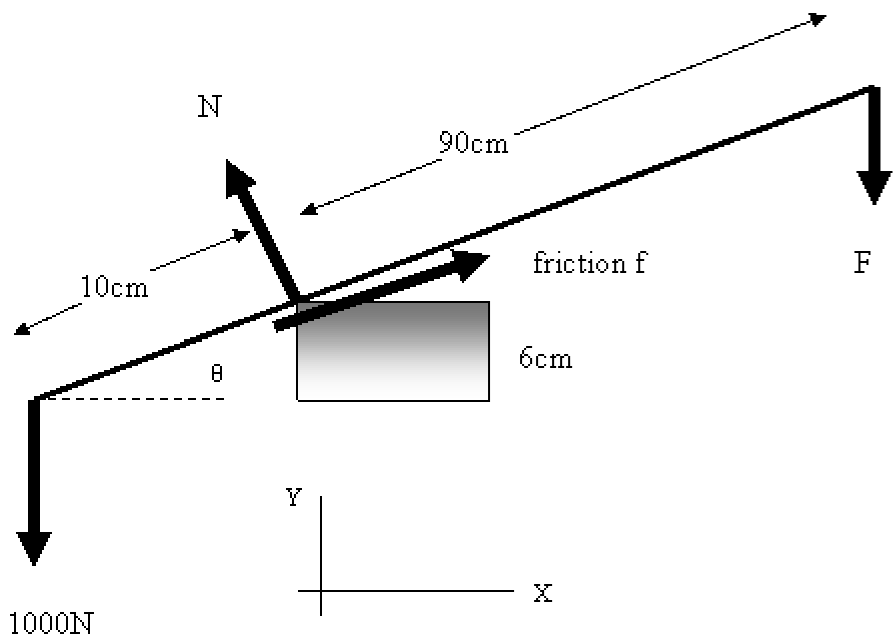
$$\sum \vec{\tau}_{iA} = 0 \Rightarrow \hat{i} \times -100\hat{j} + 1.5\hat{i} \times F_1\hat{j} = 0 \qquad (2)$$

$$-100\hat{k} + F_1 \times 1.5\hat{k} = 0$$

$$\text{or } \; 1.5F_1 = 100 \quad \Rightarrow \quad F_1 = \frac{100}{1.5} = \left(\frac{200}{3}\right)\text{N}$$

$$\text{and } \; N = 100 - F_1 = 100 - \frac{200}{3} = \frac{100}{3}\text{N}$$

Example 2: As the second illustration we take the example of a lever that you may have used sometime or the other. We are trying to lift a 1000N (~100kg mass) weight by putting a light weight but strong rod as shown in the figure using the edge of a brick as the fulcrum. The height of the brick is 6cm. The question we ask is: what is the value of the force applied in the vertical direction that is needed to lift the weight? Assume the brick corner to be rough so that it provides frictional force.



(Note: If the brick did not provide friction, the force applied cannot be only in the vertical direction as that would not be sufficient to cancel the horizontal component of N). Let us see what happens if the brick offered no friction and we applied a force in the vertical direction. The fulcrum applies a force *N* perpendicular to the rod so if we apply only a vertical force, the rod will tend to slip to the left because of the component of N in that direction. Try it out on a smooth corner and see that it does happen. However, if the friction is there then the rod will not slip. Let us apply the equilibrium conditions in such a situation. The balance of forces gives

$$\sum \vec{F} = 0 \Rightarrow F(N\sin\theta - f\cos\theta)\hat{i} + (N\cos\theta - f\sin\theta - 1000 - F)\hat{j} = 0$$
$$or \quad N\sin\theta = f\cos\theta$$
$$N\cos\theta + f\sin\theta - F - 1000 = 0$$

Let us choose the fulcrum as the point about which we balance the torque. It gives

Then

$$\sum \vec{\tau} = 0 \Rightarrow 0.9\hat{r} \times -F\hat{j} + (-0.1)\hat{r} \times -1000\hat{j} = 0$$
$$\Rightarrow (-\cdot 9\cos\theta F + 100\cos\theta)\hat{k} = 0$$
$$\text{or } F = 111\cdot 11\,\text{N}$$

The normal force and the frictional force can now be calculated with the other two equations obtained above by the force balance equation.

In the example above, we have calculated the torques and have also used normal force applied on a surface. We are going to encounter these quantities again and again in solving engineering problems. So let us study each one of them in detail.

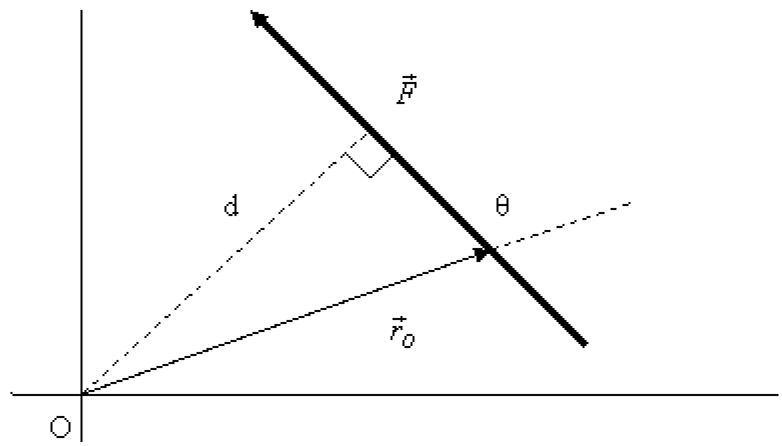
Torque due to a force: As discussed earlier, torque about a point due to a force $\vec{F}$ is obtained as the vector product

$$\vec{\tau}_O = \vec{r}_O \times \vec{F}$$
$$= (yF_z - F_y z)\hat{i} + (zF_x - xF_z)\hat{j} + (xF_y - yF_x)\hat{k}$$

where $\vec{r}_O$ is a vector from the point O to the point where the force is being applied. Actually $\vec{r}_O$ could be a vector from O to any point along the line of action of the force as we will see below. The magnitude of the torque is given as

$$|\vec{\tau}_O| = |\vec{F}||\vec{r}_O|\sin\theta$$

Thus the magnitude of torque is equal to the product of the magnitude of the force and the perpendicular distance $d = |\vec{r}_O|\sin(180^\circ - \theta) = |\vec{r}_O|\sin\theta$ from O to the line of action of the force as shown in figure 7 in the plane containing point O and the force vector. Since this distance is fixed, the torque due to a force can be calculated by taking vector $\vec{r}_O$ to be any vector from O to the line of action of the force. The unit of a torque is Newton-meter or simply Nm.



Torque is equal to the product of the magnitude of the force and its perpendicular distance d from O

Figure 7

Let us look at an example of this in 2 dimensions.

Example 3: Let there be a force of 20 N applied along the vector going from point (1,2) to point (5,3). So the force can be written as its magnitude times the unit vector from (1,2) to (5,3). Thus

$$\vec{F} = \frac{20\left(4\hat{i} + \hat{j}\right)}{\sqrt{17}}$$

Torque can be calculated about *O* by taking $\vec{r}$ to be either $\left(\hat{i} + 2\hat{j}\right)$ *or* $\left(5\hat{i} + 3\hat{j}\right)$. As argued above, the answer should be the same irrespective of which $\vec{r}$ we choose. Let us see that. By taking $\vec{r}$ to be $\left(\hat{i} + 2\hat{j}\right)$ we get

$$\vec{\tau}_O = \frac{\left(\hat{i} + 2\hat{j}\right) \times 20\left(4\hat{i} + \hat{j}\right)}{\sqrt{17}}$$

$$= \frac{20}{\sqrt{17}}(\hat{k} - 8\hat{k}) = -\frac{140\hat{k}}{\sqrt{17}}$$

On the other hand, with $\vec{r} = \left(5\hat{i} + 3\hat{j}\right)$ we get

$$\vec{\tau}_O = \frac{\left(5\hat{i} + 3\hat{j}\right) \times 20\left(4\hat{i} + \hat{j}\right)}{\sqrt{17}}$$
$$= \frac{20}{\sqrt{17}}\left(5\hat{k} - 12\hat{k}\right) = -\frac{140\hat{k}}{\sqrt{17}}$$

Which is the same as that obtained with $\vec{r} = \left(\hat{i} + 2\hat{j}\right)$. Thus we see that the torque is the same no matter where along the line of action is the force applied. This is known as the **transmissibility** of the force. So we again write that

$$\vec{\tau}_O = \vec{r} \times \vec{F}$$

where $\vec{r}$ is any vector from the origin to the line of action of the force.

If there are many forces applied on a body then the total moment about *O* is the vector sum of all other moments i.e.

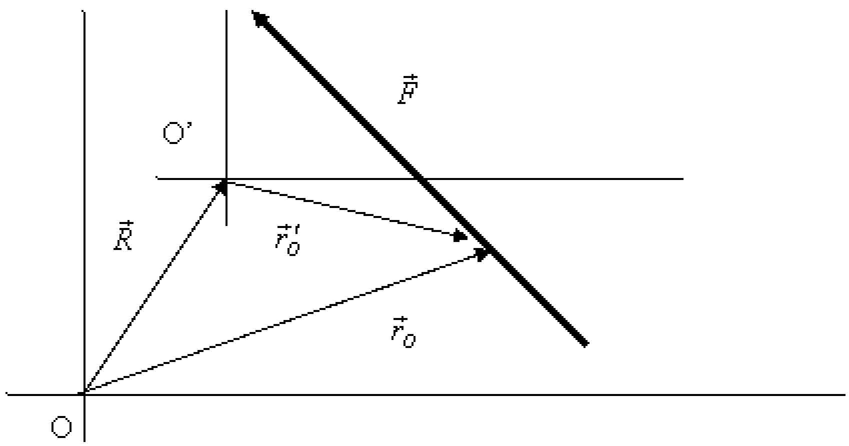
$$\vec{\tau}_O = \sum \vec{r}_{iO} \times \vec{F}_i$$

As a special case if the forces are all applied at the same point *j* then

$$\vec{\tau}_O = \sum \vec{r}_{iO} \times \vec{F}_i = \vec{r}_{jO} \times \sum_i \vec{F}_i$$
$$= \vec{r}_{jO} \times \vec{F}_{net}$$

This is known as **Varignon's theorem**. Its usefulness arises from the fact that the torque due to a given force can be calculated as the sum of torques due to its components.

As would be clear to you from the discussion so far torque depends on the location of point *O* . If for the same applied force, the torque is taken about a different point, the torque would come out to be different. However, as mentioned earlier, there is one special case when the torque is independent of the force applied and that is when the net force(vector sum of all forces) on the system is zero. Let us prove that now: Consider the torque of a force being calculated about two different points *O* and *O'* (figure 8).



Torque about two different points O and O' separated by $\vec{R}$

Figure 8

The torques about *O* and *O'* and their difference is:

$$\vec{\tau}_O = \sum \vec{r}_{iO} \times \vec{F}_i \quad \text{and} \quad \vec{\tau}_O' = \sum \vec{r}_{iO}' \times \vec{F}_i$$

$$\Rightarrow \vec{\tau}_O' - \vec{\tau}_O = \sum \left(\vec{r}_{iO}' - \vec{r}_{iO}\right) \times \vec{F}_i$$

But from the figure above

$$\vec{r}_{iO}' - \vec{r}_{iO} = \vec{R}$$

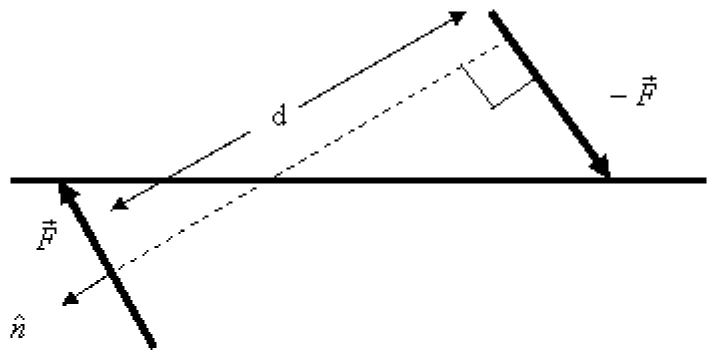
Therefore

$$\vec{\tau}_O' - \vec{\tau}_O = \sum \vec{R} \times \vec{F}_i = \vec{R} \times \sum \vec{F}_i$$

Now if the net force is zero, $\sum \vec{F}_i$ is zero and the difference between the torques about two different points also vanishes. A particular example of the net force being zero is two equal magnitude forces in directions opposite to each other and applied at a distance from one another, as in figure 5 above and also shown in figure 9 below. This is known as a couple and the corresponding torque with respect to any point is given as

$$\vec{\tau}_{couple} = (\hat{n} \times \vec{F})d.$$

where $\hat{n}$ is a unit vector perpendicular to the forces coming out of the space between them and *d* is the perpendicular distance between the forces (see figure 9).



A couple

Figure 9

Since the net force due to a couple is zero, the only action a couple has on a body is to tend to rotate it. Further the moment of a couple is independent of the origin, and so it can be applied anywhere on the body and it will have the same effect on the body. We can even change the magnitude of the force and alter the distance between them keeping the magnitude of the couple the same. Then also the effect of couple will be the same. Such vectors whose effect remains unchanged irrespective of where they are applied are known as free vectors. Free vectors have a nice property that they can be added irrespective of where they are applied without changing the effect they produce. Thus a couple is a free vector (Is force a free vector?). It is represented by the symbols

Representing a couple

Figure 10

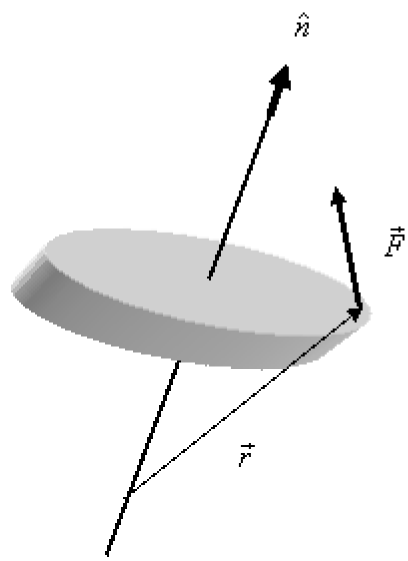
with the arrows clearly giving the sense of rotation. Keep in mind though that the direction of the couple (in the vector sense) is perpendicular to the plane in which the forces forming the couple are.

Next we focus on the moment of a force about an axis.

Moment of force about an axis: So far we have talked about moment of a force about a point only. However, many a times a body rotates about an axis. This is the situations you have bean studying in you 12th grade. For example a disc rotating about an axis fixed in two fixed ball bearings. In this case what affects the rotation is the component of the torque along the axis, where the torque is taken about a point *O* (the point can be chosen arbitrarily) on the axis as given in figure 11. Thus

$$\vec{\tau}_{about axis} = \hat{n} \cdot (\vec{r} \times \vec{F})$$

where $\hat{n}$ is the unit vector along the axis direction and $\vec{r}$ is the vector from point *O* on the axis to the force $\vec{F}$.



Disc experiencing a torque about an axis

Figure 11

Using vector identities (exercise at the end of Lecture 1), it can also be written as

$$\vec{\tau}_{about axis} = (\hat{n} \times \vec{r}) \cdot \vec{F}$$

Thus the moment of a force about an axis is the magnitude of the component of the force in the plane perpendicular to the axis times its perpendicular distance from the axis. Thus if a force is pointing towards the axis, the torque generated by this force about the axis would be zero. This can be understood as follows. When a force is applied, forces are generated at the ends of the axis being held on a one place. These forces together with $\vec{F}$ generate the torque when components along the axis by responsible for rotation of the body about the axis, in the same manner, the couple about the axis is given by the component of the couple moment in the direction for the axis. You can work it out; it is actually equal to the component of the force in the plane perpendicular to the axis times the distance $(\perp)$ of the force line of action from the axis. One point about the moment about an axis, it is independent of the origin since it depends only on the distance $\perp$ of the force the axis.

As an example let us consider a disc of radius 30 cm with its axis along the z-axis and its centre at z=0. Let a force $\vec{F} = (30\hat{i} + 20\hat{j} - 10\overleftarrow{k})N$ act on it at the point $(10\hat{i} + 10\hat{j})$ on the disc. We now find its moment about its axis. The axis has $\hat{n} = \hat{k}$. We take the origin at the centre of the disc to calculate

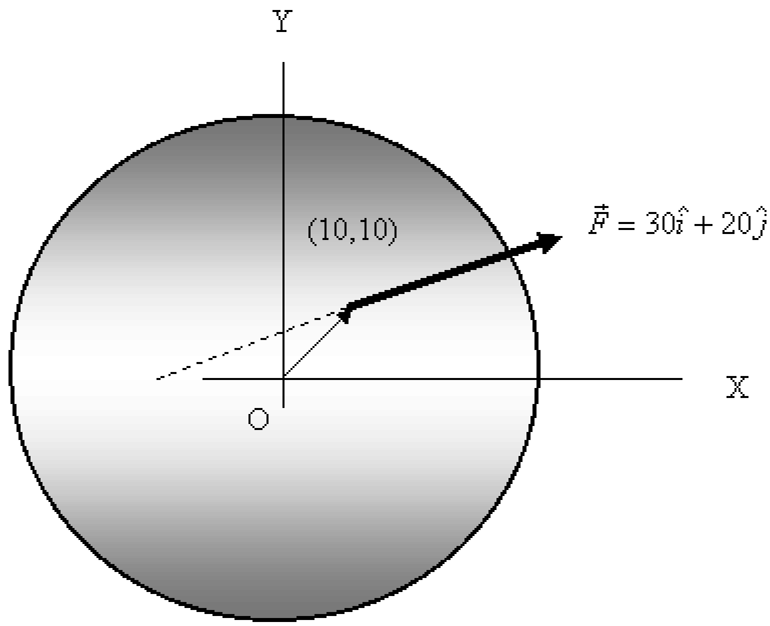
$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F} \\ &= \left(10\hat{i} + 10\hat{j}\right) \times \left(30\hat{i} + 20\hat{j} - 10\hat{k}\right) \\ &= 200\hat{k} + 100\hat{j} - 300\hat{k} - 100\hat{i} \\ &= -100\hat{i} + 100\hat{j} - 100\hat{k}\end{aligned}$$

Therefore the torque about the z-axis is

$$\vec{\tau} \cdot \hat{k} = -100Nm$$

Thus the torque about the axis is in the negative z direction which means that it would tend to rotate the disc clockwise.

Let us now see if it fits with our conventional way of calculating torque of a force about an axis. For the force $\vec{F} = 30\hat{i} + 20\hat{j} - 10\hat{k}$ the z-component of the force will not give any torque about the z-axis because it cannot rotate the body about the z-axis. So the only component of the force that gives torque about the z-axis is $\left(30\hat{i} + 20\hat{j}\right)$ that acts on the point as shown in figure 12. The magnitude of this force is $10\sqrt{13}$.



Force $\vec{F} = 30\hat{i} + 20\hat{j}$ acting at point $10\hat{i} + 10\hat{j}$ on a disc

Figure 12

The equation of line along which the force acts is

$$(y-10) = \frac{2}{3}(x-10)$$
$$\text{or } 3y = 2x + 10$$

To find the perpendicular distance of this line from the origin, we consider a line perpendicular to this line $\left(slope = -\frac{3}{2}\right)$ passing through the origin and consider the point where it intersects with $3y = 2x + 10$. The perpendicular line is

$$y = -\frac{3}{2}x \text{ or } 2y = -3x$$

Solving for the intersection point we get

$$x = -\frac{20}{13} \text{ and } y = \frac{30}{13}$$

which gives the perpendicular distance of the line of force from the centre to be

$$d = \frac{10}{\sqrt{13}}$$

Then torque about z-axis therefore is therefore $\tau = |F|.d = 10\sqrt{13} \times \frac{10}{\sqrt{13}} = 100$ clockwise, which is the same as obtained that earlier. I would like you to notice that even in this simple example using vector algebra makes life quite easy.

Let us summarize this lecture by summarizing what we have learnt:

(1) For equilibrium of a body

$$\sum \vec{F} = 0 \text{ and}$$
$$\sum \vec{\tau}_o = 0$$

are necessary and sufficient conditions.

(2) The torque about a point due to forces applied on a body, $\vec{\tau}_o = \sum_i \vec{r}_i \times \vec{F}_i$ is an origin dependent quantity but for special case of $\sum \vec{F}_i = 0$ it is origin independent.

(3) A particular case of $\sum_i \vec{F}_i = 0$ is a couple moment when two forces are equal & opposite and are separated by distance d . The couple moment is $|F|d$.

(4) Torque about an axis is given by it component along the axis. Thus y and axis $\hat{n}$ is along direction.

(5) $\tau_{aboutaxis} = \hat{n} \cdot \vec{\tau}$

Equilibrium of bodies II

In the previous lecture we have defined a couple moment. With this definition, we can now represent a force $\vec{F}$ applied on a body pivoted at a point as the sum of the same force on it at the pivot and a couple acting on it. This is shown in figure 1. Thus if the bar shown in figure 1 is in equilibrium, the pivot must be applying a force $-\vec{F}$ and a counter couple moment on it.

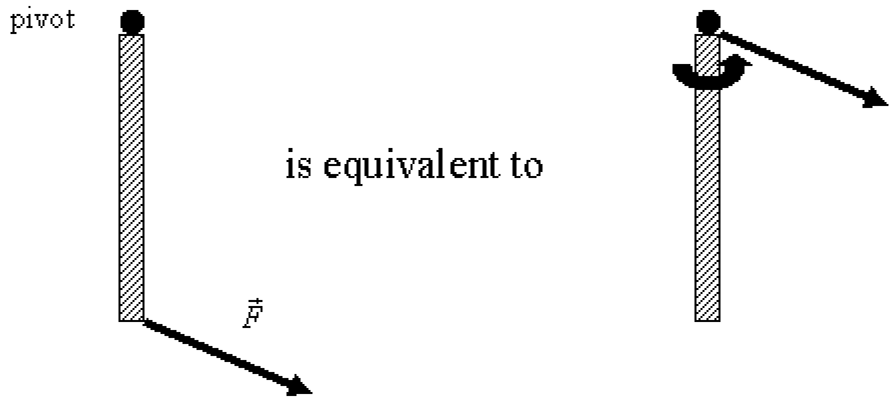


Figure 1

To see the equivalence, let us take the example above and add a zero force to the system at the pivot point. This does not really change the force applied on the system. However, the trick is to take this zero force to be made up of forces $\vec{F}$ and $-\vec{F}$ as shown in figure 2. Now the original force $\vec{F}$ and $-\vec{F}$ at the pivot are separated by distance *d* and therefore form a couple moment of magnitude *Fd*. In addition there is a force $\vec{F}$ on the body at the pivot point. The combination is therefore a force $\vec{F}$ at the pivot point P and a couple moment $\tau = Fd$. Notice that I am not saying τ about the pivot. This is because a couple is a free vector and its effect is the same no matter at which point it is specified.

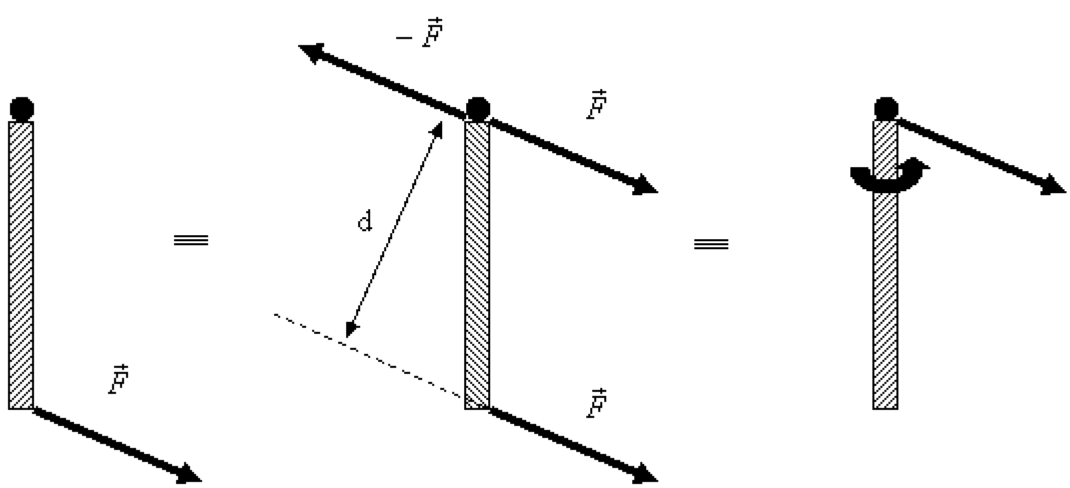


Figure 2

Example: You must have seen the gear shift handle in old buses. It is of Zigzag shape. Let it be of the shape shown in figure 3: 60cm at an angle of 45° from the x-axis, 30cm parallel to x-axis and then 30cm again at 45° from the x-axis, all in the x-y plane shown in figure 3. To change gear a driver applies a force of $\vec{F} = (-5\hat{i} + 5\hat{j} - 2\hat{k})\mathrm{N}$ on the head of the handle. We want to know what is the equivalent force and moment at the bottom i.e., at the origin of the handle.

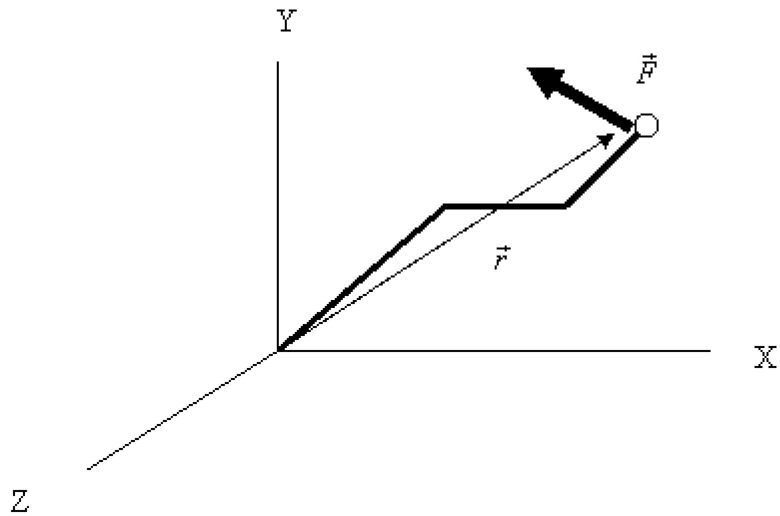


Figure 3

For this again we can apply a zero force i.e., ($\vec{F}$ and $-\vec{F}$) at the bottom so that original force and $-\vec{F}$ give a couple moment

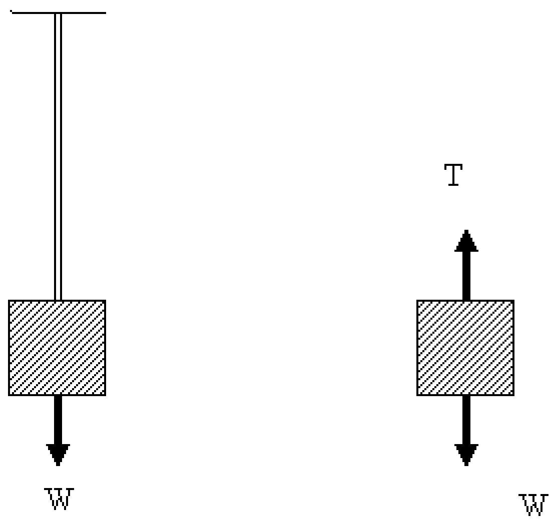
$$
\begin{aligned}
\tau &= \vec{r} \times \vec{F} \\
&= \left(\frac{60}{\sqrt{2}}\hat{i} + \frac{60}{\sqrt{2}}\hat{j} + 30\hat{i} + \frac{30}{\sqrt{2}}\hat{i} + \frac{30}{\sqrt{2}}\hat{j} \right) \times (-5\hat{i} + 5\hat{j} - 2\hat{k}) \\
&= \left(\frac{90 + 30\sqrt{2}}{\sqrt{2}}\hat{i} + \frac{90}{\sqrt{2}}\hat{j} \right) \times (-5\hat{i} + 5\hat{j} - 2\hat{k}) \\
&= \frac{450 + 150\sqrt{2}}{\sqrt{2}}\hat{k} + \frac{180 + 60\sqrt{2}}{\sqrt{2}}\hat{j} + \frac{450}{\sqrt{2}}\hat{k} - 90\sqrt{2}\hat{i} \\
&= -90\sqrt{2}\hat{i} + (90\sqrt{2} + 60)\hat{j} + 600\sqrt{2}\hat{k}
\end{aligned}
$$

Thus equivalent force system is a force $\vec{F} = (-5\hat{i} + 5\hat{j} - 2\hat{k})\mathrm{N}$ at the bottom and a couple equal to $-90\sqrt{2}\hat{i} + (90\sqrt{2} + 60)\hat{j} + 600\sqrt{2}\hat{k}$ Nm.

Having obtained equivalent force systems, next we wish to discuss what kind of forces and moments do different elements used in engineering mechanics apply on other elements.

Forces and couples generated by various elements: As we solve engineering problems, we come across many different elements that are used in engineering structures. We discuss some of them below focusing our attention on what kind of forces and torques do they give rise to.

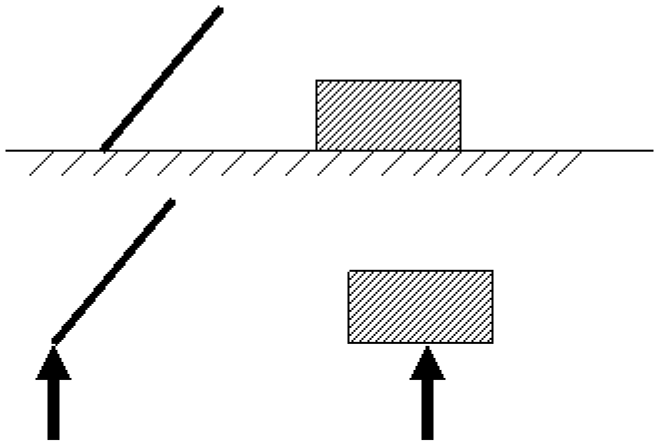
The simplest element is a string that can apply a tension. However a string can only pull by the tension generated in it but not push. For example, a string holding a weight W will develop a tension $T = W$ in it so that the net force on the weight is a tension T pulling the weight up and weight W pulling it down. Thus if the weight is in equilibrium, $T = W$. This is shown in figure 4.



A weight being held in equilibrium by the tension in the string holding it

Figure 4

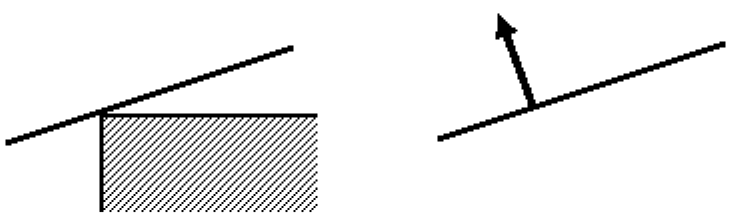
The second kind of force that is applied when two elements come in contact is that applied by a surface. A smooth surface always applies a force normal to itself. The forces on a rod and on a box applied by the surface are shown in figure 5. Thus as far as the equilibrium is concerned, for an object on a smooth surface, the surface is equivalent to a force normal to it.



A smooth surface applies a force normal to it on the objects kept on it

Figure 5

Imagine what would have happened had the force by the surface not been normal. Then an object put on a surface would start moving along the surface because of the component of the force along the surface. By the same argument if there is a smooth surface near an edge, the force on the surface due to the edge (and by Newton's III[rd] law the force on the edge due to the surface) will be normal to the surface. See figure 6.

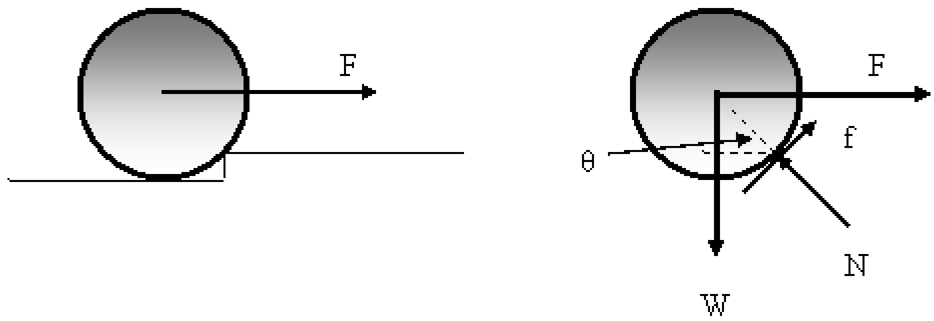


Force applied by an edge on a smooth surface

Figure 6

On the other hand if the surface is not smooth, it is then capable of applying a force along the surface also. This force is due to friction.

Let us now solve the well known example of a roller of radius *r* being pulled over a step as shown in figure 7. The height of the step is *h*. What is minimum force F required if the roller is pulled in the direction shown and is about to roll over the step. What are the normal and frictional forces at that instant?



A roller being pulled over a step and various forces on it when it is about to roll over

Figure 7

When the roller is about to roll over the step, there will be no normal reaction from the lower surface and therefore the roller will be under equilibrium under the influence of its weight *W*, the applied force *F* and the normal reaction *N* and the frictional force *f* applied by the edge of the step. To calculate the force *F*, we apply the torque equation about the edge to get

$$F(r-h) = W\sqrt{2rh-h^2}$$
$$\text{or } F = W\frac{\sqrt{2rh-h^2}}{(r-h)}$$

To find *N* and *f* we apply the force equation

$$\sum \vec{F} = 0$$

That can be written in the component form a

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

Let us look at these equations.

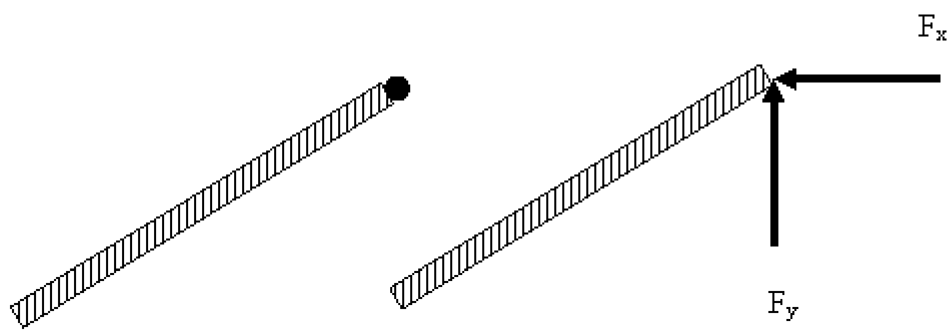
$$\sum F_x = 0 \text{ gives}$$
$$-N\cos\theta + f\sin\theta + F = 0$$
and
$$\sum F_y = 0 \text{ gives}$$
$$N\sin\theta + f\cos\theta - W = 0$$
$$\text{with} \quad \cos\theta = \frac{\sqrt{2rh - h^2}}{r}$$
$$\text{and} \quad \sin\theta = \frac{(r-h)}{r}$$

Solving these equations gives

$$f = 0 \text{ and } N = \frac{Wr}{(r-h)}$$

So in this situation, we do not require friction to keep the roller in equilibrium. On the other hand recall the problem in the previous lecture when we were trying to lift a 1000Nt weights by putting a rod on a brick edge. In that case we did require friction.

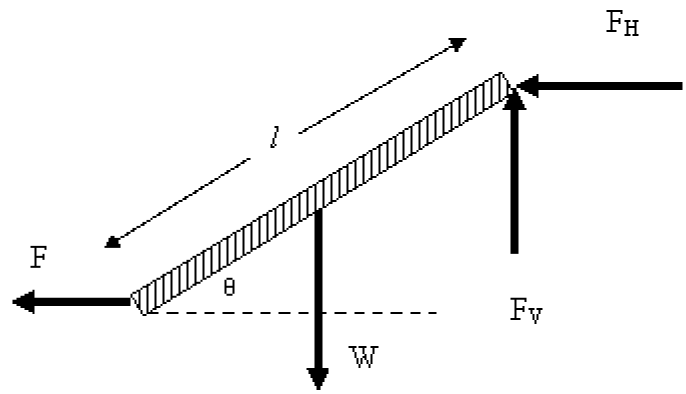
Next we consider a hinge about when an object can rotate freely. A hinge can apply a force in any direction. Thus it can apply (figure 8a) any force in X-direction and any amount of force in Y-direction but no couple.



A hinge joint and the forces applied by it

Figure 8a

To see an example, imagine lifting a train berth by pulling it horizontally. We wish to know at what angle θ from the horizontal will the berth come to equilibrium if we pull it out by a horizontal force *F* and what are the forces apply by the hinges (figure 8b).



A train berth of weight W being pulled by a horizontal force F

Figure 8b

Let the weight of the berth be *W* and its width *l*. Let the forces applied by the hinges be F_H in the horizontal direction and F_V in the vertical direction. By equilibrium conditions

$$\sum \vec{F} = 0$$

$$\sum F_X = 0 \Rightarrow -F_H - F = 0$$
$$\text{or} \quad F_H = -F$$

where the negative sign for F_H implies that it is in the direction opposite to that assumed.

Similarly

$$\sum F_y = 0 \Rightarrow F_V - W = 0 \;\text{ or }\; F_V = W$$

To find the angle we apply the moment or torque balance equation about the hinges. Weight W gives a counter clockwise torque of $W\frac{l}{2}\cos\theta$ and the force F gives a clockwise torque of *Fl*sin?

$$\sum \tau = 0 \Rightarrow W\frac{l}{2}\cos\theta - Fl\sin\ \theta = 0$$

$$\text{or} \qquad \tan\theta = \frac{W}{2F}$$

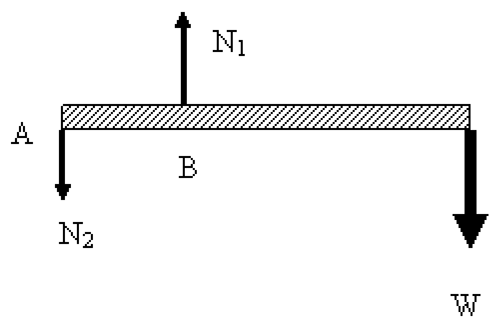
I should point put that if the hinge is not freely moving (for example due to friction) then it can produce a moment (couple) that will oppose any tendency to rotate and will have to be taken into account while considering the torque balance equation.

Next we look at a built in or fixed support as shown in the figure.

Fixed support

Figure 9

Let us analyze what happens in these cases when a load is applied. Let us look at the built-in support.



Reaction forces generated on a fixed support when a load W is applied at one end

Figure 10

As the load is put on, the beam will tend to move down on the right side pushing the inner side up. This will generate reaction forces as shown schematically in figure 10. The generated forces can be replaced by a couple and a net force either about point A or B as follows (see figure 11). Add zero force N_1 -N_1at point A then the original $N_1$and -N_1give a couple and no force and there is a net force (N_1 -N_2) at A.

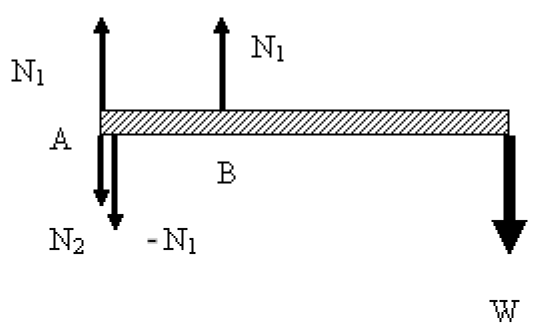
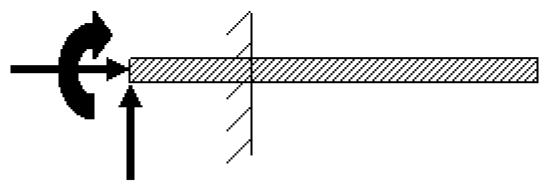


Figure 11

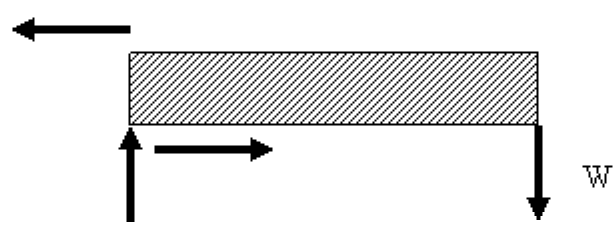
We could instead have added a zero force N_2 - N_2at B and then would have obtained an equivalent system with a different couple moment than the previous case and a net force (N_1 - N_2) at B. I leave this for you to see. You may be wondering by now at which exactly does the force really act and what is the value of the couple. Actually in the present case the two unequal forces act on the beam so the torque provided by them is not independent of the point about which it is take. In such cases, as we will learn in the later lectures, the force effectively acts at the centroid of the force and the couple moment is equal to the torque evaluated about the centroid. In any case we can say that a built-in support provides a couple and a force. We give the schematic picture above only to motivate how the forces and the couple are generated. In reality the forces are going to be distributed over the entire portion of the support that is inside the wall and it is this distribution of force that provides a net force at the centroid and a couple equal to the torque calculated about the centroid, as we will see in later lectures. Note that deeper the support is fixed into a wall, larger would be the couple provided by it. Hence whereas to hang a light photo-frame or a painting on a wall a small nail would suffice, a longer nail would be better if the frame is heavy. In addition to providing a force perpendicular to the support and a couple, a fixed support also provides a force in the direction parallel to itself. Thus if you try to pull out the support or try to push it in, it does not move easily. The forces and couple provided by a fixed support are therefore as shown in figure 12.



Forces and couple provided by a fixed support

Figure 12

Let us now look at the support welded/ glued to the well. In that case suppose we put a load *W* at the end of the beam, you will see that the forces generated will be as shown below in figure 13.



Reaction forces generated at a glued support

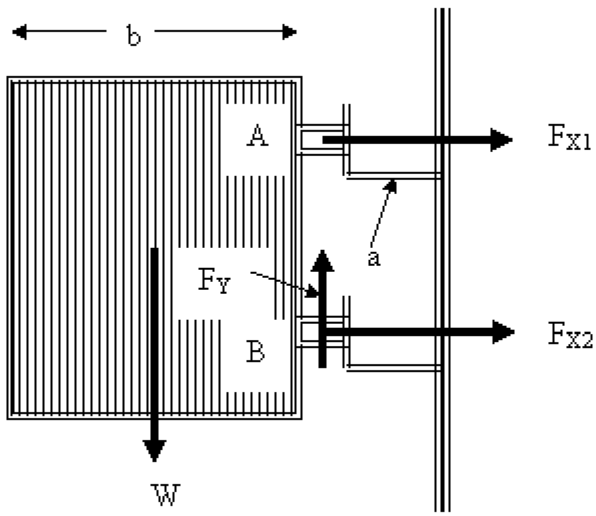
Figure 13

where in this particular case the horizontal forces must be equal so as to satisfy

$$\sum F_x = 0$$

Thus the horizontal forces provide a couple and the beam can be said to provide a couple a force in the direction perpendicular to the support. Further a glued support also cannot be pulled out or pushed in. Therefore it too is capable of providing a horizontal nonzero reaction force. Thus a welded or glued support can also be represented as shown in figure 12. Note that wider the support, larger moment it is capable of providing. Let us now solve an example of this.

Example: You must have seen gates being supported on two supports (see figure 14). Suppose the weight of the gate is *W* and its width *b*. The supports are protruding out of the wall by *a* and the distance between them is *h*. If the weight of the gate is supported fully by the lower support, find the horizontal forces, vertical forces and the moment load on both the supports.



A gate supported on two fixed supports

Figure 14

To solve this problem, let us first find out what are the forces required to keep the gate in balance. The forces applied by the supports on the gate are shown in figure 14. Since the weight of the gate is fully supported on the lower support all the vertical force is going to be provided by the lower support only. Thus

$$\sum F_y = 0 \Rightarrow F_Y - W = 0 \text{ which means } F_Y = W$$

Similarly

$$\sum F_x = 0 \Rightarrow F_{X1} + F_{X2} = 0 \text{ or } F_{X2} = -F_{X1}$$

To find, F_{X1} or F_{X2}, let us take moment about point A or B

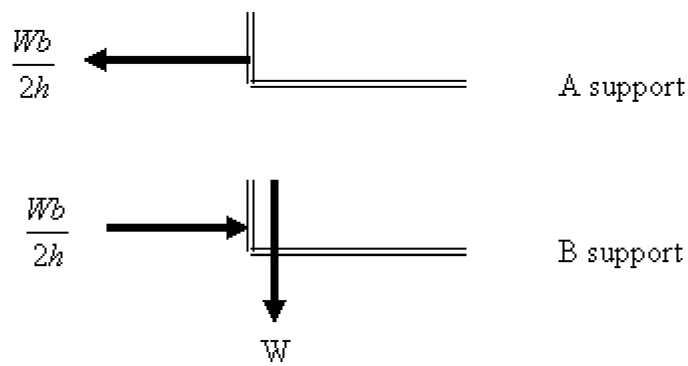
Let us make $\sum \tau_B = 0$. This gives (following the convention that counterclockwise torque is positive and clockwise torque is negative)

$$-F_{X1}h + W\frac{b}{2} = 0$$

$$\text{or } F_{X1} = \left(\frac{Wb}{2h}\right)$$

$$\text{and } F_{X2} = -\left(\frac{Wb}{2h}\right)$$

The negative sign for F_{X2}means that the force's direction is opposite to what it was taken to be in figure 14. We also wish to find the forces and couple on the support. By Newton 'sIII[rd] Law, forces on the support are opposite to those on the gate. Thus the forces on the two supports are:



Forces on the two supports

Figure 15

You see that support A is being pulled out whereas support B is being pushed in (we observe an effect of this at our houses all the time: the upper hinges holding a door tend to come out of the doorjamb). Now the force by the wall on support A will be $\left(\frac{Wb}{2h}\right)$- to the right to keep it fixed in its place. On the other hand the situation for the lower support is more involved. The lower support will be kept in its place by the wall providing it horizontal and vertical forces and a torque. The net horizontal force is $\left(\frac{Wb}{2h}\right)$- to the left and the net vertical force is W pointing up. The lower support also balances a torque. Taking the torques about the point where it enters the wall, its value comes out to be

$$\tau = Wa$$

If we assume that the net vertical force and the torque is provided by only two reaction forces at two points as in figure 10, these two reaction forces can be calculated easily if we know the length of the portion inside the wall. I leave it as an exercise. In solving this, you will notice that the reaction forces are smaller if the support is deeper inside the wall. As pointed out earlier, in reality the force is going to be distributed over the entire portion of the support inside the wall. So a more realistic calculation is a little more involved.

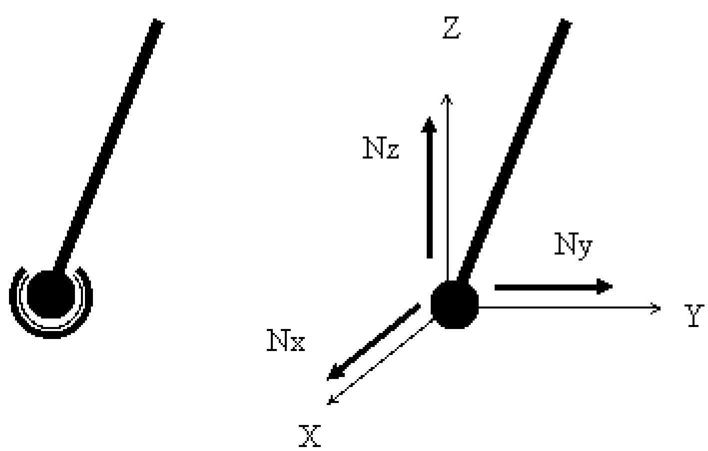
To summarize this lecture, we have looked at some simple engineering elements and have outlined what kind of forces and torques are they capable of applying. In the next lecture we are going discuss forces in three dimensions. We are also going to look at conditions that forces with certain geometric relations should satisfy for providing equilibrium.

Equilibrium of bodies III

In the previous lecture I have been talking about equilibrium in a plane. We now move on to three dimensional (*3-d*) cases. In three dimensional cases the equilibrium conditions lead to balance along all three axes. Then

$$\sum \vec{F} = 0 \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases} \quad \text{and} \quad \sum \vec{\tau} = 0 \Rightarrow \begin{cases} \sum \tau_x = 0 \\ \sum \tau_y = 0 \\ \sum \tau_z = 0 \end{cases}$$

We now have to take care of components of forces and torque in all three dimensions. The engineering elements that we considered earlier are now considered as *3-d* case. Thus consider a ball-socket joint in which a ball is supported in a socket (figure 1).

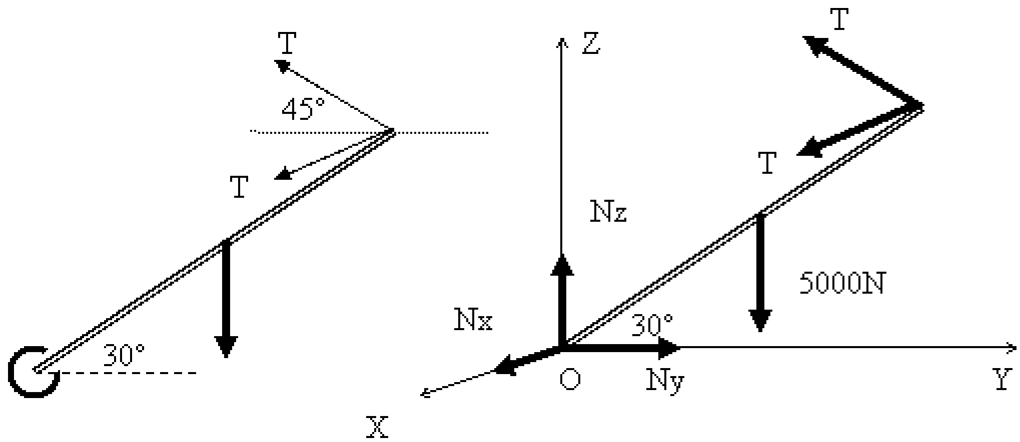


A ball-socket joint (left) is capable of applying forces in all three directions (right).

Figure 1

A ball-socket joint provides reaction forces *Nx*, *Ny*and *Nz*in all three directions (figure 1) but it cannot apply any torque. This is a little like a hinge joint in *2d* . Let me solve an example using such a joint.

Example 1: To balance a heavy weight of 5000 *N*, two persons dig a hole in the ground and put a pole of length *l* in it so that the hole acts as a socket. The pole makes an angle of 30° from the ground. The weight is tied at the mid point of the pole and the pole is pulled by two horizontal ropes tied at its ends as shown in figure 2. Find the tension in the two ropes and the reaction forces of the ground on the pole.



A pole balancing a weight on it (left). Forces acting on it are shown on the right.

Figure 2

To solve this problem, let me first choose a co-ordinate system. I choose it so that the pole is over the y-axis in the *(y-z)* plane (see figure 2).

The ropes are in *(xy)* direction with tension *T* in each one of them so that tension in each is written as

$$\left(\frac{T}{\sqrt{2}}\hat{i} - \frac{T}{\sqrt{2}}\hat{j}\right) \quad \text{and} \quad \left(-\frac{T}{\sqrt{2}}\hat{i} - \frac{T}{\sqrt{2}}\hat{j}\right)$$

You may be wondering why I have taken the tension to be the same in the two ropes. Actually it arises from the torque balance equation; if the tensions were not equal, their component in the x-direction will give a nonzero torque.

Let the normal reaction of the ground be (*Nx* ,*Ny* , *Nz*). Then the force balance equation gives

$$\sum F_x = 0 \Rightarrow N_x + \frac{T}{\sqrt{2}} - \frac{T}{\sqrt{2}} = 0 \Rightarrow N_x = 0$$
$$\sum F_y = 0 \Rightarrow N_y - \frac{2T}{\sqrt{2}} = 0 \Rightarrow N_y = T\sqrt{2}$$
$$\sum F_z = 0 \Rightarrow N_z - 5000 = 0 \Rightarrow N_z = 5000N$$

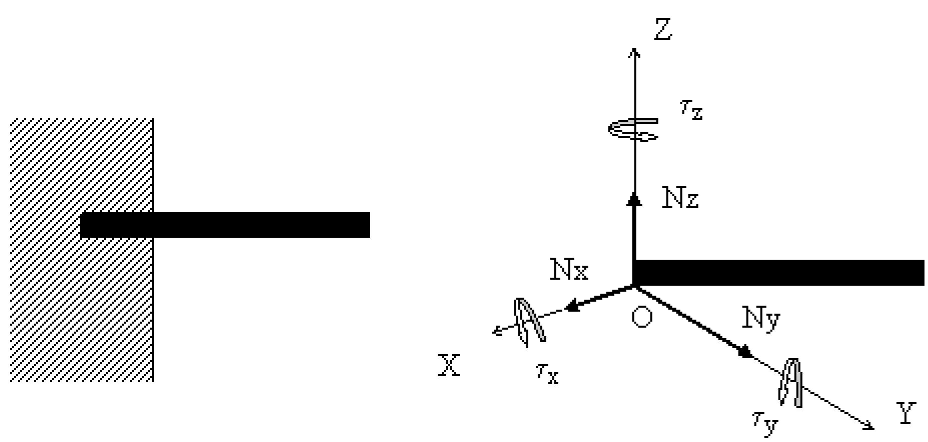
Taking torque about point O and equating it to zero, we get

$$\left(l\cos 30^o\,\hat{j} + l\sin 30^o\,\hat{k}\right)\times\left(-T\sqrt{2}\,\hat{j}\right) + \left(\frac{l}{2}\cos 30^o\,\hat{j} + \frac{l}{2}\sin 30^o\,\hat{k}\right)\times\left(-5000\hat{k}\right) = 0$$
$$Tl\sqrt{2}\frac{1}{2}\hat{i} - 2500\cdot l\cdot\frac{\sqrt{3}}{2}\hat{i} = 0 \quad\Rightarrow\quad T\sqrt{2} = 2500\sqrt{3}$$

which gives

$$\boxed{T = 3062N. \qquad N_y = 4330N}$$

Next, if I consider a fixed connection, say in a wall, it is capable of providing force along all the three axes and also of providing torques about the three axes, Thus in *3-d* it will be represented as shown in figure 3.



A fixed joint (left) is capable of producing reaction forces and reaction torques along all three axes (right).

Figure 3

This is a generalization of the fixed or welded/glued support in *2-d* . How are these torques etc. generated? Recall what I did for a fixed support in *2-d* and carry out a similar analysis in *3-d.*

Hopefully by the analysis carried out so far, you would be able to recognize what all reactions a given element of a mechanical system can provide. For example look at the support shown in figure 4 where the shaft can not move through the hole in the fixed block, but it is free to rotate. Can you tell the reaction forces and torques that this support provides?

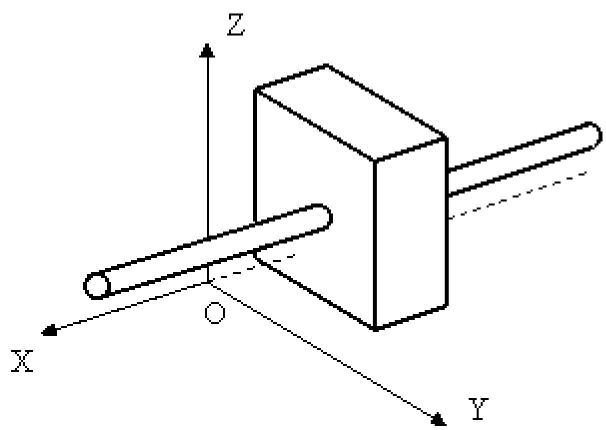


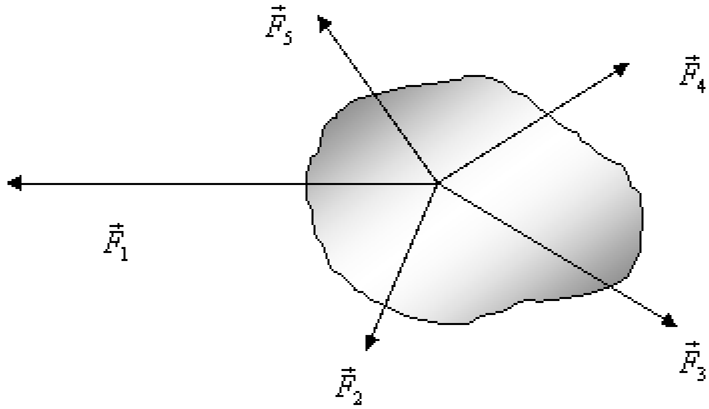
Figure 4

Having discussed the elements that apply different kinds of forces, let us look at some situations there due to the geometry of forces applied, some of the equilibrium equations are automatically satisfied. If we recognize this, it saves us from doing extra calculations involving that particular condition.

If all forces are **concurrent at a point** (see figure 5), i.e., they all cross each other at one point O then torques of all the forces is identically zero about O . Thus the only equilibrium condition is

$$\sum \vec{F} = 0 \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$

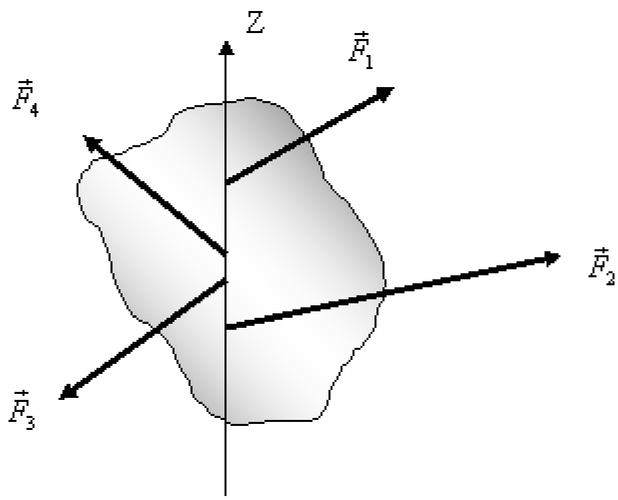
Recall that if the sum of all forces on a system is zero, torque is independent of the origin. Thus although in the beginning I used the fact that torque about the point of concurrence is zero, it is true about any point once the force equation for equilibrium is satisfied.



Forces concurrent at a point

Figure 5

Next consider the case when all **forces intersect one particular line**, call it the z-axis without any loss of generality (see figure 6).



Forces intersecting one line

Figure 6

Using transmissibility of the force, in this case we can take the force $\vec{F}_i \; (i = 1, 2, \cdots)$ to be acting at point $z_i\hat{k}$. Then the torque due to all these forces will be

$$\vec{\tau} = \sum_i Z_i\hat{k} \times \left(F_{ix}\hat{i} + F_{iy}\hat{j} + F_{iz}\hat{k}\right)$$
$$= \sum_i \left(Z_i F_{ix}\hat{j} - Z_i F_{iy}\hat{i}\right)$$

Thus the *Z* component of the torque is automatically zero. In general when the forces intersect a line, the torque component along that line vanishes. Under these circumstances, if we take that line to be the z-axis, the equilibrium conditions are

$$\sum \vec{F} = 0 \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases} \quad \text{and} \quad \sum \vec{\tau} = 0 \Rightarrow \begin{cases} \sum \tau_x = 0 \\ \sum \tau_y = 0 \end{cases}$$

Next I discuss what happens if all the applied forces are parallel, say to the *Z* axis. Then the forces do not have any *x or y* components. Further, by the z-component of the torque also vanishes (left as an exercise for you to show). The equilibrium conditions in this case reduce to

$$\sum \vec{F} = 0 \Rightarrow \sum F_z = 0 \quad \text{and} \quad \sum \vec{\tau} = 0 \Rightarrow \begin{cases} \sum \tau_x = 0 \\ \sum \tau_y = 0 \end{cases}$$

In general of course we have all the six condition.

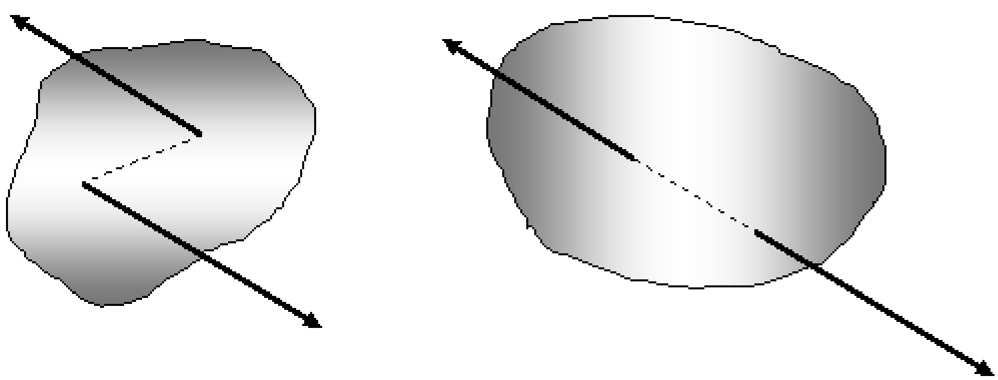
$$\sum \vec{F} = 0 \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases} \quad \text{and} \quad \sum \vec{\tau} = 0 \Rightarrow \begin{cases} \sum \tau_x = 0 \\ \sum \tau_y = 0 \\ \sum \tau_z = 0 \end{cases}$$

Let me now summarize what all you have learnt so far in considering the equilibrium of engineering structures. In the process I also introduce you to a term called the **Free Body Diagram**. I have actually been using it without calling it so. Now, let us formalize it.

In talking about the equilibrium of a body we consider all the external forces applied on it and the interaction of the body with other objects around it. This interaction produces more forces and torques on the body. Thus when we single out a body in equilibrium, objects like hinges, ball-socket joint, fixed supports around it are replaced elements by the corresponding forces & torques that they generate. This is what is called a free-body diagram. Making a free-body diagram allows us to focus our attention only on the information relevant to the equilibrium of the body, leaving out unnecessary details. Thus making a free-body diagram is pretty much like Arjuna - when asked to take an aim on the eye of a bird - seeing only the eye and nothing else. The diagrams made on the right side of figures 1, 2 and 3 are all free-body diagrams.

In the coming lecture we will be applying the techniques learnt so far to a very special structure called the truss. To prepare you for that, in the following I consider the special case of a system in equilibrium under only two forces. For completeness I will also take up equilibrium under three forces.

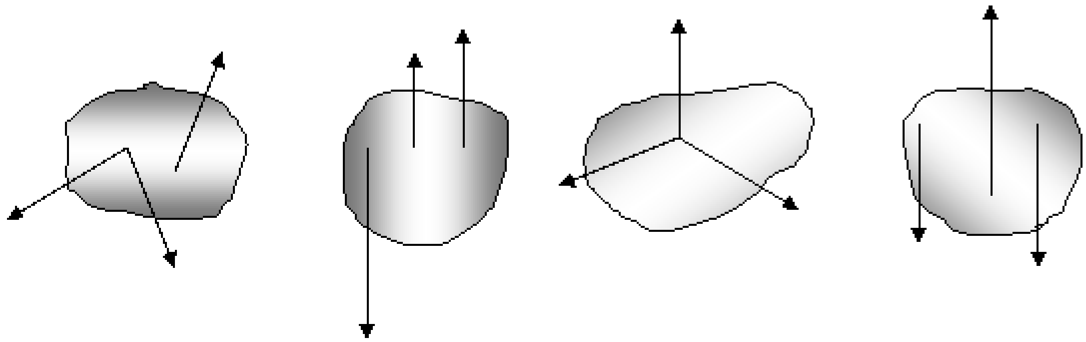
When only two forces are applied, no matter what the shape or the size of the object in equilibrium is, the forces must act along the same line, in directions opposite to each other, and their magnitudes must be the same. That the forces act in directions opposite to each other and have equal magnitude follows from the equilibrium conditions $\vec{F}_1 + \vec{F}_2 = 0$, which implies that $\vec{F}_2 = -\vec{F}_1$. Further, if the forces are not along the same line then they will form a couple that will tend to rotate the body. Thus $\vec{\tau} = 0$ implies that the forces act along the same line, i.e. they be collinear (see figure 7).



Two bodies being applied two equal and opposite forces. The body on the left is not in equilibrium whereas that on the right is.

Figure 7

Similarly if there are three forces acting on a body that is in equilibrium then the three forces must be in the same plane and concurrent. If there are not concurrent then they must be parallel (of course remaining in the same plane). This can be understood as follows. Any two members of the three applied forces form a plane. If the third force is not in the same plane, it will have a non-vanishing component perpendicular to the plane; and that component does not get cancelled. Thus unless all three forces are in the same plane, they cannot add up to zero. So to satisfy the equation $\sum \vec{F} = 0$, the forces must be in the same plane, i.e. they must be coplanar. For equilibrium the torque about any point must also be zero. Since the forces are in the same plane, any two of them will intersect at a certain point *O*. These two forces will also have zero moment about *O*. If the third force does not pass through *O*, it will give a non-vanishing torque (see figure 8). So to satisfy the torque equation, the forces have to be concurrent. Zero torque condition can also be satisfied if the three forces are parallel forces (see figure 8); that is the other possibility for equilibrium under three forces.

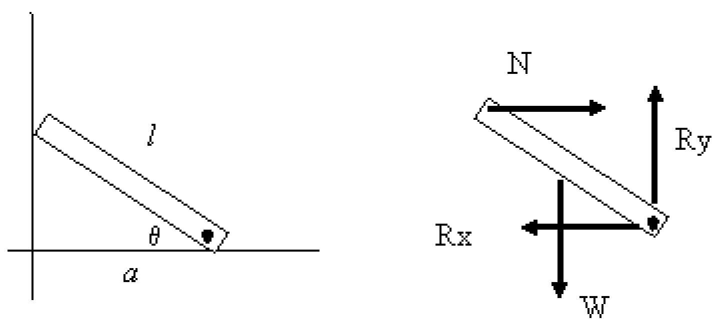


Four bodies being applied three coplanar forces. Two bodies on the left are not in equilibrium whereas the two on the right are.

Figure 8

In the end, I now discuss one more concept about equilibrium of bodies, that of **staticaldeterminacy** . Along the way I also introduce some connected concepts like constraints, degree of redundancy and redundant support. On constraints, I will discuss more in the lecture on Method of virtual work.

To introduce the terms used above, I consider a rod of length *l* and weight *W* held at a pin-joint on a floor at a distance of *a* from a wall, on which its other end is. This is shown in figure 9 along with the free-body diagram of the rod.



A rod of length l and weight W held at a pin-joint (left). Its free body diagram is drawn on the right.

Figure 9

There are three unknowns - Rx, Ry and N - in the problem and three equations of equilibrium that will determine the unknowns. Specifically:

$$R_y - W = 0 \Rightarrow R_y = W$$
$$R_x - N = 0 \Rightarrow R_x = N$$

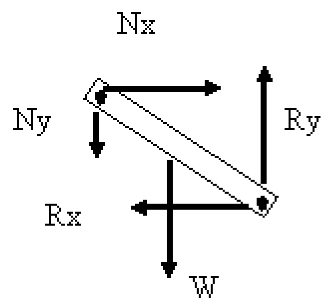
Taking moment about the pin, we get

$$N\sqrt{l^2 - a^2} = W \cdot \frac{a}{2} \quad \Rightarrow \quad N = \frac{Wa}{2\sqrt{l^2 - a^2}}$$

This gives

$$\boxed{R_x = N = \frac{Wa}{2\sqrt{l^2 - a^2}} \quad \text{and} \quad R_y = W}$$

In this case, the constraints or the external supports we apply are just sufficient to keep the system in equilibrium. Such systems are known as **statically determinate** systems. Now suppose we apply one more support. Let us support the rod at both ends by pin joints. The free-body diagram will then look like that shown in figure 10.



Free body diagram of the rod when it is supported by pin joints at both its ends.

Figure 10

Now the pin on top end is also applying a force on the rod. Thus the equations of equilibrium read as

$$-R_x + N_x = 0 \qquad R_y - N_y = W$$

$$-N_x\sqrt{l^2 - a^2} + N_y a + \frac{Wa}{2} = 0$$

The situation on hand is that we have four unknowns - Rx, Ry, Nx and Ny - and only three equations. Thus one of the unknown cannot be determined. In particular only $R_y - N_y = W$ is known and what are individual R_y and N_y cannot be determined unless some additional information is also given. Such systems are known as **statically indeterminate** systems. In such systems we are applying more constraints than are needed to keep the system in equilibrium. Even if we remove one of the constraints - in this case replace the upper pin by a plane surface - the system is capable of remaining in equilibrium. Such supports that can be removed without disturbing the equilibrium are known as redundant supports. And the number of redundant supports is the **degree of staticalindeterminacy** .

After introducing you to the concepts discussed above, we will be studying trusses in the next lecture.